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EXISTENCE OF POSITIVE SOLUTIONS FOR A CLASS OF QUASILINEAR SINGULAR ELLIPTIC SYSTEMS INVOLVING CAFFARELLI-KOHN-NIRENBERG EXPONENT WITH SIGN-CHANGING WEIGHT FUNCTIONS

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The paper deals with the existence of weak positive solutions for a new class of quasilinear singular elliptic systems involving critical Caffarelli–Kohn–Nirenberg exponent with sign-changing weight functions using the method of sub-super solutions. Our results are natural extensions from the previous ones in [3].

Key words : Caffarelli–Kohn–Nirenberg exponents; elliptic system, sub-supersolution method, sign-changing.

1. Introduction

During the past few years, the treatise of positive solutions of singular partial differential equations or systems has been an extremely active research area. The singular nonlinear problems emerge naturally and they take a main role in the interdisciplinary field between analysis, biology, geometry, mathematical physics, elasticity, etc.

This article deals with the existence of positive solutions of the following boundary value problem

$$\begin{cases} -\mathrm{div}\left(|x|^{-ap}\,|\nabla u|^{p-2}\,\nabla u\right) = \lambda\,|x|^{-(a+1)p+c_1}\left[g(x)\,A(u)+f(v)\right], \text{in } \Omega, \\ -\mathrm{div}\left(|x|^{-bq}\,|\nabla v|^{q-2}\,\nabla v\right) = \lambda\,|x|^{-(b+1)q+c_2}\left[g(x)\,B(v)+h(u)\right], \text{in } \Omega, \\ \\ u=v=0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^{\mathbb{N}}$ and

$$1 < p,\ q < N,\ 0 \leq a \leq \frac{N-p}{N},$$

with

$$0 \leq b \leq \frac{N-q}{N}, c_1, c_2 \text{ are positive parameters.}$$

$g(x)$ is a C^1 sign-changing the weight function, that possibly negative nearby the boundary and $f,\ h, A, B$ are C^1 nondecreasing functions satisfy

$$A(0) \geq 0, B(0) \geq 0.$$

The study of this kind of problems is motivated by its different applications, for example, population genetics, in fluid mechanics, Newtonian fluids, glaciology and flow through porous media (see for more detail [4, 9, 16, 21]).

On the other hand, there is an extensive practical background for quasilinear elliptic systems have. They are described in the multiplicative chemical reaction stimulated by the catalyst grains under variant temperature or constant, in the quasi-regular and quasi-conformal theory mappings in Riemannian manifolds with boundary, or in the description of many physical phenomena such as the pulses propagation in Kerr-like photorefractive media and birefringent optical fibers (see [19, 30]). Moreover, for additional results on elliptic problems, see ([1, 3, 6, 20, 24]). For the regular case, the quasilinear elliptic equation has been intensively studied by many authors where $c_1 = p$, $c_2 = q$ and $a = b = 0$, (see for example [2]). In the current work we concentrate on further extending the study in [5] for the quasilinear elliptic systems involving singularity. The extensions are nontrivial and challenging due to the singularity in the weights. Our approach is based on the method of sub and super solutions.

2. Technical Assumptions and Auxiliary Results

Let Ω be a bounded domain in $\mathbb{R}^N$ with its smooth boundary, $0 \in \Omega$ and $W_0^{1,p}\left(\Omega, |x|^{-ap}\right)$ denote the completion of $C_0^\infty(\Omega)$ with the norm

$$\|u\| = \left(\int_\Omega |x|^{-ap} |\nabla u|^p \, dx\right)^{\frac{1}{p}}.$$

We consider the following nonlinear eigenvalue problem

$$\begin{cases} -\text{div}\left(|x|^{-sr} |\nabla \phi|^{r-2} \nabla \phi\right) = \lambda |x|^{-(s+1)r+t} |\phi|^{r-2} \phi, x \in \Omega, \\ \phi = 0,\ x \in \partial\Omega. \end{cases} \tag{2.1}$$

For $t = c_1$ and $r = p, s = a,$ we assume $\phi_{1,p}$ the eigenfunction corresponding to the first eigenvalue $\lambda_{1,p}$ of problem (2.1) where

$$\phi_{1,p} > 0 \text{ in } \Omega \text{ and } \|\phi_{1,p}\|_\infty = 1.$$

We consider the following assumptions similar to that in [26]:

For $s = b$, $r = q$, and $t = c_2$, we assume $\phi_{1,q}$ the eigenfunction corresponding to the first eigenvalue $\lambda_{1,q}$ of problem (2.1) where

$$\phi_{1,q} > 0 \text{ in } \Omega \text{ and } \|\phi_{1,q}\|_\infty = 1.$$

The maximum principle gives that

$$\frac{\partial \phi_r}{\partial n} < 0 \text{ on } \partial\Omega \text{ for } r \in \{p, q\},$$

where n is the outward normal. Then, there are positive constants m_0, δ and $\sigma_p, \sigma_q \in (0,1)$ such that

$$\begin{cases} \lambda_{1,r} |x|^{-(s+1)r+t} \phi_{1,r}^r - |x|^{-sr} |\nabla \phi_{1,r}|^r \leq -m_0, x \in \overline{\Omega}_\delta. \\ \phi_{1,r} \geq \sigma_r, x \in \Omega \backslash \overline{\Omega}_\delta, \end{cases} \tag{2.2}$$

with

$$r \in \{p, q\}, s \in \{a, b\}, t \in \{c_1, c_2\} \text{ and } \overline{\Omega}_\delta = \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}.$$

We also assume the unique solution $(\zeta_p(x), \zeta_q(x)) \in W_0^{1,p}(\Omega, |x|^{-ap}) \times W_0^{1,q}\left(\Omega, |x|^{-bq}\right)$ of the following quasilinear singular system

$$\begin{cases} -\text{div}\left(|x|^{-ap} |\zeta_p|^{p-2} \nabla \zeta_p\right) = |x|^{-(a+1)p+c_1}, x \in \Omega, \\ -\text{div}\left(|x|^{-bq} |\zeta_q|^{q-2} \nabla \zeta_q\right) = |x|^{-(b+1)q+c_2}, x \in \Omega, \\ \zeta_p = \zeta_q = 0, \ x \in \ \partial\Omega, \end{cases} \tag{2.3}$$

where

$$\|\zeta_p\|_\infty = \mu_p \text{ and } \|\zeta_q\|_\infty = \mu_q.$$

Then, according [26], we have $\zeta_r > 0$ in and $\frac{\partial \phi_r}{\partial n} < 0$ on $\partial\Omega$ for $r \in \{p, q\}$.

Throughout this work, we consider that the weight function $g(x)$ hold negative values in $\overline{\Omega}_{\delta}$. However, it requires to be strictly positive in $\Omega\backslash\overline{\Omega}_{\delta}$. Precisely, we assume that there exist a positive constants β and η satisfy

$$g(x) \geq -\beta \text{ on } \overline{\Omega}_{\delta} \text{ and } g(x) \geq \eta \text{ on } \Omega\backslash\overline{\Omega}_{\delta}. \tag{2.4}$$

Let $s_0 \geq 0$ such that

$$\eta A(s) + f(s) > 0, \eta B(s) + h(s) > 0 \text{ for } s > s_0 \tag{2.5}$$

and

$$f_0 = \max\{0, -f(0)\}, h_0 = \max\{0, -h(0)\}. \tag{2.6}$$

3. Main Result

Putting

$$X = W_0^{1,p}\left(\Omega, |x|^{-ap}\right) \times W_0^{1,q}\left(\Omega, |x|^{-bq}\right).$$

We give the following definition of weak solution and sub-super solution of the problem (1.1):

Definition 1 — A pair of nonnegative functions $(\psi_1, \psi_2), (z_1, z_2)$ in X are called a weak subsolution and supersolution of (1.1) if they satisfy: for $(\psi_1, \psi_2) = (z_1, z_2) = (0, 0)$ on $\partial\Omega$

$$\int_{\Omega} |x|^{-ap} |\nabla\psi_1|^{p-2} \nabla\psi_1 \nabla\omega_1 dx \leq \lambda \int_{\Omega} \lambda |x|^{-(a+1)p+c_1} \left[g(x) A(\psi_1) + f(\psi_2)\right] \omega_1 dx,$$

$$\int_{\Omega} |x|^{-bq} |\nabla\psi_2|^{q-2} \nabla\psi_2 \nabla\omega_2 dx \leq \lambda \int_{\Omega} |x|^{-(b+1)q+c_2} \left[g(x) B(\psi_2) + h(\psi_1)\right] \omega_2 dx$$

and

$$\int_{\Omega} |x|^{-ap} |\nabla z_1|^{p-2} \nabla z_1 \nabla\omega_1 dx \geq \lambda \int_{\Omega} |x|^{-(a+1)p+c_1} \left[g(x) A(z_1) + f(z_2)\right] \omega_1 dx,$$

$$\int_{\Omega} |x|^{-bq} |\nabla z_2|^{q-2} \nabla z_2 \nabla\omega_2 dx \geq \lambda \int_{\Omega} |x|^{-(b+1)q+c_2} \left[g(x) B(z_2) + h(z_1)\right] \omega_2 dx,$$

for all test functions

$$\omega_1(x) \in W_0^{1,p}\left(\Omega, |x|^{-ap}\right)$$

and

$$\omega_2(x) \in W_0^{1,p}\left(\Omega, |x|^{-bq}\right),$$

with $\omega_1, \omega_2 \geq 0$.Then the following result holds:

Lemma 1 — (ref. [20]). Suppose there exist sub and super-solutions (ψ_1, ψ_2) and (z_1, z_2) respectively of (1.1) such that $(\psi_1, \psi_2) \leq (z_1, z_2)$. Then (1.1) has a weak solution (u, v) such that

$$(u, v) \in [(\psi_1, \psi_2), (z_1, z_2)] .$$

In order to give the main result of this paper, we consider the following assumptions:

$(A1)$ we have for every constant $K > 0$:

$$\lim_{s \to +\infty} \frac{f\left(K\left(h(s)^{\frac{1}{q-1}}\right)\right)}{s^{p-1}} = 0.$$

$(A2)$

$$\lim_{s \to +\infty} f(s) = \lim_{s \to +\infty} h(s) = +\infty.$$

$(A3)$

$$\lim_{s \to +\infty} \frac{A(s)}{s^{p-1}} = \lim_{s \to +\infty} \frac{B(s)}{s^{p-1}} = 0.$$

$(A4)$ If $\alpha_p = \frac{p-1}{p}\sigma^{\frac{p}{p-1}}$, $\alpha_q = \frac{q-1}{q}\sigma^{\frac{q}{q-1}}$, and $\overline{\alpha} = \min\{\alpha_p, \alpha_q\}$ then there exists $\gamma > \frac{s_0}{\overline{\alpha}}$ such that

$$\max\left\{\frac{\gamma\lambda_{1,p}}{\eta A\left(\gamma^{\frac{1}{p-1}}\alpha_p\right) + f\left(\gamma^{\frac{1}{q-1}}\alpha_q\right)}, \frac{\gamma\lambda_{1,q}}{\eta B\left(\gamma^{\frac{1}{q-1}}\alpha_q\right) + h\left(\gamma^{\frac{1}{p-1}}\alpha_p\right)}\right\}$$
$$< \min\left\{\frac{m_0\gamma}{\beta A\left(\gamma^{\frac{1}{p-1}}\right) + f_0}, \frac{m_0\gamma}{\beta B\left(\gamma^{\frac{1}{q-1}}\right) + h_0}\right\}.$$

We recall that m_0, σ_p and σ_q are introduced in relation (2.2) while s_0 is defined in (2.5). We now state our main result for the problem (1.1).

***Theorem* 1** — *Suppose that* $(A1) - (A4)$ *hold, then for every* $\lambda \in [A, B]$, *system (1.1) has at least one positive weak solution.*

PROOF : Choose $r > 0$ such that $r \leq \min\left\{|x|^{-(a+1)p+c_1}, |x|^{-(b+1)q+c_2}\right\}$ in $\overline{\Omega}_\delta$.

We take $\gamma > \frac{s_0}{\overline{\alpha}}$ as in hypothesis $(A4)$. Define

$$A = \max\left\{\frac{\gamma\lambda_{1,p}}{\eta A\left(\gamma^{\frac{1}{p-1}}\alpha_p\right) + f\left(\gamma^{\frac{1}{q-1}}\alpha_q\right)}, \frac{\gamma\lambda_{1,q}}{\eta B\left(\gamma^{\frac{1}{q-1}}\alpha_q\right) + h\left(\gamma^{\frac{1}{p-1}}\alpha_p\right)}\right\}$$

and

$$B=\min\left\{\frac{m_0\gamma}{\beta A\left(\gamma^{\frac{1}{p-1}}\right)+f_0},\frac{m_0\gamma}{\beta B\left(\gamma^{\frac{1}{q-1}}\right)+h_0}\right\}.$$

Setting

$$\psi_1=(\gamma r)^{\frac{1}{p-1}}\frac{p-1}{p}\phi_{1,p}^{\frac{p}{p-1}} \text{ and } \psi_2=(\gamma r)^{\frac{1}{q-1}}\frac{q-1}{q}\phi_{1,q}^{\frac{q}{q-1}}.$$

We will check that (ψ_1,ψ_2) is a sub-solution of (1.1) for $\lambda\in[A,B]$.

Indeed, let ω_1 with $\omega_1\geq 0$ in. Then, it can be shown that

$$\begin{cases}\int\limits_{\Omega}|x|^{-ap}|\nabla\psi_1|^{p-2}\nabla\psi_1\nabla\omega_1dx=\gamma r\int\limits_{\Omega}|x|^{-ap}\phi_{1,p}|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}.\nabla\omega_1dx\\ =\gamma r\left\{\int\limits_{\Omega}|x|^{-ap}|\nabla\phi_{1,p}|^{p-2}\nabla\phi_{1,p}.\left[\nabla\left(\phi_{1,p}\omega_1\right)-\nabla\phi_{1,p}\omega_1\right]dx\right\}\\ =\gamma r\int\limits_{\Omega}\left[\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p-|x|^{-ap}|\nabla\phi_{1,p}|^p\right]\omega_1dx.\end{cases}\tag{3.1}$$

Similarly we get □

$$\int\limits_{\Omega}|x|^{-bq}|\nabla\psi_2|^{q-2}\nabla\psi_2\nabla\omega_2dx=\gamma r\int\limits_{\Omega}\left[\lambda_{1,q}|x|^{-(b+1)q+c_2}\phi_{1,q}^q-|x|^{-bq}|\nabla\phi_{1,q}|^q\right]\omega_2dx.\tag{3.2}$$

Now, on $\overline{\Omega}_\delta$ by relation (2.2) we have

$$\gamma\left(\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p-|x|^{-ap}|\nabla\phi_{1,p}|^p\right)\leq-m_0\gamma.$$

Since $\lambda\leq B$ then we have

$$\lambda\leq\frac{m_0\gamma}{\beta A\left(\gamma^{\frac{1}{p-1}}\right)+f_0},$$

thus

$$\begin{cases}\gamma\left(\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p-|x|^{-ap}|\nabla\phi_{1,p}|^p\right)\leq-m_0\gamma\\ \leq\lambda\left(-\beta A\left(\gamma^{\frac{1}{p-1}}\right)-f_0\right)\\ \leq\lambda\left(g(x)A\left(\gamma^{\frac{1}{p-1}}\frac{p-1}{p}\phi_{1,p}^{\frac{p}{p-1}}\right)+f\left(\gamma^{\frac{1}{q-1}}\frac{q-1}{q}\phi_{1,q}^{\frac{q}{q-1}}\right)\right),\end{cases}$$

then

$$\gamma r\left(\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p-|x|^{-ap}|\nabla\phi_{1,p}|^p\right)\leq\lambda|x|^{-(a+1)p+c_1}\left[g(x)A(\psi_1)+f(\psi_2)\right].$$

By using (3.1)

$$\int_{\overline{\Omega}_\delta} |x|^{-ap} |\nabla\psi_1|^{p-2} \nabla\psi_1 \nabla\omega_1 dx \le \lambda \int_{\overline{\Omega}_\delta} |x|^{-(a+1)p+c_1} \left[g(x) A(\psi_1) + f(\psi_2)\right] \omega_1 dx. \tag{3.3}$$

Similarly, we shows that

$$\int_{\overline{\Omega}_\delta} |x|^{-bq} |\nabla\psi_2|^{q-2} \nabla\psi_2 \nabla\omega_2 dx \le \lambda \int_{\overline{\Omega}_\delta} |x|^{-(b+1)q+c_2} \left[g(x) B(\psi_2) + h(\psi_1)\right] \omega_2 dx. \tag{3.4}$$

Next, on $\Omega\backslash\overline{\Omega}_\delta$. Since $\lambda \ge A$, then

$$\lambda \ge \frac{\gamma\lambda_{1,p}}{\eta A\left(\gamma^{\frac{1}{p-1}}\alpha_p\right) + f\left(\gamma^{\frac{1}{q-1}}\alpha_q\right)},$$

thus, we have

$$\begin{aligned} \gamma\left(\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p - |x|^{-ap}|\nabla\phi_{1,p}|^p\right) &\le \gamma\lambda_{1,p} \\ &\le \lambda\left[\eta A\left(\gamma^{\frac{1}{p-1}}\alpha_p\right) + f\left(\gamma^{\frac{1}{q-1}}\alpha_q\right)\right] \end{aligned}$$

and

$$\gamma r\left(\lambda_{1,p}|x|^{-(a+1)p+c_1}\phi_{1,p}^p - |x|^{-ap}|\nabla\phi_{1,p}|^p\right) \le \lambda|x|^{-(a+1)p+c_1}\left[g(x) A(\psi_1) + f(\psi_2)\right], \tag{3.5}$$

so by (3.1)

$$\int_{\Omega\backslash\overline{\Omega}_\delta} |x|^{-ap} |\nabla\psi_1|^{p-2} \nabla\psi_1 \nabla\omega_1 dx \le \lambda \int_{\Omega\backslash\overline{\Omega}_\delta} |x|^{-(a+1)p+c_1} \left[g(x) A(\psi_1) + f(\psi_2)\right] \omega_1 dx. \tag{3.6}$$

Similarly, we shows that for all $\lambda \in [A, B]$

$$\int_{\Omega\backslash\overline{\Omega}_\delta} |x|^{-bq} |\nabla\psi_2|^{q-2} \nabla\psi_2 \nabla\omega_2 dx \le \lambda \int_{\Omega\backslash\overline{\Omega}_\delta} |x|^{-(b+1)q+c_2} \left[g(x) B(\psi_2) + h(\psi_1)\right] \omega_2 dx \tag{3.7}$$

(3.3) and (3.6) give:

$$\int_{\Omega} |x|^{-ap} |\nabla\psi_1|^{p-2} \nabla\psi_1 \nabla\omega_1 dx \le \lambda \int_{\Omega} |x|^{-(a+1)p+c_1} \left[g(x) A(\psi_1) + f(\psi_2)\right] \omega_1 dx. \tag{3.8}$$

Similarly, we shows that

$$\int_{\Omega} |x|^{-bq} |\nabla\psi_2|^{q-2} \nabla\psi_2 \nabla\omega_2 dx \le \lambda \int_{\Omega} |x|^{-(b+1)q+c_2} \left[g(x) B(\psi_2) + h(\psi_1)\right] \omega_2 dx. \tag{3.9}$$

From (3.8) and (3.9), we deduce that (ψ_1,ψ_2) is a sub-solution of (1.1). However, we have $\psi_1>0$ and $\psi_2>0$ in Ω.

Next, we introduce a supersolution of problem (1.1). For this intent, we can prove that there exists a large enough positive constant C so that

$$(z_1,z_2)=\left(\frac{C}{\mu_p}\lambda^{\frac{1}{p-1}}\zeta_p,\left[2h\left(C\lambda^{\frac{1}{q-1}}\right)\right]^{\frac{1}{q-1}}\lambda^{\frac{1}{q-1}}\zeta_q\right).$$

Let $\omega_1\in W_0^{1,p}\left(\Omega,|x|^{-ap}\right)$ with $\omega_1\geq 0$.

For sufficient C large

$$\frac{\mu_p^{p-1}\left[\|g\|_\infty A\left(C\lambda^{\frac{1}{p-1}}\right)+f\left(2h\left(C\lambda^{\frac{1}{q-1}}\right)\right)^{\frac{1}{q-1}}\lambda^{\frac{1}{q-1}}\mu_q\right]}{C^{p-1}}\leq 1,$$

then

$$\int_\Omega |x|^{-ap}\left|\nabla z_1\right|^{p-2}\nabla z_1\nabla\omega_1dx=\lambda\left(\frac{C}{\mu_p}\right)^{p-1}\int_\Omega |x|^{-ap}\left|\nabla\zeta_p\right|^{p-2}\nabla\zeta_p\nabla\omega_1dx$$

$$=\lambda\left(\frac{C}{\mu_p}\right)^{p-1}\int_\Omega |x|^{-(a+1)p+c_1}\omega_1dx$$

$$\geq\lambda\int_\Omega |x|^{-(a+1)p+c_1}\left[\|g\|_\infty A\left(C\lambda^{\frac{1}{p-1}}\right)+f\left(2h\left(C\lambda^{\frac{1}{q-1}}\right)\right)^{\frac{1}{q-1}}\lambda^{\frac{1}{q-1}}\mu_q\right]\omega_1dx \tag{3.10}$$

$$\geq\lambda\int_\Omega |x|^{-(a+1)p+c_1}\left[\|g\|_\infty A\left(C\lambda^{\frac{1}{p-1}}\frac{\zeta_p}{\mu_p}\right)+f\left(2h\left(C\lambda^{\frac{1}{q-1}}\right)\right)^{\frac{1}{q-1}}\lambda^{\frac{1}{q-1}}\zeta_q\right]\omega_1dx$$

$$=\lambda\int_\Omega |x|^{-(a+1)p+c_1}\left[g(x)A(z_1)+f(z_2)\right]\omega_1dx.$$

Similarly, we choose C large so that

$$\frac{\|g\|_\infty\left(B\left(2h\left(C\lambda^{\frac{1}{p-1}}\right)\right)^{\frac{1}{q-1}}\lambda^{\frac{1}{q-1}}\mu_q\right)}{h\left(C\lambda^{\frac{1}{p-1}}\right)}\leq 1.$$

Let $\omega_1 \in W_0^{1,p}\left(\Omega, |x|^{-ap}\right)$ with $\omega_1 \geq 0$, then

$$\begin{aligned}
&\int_\Omega |x|^{-bq} \left|\nabla z_2\right|^{q-2} \nabla z_2 \nabla \omega_2 dx = 2\lambda h\left(C\lambda^{\frac{1}{p-1}}\right) \int_\Omega |x|^{-bq} \left|\nabla \zeta_q\right|^{p-2} \nabla \zeta_q \nabla \omega_2 dx \\
&= 2\lambda h\left(C\lambda^{\frac{1}{p-1}}\right) \int_\Omega |x|^{-(b+1)q+c_2} \omega_2 dx \\
&\geq \lambda \int_\Omega |x|^{-(b+1)q+c_2} \|g\|_\infty \left(B\left(2h\left(C\lambda^{\frac{1}{p-1}}\right)\right)^{\frac{1}{q-1}} \lambda^{\frac{1}{q-1}} \mu_q\right) \omega_2 dx \\
&= \lambda \int_\Omega |x|^{-(b+1)q+c_2} \left[g(x) B(z_2) + h(z_1)\right] \omega_2 dx.
\end{aligned} \tag{3.11}$$

From (3.10) and (3.11) yield that (z_1, z_2) is a super-solution of problem (1.1) with $\psi_1 \leq z_1$ and $\psi_2 \leq z_2$ for $C > 0$ large.

Hence by Lemma (1.1), there exist a positive solution (u, v) of (1.1), where

$$(\psi_1, \psi_2) \leq (u, v) \leq (z_1, z_2).$$

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