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EXISTENCE AND CONCENTRATION OF SOLUTIONS FOR SUBLINEAR SCHRÖDINGER-POISSON EQUATIONS

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We concern the sublinear Schrödinger-Poisson equations

$$\begin{cases} -\triangle u + \lambda V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\triangle \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases}$$

where $\lambda > 0$ is a parameter, $V \in C(\mathbb{R}^3, [0, +\infty))$, $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ and $V^{-1}(0)$ has nonempty interior. We establish the existence of solution and explore the concentration of solutions on the set $V^{-1}(0)$ as $\lambda \to \infty$ as well. Our results improve and extend some related works.

Key words : Schrödinger-Poisson problem; Sublinear; concentration of solutions.

1. Introduction and Main Results

In this paper, we are concerned with sublinear Schrödinger-Poisson system

$$\begin{cases} -\triangle u + \lambda V(x)u + \phi u = f(x,u) & \text{in } \mathbb{R}^3, \\ -\triangle \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\lambda > 0$, $V \in C(\mathbb{R}^3, [0, +\infty))$ and $f \in C(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$. Problem (1.1) arises in applications from mathematical physics. For more mathematical and physical interpretation, we refer to [1, 2, 3, 11, 30] and the references therein.

In recent years, there has been increasing attention to systems like (1.1) in the superlinear case, see [5-8, 28, 29] and [4, 12-16] for related sublinear case and the existence and multiplicity of solutions,

[1]Supported NSFC (11471187, 11571197) and SNSFC (ZR2014AM034)

see [17, 19-21, 26]. [19] investigated the existence of solutions of (1.1) by using the variant fountain theorem established in [10], under the following conditions:

(V_1') $V(x) \in C(\mathbb{R}^3)$ and $\inf_{x\in\mathbb{R}^3} V(x) \geq a > 0$.

(V_2') For any constants $b > 0$, $meas\{x \in \mathbb{R}^3 : V(x) \leq b\} < +\infty$, where $meas$ denotes the Lebesgue measure in $\mathbb{R}^3$.

(h_1) $F(x,u) = b(x)|u|^{p+1}$, where $F(x,u) = \int_0^u f(x,s)ds$, $b : \mathbb{R}^3 \to (0,+\infty)$ is a positive continuous function such that $b \in L^{\frac{2}{1-p}}(\mathbb{R}^3)$ and $0 < p < 1$ is a constant.

$(V_1')(V_2')$ also appeared in [17-26]. [20] considered the existence and multiplicity of solutions of (1.1) by using the minimizing theorem and the dual fountain theorem respectively, [21] established the existence and multiplicity of negative energy solutions for the above problem via the genus properties in critical point theory, [26] established some existence criteria to guarantee that problem has at least one or infinitely many nontrivial solutions by using the genus properties in critical point theory.

Motivated by the above papers, we continue to consider problem (1.1) with steep well potential and establish the existence of nontrivial solution and concentration results (as $\lambda \to \infty$) under some mild assumptions (where, $f(x,u)$ is sublinear and indefinite) different from those studied previously. We make the following assumptions.

(V_1) $V(x) \in C(\mathbb{R}^3)$ and $V(x) \geq 0$ on $\mathbb{R}^3$.

(V_2) There is a constant $d > 0$ such that $V_d := \{x \in \mathbb{R}^3 | V(x) < d\}$ is nonempty and has finite measure.

(V_3) $\Omega = int\{V^{-1}(0)\}$ is nonempty and has smooth boundary with $\bar{\Omega} = V^{-1}(0)$.

(f_1) $f \in C(\mathbb{R}^3, \mathbb{R})$ and there exist constants $0 < \gamma_1 < \gamma_2 < \cdots < \gamma_m < 1$ and functions $K_i(x) \in L^{\frac{2}{1-\gamma_i}}(\mathbb{R}^3, (0,+\infty))$ such that

$$|f(t,u)| \leq \sum_{i=1}^{m} (\gamma_i + 1) K_i |u|^{\gamma_i}, \quad \forall\, (x,u) \in \mathbb{R}^3 \times \mathbb{R}.$$

(f_2) There exist constants $\eta, \delta > 0$, $\gamma_0 \in (1,2)$ such that

$$F(t,u) \triangleq \int_0^u f(x,s)ds \geq \eta |u|^{\gamma_0}, \quad \text{for all } x \in \Omega \text{ and all } |u| \leq \delta.$$

***Theorem* 1.1** — *Assume that the conditions (V_1)-(V_3) and (f_1)-(f_2) hold. Then, for every $\lambda > 0$, problem (1.1) has at least one nontrivial solution u_λ.*

***Theorem* 1.2** — *Let u_λ be a solution of problem (1.1), then $u_\lambda \to u_0$ in $H^1(\mathbb{R}^3)$ as $\lambda \to \infty$, where $u_0 \in H_0^1(\Omega)$ is a nontrivial solution of the equation*

$$\begin{cases} -\triangle u + \phi u = f(x,u) & in\ \Omega, \\ -\triangle \phi = u^2 & in\ \Omega. \end{cases} \tag{1.2}$$

Remark 1.1 : (f_1) is weaker than the condition (h_1) and (V_1)-(V_2) are weaker than$(V_1')(V_2')$ which were introduced by Bartsch and Wang [23] (see also [24]) in order to guarantee the compact embedding of the functional space (see [21, Remark 3.5]). Thus, the (PS)-condition can not be directly got as done in the literature, which makes the problem more complicated. To overcome this difficulty, we adopt different method.

Remark 1.2 : The novelty of this paper is to investigate the concentration phenomenon of solutions on the set $V^{-1}(0)$ as $\lambda \to \infty$. (V_3) is used in deriving concentration phenomenon of solutions for the solutions of problem (1.1). Generally speaking, there may exist some behaviours and phenomenons for the solutions of problem (1.1) under (V_3). To the best of our knowledge, few works concern on this up to now.

2. Variational Setting and Proof of Theorem 1.1

Denote the usual L^q-norm with the norm $|\cdot|_q$ for $1 \leq q \leq \infty$, c_i, C, C_i stand for different positive constants. It is well known that $\mathcal{D}^{1,2}(\mathbb{R}^3) \hookrightarrow L^{2^*}(\mathbb{R}^3)$, where $2^* = 6$ is the critical Sobolev exponent of $\mathbb{R}^3$. Let S be the best embedding constant of this embedding,

$$|u|_6^2 \leq S^{-1}\|u\|_{\mathcal{D}^{1,2}}^2. \tag{2.1}$$

Let $X = \left\{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3}(|\nabla u|^2 + V(x)u^2)dx < \infty\right\}$ be equipped with the inner product and the norm

$$(u,v) = \int_{\mathbb{R}^3}(\nabla u \nabla v + V(x)uv)dx \quad \|u\| = (u,u)^{1/2}, \quad u,v \in X.$$

For $\lambda > 0$, we need the following inner product

$$(u,v)_\lambda = \int_{\mathbb{R}^3}(\nabla u \nabla v + \lambda V(x)uv)dx, \quad \|u\|_\lambda^2 = (u,v)_\lambda\ u,v \in X$$

Set $X_\lambda = (X, \|u\|_\lambda)$, then X_λ is a Hilbert space. By using (V_1)-(V_3), it is easy to check that there exists positive constant c_0 (independent of λ) such that

$$\|u\|_{H^1(\mathbb{R}^3)} \leq c_0\|u\|_\lambda, \quad \text{for all } u \in X_\lambda.$$

The embedding $X_\lambda \hookrightarrow L^p(\mathbb{R}^3)$ is continuous for $p \in [2,6]$, and $X_\lambda \hookrightarrow L^p_{loc}(\mathbb{R}^3)$ is compact for $p \in [2,6)$, i.e., there are constants $c_p > 0$ such that

$$|u|_p \leq c_p \|u\|_{H^1(\mathbb{R}^3)} \leq c_p c_0 \|u\|_\lambda, \text{ for all } u \in X_\lambda,\ 2 \leq p \leq 6. \tag{2.2}$$

For any given $u \in H^1(\mathbb{R}^3)$, the Lax-Milgram theorem implies that there exists a unique $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ such that

$$-\triangle \phi_u = u^2. \tag{2.3}$$

Lemma 2.1 — Let $\phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$ be the unique solution of $-\triangle\phi_u = u^2$. Then we have

(1) $\phi_u(x) \geq 0,\ \ x \in \mathbb{R}^3$.

(2) For $u \in H^1(\mathbb{R}^3)$, one has $\|\phi_u\|_{\mathcal{D}^{1,2}} \leq c|u|^2_{\frac{12}{5}}$.

(3) If $u_n \to u$ strongly in $L^{\frac{12}{5}}(\mathbb{R}^3)$, then $\phi_{u_n} \to \phi_u$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^3)$.

Substituting $\phi = \phi_u$ into system (SP), we can rewrite system (SP) as the single equation

$$-\triangle u + \lambda V(x)u + \phi_u u = f(x,u),\ \ u \in X_\lambda.$$

We define the energy functional on X_λ by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{\lambda}{2}\int_{\mathbb{R}^3} V(x)u^2 dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} F(x,u)dx. \tag{2.4}$$

Lemma 2.2 — Assume that (V_1)-(V_2) and (f_1) hold. Then $I_\lambda : X_\lambda \to \mathbb{R}$ is of class $C^1(X_\lambda, \mathbb{R})$ and

$$\langle I'_\lambda(u), v\rangle = \int_{\mathbb{R}^3} \nabla u \nabla v dx + \lambda \int_{\mathbb{R}^3} V(x)uv dx + \int_{\mathbb{R}^3} \phi_u uv dx - \int_{\mathbb{R}^3} f(x,u)v dx. \tag{2.5}$$

Moreover, the critical points of I_λ are solutions of problem (1.1).

PROOF : From (f_1), we have

$$|F(x,u)| \leq \sum_{i=1}^m K_i(x)|u|^{\gamma_i+1}, \text{ for all } (x,u) \in \mathbb{R}^3 \times \mathbb{R}. \tag{2.6}$$

For any $u \in E$, we obtain from (V_1)-(V_2), (f_1), (2.6) and the Hölder inequality that

$$\begin{aligned}\int_{\mathbb{R}^3} |F(x,u(x))|dx &\leq \int_{\mathbb{R}^3} \sum_{i=1}^m K_i(x)|u(x)|^{\gamma_i+1} dx \\ &\leq \sum_{i=1}^m \left(\int_{\mathbb{R}^3} |K_i(x)|^{\frac{2}{1-\gamma_i}} dx\right)^{\frac{1-\gamma_i}{2}} \left(\int_{\mathbb{R}^3} |u(x)|^2 dx\right)^{\frac{1+\gamma_i}{2}} \\ &\leq \sum_{i=1}^m (c_2 c_0)^{1+\gamma_i} |K_i(x)|_{\frac{2}{1-\gamma_i}} \|u(x)\|_\lambda^{1+\gamma_i} < +\infty,\end{aligned} \tag{2.7}$$

and by Lemma 2.1 and (2.2), we have, for any position C,

$$\int_{\mathbb{R}^3} \phi_u u^2 dx \leq C\|u\|_\lambda^4. \tag{2.8}$$

As mentioned above, I_λ is well defined on X_λ. By Lebesgue's theorem and the Hölder inequality, it is easy to obtain the claims.

Lemma 2.3 — (see [9]). Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy the (PS)-condition. If I is bounded from below, then $c = \inf_E I$ is a critical value of I.

Lemma 2.4 — Suppose that (V_1)-(V_3) and (f_1)-(f_2) are satisfied. There I_λ is bounded from below.

PROOF : The proof is standard, and we omit it. □

Lemma 2.5 — Suppose that (V_1)-(V_3) and (f_1)-(f_2) are satisfied. Then I_λ satisfies the (PS)-condition for each $\lambda > 0$.

PROOF : Assume that $\{u_n\}$ is a sequence such that $\{I_\lambda(u_n)\}$ is bounded and $I'_\lambda(u_n) \to 0$ as $n \to \infty$. By Lemma 2.4, it is clear that $\{u_n\}$ is bounded in X_λ. Thus, there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$

$$|u_n|_p \leq c_p c_0 \|u_n\|_\lambda \leq C, \quad 2 \leq p \leq 6. \tag{2.9}$$

Passing to a subsequence if necessary, we may assume that $u_n \rightharpoonup u$ in X_λ. For any $\varepsilon > 0$, by $K_i(x) \in L^{\frac{2}{1-\gamma_i}}(\mathbb{R}^3, \mathbb{R}^+)$, we can choose $R_\epsilon > 0$ such that

$$\left(\int_{\mathbb{R}^3 \setminus B_{R_\epsilon}} |K_i(x)|^{\frac{2}{1-\gamma_i}} dx \right)^{\frac{1-\gamma_i}{2}} < \epsilon, \quad i = 1, 2, \cdots, m. \tag{2.10}$$

Since

$$\lim_{n\to\infty} \int_{B_{R_\epsilon}} |u_n - u|^2 dx = 0, \tag{2.11}$$

there exists $N_0 \in \mathbb{N}$ such that

$$\int_{B_{R_\epsilon}} |u_n - u|^2 dx < \epsilon^2, \quad \text{for } n \geq N_0. \tag{2.12}$$

Hence, by (f_1), (2.9), (2.12) and the Hölder inequality, we have, for $n \geq N_0$,

$$\begin{aligned}
&\int_{B_{R_\epsilon}} |f(x,u_n)-f(x,u)||u_n-u|dx \leq \left(\int_{B_{R_\epsilon}} |f(x,u_n)-f(x,u)|^2 dx\right)^{1/2} \left(\int_{B_{R_\epsilon}} |u_n-u|^2 dx\right)^{1/2} \\
&\leq \left(\int_{B_{R_\epsilon}} 2(|f(x,u_n)|^2+|f(x,u)|^2)dx\right)^{1/2} \epsilon \\
&\leq \left\{\int_{B_{R_\epsilon}} 2m\big[\sum_{i=1}^m (\gamma_i+1)^2 K_i^2(x)|u_n|^{2\gamma_i} + \sum_{i=1}^m (\gamma_i+1)^2 K_i^2(x)|u|^{2\gamma_i}\big]dx\right\}^{1/2} \epsilon \\
&\leq \sqrt{2m}\big[\sum_{i=1}^m (\gamma_i+1)^2 |K_i(x)|^2_{\frac{2}{1-\gamma_i}} (C^{2\gamma_i}+|u|_2^{2\gamma_i})\big]^{1/2}\epsilon.
\end{aligned} \tag{2.13}$$

On the other hand, by (f_1), (2.9), (2.10) and the Hölder inequality, we have for $n \in N$,

$$\begin{aligned}
&\int_{\mathbb{R}^3\setminus B_{R_\epsilon}} |f(x,u_n)-f(x,u)||u_n-u|dx \\
&\leq \int_{\mathbb{R}^3\setminus B_{R_\epsilon}} \sum_{i=1}^m (\gamma_i+1)K_i(|u_n|^{\gamma_i+1}+|u|^{\gamma_i+1}+|u|^{\gamma_i}|u_n|+|u_n|^{\gamma_i}|u|)dx \\
&\leq \sum_{i=1}^m (\gamma_i+1)\left(\int_{\mathbb{R}^3\setminus B_{R_\epsilon}} |K_i|^{\frac{2}{1-\gamma_i}}dx\right)^{\frac{1-\gamma_i}{2}} \big(|u_n|_2^{1+\gamma_i}+|u|_2^{1+\gamma_i}+|u_n|_2^{\gamma_i}|u|_2 \\
&+|u|_2^{\gamma_i}|u_n|_2\big) \\
&\leq \epsilon\sum_{i=1}^m (\gamma_i+1)\Big(|u_n|_2^{1+\gamma_i}+|u|_2^{1+\gamma_i}+|u_n|_2^{\gamma_i}|u|_2+|u|_2^{\gamma_i}|u_n|_2\Big) \\
&\leq \epsilon\sum_{i=1}^m (\gamma_i+1)\Big(C^{1+\gamma_i}+|u|_2^{1+\gamma_i}+C^{\gamma_i}|u|_2+C|u|_2^{\gamma_i}\Big).
\end{aligned} \tag{2.14}$$

Since ϵ is arbitrary, combining (2.13) with (2.14), we have

$$\int_{\mathbb{R}^3} |f(x,u_n)-f(x,u)||u_n-u|dx \to 0 \quad as \ \ n\to\infty. \tag{2.15}$$

Recall that

$$(xy)^{1/2}(x+y) \leq x^2+y^2, \ \ \forall x,y\geq 0. \tag{2.16}$$

Hence we obtain, by Lemma 2.1 and the Hölder's inequality,

$$\begin{aligned}
&\int_{\mathbb{R}^3}(\phi_{u_n}u_nu+\phi_uu_nu)dx\\
&\le\left(\int_{\mathbb{R}^3}\phi_{u_n}u_n^2dx\right)^{1/2}\left(\int_{\mathbb{R}^3}\phi_{u_n}u^2dx\right)^{1/2}+\left(\int_{\mathbb{R}^3}\phi_uu_n^2dx\right)^{1/2}\left(\int_{\mathbb{R}^3}\phi_uu^2dx\right)^{1/2}\\
&=\left(\int_{\mathbb{R}^3}\nabla\phi_{u_n}\nabla\phi_udx\right)^{1/2}\left(\|\phi_{u_n}\|_{\mathcal{D}^{1,2}}+\|\phi_u\|_{\mathcal{D}^{1,2}}\right)\\
&\le\left(\int_{\mathbb{R}^3}|\nabla\phi_{u_n}|^2dx\right)^{1/4}\left(\int_{\mathbb{R}^3}|\nabla\phi_u|^2dx\right)^{1/4}\left(\|\phi_{u_n}\|_{\mathcal{D}^{1,2}}+\|\phi_u\|_{\mathcal{D}^{1,2}}\right)\\
&\le\|\phi_{u_n}\|^2_{\mathcal{D}^{1,2}}+\|\phi_u\|^2_{\mathcal{D}^{1,2}}\\
&=\int_{\mathbb{R}^3}(\phi_{u_n}u_n^2+\phi_uu^2)dx
\end{aligned}$$

which implies that

$$\int_{\mathbb{R}^3}(\phi_{u_n}u_n-\phi_uu)(u_n-u)dx\ge 0. \tag{2.17}$$

By (2.5), (2.17), one yields

$$\begin{aligned}
\|u_n-u\|^2_\lambda=&\langle I'_\lambda(u_n)-I'_\lambda(u),u_n-u\rangle-\int_{\mathbb{R}^3}(\phi_{u_n}u_n-\phi_uu)(u_n-u)dx\\
&+\int_{\mathbb{R}^3}(f(x,u_n)-f(x,u))(u_n-u)dx\\
\le&\langle I'_\lambda(u_n)-I'_\lambda(u),u_n-u\rangle+\int_{\mathbb{R}^3}|f(x,u_n)-f(x,u)||u_n-u|dx.
\end{aligned} \tag{2.18}$$

Since $\langle I'_\lambda(u_n)-I'_\lambda(u),u_n-u\rangle\to 0\ \ n\to+\infty$, it follows from (2.15) and (2.18) that $u_n\to u$ in X_λ. Hence, I_λ satisfies the (PS)-condition. The proof is complete. □

PROOF OF THEOREM 1 : By Lemmas $(2.2)-(2.5)$, we know that $c_\lambda=\inf_{X_\lambda}I_\lambda(u)$ is a critical value of I_λ, that is, there exists a critical point $u_\lambda\in X_\lambda$ such that $I_\lambda(u_\lambda)=c_\lambda$. Next, we shows that $u_\lambda\ne 0$. Let $u^*\in H^1_0(\Omega)\setminus\{0\}$ and $\|u^*\|_\infty\le 1$, then by (f_2) (2.4) and (2.8), we have

$$\begin{aligned}
I_\lambda(tu^*)&=\frac{t^2}{2}\|u^*\|_\lambda+\frac{t^4}{4}\int_{\mathbb{R}^3}\phi_{u^*}(u^*)^2dx-\int_{\mathbb{R}^3}F(x,tu^*)dx\\
&\le\frac{t^2}{2}\|u^*\|^2_\lambda+\frac{Ct^4}{4}\|u^*\|^4_\lambda-\eta t^{\gamma_0}\int_\Omega|u^*|^{\gamma_0}dx,
\end{aligned} \tag{2.19}$$

where $0<t<\delta$, δ be given in (f_2). Since $1<\gamma_0<2$, it follows from (2.19) that $I_\lambda(tu^*)<0$ for $t>0$ small enough. Hence, $I_\lambda(u_\lambda)=c_\lambda<0$, therefore, u_λ is a nontrivial solution of problem (1.1). The proof is finished. □

3. Concentration of Solutions

Define $\tilde{c} = \inf_{H_0^1(\Omega)} I_\lambda$. From the proof of Theorem (1.1), $\tilde{c} < 0$ can be achieved. Since $H_0^1(\Omega) \subset X_\lambda$ for all $\lambda > 0$, we get $\inf_{X_\lambda} I_\lambda \leq \inf_{H_0^1(\Omega)} I_\lambda < 0$, hence, $c_\lambda \leq \tilde{c} < 0$.

Proof of Theorem 1.2. : We follow the arguments in [24]. For any sequence $\lambda_n \to \infty$, let $u_n := u_{\lambda_n}$ be the critical points of I_{λ_n} obtained in Theorem 1.1. Thus

$$I_{\lambda_n}(u_n) \leq \tilde{c} < 0 \tag{3.1}$$

which implies

$$\|u_n\|_{\lambda_n} \leq c_1, \tag{3.2}$$

for some constant c_1 which is independent of (λ_n). Therefore, we may assume that $u_n \rightharpoonup u_0$ in X_λ and $u_n \to u_0$ in $L^p_{loc}(\mathbb{R}^3)$ for $2 \leq p < 2^*$. From Fatou's lemma, we have

$$\int_{\mathbb{R}^3} V(x)|u_0|^2 dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^3} V(x)|u_n|^2 dx \leq \liminf_{n\to\infty} \frac{\|u_n\|^2_{\lambda_n}}{\lambda_n} = 0,$$

which implies that $u_0 = 0$ a.e. in $\mathbb{R}^3 \backslash V^{-1}(0)$ and $u_0 \in H_0^1(\Omega)$ by (V_3).

By Lions vanishing lemma [10], we can verify that $u_n \to u_0$ in $L^p(\mathbb{R}^3)$ for $2 \leq p < 6$. Next, for any $\varphi \in C_0^\infty(\Omega)$, since $\langle I'_{\lambda_n}(u_n), \varphi\rangle = 0$ and (V_3), it is easy to verify that

$$\int_\Omega \nabla u_0 \nabla \varphi dx + \int_\Omega \phi_{u_0} u_0 \varphi = \int_\Omega f(x, u_0)\varphi dx. \tag{3.3}$$

By the density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$, (3.3) implies that u_0 is a weak solution of problem (1.2).

Finally, we prove that $u_n \to u_0$ in $H^1(\mathbb{R}^3)$. Since $\langle I'_{\lambda_n}(u_n), u_n\rangle = \langle I'_{\lambda_n}(u_n), u_0\rangle = 0$, we have

$$\|u_n\|^2_{\lambda_n} + \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 dx = \int_{\mathbb{R}^3} f(x, u_n)u_n dx,$$
$$\langle u_n, u_0\rangle_{\lambda_n} + \int_{\mathbb{R}^3} \phi_{u_n} u_n u_0 dx = \int_{\mathbb{R}^3} f(x, u_n)u_0 dx,$$
$$\int_{\mathbb{R}^3} (\phi_{u_n} u_n^2 - \phi_{u_n} u_n u_0) dx \to 0,$$

so

$$\lim_{n\to\infty} \|u_n\|^2_{\lambda_n} = \lim_{n\to\infty} (u_n, u_0)_{\lambda_n} = \lim_{n\to\infty} (u_n, u_0) = \|u_0\|^2,$$

which together with $\|u\| \leq \|u\|_\lambda$ (for $\lambda \geq 1$) imply that

$$\limsup_{n\to\infty} \|u_n\|^2 \leq \|u_0\|^2.$$

On the other hand, the weakly lower semi-continuity of norm yields that $\|u_0\|^2 \leq \liminf_{n\to\infty} \|u_n\|^2$. Hence, $u_n \to u_0$ in $H^1(\mathbb{R}^3)$. From (3.1), we have

$$\frac{1}{2}\int_\Omega |\nabla u_0|^2 dx + \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 dx - \int_\Omega F(x, u_0) dx \leq \tilde{c} < 0,$$

which implies that $u_0 \neq 0$. This completes the proof. □

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