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CENTRALIZERS IN LIE ALGEBRAS

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We determine the number of centralizers of different non-abelian finite dimensional Lie algebras over a specific field. Also, the concept of Lie algebras with abelian centralizers are studied and using a result of Bokut and Kukin [5], for a given residually free Lie algebra L, it is shown that L is fully residually free if and only if every centralizer of non-zero elements of L is abelian.

Key words : Centralizer; n-centralizer Lie algebras; CT Lie algebra; free Lie algebra.

1. Introduction and Preliminaries

Let L be a finite dimension Lie algebra over the fixed field F. Then for any element $x \in L$, the set $C_L(x) = \{y \in L \mid [x,y] = 0\}$ is called the *centralizer* of x in L. The set of all centralizers in L is denoted by $Cent(L)$ and $|Cent(L)|$ denotes the number of distinct centralizers in L. A Lie algebra L is called n-*centralizer* if $|Cent(L)| = n$ and L is called *primitive* n-*centralizer* if $|Cent(L/Z(L))| = |Cent(L)| = n$, where $Z(L)$ is the centre of L. A subalgebra K of L is called *proper centralizer* of L if $K = C_L(x)$, for some $x \in L \setminus Z(L)$.

Similar to group theory, it is clear that L is abelian if and only if $|Cent(L)| = 1$.

Lemma 1.1 — Let L_1 and L_2 be Lie algebras, then

$$Cent(L_1 \oplus L_2) = Cent(L_1) \oplus Cent(L_2).$$

PROOF : Clearly, the Lie product of elements of $L_1 \oplus L_2$ is defined by $[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2])$, for all $x_i, y_i \in L_i$ $(i = 1, 2)$. Now the result follows from the property that $C_{(L_1 \oplus L_2)}(x_1, x_2) = C_{L_1}(x_1) \oplus C_{L_2}(x_2)$, for all $x_1 \in L_1$ and $x_2 \in L_2$. □

Lemma 1.2 — For a Lie algebra L, the centre $Z(L)$, is the intersection of all centralizers in L, i.e. $Z(L) = \bigcap_{x \in L} C_L(x)$, for all $x \in L$.

PROOF : Clearly, $Z(L) \subseteq \bigcap_{x \in L} C_L(x)$. Now, suppose that $l \in \bigcap_{x \in L} C_L(x)$, then $[x, l] = 0$, for all $x \in L$ and so $l \in Z(L)$, which gives the claim. □

Lemma 1.3 — If L is a non-abelian Lie algebra, then L is the union of centralizers of all non-central elements of L.

PROOF : Clearly, $\bigcup_{x \in L - Z(L)} C_L(x) \subseteq L$. Let $l \in Z(L)$, then by using Lemma 1.2, $l \in C_L(x)$ for all $x \in L$ and since $l \in C_L(l)$ it follows that $l \in \bigcup_{x \in L - Z(L)} C_L(x)$. Therefore $L \subseteq \bigcup_{x \in L - Z(L)} C_L(x)$ and the proof is complete. □

Lemma 1.4 — A Lie algebra L can not are written as a union of two proper Lie subalgebras.

PROOF : Suppose H and K are two proper Lie subalgebras of L such that $L = H \cup K$. Let $h \in H - K$ and $k \in K - H$, then either $h + k \in H$ or $h + k \in K$, which imply $k \in H$ or $h \in K$, respectively. This gives a contradiction. Therefore $h + k \notin L$, which gives the lemma. □

***Theorem* 1.5** — *Let L be a non-abelian Lie algebra, then $|Cent(L)| \geq 4$.*

PROOF : By Lemma 1.3, L is the union of its proper centralizers. Since L is non-abelian, we have $|Cent(L)| > 1$. If $|Cent(L)| = 2$, then L is the proper Lie subalgebra of itself, which is impossible. Suppose $|Cent(L)| = 3$, then $Cent(L) = \{L, C_L(x), C_L(y)\}$, where $C_L(x)$ and $C_L(y)$ are proper centralizers of L. Therefore $L = C_L(x) \cup C_L(y)$, which is impossible by Lemma 1.4. Hence, $|Cent(L)| \geq 4$. □

2. Counting Centralizers in Lie Algebras

In this section, we study the centralizers of low-dimensional Lie algebras over the Galois field of p elements, $\mathbb{Z}_p$, for any prime number p.

Lemma 2.1 — Let L_i's be finite dimensional Lie algebras with $|Cent(L_i)| = n_i$, for $i = 1, 2, ..., r$. Then $|Cent(L_1 \oplus L_2 \oplus ... \oplus L_r)| = \prod_{i=1}^{r} n_i$.

PROOF : Assume $L = L_1 \oplus L_2 \oplus ... \oplus L_r$. Using Lemma 1.1, we have

$$C_L(x_1, x_2, ..., x_r) = C_{L_1}(x_1) \oplus C_{L_2}(x_2) \oplus ... \oplus C_{L_r}(x_r),$$

for all $(x_1, x_2, ..., x_r) \in L$. It follows that $C_L(x_1, x_2, ..., x_r) = C_L(y_1, y_2, ..., y_r)$ if and only if $C_{L_i}(x_i) = C_{L_i}(y_i)$, for all $1 \leq i \leq r$. This implies that $|Cent(L_1 \oplus L_2 \oplus ... \oplus L_r)| = \prod_{i=1}^{r} n_i$. □

Lemma 2.2 — Let K be a subalgebra of a finite dimensional Lie algebra L. Then $|Cent(K)| \leq |Cent(L)|$.

PROOF : Let $k_1, k_2, ..., k_m$ be a basis of K, and $C_K(k_1), ..., C_K(k_m)$ be the distinct centralizers in K. On the other hand, $C_K(k_i) = K \cap C_L(k_i)$ then $C_L(k_i) \neq C_L(k_j)$, for all $i \neq j$, where $1 \leq i, j \leq m$, and hence the lemma is obtained. □

Lemma 2.3 — Let L be n-centralizer Lie algebra with $L^2 \cap Z(L) = 0$. Then L is a primitive n-centralizer.

PROOF : Suppose that $Cent(L) = \{C_L(x_1), C_L(x_2), ..., C_L(x_n)\}$ is the set of all distinct centralizers in L. One can easily check that $C_{L/Z(L)}(x + Z(L)) = C_L(x)/Z(L)$. Hence it is enough to show that for any $1 \leq i \neq j \leq n$, $C_{L/Z(L)}(x_i + Z(L)) \neq C_{L/Z(L)}(x_j + Z(L))$. So assume there exist some $1 \leq i \neq j \leq n$ such that $C_{L/Z(L)}(x_i + Z(L)) = C_{L/Z(L)}(x_j + Z(L))$. Suppose $y \in C_L(x_i)$, then $y + Z(L) \in C_{L/Z(L)}(x_i + Z(L)) = C_{L/Z(L)}(x_j + Z(L))$ and by the assumption we have $[y, x_j] = 0$, i.e., $C_L(x_i) \subseteq C_L(x_j)$. Using similar argument, we have $C_L(x_j) \subseteq C_L(x_i)$ which gives a contradiction. Thus $|Cent(L/Z(L))| = n$ and hence L is a primitive n-centralizer. □

In the following, we determine the number of centralizers of 2-dimension non-abelian Lie algebra over the Galois field of p elements.

***Theorem* 2.4** — *Let L be a 2-dimension non-abelian Lie algebra over the field $\mathbb{Z}_p$. Then $\left|Cent(L)\right| = p + 2$.*

PROOF : Clearly there exists a unique 2-dimension non-abelian Lie algebra over any field. The centre of this Lie algebra is trivial and the Lie algebra has a basis $\{x, y\}$ such that its Lie bracket is described by $[x, y] = x$. Clearly

$$C_L(x) = \langle x \rangle, C_L(y) = \langle y \rangle, C_L(\alpha x + \beta y) = \langle \alpha x + \beta y \rangle,$$

and the number of distinct $C_L(\alpha x + \beta y) = \langle \alpha x + \beta y \rangle$ is equal to $\frac{(p-1)^2}{p-1} = p - 1$, for all non-zero $\alpha, \beta \in \mathbb{Z}_p$. Now adding the centralizers $C_L(x)$, $C_L(y)$ and L, we have $\left|Cent(L)\right| = p + 2$. □

Definition 2.5 — Let L be a 3-dimension non-abelian Lie algebra over a field F, with L^2 to be 1-dimension so that L^2 is contained in $Z(L)$. Such a Lie algebra is known as *Heisenberg Lie algebra*.

***Theorem* 2.6** — *Let L be Heisenberg Lie algebra over the field $\mathbb{Z}_p$, then $|Cent(L)| = p+2$.*

PROOF : Clearly there is a unique such a Lie algebra, and it has a basis $\{f, g, z\}$, where $[f, g] = z$ and z lies in $Z(L)$. Hence for every $y \in C_L(f)$ there exist $\alpha, \beta, \gamma \in \mathbb{Z}_p$ such that $y = \alpha f + \beta g + \gamma z$, then

$$0 = [f, y] = [f, \alpha f + \beta g + \gamma z] = \beta[f, g] = \beta z,$$

and so $\beta = 0$. Thus $C_L(f) = \langle \alpha f + \gamma z \rangle$. Similarly $C_L(g) = \langle \beta g + \gamma z \rangle$, $C_L(\alpha f + \beta g) = \langle \alpha f + \beta g + \gamma' z \rangle$, $C_L(f + z) = C_L(f)$, $C_L(g + z) = C_L(g)$, $C_L(\alpha f + \beta g + \gamma z) = C_L(\alpha f + \beta g)$ and clearly $C_L(z) = L$. Now for any non-zero elements $\alpha, \beta \in \mathbb{Z}_p$, we have $\frac{(p-1)^2}{p-1} = p-1$ distinct centralizers of the form $C_L(\alpha f + \beta g)$. So $|Cent(L)| = p - 1 + 3 = p + 2$. □

Example 2.7 : Let $L = n(3, \mathbb{Z}_p) = \langle e_{12}, e_{13}, e_{23} \rangle$ be the Lie algebra of non-zero strictly upper triangular matrices, then $[e_{12}, e_{23}] = e_{13}$ and $L^2 = Z(L)$. Hence, the above theorem implies that $\big|Cent(L)\big| = p + 2$.

As in Theorem 3.2 [8], there exists a unique 3-dimensional Lie algebra over a field F such that L^2 is 1-dimension and $L^2 \not\subseteq Z(L)$. Hence such a Lie algebra is the direct sum of the non-abelian 2-dimension with 1-dimension Lie subalgebras.

***Theorem* 2.8** — *Let L be the 3-dimensional Lie algebra as above over the field $\mathbb{Z}_p$. Then $|Cent(L)| = p+2$.*

PROOF : By the assumption we may write $L = L_2 \oplus L_1$, where L_2 is 2-dimensional non-abelian and L_1 is 1-dimensional Lie algebra over $\mathbb{Z}_p$. By Lemma 2.1, $\big|Cent(L)\big| = \big|Cent(L_2)\big|\big|Cent(L_1)\big| = (p+2) \times 1 = p+2$. □

The following lemma is very useful for our further study.

Lemma 2.9 — ([8], Lemma 3.3). Let L be a Lie algebra with dimension 3 and $dimL^2 = 2$. Then

(i) L^2 is abelian;

(ii) the map $adx : L^2 \to L^2$ is an isomorphism, for all $x \in L - L^2$.

***Theorem* 2.10** — *Let L be a Lie algebra with dimension 3 over the field $\mathbb{Z}_p$ and $dimL^2 = 2$. Then $\big|Cent(L)\big| = p^2 + 2$.*

PROOF : Let $\{y, z\}$ be a basis for the derived subalgebra L^2 and extend it to a basis $\{x, y, z\}$ for L. Then the derived subalgebra L^2 is abelian, and hence $[y, z] = 0$. Therefore we have $C_L(y) = \{l = \alpha x + \beta y + \gamma z \mid [y, l] = 0\} = \langle \beta y + \gamma z \rangle$, $C_L(z) = C_L(y)$, $C_L(x) = \langle x \rangle$, $C_L(x + y) = \langle \alpha x + \alpha y \rangle$,

$C_L(x+z) = \langle \alpha x + \alpha z\rangle$, $C_L(y+z) = C_L(z) = C_L(y)$, $C_L(x+y+z) = \langle \alpha x + \alpha y + \alpha z\rangle$, $C_L(\alpha x + \beta y) = \langle \alpha x + \beta y\rangle$, $C_L(\alpha x + \beta z) = \langle \alpha x + \beta z\rangle$, $C_L(\alpha x + \beta y + \gamma z) = \langle \alpha x + \beta y + \gamma z\rangle$.

Clearly, each cases $\langle \alpha x, \beta y\rangle$ and $\langle \alpha x + \beta z\rangle$, has $(p-1)^2$ distinct set of centralizers so that every set contains $p-1$ elements. Hence each of these have only $\frac{(p-1)^2}{p-1} = p-1$ distinct sets. Also for $C_L(\alpha x + \beta y + \gamma z) = \langle \alpha x + \beta y + \gamma z\rangle$ we have $\frac{(p-1)^3}{p-1} = (p-1)^2$ distinct sets. Therefore the number of $Cent(L)$ is equal to $(p-1)^2 + 2(p-1) + 3 = p^2 + 2$. □

The following example justifies the above theorem.

Example 2.11 : Consider the 3-dimension Lie algebra L with the basis $\{x, y, z\}$ as in Theorem 2.10 over the field $\mathbb{Z}_5$, then one may calculate all the centralizers of L in the following way:

$C_L(x) = \{0, x, 2x, 3x, 4x\},\ C_L(y) = \{0, y, 2y, 3y, 4y\},\ C_L(0) = L,$
$C_L(x+y) = \{0, x+y, 2(x+y), 3(x+y), 4(x+y)\},$
$C_L(x+z) = \{0, x+z, 2(x+z), 3(x+z), 4(x+z)\},$
$C_L(y+z) = C_L(z) = C_L(y),$
$C_L(x+y+z) = \{0, x+y+z, 2(x+y+z), 3(x+y+z), 4(x+y+z)\},$
$C_L(2x+y) = \{0, 2x+y, 4x+2y, x+3y, 3x+4y)\},$
$C_L(3x+y) = \{0, 3x+y, x+2y, 4x+3y, 3x+4y)\},$
$C_L(4x+y) = \{0, 4x+y, 3x+2y, 2x+3y, x+4y)\},$
$C_L(2x+z) = \{0, 2x+z, 4x+2z, x+3z, 3x+4z)\},$
$C_L(3x+z) = \{0, 3x+z, x+2z, 4x+3z, 2x+4z)\},$
$C_L(4x+z) = \{0, 4x+z, 3x+2z, 2x+3z, x+4z)\},$
$C_L(2x+y+z) = \{0, 2x+y+z, 4x+2y+2z, x+3y+3z, 3x+4y+4z)\},$
$C_L(3x+y+z) = \{0, 3x+y+z, x+2y+2z, 4x+3y+3z, 3x+4y+4z)\},$
$C_L(4x+y+z) = \{0, 4x+y+z, 3x+2y+2z, 2x+3y+3z, x+4y+4z)\},$
$C_L(x+2y+z) = \{0, x+2y+z, 2x+4y+2z, 3x+y+3z, 4x+3y+4z)\},$
$C_L(x+3y+z) = \{0, x+3y+z, 2x+y+2z, 3x+4y+3z, 4x+2y+4z)\},$
$C_L(x+4y+z) = \{0, x+4y+z, 2x+3y+2z, 3x+2y+3z, 4x+y+4z)\},$
$C_L(x+y+2z) = \{0, x+y+2z, 2x+2y+4z, 3x+3y+z, 4x+4y+3z\},$
$C_L(x+y+3z) = \{0, x+y+3z, 2x+2y+z, 3x+3y+4z, 4x+4y+2z\},$
$C_L(x+y+4z) = \{0, x+y+4z, 2x+2y+3z, 3x+3y+2z, 4x+4y+z\},$
$C_L(x+2y+3z) = \{0, x+2y+3z, 2x+4y+z, 3x+y+4z, 4x+3y+2z)\},$
$C_L(x+3y+4z) = \{0, x+3y+4z, 2x+y+3z, 3x+4y+2z, 4x+2y+z)\},$
$C_L(2x+3y+4z) = \{0, 2x+3y+4z, 4x+y+3z, x+4y+2z, 3x+2y+z)\},$
$C_L(x+2y+4z) = \{0, x+2y+4z, 2x+4y+3z, 3x+y+2z, 4x+3y+z)\},$

$C_L(x+3y+2z) = \{0, x+3y+2z, 2x+y+4z, 3x+4y+z, 4x+2y+3z\}$.

One observes that $|Cent(L)| = 27$. On the other hand, by Theorem 2.10, the number of distinct centralizers must be $5^2 + 2 = 27$.

***Theorem* 2.12** — *Let L be a Lie algebra over the field $\mathbb{Z}_p$ such that $dimL = 3$ and $dimL^2 = 3$. Then $\big|Cent(L)\big| = p^2 + p + 2$.*

PROOF : Let x be a non-zero element of the Lie algebra L, then extend x to a basis of L, say $\{x, y, z\}$. Clearly L^2 is spanned by $\{[x, y], [x, z], [y, z]\}$ and this set must be linearly independent. So we have $C_L(x) = \langle x\rangle$, $C_L(y) = \langle y\rangle$, $C_L(z) = \langle z\rangle$, $C_L(\alpha x + \beta y) = \langle \alpha x + \beta y\rangle$, $C_L(\alpha x + \gamma z) = \langle \alpha x + \gamma z\rangle$, $C_L(\beta y + \gamma z) = \langle \beta y + \gamma z\rangle$, $C_L(\alpha x + \beta y + \gamma z) = \langle \alpha x + \beta y + \gamma z\rangle$. Now for all non-zero elements $\alpha, \beta \in \mathbb{Z}_p$, we can write $(p-1)^3$ sets of centralizers of the form $C_L(\alpha x + \beta y + \gamma z) = \langle \alpha x + \beta y + \gamma z\rangle$, but every set contains $p-1$ elements and so we have $\frac{(p-1)^3}{p-1} = (p-1)^2$ distinct sets of the form $C_L(\alpha x + \beta y + \gamma z)$. Similarly, we have $\frac{(p-1)^2}{p-1} = p-1$ distinct sets of each $C_L(\alpha x + \beta y)$, $C_L(\alpha x + \gamma z)$ and $C_L(\beta y + \gamma z)$. So summing up all together we obtain $|Cent(L)| = (p-1)^2 + 3(p-1) + 4 = p^2 + p + 2$. □

The following example justifies the above theorem.

Example 2.13 : The distinct centralizers of the 3-dimension Lie algebra in Theorem 2.12 over the field $\mathbb{Z}_3$ are as follows:

$C_L(x) = \{0, x, 2x\}$, $C_L(y) = \{0, y, 2y\}$, $C_L(z) = \{0, z, 2z\}$, $C_L(0) = L$,
$C_L(x+y) = \{0, x+y, 2x+2y\}$,
$C_L(2x+y) = \{0, 2x+y, x+2y\}$,
$C_L(x+z) = \{0, x+z, 2x+2z\}$,
$C_L(2x+z) = \{0, 2x+z, x+2z\}$,
$C_L(y+z) = \{0, y+z, 2y+2z\}$,
$C_L(2y+z) = \{0, 2y+z, y+2z\}$,
$C_L(x+y+z) = \{0, x+y+z, 2x+2y+2z\}$,
$C_L(2x+y+z) = \{0, 2x+y+z, x+2y+2z\}$,
$C_L(x+2y+z) = \{0, x+2y+z, 2x+y+2z\}$,
$C_L(x+y+2z) = \{0, x+y+2z, 2x+2y+z\}$.

So $|Cent(L)| = 14$ and using Theorem 2.12, we get the same number, i.e. $|Cent(L)| = 3^2 + 3 + 2 = 14$.

3. Lie Algebras with Abelian Centralizers

In this section we study Lie algebras L, in which every centralizer of non-zero elements of L is abelian. Such Lie algebras are equivalent to commutative transitive Lie algebras (see Lemma 3.2). The concept of commutative transitive groups was first introduced and studied by Weisner [11] in 1925.

Definition 3.1 — A Lie algebra L is *commutative transitive* (henceforth CT), if $[x, y] = 0$ and $[y, z] = 0$ imply that $[x, z] = 0$, for any non-zero elements x, y, z in L.

The property of CT is clearly subalgebra closed, while it is not quotient closed, as every free Lie algebra is CT (see [9], Example 4.4 for more detail).

The Frattini subalgebra $\Phi(L)$ of a Lie algebra L, is the intersection of all maximal subalgebras of L or it is L itself, when there are no maximal subalgebras (see also [10]).

In this section, we study the concept of commutative transitive Lie algebras and among other results, their relationships with fully residually free Lie algebras are established.

Here, we introduce some basic notion and then prove our main results of this section.

Lemma 3.2 — For any Lie algebra L, the following statements are equivalent:

(i) L is CT Lie algebra;

(ii) The centralizers of non-zero elements of L are abelian.

PROOF : $(i) \Rightarrow (ii)$ Let L be a CT Lie algebra. For any non-zero element $x \in L$, if $y, z \in C_L(x)$ we have $[y, x] = 0$ and $[x, z] = 0$. The definition of CT implies that $[y, z] = 0$. Hence $C_L(x)$ is abelian.

$(ii) \Rightarrow (i)$ Assume x, y, z are non-zero elements of L, with $[x, y] = 0$ and $[y, z] = 0$. Obviously $x, z \in C_L(y)$. By the assumption $C_L(y)$ is abelian and hence $[x, z] = 0$. Thus L is commutative transitive. □

The proof of the following lemma is a routine argument by using Zorn's Lemma.

Lemma 3.3 — Every abelian subalgebra K of a given Lie algebra L is contained in a maximal abelian subalgebra.

PROOF : Consider the collection of all abelian subalgebras of L containing K, ordered by inclusion. We first show that in this partially ordered set, every chain has an upper bound. Indeed, given an ascending chain of abelian subalgebras and consider their union. We need to show that this is again abelian. Given an ascending chain of subalbebras then for any two elements x and y in their union, one has $[x, y] = 0$. Thus the partially ordered collection of abelian subalgebras containing K satisfies the condition that every chain has an upper bound in the collection. Zorn's lemma yields that there exists a maximal element in the partially ordered collection M, say. Hence K is contained in the maximal abelian subalgebra M of L. □

The following fact is needed in proving our main results.

Proposition 3.4 — Let L be a non-abelian Lie algebra with $\Phi(L) \neq 0$. Then L has one maximal abelian subalgebra.

PROOF : Let L be a non-abelian Lie algebra with non-zero Frattini subalgebra, $\Phi(L)$. Without loss of generality, we may assume that M_1 and M_2 are maximal abelian subalgebras with $M_1 \neq M_2$. Assume there exists an element $m_1 \in M_1 \setminus M_2$, then clearly

$$M_2 \subseteq M_2 \oplus \langle m_1 \rangle \subseteq L.$$

If $M_2 \oplus \langle m_1 \rangle = L$, then L is abelian which contradicts our assumption. Hence m_1 must be in M_2 and so $M_1 = M_2$. □

Using the above proposition and Lemma 3.3, we obtain the following useful result.

***Theorem* 3.5** — *Every non-abelian Lie algebra L with non-zero Frattini subalgebra is CT.*

PROOF : Let L be a non-abelian Lie algebra, for which $\Phi(L) \neq 0$ and assume that $[x, y] = 0$, $[y, z] = 0$, for non-zero elements x, y, z in L. Suppose that M_1 and M_2 are maximal abelian subalgebras in L, which contain two abelian ideals $I_1 = \langle x, y \rangle$ and $I_2 = \langle y, z \rangle$, respectively. Then Proposition 3.4 implies that $M_1 = M_2$, which gives $[x, z] = 0$ and so L is CT. □

In 2010, Klep and Moravec [9] classified all finite dimensional commutative transitive Lie algebras over an algebraically closed field of characteristic 0. They proved that these Lie algebras are either simple or soluble, where the only simple such Lie algebra is sl_2. Also, they showed that in the soluble case, Lie algebras are either abelian or a one-dimensional split extension of abelian Lie algebra (see [9] for more information).

Now, using Theorem 3.5 one can easily see that every non-abelian Lie algebra with $\Phi(L) \neq 0$ is either simple or soluble. One notes that all the results on CT Lie algebras in [9], carried out the assumption of non-triviality of Frattini subalgebras.

In the following, we focus on non-abelian CT Lie algebras and give some structural results.

***Theorem* 3.6** — *The centre of a non-abelian CT Lie algebra is trivial.*

PROOF : Assume L is a non-abelian Lie algebra with non-zero centre and z is a non-zero element in $Z(L)$. Clearly, for every non-zero elements $x, y \in L$

$$[x, z] = 0, [z, y] = 0,$$

then the definition of CT Lie algebras implies that $[x, y] = 0$. Hence L is abelian Lie algebra and this contradiction gives the result. □

A *derivation* of a Lie algebra L over a field F is an F-linear transformation $d : L \longrightarrow L$ such that

$$d([x, y]) = [d(x), y] + [x, d(y)],$$

for all $x, y \in L$. We denote by $Der(L)$ the vector space of derivations of L, which forms a Lie algebra with respect to the bracket of linear transformations, called the *derivation algebra* of L. Clearly, the space

$$ad_L = \{ad_x | x \in L\}$$

of *inner derivations* is an ideal of $Der(L)$.

***Theorem* 3.7** — *Let L be a non-abelian CT Lie algebra, then $Z(Der(L)) = 0$.*

PROOF : It is clear that L is centre less Lie algebra. Assume that $d \in Z(Der(L))$. Then in particular we have $dad_x(y) = ad_x d(y)$, and hence $d([x, y]) = [x, d(y)]$, for all $x, y \in L$. Hence by the definition of derivation, $[d(x), y] = 0$, for all $x, y \in L$. Since L has trivial centre, we obtain $d(x) = 0$, i.e., $d = 0$. Therefore $Z(Der(L)) = 0$. □

Let χ be a class of Lie algebras. Then a Lie algebra L is *residually* χ if for every non-zero element $x \in L$, there exists a homomorphism $\phi : L \to K$, where K is a χ-Lie algebra such that $\phi(x) \neq 0$. Also a Lie algebra L is *fully residually* χ, if for finitely many non-zero elements $x_1, ..., x_n$ in L there exists a homomorphism $\phi : L \to K$, where K is a χ-Lie algebra such that $\phi(x_i) \neq 0$, for all $i = 1, ..., n$.

In 1967, Baumslag [3] introduced the notion of fully residually free groups and proved that a residually free group is fully residually free if and only if it is commutative transitive. A group G is commutative transitive, if $[x, y] = 1$ and $[y, z] = 1$ implies that $[x, z] = 1$, for non-trivial elements x, y, z in G.

Lemma 3.8 — (Bokut and Kukin [5], Lemma 4.16.2). A Lie algebra L is fully residually free if and only if, for every two linearly independent elements x_1 and x_2 in L, there exists a homomorphism ϕ from the Lie algebra L into a free Lie algebra $\mathcal{F}$ such that the elements $\phi(x_1)$ and $\phi(x_2)$ are linearly independent in $\mathcal{F}$.

Now, using the above lemma we give the following result concerning free Lie algebras.

***Theorem* 3.9** — *Let L be a residually free Lie algebra. Then L is fully residually free, if and only if L is CT.*

PROOF : Without loss of generality we may assume that L is non-abelian Lie algebra. Now let L be a non-abelian residually free CT Lie algebra over a field F. Then we show that for given non-zero linearly independent elements x_1 and x_2 in L, there exists a homomorphism $\phi : L \to \mathcal{F}$ such that $\mathcal{F}$ is a free Lie algebra and $\phi(x_1)$ and $\phi(x_2)$ are linearly independent. Hence Lemma 3.8 implies that L is fully residually free.

For every non-zero element x_1 in L, there exits a homomorphism $\phi : L \to \mathcal{F}$ such that $\phi(x_1) \neq 0$, as by the assumption L is residually free Lie algebra. On the other hand, Theorem 3.6 implies that $Z(L) = 0$. Hence, $[x_1, x_2] \neq 0$ for some x_2 in L. So x_1 and x_2 are linearly independent in L. Clearly, $[\phi(x_1), \phi(x_2)] \neq 0$, as $\mathcal{F}$ is free Lie algebra. Then $\phi(x_1)$ and $\phi(x_2)$ are linearly independent and hence L is fully residually free.

Conversely, let L be a fully residually free Lie algebra such that $[x_1, x_2] = 0$ and $[x_2, x_3] = 0$, for any non-zero elements x_1, x_2 and x_3 in L. Assume that L is not CT and $x_4 = [x_1, x_3] \neq 0$, then there exits a homomorphism $\phi : L \to \mathcal{F}$, where $\mathcal{F}$ is a free Lie algebra and $\phi(x_i) \neq 0$ for $i = 1, 2, 3, 4$, as by the assumption L is fully residually free Lie algebra. Hence, $\phi(x_4) = [\phi(x_1), \phi(x_3)] \neq 0$. Now, to prove our claim it is enough to show that either $[\phi(x_1), \phi(x_2)] \neq 0$ or $[\phi(x_2), \phi(x_3)] \neq 0$. But both of which contradict the assumptions $[x_1, x_2] = 0$ and $[x_2, x_3] = 0$, respectively. Thus $[x_1, x_3] = 0$ and L is CT Lie algebra. □

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