Indian J. Pure Appl. Math., **47**(3): 491-500, September 2016

DOI: 10.1007/s13226-016-0180-9

ON A MATRIX TRACE INEQUALITY DUE TO ANDO, HIAI AND OKUBO

Lucijan Plevnik[1]

Institute of Mathematics, Physics and Mechanics, Jadranska 19,
SI-1000 *Ljubljana, Slovenia*
e-mail: lucijan.plevnik@fmf.uni-lj.si

(*Received* 04 *October* 2015; *accepted* 17 *December* 2015)

Ando *et al*. have proved that inequality $\mathfrak{Re}\,\mathrm{tr}\,A^{p_1}B^{q_1}\cdots A^{p_k}B^{q_k} \le \mathrm{tr}\,A^{p_1+\cdots+p_k}B^{q_1+\cdots+q_k}$ is valid for all positive semidefinite matrices A, B and those nonnegative real numbers $p_1, q_1, \ldots, p_k, q_k$ which satisfy certain additional conditions. We give an example to show that this inequality is not valid for all collections of $p_1, q_1, \ldots, p_k, q_k \ge 0$. We also study related trace inequalities.

Key words : Inequality; positive semidefinite matrix; log-convexity

1. Introduction

If A is a complex square matrix, then we write $A \ge 0$ if A is positive semidefinite. The famous inequality due to Lieb and Thirring reads as

$$\mathrm{tr}\,AB\ldots AB = \mathrm{tr}\,(AB)^n \le \mathrm{tr}\,A^nB^n$$

for all $A, B \ge 0$ and natural numbers n. It has been of interest in the literature whether the trace of some other word in A and B in which each of the letters occurs n times can be bounded with $\mathrm{tr}\,A^nB^n$ from above and $\mathrm{tr}\,(AB)^n$ from below, see e. g., [11]. But in these estimates we need to consider either a real part or an absolute value of the trace of a word, as the latter needs not be neither positive [15] nor real [11, Remark 2.7]. Generally, Ando *et al*. considered in [2] the inequalities of the form

$$\mathfrak{Re}\,\mathrm{tr}\,A^{p_1}B^{q_1}\cdots A^{p_k}B^{q_k} \le \mathrm{tr}\,A^{p_1+\cdots+p_k}B^{q_1+\cdots+q_k}, \tag{1.1}$$

[1]Currently at Institute Jožef Stefan, Jamova cesta 39, SI-1000 Ljubljana, Slovenia.

where $k \geq 2$ is a natural number and $p_1, q_1, \dots, p_k, q_k$ are real numbers. Moreover, here and throughout the paper, $|||\cdot|||$ will denote an arbitrary unitarily invariant norm. In [1], Ando and Hiai were studying related inequalities of the form

$$|||A^{p_1}B^{q_1}\cdots A^{p_k}B^{q_k}||| \leq \left|\left|\left|A^{p_1+\ldots+p_k}B^{q_1+\ldots+q_k}\right|\right|\right|. \tag{1.2}$$

They showed that inequalities (1.1) and (1.2) are valid whenever $p_1, q_1, \dots, p_k, q_k$ are nonnegative and satisfy some additional restrictions. They pointed out that the problems were subtle for general p_1, q_1, …, p_k, $q_k \geq 0$ and that they did not have any counterexamples. It is natural to ask what is the complete range of validity of these inequalities. In this paper we find an example which shows that neither (1.1) nor (1.2) holds for all nonnegative $p_1, q_1, \dots, p_k, q_k$. Moreover, we will provide an example which disproves an estimate in the other direction

$$\operatorname{tr}\left(A^{\frac{p_1+\ldots+p_k}{k}}B^{\frac{q_1+\ldots+q_k}{k}}\right)^k \leq \left|\operatorname{tr} A^{p_1}B^{q_1}\cdots A^{p_k}B^{q_k}\right|,$$

as well.

Trace inequalities of type (1.1) have also been studied in another context. For nonnegative real numbers a and b and $0 \leq t \leq 1$, their Heinz mean equals $H_t(a,b) = \frac{a^{1-t}b^t + a^t b^{1-t}}{2}$. These means have been studied a lot in the literature and their fundamental property is that they interpolate geometric and arithmetic mean in the following sense:

$$H_{\frac{1}{2}}(a,b) = \sqrt{ab} \leq H_t(a,b) \leq \frac{a+b}{2} = H_0(a,b) = H_1(a,b). \tag{1.3}$$

In accordance with the definition for nonnegative real numbers, for $A, B \geq 0$ and $0 \leq t \leq 1$, we set $H_t(A,B) = \frac{1}{2}\left(A^{1-t}B^t + A^t B^{1-t}\right)$. Several matrix versions of (1.3) have been established, two of the most general can be found in [5, Corollary IX.4.10] and [3, Theorem 2] which tell in particular that

$$\left|\left|\left|A^{\frac{1}{2}}B^{\frac{1}{2}}\right|\right|\right| \leq |||H_t(A,B)||| \leq \left|\left|\left|\frac{A+B}{2}\right|\right|\right|, \quad 0 \leq t \leq 1.$$

In [10], Bourin asked whether a related inequality holds:

$$|||A^pB^q + B^pA^q||| \leq \left|\left|\left|A^{p+q} + B^{p+q}\right|\right|\right|, \quad A, B \geq 0, \quad p, q \geq 0. \tag{1.4}$$

In an attempt to answer this question positively, Hayajneh and Kittaneh [12] have conjectured that

$$|||A^pB^q + B^pA^q||| \leq |||A^pB^q + A^qB^p|||, \quad A, B \geq 0, \quad p, q \geq 0. \tag{1.5}$$

They also proved a very special case: when $|||\cdot|||$ is the Frobenius norm $\|\cdot\|_2$, $p = 1$, 2, or 3, and $q = 1$, then (1.5) holds. In order to prove (1.5), one may assume that $p + q = 1$ after considering

$A^{\frac{1}{p+q}}$ and $B^{\frac{1}{p+q}}$ instead of A and B, respectively. Then a straightforward computation shows that in the case of the Frobenius norm, (1.5) is equivalent to

$$\mathfrak{Re}\,\mathrm{tr}\,A^tB^{1-t}A^{1-t}B^t \leq \mathrm{tr}\,AB, \quad 0 \leq t \leq 1, \tag{1.6}$$

which is a special case of inequality (1.1). Bhatia [6] has shown that (1.6) holds when $\frac{1}{4} \leq t \leq \frac{3}{4}$. Recently, Bottazzi *et al*. [7] have proved that the inequality actually holds whenever t belongs to a vertical strip in a complex plane containing the interval $[\frac{1}{4}, \frac{3}{4}]$. Moreover, they have found a counterexample which shows that (1.5) does not hold for all $0 \leq t \leq 1$ if $|||\cdot|||$ is the operator norm.

As mentioned before, in the papers [11] and [12] some inequalities of the type (1.1) were studied. For example, for all $A, B \geq 0$ and $p, q \geq 0$ we have

$$\mathrm{tr}\,\left(AB^{\frac{p+q}{2}}\right)^2 \leq \mathrm{tr}\,AB^pAB^q \leq \mathrm{tr}\,A^2B^{p+q}. \tag{1.7}$$

In fact, [9, Theorem 1.2] and [8, Theorem 1] give even stronger version of the second inequality:

$$\mathrm{tr}\,Af(B)\,Ag(B) \leq \mathrm{tr}\,A^2f(B)\,g(B)$$

for nonnegative nondecreasing functions f and g. In Section 3, we will generalize (1.7) in two different directions.

There is another interesting question posed by Bourin in [10]. Motivated by results on subadditivity, he asked whether we have

$$\left|\left|\left|A^{p+q} + B^{p+q}\right|\right|\right| \leq \left|\left|\left|(A^p + B^p)^{\frac{1}{2}}(A^q + B^q)(A^p + B^p)^{\frac{1}{2}}\right|\right|\right| \tag{1.8}$$

for all $A, B \geq 0$ and $p, q \geq 0$. Again, it suffices to check if

$$|||A + B||| \leq \left|\left|\left|(A^t + B^t)^{\frac{1}{2}}(A^{1-t} + B^{1-t})(A^t + B^t)^{\frac{1}{2}}\right|\right|\right|, \quad 0 \leq t \leq 1, \tag{1.9}$$

holds. Hayajneh and Kittaneh showed in [13] that the answer is affirmative in the case when $|||\cdot|||$ is either the trace norm $\|\cdot\|_1$ or the Frobenius norm $\|\cdot\|_2$. However, after examining the proof, one sees that at least for these two norms it is natural to seek a stroger inequality. Indeed, for the trace norm, the left-hand side in (1.9) equals $\mathrm{tr}\,(A + B)$, while the right-hand side equals $\mathrm{tr}\,(A + B)$ plus additional terms $\mathrm{tr}\,A^tB^{1-t} + \mathrm{tr}\,A^{1-t}B^t$. In the case of the Frobenius norm, we get quite a few additional terms, see [13]. In the last section we propose what we find a natural candidate for the strengthening of (1.9) and give positive results in the cases of the trace and Frobenius norms. The general problem remains open, but a recent result of Audenaert [4, Theorem 3.1] yields as a corollary a weaker version of (1.8) which reads like

$$\left|\left|\left|A^{p+q} + B^{p+q}\right|\right|\right| \leq |||(A^p + B^p)(A^q + B^q)|||.$$

For any $n \times n$ matrix A denote its eigenvalues by $\lambda_j(A)$, arranged such that $|\lambda_1(A)| \geq \ldots \geq |\lambda_n(A)|$. The vector of eigenvalues $(\lambda_1(A), \ldots, \lambda_n(A))$ will be denoted by $\lambda(A)$. We use the standard notation for majorization, see e. g., [5, Chapter II]. For vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ with $x_1 \geq \ldots \geq x_n \geq 0$ and $y_1 \geq \ldots \geq y_n \geq 0$ we write

$$x \prec_{\text{w log}} y$$

if

$$\prod_{j=1}^{k} x_j \leq \prod_{j=1}^{k} y_j, \quad k = 1, \ldots, n.$$

Such a majorization implies a weak majorization $x \prec_w y$, that is

$$\sum_{j=1}^{k} x_j \leq \sum_{j=1}^{k} y_j, \quad k = 1, \ldots, n.$$

2. Inequalities of Type (1.1) and (1.2)

We start the section with the example which shows that (1.1) is not valid for all collections $p_1, q_1, \ldots, p_k, q_k \geq 0$.

Example 2.1 : Let

$$A = \begin{bmatrix} 76 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 20 & -14 & 13 \\ -14 & 2880 & 3100 \\ 13 & 3100 & 3380 \end{bmatrix}.$$

Then we have $A, B \geq 0$ and

$$\operatorname{tr} A^4BAB^4 = 7608677695167720100 > 7566365725138281700 = \operatorname{tr} A^5B^5. \tag{2.1}$$

Remark 2.2 : Example 2.1 shows that (1.6) does not hold for matrices $A^5, B^5 \geq 0$ and $t = \frac{4}{5}$.

However, we haven't been able to answer the following natural question.

Question 2.3 : Is the interval $\left[\frac{1}{4}, \frac{3}{4}\right]$ the maximal one on which the inequality (1.6) holds?

Corollary 2.4 — If A and B are as in Example 2.1, then

$$\left\|A^{\frac{3}{2}}BAB^{\frac{3}{2}}\right\|_2 \;>\; \left\|A^{\frac{5}{2}}B^{\frac{5}{2}}\right\|_2.$$

In particular, (1.2) does not hold for all collections $p_1, q_1, \ldots, p_k, q_k \geq 0$.

PROOF : It is well-known that for any two matrices X and Y of appropriate sizes we have

$$|\operatorname{tr} X^*Y| \le \frac{\operatorname{tr} X^*X + \operatorname{tr} Y^*Y}{2}.$$

By applying this equality for $X = A^{\frac{5}{2}}B^{\frac{5}{2}}$ and $Y = A^{\frac{3}{2}}BAB^{\frac{3}{2}}$ we get

$$|\operatorname{tr} A^4BAB^4| = |\operatorname{tr} X^*Y| \le \frac{\left\|A^{\frac{5}{2}}B^{\frac{5}{2}}\right\|_2^2 + \left\|A^{\frac{3}{2}}BAB^{\frac{3}{2}}\right\|_2^2}{2}.$$

Since $\operatorname{tr} A^5B^5 = \left\|A^{\frac{5}{2}}B^{\frac{5}{2}}\right\|_2^2$, Example 2.1 now yields

$$\left\|A^{\frac{5}{2}}B^{\frac{5}{2}}\right\|_2^2 < \frac{\left\|A^{\frac{5}{2}}B^{\frac{5}{2}}\right\|_2^2 + \left\|A^{\frac{3}{2}}BAB^{\frac{3}{2}}\right\|_2^2}{2},$$

which implies the required result. □

Example 2.5 : Let

$$C = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 2 & -2 \\ 2 & 6 & 6 \\ -2 & 6 & 15 \end{bmatrix}.$$

Then we have $C, D \ge 0$ and

$$\operatorname{tr}\ \left(C^3D^3\right)^2 = 52607744 > 25527680 = \operatorname{tr} C^5DCD^5.$$

We finish the section with two propositions which both generalize (1.7).

Proposition 2.6 — Let $A, B \ge 0$ and $p, q \ge 0$. Then we have

$$\lambda\left(AB^{\frac{p+q}{2}}\right)^2 \prec_{\text{w log}} \lambda\left(AB^pAB^q\right) \prec_{\text{w log}} \lambda\left(A^2B^{p+q}\right).$$

In particular,

$$\operatorname{tr}\ \left(AB^{\frac{p+q}{2}}\right)^{2r} \le \operatorname{tr}\ \left(AB^pAB^q\right)^r \le \operatorname{tr}\ \left(A^2B^{p+q}\right)^r$$

for all $r \ge 0$.

PROOF : It is enough to show that

$$\lambda_1\left(AB^{\frac{p+q}{2}}\right)^2 \le \lambda_1\left(AB^pAB^q\right) \le \lambda_1\left(A^2B^{p+q}\right) \tag{2.2}$$

and then we can apply a standard argument using antisymmetric tensors, see e.g. proof of [5, Theorem IX.2.9]. There is nothing to prove if $p = q = 0$, so assume that $p+q > 0$. Let us first prove the second inequality in (2.2). After multiplying A and B with the same positive scalar, if necessary, we may assume that $\lambda_1\left(A^2B^{p+q}\right)=1$, that is $B^{p+q}\leq A^{-2}$. Then we have $B^p\leq A^{-\frac{2p}{p+q}}$ and $A^{\frac{2q}{p+q}}\leq B^{-q}$, so

$$\lambda_1\left(AB^pAB^q\right)=\lambda_1\left(B^{\frac{q}{2}}AB^pAB^{\frac{q}{2}}\right)\leq\lambda_1\left(B^{\frac{q}{2}}A^{\frac{2q}{p+q}}B^{\frac{q}{2}}\right)\leq 1,$$

as desired. We proceed with the proof of the first inequality in (2.2). If $\|\cdot\|$ is the operator norm, then we have

$$\lambda_1\left(AB^{\frac{p+q}{2}}\right)^2=\lambda_1\left(B^{\frac{p+q}{4}}AB^{\frac{p+q}{4}}\right)^2=\left\|B^{\frac{p+q}{4}}AB^{\frac{p+q}{4}}\right\|^2.$$

Because $B^{\frac{p+q}{4}}AB^{\frac{p+q}{4}}$ is Hermitian, the latter is not larger than

$$\left\|B^{\frac{q}{2}}AB^{\frac{p}{2}}\right\|^2=\left\|\left(B^{\frac{q}{2}}AB^{\frac{p}{2}}\right)\left(B^{\frac{p}{2}}AB^{\frac{q}{2}}\right)\right\|=\lambda_1\left(B^{\frac{q}{2}}AB^pAB^{\frac{q}{2}}\right)=\lambda_1\left(AB^pAB^q\right). \quad \square$$

Recall that positive function f is log-convex if $\log\circ f$ is convex and that this condition is stronger than convexity.

Proposition 2.7 — Let $A,B\geq 0$. Then the function $f:[0,\infty)\times[0,\infty)\to[0,\infty)$, defined by

$$f\left(p,q\right)=|||B^qAB^pAB^q|||,\quad p,q\in[0,\infty),$$

is log-convex.

PROOF : Let $p_1,p_2,q_1,q_2\in[0,\infty)$. Then

$$f\left(\frac{p_1+p_2}{2},\frac{q_1+q_2}{2}\right)=\left|\left|\left|B^{\frac{q_1+q_2}{2}}AB^{\frac{p_1+p_2}{2}}AB^{\frac{q_1+q_2}{2}}\right|\right|\right|\leq\left|\left|\left|\left(B^{q_1}AB^{\frac{p_1}{2}}\right)\left(B^{\frac{p_2}{2}}AB^{q_2}\right)\right|\right|\right|,$$

because $B^{\frac{q_1+q_2}{2}}AB^{\frac{p_1+p_2}{2}}AB^{\frac{q_1+q_2}{2}}$ is Hermitian. By setting $X=B^{\frac{p_1}{2}}AB^{q_1}$ and $Y=B^{\frac{p_2}{2}}AB^{q_2}$ in Cauchy-Schwarz inequality

$$|||X^*Y|||\leq\sqrt{|||XX^*|||\,|||YY^*|||}=\sqrt{|||X^*X|||\,|||Y^*Y|||}\tag{2.3}$$

[5, Theorem IX.5.1], we consequently get

$$f\left(\frac{p_1+p_2}{2},\frac{q_1+q_2}{2}\right)\leq\sqrt{f\left(p_1,q_1\right)f\left(p_2,q_2\right)},$$

as desired. $\square$

Remark 2.8 : Proposition 2.7 is a generalization of (1.7), because it tells in particular that for $p,q\geq 0$, the function

$$f\left(r,s\right)=\left\|\left(B^{\frac{1}{2}}\right)^sAB^rA\left(B^{\frac{1}{2}}\right)^s\right\|_1=\operatorname{tr}AB^rAB^s$$

is log-convex on the line segment $\{((1-t)(p+q), t(p+q)) \,:\, 0 \le t \le 1\}$. Since it is symmetric with respect to the middle of the segment $\left(\frac{p+q}{2}, \frac{p+q}{2}\right)$, it attains minimum there, while it attains maximums in the edges $(p+q, 0)$, $(0, p+q)$.

3. Another Bourin Question

In this section we will propose our candidate for the strengthening of (1.9). In order to justify our choice, we will first prove a reverse inequality. Fix arbitrary $A, B \ge 0$ and denote

$$F(t) = (A^t + B^t)^{\frac{1}{2}}(A^{1-t} + B^{1-t})(A^t + B^t)^{\frac{1}{2}}, \quad 0 \le t \le 1.$$

Note that $F(t)$ and $F(1-t)$ are unitarily equivalent for every $t \in [0,1]$.

The following proposition was proven in [3], but we present a different proof here.

Proposition 3.1 — For any $j \ge 1$ and $t \in [0,1]$, we have

$$\lambda_j(F(t)) \le 2\lambda_j(A+B) = \lambda_j(F(0)) = \lambda_j(F(1)).$$

Proof : The map $x \mapsto x^s$, $x \ge 0$, is matrix concave for $0 \le s \le 1$, see [5, Chapter V]. Hence,

$$\left(\frac{A+B}{2}\right)^s \ge \frac{A^s + B^s}{2},$$

which yields

$$A^s + B^s \le 2^{1-s}(A+B)^s.$$

We apply this equation twice and get

$$\begin{aligned}
\lambda_j(F(t)) &\le 2^t \lambda_j\left((A^t + B^t)^{\frac{1}{2}} (A+B)^{1-t} (A^t + B^t)^{\frac{1}{2}}\right) \\
&= 2^t \lambda_j\left((A+B)^{\frac{1-t}{2}} (A^t + B^t) (A+B)^{\frac{1-t}{2}}\right) \\
&\le 2^t 2^{1-t} \lambda_j\left((A+B)^{\frac{1-t}{2}} (A+B)^t (A+B)^{\frac{1-t}{2}}\right) = 2\lambda_j(A+B).
\end{aligned}$$

□

The last proposition tells that the function $\lambda_j(F(t))$ has maximums at $t = 0$ and $t = 1$. In particular, for every unitarily invariant norm, the function $|||F(t)|||$ has maximums at $t = 0$ and $t = 1$. This brings us to a natural question.

Question 3.2 : Do functions $\lambda_j(F(t))$, $j \ge 1$, or $|||F(t)|||$ have a minimum at $t = \frac{1}{2}$? Moreover, are they convex or log-convex?

Since $\lambda_j(F(1-t)) = \lambda_j(F(t))$ for any $j \ge 1$ and $t \in [0,1]$, the convexity condition is indeed stronger than the condition about having minimum at $t = \frac{1}{2}$. But the latter condition is stronger than

(1.9). Indeed, if we set $g(t) = \left|\left|\left|(A^t + B^t)^{\frac{1}{t}}\right|\right|\right|$, then $|||A + B||| = g(1)$ and $\left|\left|\left|F\left(\frac{1}{2}\right)\right|\right|\right| = g\left(\frac{1}{2}\right)$. But we have $g\left(1\right) \leq g\left(\frac{1}{2}\right)$, because $t \mapsto g\left(t\right)$ is a decreasing function on $(0, 1]$, as was shown in [14].

We will consider the question about log-convexity for the trace norm and the Frobenius norm. When doing that, we will need the following proposition.

Proposition 3.3 — Let $A, B \geq 0$ and $a, b, c, d \in \mathbb{R}$. Then the function

$$f\left(t\right) = \left|\left|\left|A^{at+c}B^{bt+d}A^{at+c}\right|\right|\right|$$

is log-convex on the region of its definiteness D_f.

PROOF : Note that if A and B are invertible, then $D_f = \mathbb{R}$. If A (resp. B) is not invertible, then f is defined only where $at + c \geq 0$ (resp. $bt + d \geq 0$). In these cases, D_f is either an interval or a point or an empty set.

For every $t \in D_f$ set $X_t = B^{\frac{bt+d}{2}}A^{at+c}$ and suppose that D_f contains at least two points, say t and s. We also have $\frac{t+s}{2} \in D_f$, so one can compute $A^{a\frac{t+s}{2}+c}B^{b\frac{t+s}{2}+d}A^{a\frac{t+s}{2}+c}$ which is a hermitian matrix. Thus,

$$f\left(\frac{t+s}{2}\right) = \left|\left|\left|A^{a\frac{t+s}{2}+c}B^{b\frac{t+s}{2}+d}A^{a\frac{t+s}{2}+c}\right|\right|\right| \leq \left|\left|\left|A^{at+c}B^{b\frac{t+s}{2}+d}A^{as+c}\right|\right|\right| = |||X_t^*X_s|||,$$

which is by (2.3) bounded from above by $\sqrt{|||X_t^*X_t||| |||X_s^*X_s|||} = \sqrt{f\left(t\right)f\left(s\right)}$, as desired. □

In the sequel we will need an obvious fact that a positive power of a log-convex function is log-convex and a well-known fact that the sum of two log-convex functions is log-convex.

Corollary 3.4 — Let $A, B \geq 0$, $p \geq 1$, and $a, b, c, d \in \mathbb{R}$. Then the function

$$t \mapsto \operatorname{tr}\left(A^{at+c}B^{bt+d}\right)^p, \quad 0 \leq t \leq 1, \tag{3.1}$$

is log-convex on the region of its definiteness.

PROOF : By Proposition 3.3, the function from (3.1) is a p-th power of log-convex function

$$t \mapsto \left\|A^{\frac{at+c}{2}}B^{bt+d}A^{\frac{at+c}{2}}\right\|_p.$$

***Theorem* 3.5** — *Function*

$$t \mapsto \|F(t)\|_p, \quad t \in [0, 1],$$

is log-convex for $p \in \{1, 2\}$.

PROOF : We have

$$\|F(t)\|_1 = \text{tr }(A+B) + \text{tr }\left(A^t B^{1-t}\right) + \text{tr }\left(A^{1-t} B^t\right)$$

and

$$\begin{aligned}
||F(t)||_2^2 &= \text{tr }\left(A+B+A^tB^{1-t}+B^tA^{1-t}\right)^2 \\
&= \text{tr }(A+B)^2 + 2\text{tr }(A+B)\left(A^tB^{1-t}+B^tA^{1-t}\right) + \text{tr }\left(A^tB^{1-t}+B^tA^{1-t}\right)^2 \\
&= \text{tr }(A+B)^2 + 2\text{tr}\, A^{1+t}B^{1-t} + 2\text{tr}\, A^{2-t}B^t + 2\text{tr}\, B^{1+t}A^{1-t} \\
&\quad +2\text{tr}\, B^{2-t}A^t + \text{tr }\left(A^tB^{1-t}\right)^2 + \text{tr }\left(A^{1-t}B^t\right)^2 + 2\text{tr}\, AB,
\end{aligned}$$

so the claim follows from Corollary 3.4. □

Corollary 3.6 — Functions

$$t \mapsto \text{tr }\left(A^{1-t}B^t + A^tB^{1-t}\right), \quad t \in [0,1],$$

and

$$t \mapsto \text{tr }\left(A+B+A^tB^{1-t}+B^tA^{1-t}\right)^2, \quad t \in [0,1],$$

are decreasing on $\left[0, \frac{1}{2}\right]$ and increasing on $\left[\frac{1}{2}, 1\right]$.

ACKNOWLEDGEMENT

This paper is a result of the author's research visit at Indian Statistical Institute in 2014 which was funded by Slovene human resources development and scholarship fund. The author would like to thank:

- Professor Rajendra Bhatia for his hospitality during the visit, for introduction into this area of research, and for discussions,
- Professor Ajit Iqbal Singh for her suggestions which simplified computations,
- Professor Jean-Christophe Bourin for his information about the results from papers [4], [9] and [8], and for suggesting to generalize convexity results to log-convexity.

REFERENCES

1. T. Ando and F. Hiai, Log majorization and complementary Golden-Thompson type inequalities, *Linear Algebra Appl.*, **198** (1994), 113-131.

2. T. Ando, F. Hiai and K. Okubo, Trace inequalities for multiple products of two matrices, *Math. Inequal. Appl*., **3** (2000), 307-318.

3. K. Audenaert, A singular value inequality for Heinz means, *Linear Algebra Appl*., **422** (2007), 279-283.

4. K. Audenaert, A norm inequality for pairs of commuting positive semidefinite matrices, *Electron. J. Linear Algebra*, **30** (2015), DOI: http://dx.doi.org/10.13001/1081-3810.2829.

5. R. Bhatia, Matrix Analysis, *Springer*, Berlin, 1997.

6. R. Bhatia, Trace inequalities for products of positive definite matrices, *J. Math. Phys*., **55** (2014).

7. T. Bottazzi, R. Elencwajg, G. Larotonda and A. Varela, Inequalities related to Bourin and Heinz means with a complex parameter, *J. Math. Anal. Appl*., **426** (2015), 765-773.

8. J.-C. Bourin, Some inequalities for norms on matrices and operators, *Linear Algebra Appl*., **292** (1999), 139-154.

9. J.-C. Bourin, Matrix versions of some classical inequalities, *Linear Algebra Appl*., **416** (2006), 890-907.

10. J.-C. Bourin, Matrix subadditivity inequalities and block-matrices, *Int. J. Math*., **20** (2009), 679-691.

11. S. Furuichi, K. Kuriyama and K. Yanagi, Trace inequalities for products of matrices, *Linear Algebra Appl*., **430** (2009), 2271-2276.

12. S. Hayajneh and F. Kittaneh, Lieb-Thirring trace inequalities and a question of Bourin, *J. Math. Phys*., **54**, 033504 (2013); doi: 10.1063/1.4793993.

13. S. Hayajneh and F. Kittaneh, Trace inequalities and a question of Bourin, *Bull. Aust. Math. Soc*., **88** (2013), 384-389.

14. F. Hiai and X. Zhan, Inequalities involving unitarily invariant norms and operator monotone functions, *Linear Algebra Appl*., **341** (2002), 151-169.

15. C. J. Hillar and C. R. Johnson, Eigenvalues of words in two positive definite letters, *SIAM J. Matrix Anal. Appl*., **23** (2002), 916-928.