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ESTIMATES FOR WALLIS' RATIO AND RELATED FUNCTIONS

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We present improvements of approximation formula for Wallis ratio related to a class of inequalities stated in [D.-J. Zhao, On a two-sided inequality involving Wallis's formula, *Math. Practice Theory*, **34** (2004), 166-168], [Y. Zhao and Q. Wu, Wallis inequality with a parameter, *J. Inequal. Pure Appl. Math.*, **7**(2) (2006), Art. 56] and [C. Mortici, Completely monotone functions and the Wallis ratio, *Applied Mathematics Letters*, **25** (2012), 717-722]. Some sharp inequalities are obtained as a result of monotonicity of some functions involving gamma function.

Key words : Wallis ratio; Gamma function; polygamma function; complete monotonicity; speed of convergence; inequalities.

1. Introduction and Motivation

The Wallis ratio defined for every integer $n \geq 1$ by

$$P_n = \frac{1 \cdot 3 \cdot ... \cdot (2n-1)}{2 \cdot 4 \cdot ... \cdot 2n}$$

has many applications in pure and applied mathematics as in other branches of science and in consequence it is studied by a large number of authors. This ratio is closely related to the Euler gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$$

since

$$P_n = \frac{1}{\sqrt{\pi}} \cdot \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)}. \tag{1}$$

First Kazarinoff [6] proved that the following inequality holds for every integer $n \geq 1$:

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < P_n \leq \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}. \tag{2}$$

Zhao [16] improved Kazarinoff's result giving the following inequalities for every integer $n \geq 1$:

$$\frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}} < P_n \leq \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{3}}\right)}}. \tag{3}$$

Afterwards Zhao and Wu [17] got an accurate upper bound of (3), proving that for every $0 < \varepsilon < \frac{1}{2}$:

$$P_n \leq \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}+\varepsilon}\right)}}, \tag{4}$$

whenever $n \geq n^*(\varepsilon)$, where $n^*(\varepsilon)$ is the maximal solution of the equation

$$32\varepsilon n^2 + 4\varepsilon^2 n + 32\varepsilon n - 17n + 4\varepsilon^2 - 1 = 0.$$

Interesting results can be also read in [2, 3, 7, 9, 15].

Recently Mortici [10] established the following double inequality for every integer $n \geq 1$:

$$\frac{\alpha}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}} < P_n \leq \frac{\beta}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}}, \tag{5}$$

where $\alpha = 1$ and $\beta = \frac{3\sqrt{7\pi}}{14} = 1,00049...$ are the best possible constants.

Numerical computations we made show that for large values of n, the right-hand side expression in (5) moves away from P_n. In the same time, the left-hand side expression in (5) becomes increasingly closed to P_n. In fact, this is normal if we take into account that (5) is a consequence of the complete monotonicity on $[1, \infty)$ of the function

$$h(x) = \ln \frac{\Gamma\left(x+\frac{1}{2}\right)}{\Gamma(x+1)} \sqrt{x\left(1+\frac{1}{4x-\frac{1}{2}}\right)}.$$

More precisely, (5) follows from $0 = h(\infty) < h(x) \leq h(1) = \ln \frac{3\sqrt{7\pi}}{14}$.

According to this remark, we deduce that accurate approximations of the form

$$P_n \approx \frac{\lambda(n)}{\sqrt{\pi n \left(1 + \frac{1}{4n - \frac{1}{2}}\right)}} \tag{6}$$

are obtained for sequences $\lambda(n) \to 1$, as n approaches infinity. In the next section we find such sequences of the form

$$\lambda(n) = \exp\left(\frac{a}{n^3} + \frac{b}{n^5}\right), \tag{7}$$

where a, b are certain real numbers.

2. A Family of Approximations for Wallis Ratio

In the first part of this section we prove that

$$P_n \approx \frac{\exp\left(\frac{3}{512n^3} - \frac{51}{32768n^5}\right)}{\sqrt{\pi n \left(1 + \frac{1}{4n - \frac{1}{2}}\right)}}, \quad as \ \ n \to \infty \tag{8}$$

is the most accurate approximation among all approximations of the form (6), with $\lambda(n)$ given by (7).

In order to find the best approximation (6), we introduce the relative error sequence w_n by the following formulas for every integer $n \geq 1$:

$$P_n \approx \frac{\exp\left(\frac{a}{n^3} + \frac{b}{n^5}\right)}{\sqrt{\pi n \left(1 + \frac{1}{4n - \frac{1}{2}}\right)}} \exp w_n,$$

and we consider an approximation (6) better as the speed of convergence of w_n is higher. Since $\frac{P_n}{P_{n+1}} \to 1$, we must have

$$\begin{aligned} w_n - w_{n+1} &= \ln \frac{2n+2}{2n+1} + \frac{1}{2} \ln \frac{n(8n+1)(8n+7)}{(n+1)(8n+9)(8n-1)} \\ &\quad - \left(\frac{a}{n^3} + \frac{b}{n^5}\right) + \left(\frac{a}{(n+1)^3} + \frac{b}{(n+1)^5}\right), \end{aligned}$$

or using Maple software

$$\begin{aligned} w_n - w_{n+1} &= 3\left(\frac{3}{512} - a\right)\frac{1}{n^4} + 6\left(a - \frac{3}{512}\right)\frac{1}{n^5} + 5\left(\frac{333}{32\,768} - 2a - b\right)\frac{1}{n^6} \\ &\quad + 15\left(a + b - \frac{141}{32768}\right)\frac{1}{n^7} + 7\left(\frac{23031}{2097152} - 3a - 5b\right)\frac{1}{n^8} + O\left(\frac{1}{n^9}\right). \end{aligned} \tag{9}$$

As a consequence of a much used lemma of Cesaro-Stolz type in the recent past (we refer to [10-14]; for proof see [14]), the sequence w_n is fastest possible, when $w_n - w_{n+1}$ is fastest possible; that is when the first coefficients in (9) vanish. We get $a = \frac{3}{512}$, $b = -\frac{51}{32\,768}$. In this case,

$$w_n - w_{n+1} = \frac{17\,409}{2097\,152 n^8} + O\left(\frac{1}{n^9}\right).$$

As we explained, (8) is the best approximation among all approximations (6). Related to approximation (8), we present the following bounds of Wallis ratio.

***Theorem* 1** — *The following inequality holds for every integer* $n \geq 1$:

$$\frac{\exp\left(\frac{3}{512n^3} - \frac{51}{32768n^5}\right)}{\sqrt{\pi n\left(1 + \frac{1}{4n-\frac{1}{2}}\right)}} < P_n < \frac{\exp \frac{3}{512n^3}}{\sqrt{\pi n\left(1 + \frac{1}{4n-\frac{1}{2}}\right)}}. \tag{10}$$

This double inequality improves much the results of Kazarinoff (2), Zhao [16], Zhao and Wu [17] and Mortici [10].

PROOF OF THEOREM 1 : Inequality (10) reads as $a_n > 0$ and $b_n < 0$, where

$$a_n = \ln P_n + \frac{1}{2} \ln\left[\pi n\left(1 + \frac{1}{4n - \frac{1}{2}}\right)\right] - \left(\frac{3}{512n^3} - \frac{51}{32768n^5}\right)$$

and

$$b_n = \ln P_n + \frac{1}{2} \ln\left[\pi n\left(1 + \frac{1}{4n - \frac{1}{2}}\right)\right] - \frac{3}{512n^3}.$$

As a_n and b_n converge to zero as $n \to \infty$, it suffices to show that a_n is strictly decreasing and b_n is strictly increasing. In this sense, we have $a_{n+1} - a_n = f(n)$ and $b_{n+1} - b_n = g(n)$, where

$$\begin{aligned} f(x) &= \ln \frac{2x+1}{2x+2} + \frac{1}{2} \ln \frac{(x+1)\left(1 + \frac{1}{4(x+1)-\frac{1}{2}}\right)}{x\left(1 + \frac{1}{4x-\frac{1}{2}}\right)} \\ &\quad - \left(\frac{3}{512(x+1)^3} - \frac{51}{32768(x+1)^5}\right) + \left(\frac{3}{512x^3} - \frac{51}{32768x^5}\right) \end{aligned}$$

and

$$g(x) = \ln \frac{2x+1}{2x+2} + \frac{1}{2} \ln \frac{(x+1)\left(1 + \frac{1}{4(x+1)-\frac{1}{2}}\right)}{x\left(1 + \frac{1}{4x-\frac{1}{2}}\right)} - \frac{3}{512(x+1)^3} + \frac{3}{512x^3}.$$

We have

$$f'(x) = \frac{3P(x-1)}{32768x^6(x+1)^6(8x+1)(8x+9)(2x+1)(8x-1)(8x+7)} > 0$$

and

$$g'(x) = -\frac{9Q(x-1)}{512x^4(x+1)^4(8x+1)(8x+9)(2x+1)(8x-1)(8x+7)} < 0,$$

where

$$\begin{aligned} P(x) &= 666\,366\,876x + 1648\,214\,691x^2 + 2304\,768\,222x^3 \\ &\quad +1993\,818\,699x^4 + 1093\,112\,028x^5 + 371\,032\,316x^6 \\ &\quad +71\,307\,264x^7 + 5942\,272x^8 + 116\,541\,057 \end{aligned}$$

$$\begin{aligned} Q(x) &= 815\,730x + 1472\,366x^2 + 1387\,824x^3 \\ &\quad +720\,904x^4 + 195\,840x^5 + 21\,760x^6 + 184\,301. \end{aligned}$$

In consequence, f is strictly increasing on $[1,\infty)$, g is strictly decreasing on $[1,\infty)$, with $f(\infty) = g(\infty) = 0$, so $f < 0$ and $g > 0$ on $[1,\infty)$. The proof is now completed. □

3. The Estimates in Continuous Version

Taking into account (1), the inequality (10) reads as

$$\frac{\exp\left(\frac{3}{512n^3} - \frac{51}{32768n^5}\right)}{\sqrt{n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}} < \frac{\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} < \frac{\exp\frac{3}{512n^3}}{\sqrt{n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}}. \tag{11}$$

Usually, whenever an approximation formula $u(n) \approx v(n)$ is given it is introduced the function $F(x) = u(x)/v(x)$ to derive its monotonicity. Many times, F (eventually $-F$) is logarithmically completely monotone.

We prove the following result.

Theorem 2 — *For the functions* $F, G : [1,\infty) \to \mathbb{R}$ *given by*

$$F(x) = \ln\Gamma\left(x+\frac{1}{2}\right) - \ln\Gamma(x+1) + \frac{1}{2}\ln x + \frac{1}{2}\ln\frac{8x+1}{8x-1} - \frac{3}{512x^3} + \frac{51}{32768x^5},$$

$$G(x) = \ln\Gamma\left(x+\frac{1}{2}\right) - \ln\Gamma(x+1) + \frac{1}{2}\ln x + \frac{1}{2}\ln\frac{8x+1}{8x-1} - \frac{3}{512x^3}$$

the following assertions hold true:

i) F *is strictly convex, strictly decreasing, with* $F(\infty) = 0$.

ii) G *is strictly concave, strictly increasing, with* $G(\infty) = 0$.

Being strictly decreasing with $F(\infty) = 0$, the function F is positive on $[1, \infty)$. Similarly, G is negative on $[1, \infty)$. Inequalities $F > 0$ and $G < 0$ can be arranged as the following continuous version of (10).

Corollary 1 — The following inequality holds true for every real $x \geq 1$:

$$\frac{\exp\left(\frac{3}{512x^3} - \frac{51}{32768x^5}\right)}{\sqrt{\pi x\left(1 + \frac{1}{4x - \frac{1}{2}}\right)}} < \frac{1}{\sqrt{\pi}} \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} < \frac{\exp \frac{3}{512x^3}}{\sqrt{\pi x\left(1 + \frac{1}{4x - \frac{1}{2}}\right)}}. \tag{12}$$

Moreover, the following inequalities for every real $x \geq 1 : F(\infty) < F(x) \leq F(1) = \ln \frac{3\sqrt{7\pi}}{14} - \frac{141}{32\,768}$ and $G(\infty) > G(x) \geq G(1) = \ln \frac{3\sqrt{7\pi}}{14} - \frac{3}{512}$ can be used to obtain the following stronger result.

Corollary 2 — a) For every real number $x \geq 1$, it is asserted that

$$\alpha \cdot \frac{\exp \frac{3}{512x^3}}{\sqrt{\pi x\left(1 + \frac{1}{4x - \frac{1}{2}}\right)}} \leq \frac{1}{\sqrt{\pi}} \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} < \frac{\exp \frac{3}{512x^3}}{\sqrt{\pi x\left(1 + \frac{1}{4x - \frac{1}{2}}\right)}},$$

where the constant $\alpha = \frac{3\sqrt{7\pi}}{14} \exp\left(-\frac{3}{512}\right) = 0.999\,016...$ is sharp.

b) For every real number $x \geq 1$, it is asserted that

$$\frac{\exp\left(\frac{3}{512x^3} - \frac{51}{32768x^5}\right)}{\sqrt{\pi x\left(1 + \frac{1}{4x - \frac{1}{2}}\right)}} < \frac{1}{\sqrt{\pi}} \frac{\Gamma(x + \frac{1}{2})}{\Gamma(x + 1)} \leq \beta \cdot \frac{\exp\left(\frac{3}{512x^3} - \frac{51}{32768x^5}\right)}{\sqrt{\pi x\left(1 + \frac{1}{4x - \frac{1}{2}}\right)}},$$

where the constant $\beta = \frac{3\sqrt{7\pi}}{14} \exp\left(-\frac{141}{32\,768}\right) = 1.\,000\,572...$ is sharp.

In what follows we essentially use a result of Alzer [1], who proved that the following functions are completely monotonic on $(0, \infty)$ for every integer $n \geq 1$:

$$F_n(x) = \ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x + x - \ln \sqrt{2\pi} - \sum_{i=1}^{2n} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}$$

$$G_n(x) = -\ln \Gamma(x) + \left(x - \frac{1}{2}\right) \ln x - x + \ln \sqrt{2\pi} + \sum_{i=1}^{2n-1} \frac{B_{2i}}{2i(2i-1)x^{2i-1}}.$$

Here B_i's are the Bernoulli numbers defined by

$$\frac{x}{e^x - 1} = \sum_{i=0}^{\infty} B_i \frac{x^i}{i!}, \quad |x| < 2\pi.$$

In particular, from $F_4'' > 0$ and $G_5'' > 0$, we get the following bounds for ψ' function for every real $x > 0$:

$$a(x) < \psi'(x) < b(x), \tag{13}$$

where

$$a(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7} - \frac{1}{30x^9}$$
$$b(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{1}{42x^7}.$$

PROOF OF THEOREM 2 : Standard computations lead us to

$$F''(x) = \psi'\left(x + \frac{1}{2}\right) - \psi'(x+1) - \frac{1}{2x^2} + \frac{32}{(8x-1)^2} - \frac{32}{(8x+1)^2} - \frac{9}{128x^5} + \frac{765}{16\,384x^7}$$

$$G''(x) = \psi'\left(x + \frac{1}{2}\right) - \psi'(x+1) - \frac{1}{2x^2} - \frac{32}{(8x+1)^2} + \frac{32}{(8x-1)^2} - \frac{9}{128x^5}.$$

Using (13), we get $F''(x) > u(x)$, $G''(x) < v(x)$, where

$$u(x) = a\left(x + \frac{1}{2}\right) - b(x+1) - \frac{1}{2x^2} + \frac{32}{(8x-1)^2} - \frac{32}{(8x+1)^2} - \frac{9}{128x^5} + \frac{765}{16\,384x^7}$$

$$v(x) = b\left(x + \frac{1}{2}\right) - a(x+1) - \frac{1}{2x^2} - \frac{32}{(8x+1)^2} + \frac{32}{(8x-1)^2} - \frac{9}{128x^5}.$$

But

$$u(x) = \frac{A(x-1)}{1720\,320x^7(64x^2-1)^2(2x+1)^7(x+1)^9} > 0$$

$$v(x) = -\frac{B(x-1)}{13440x^5(64x^2-1)^2(x+1)^9(2x+1)^7} < 0,$$

where

$$A(x) = 119\,333\,322\,752x^{18} + 4602\,664\,779\,776x^{17} + \cdots$$

$$B(x) = 329\,011\,200x^{18} + 10\,034\,841\,600x^{17} + \cdots$$

are polynomials with all coefficients positive.

It follows that F is strictly convex, while G is strictly concave on $[1, \infty)$.

Furthermore, F' is strictly increasing and G' is strictly decreasing on $[1, \infty)$ and $F'(\infty) = G'(\infty) = 0$, so $F' < 0$ and $G' > 0$. Finally, F is strictly decreasing and G is strictly increasing with $F(\infty) = G(\infty) = 0$ and the proof is completed. □

4. Comparison Tests

As we mentioned in the previous sections, the inequalities (12) improve considerably the results stated in [6, 10, 16, 17].

Recently, being preocupied to find approximation formulas for Wallis' ratio involving roots of higher order, Mortici [13, Theorem 3.1] presented a better inequality for every real $x \geq 2$:

$$\sqrt[4]{x^2+\frac{1}{2}x+\frac{1}{8}-\frac{1}{128x}}<\frac{\Gamma(x+1)}{\Gamma(x+\frac{1}{2})}<\sqrt[4]{x^2+\frac{1}{2}x+\frac{1}{8}}. \tag{14}$$

We show in the next Proposition 1 that our result (12) from Corollary 1 is stronger than (14).

Proposition 1 — The following inequalities hold true for every real $x \geq 2$:

$$\sqrt[4]{x^2+\frac{1}{2}x+\frac{1}{8}-\frac{1}{128x}}<\frac{\sqrt{x\left(1+\frac{1}{4x-\frac{1}{2}}\right)}}{\exp\frac{3}{512x^3}}$$

and

$$\frac{\sqrt{x\left(1+\frac{1}{4x-\frac{1}{2}}\right)}}{\exp\left(\frac{3}{512x^3}-\frac{51}{32768x^5}\right)}<\sqrt[4]{x^2+\frac{1}{2}x+\frac{1}{8}}.$$

Proof : The requested inequalities can be written as $r<0$ and $s>0$, where

$$r(x)=\frac{1}{4}\ln\left(x^2+\frac{1}{2}x+\frac{1}{8}-\frac{1}{128x}\right)-\frac{1}{2}\ln\left[x\left(1+\frac{1}{4x-\frac{1}{2}}\right)\right]+\frac{3}{512x^3}$$

$$s(x)=\frac{1}{4}\ln\left(x^2+\frac{1}{2}x+\frac{1}{8}\right)-\frac{1}{2}\ln\left[x\left(1+\frac{1}{4x-\frac{1}{2}}\right)\right]+\frac{3}{512x^3}-\frac{51}{32768x^5}.$$

We have

$$r'(x)=\frac{3C(x-2)}{512x^4(64x^2-1)(16x+64x^2+128x^3-1)}>0$$

$$s'(x)=-\frac{D(x-2)}{32768x^6(64x^2-1)(8x^2+4x+1)}<0,$$

where

$$C(x)=229\,936x+343\,936x^2+226\,560x^3+69\,632x^4+8192x^5+44\,637$$

$$D(x)=6470\,620x+7706\,104x^2+4388\,864x^3+1212\,416x^4+131\,072x^5+2023\,639.$$

Now r is strictly increasing and s is strictly decreasing on $[2,\infty)$, with $r(\infty)=s(\infty)=0$, so $r<0$ and $s>0$ on $[2,\infty)$ and the conclusion follows. □

Finally remark that our approximation formula

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\approx\gamma_n:=\frac{\sqrt{n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}}{\exp\left(\frac{3}{512n^3}-\frac{51}{32768n^5}\right)}$$

gives much better results then the following formula involving roots of sixth order presented in [13]:

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}\approx\omega_n:=\sqrt[6]{n^3+\frac{3}{4}n^2+\frac{9}{32}n+\frac{5}{128}},$$

as we can see from the following table.

n	$\omega_n-\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}$	$\gamma_n-\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})}$
10	$4.924\,910\,350\times10^{-7}$	$3.744\,611\,952\times10^{-10}$
100	$1.693\,999\,241\times10^{-10}$	$1.187\,208\,943\times10^{-16}$
250	$6.893\,250\,304\times10^{-12}$	$3.073\,570\,023\times10^{-19}$
500	$6.103\,543\,791\times10^{-13}$	$3.395\,055\,792\times10^{-21}$
1000	$5.399\,553\,331\times10^{-14}$	$3.750\,343\,193\times10^{-23}$
3000	$1.155\,279\,331\times10^{-15}$	$2.908\,056\,841\times10^{-26}$

5. Final Remarks

We start this section by giving the motivation of the constants from (7). We obtained these constants by using some computer software, but formula (7) is a part of an entire asymptotic formula.

In this sense, remark that Lin *et al.* [8, Rel. (1.6)] noticed that

$$W_n:=\prod_{k=1}^{n}\frac{4k^2}{4k^2-1}\sim\frac{\pi}{2}\left(1+\frac{1}{2n}\right)^{-1}\exp\left(\sum_{j=0}^{\infty}\frac{c_j}{n^{2j+1}}\right),\quad(n\to\infty),$$

where c_j are given by

$$\tanh\frac{x}{4}=\sum_{j=0}^{\infty}c_j\frac{x^{2j+1}}{(2j)!}.$$

See also [4] and [5]. As

$$P_n=(2n+1)^{-1/2}\,W_n^{-1/2},$$

we can easily get

$$P_n \sim \left(\frac{1}{n\pi}\right)^{1/2} \exp\left(-\sum_{j=0}^{\infty} \frac{c_j}{2n^{2j+1}}\right), \quad (n \to \infty).$$

Finally, we can deduce a complete formula (6) of type

$$P_n \sim \frac{1}{\sqrt{\pi n\left(1+\frac{1}{4n-\frac{1}{2}}\right)}} \exp\left(\sum_{j=0}^{\infty} \frac{d_j}{n^{2j+1}}\right), \tag{15}$$

where the coefficients d_j are given in terms of c_j and using the expansion of

$$\frac{1}{2}\ln\left(1+\frac{1}{4n-\frac{1}{2}}\right)$$

as a series of n^{-1}.

Personal computations we made lead us to $d_0 = 0$, while $d_1 = a$, $d_2 = b$ from (7). More precisely, $a = \frac{3}{512}$, $b = -\frac{51}{32768}$.

As these computations are long to be listed here, we omit them.

Having available the entire asymptotic formula (15), inequalities stated in Theorem 1 and those following can be extended to new inequalities obtained by truncation the series (15) at any jth term. In our opinion, this is not an easy task, as the coefficients d_j have a complicated form. In consequence, we put up this proposal as an open problem.

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