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SPECTRAL ZETA FUNCTION ON PSEUDO H-TYPE NILMANIFOLDS[1]

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Dedicated to Professor Kalyan B. Sinha on the occasion of his 70th birthday.

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We explain the explicit integral form of the heat kernel for the sub-Laplacian on two step nilpotent Lie groups G based on the work of Beals, Gaveau and Greiner. Using such an integral form we study the heat trace of the sub-Laplacian on nilmanifolds $L\backslash G$ where L is a lattice. As an application a common property of the spectral zeta function for the sub-Laplacian on $L\backslash G$ is observed. In particular, we introduce a special class of nilpotent Lie groups, called pseudo H-type groups which are generalizations of groups previously considered by Kaplan. As is known such groups always admit lattices. Here we aim to explicitly calculate the heat trace and the spectrum of the (sub)-Laplacian on various low dimensional compact nilmanifolds including several pseudo H-type nilmanifolds $L\backslash G$, i.e. where G is a pseudo H-type group. In an appendix we discuss a zeta function which typically appears as the Mellin transform for these heat traces.

Key words : Sub-Laplacian; sub-Riemannian manifold; nilmanifold; heat kernel; spectral zeta function; pseudo H-type group; admissible module.

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1. Introduction

During the last century there has been an increasing interest in studying the global theory of elliptic operators, in particular, in relation with manifold theory, theoretical physics and analysis. Even nowadays the research is far from being complete and branches out to various directions.

The notion of a Riemannian manifold forms the underlying concept for developing elliptic operator theory. In various contexts spectral invariants are studied and from the view point of spectral analysis they form an interesting subject themselves.

The main topic of the present note forms the study of spectral zeta functions induced by an intrinsic type of operators acting on a class of nilmanifolds. In general, the underlying geometric structure is of non-holonomic type (or sub-Riemannian) and instead of the Laplacian which is elliptic and appears in the framework of Riemannian manifolds we deal with a sub-Laplacian.

The latter one is not elliptic, but is a "sub-elliptic operator". It satisfies an "a priori estimate" with a loss of derivatives, a weaker version of the standard "a priori estimate". In several aspects this estimate enables us to treat sub-elliptic operator similarly to elliptic operators. For example, a proof of sub-elliptic estimates implies the hypo-ellipticity of the sub-Laplacian analogous to the elliptic case and it implies that the resolvent of the sub-Laplacian is compact if the manifold is compact. Recall that for elliptic operators the existence of an "a priori estimate" is equivalent to the non-vanishing of the principal symbol at any point of the punctured cotangent bundle $T_0^*(M) = T^*(M)\backslash\{0\}$. However, there is no such characterization of "sub-ellipticity" solely in terms of the principal symbol. This difference causes an obstruction in developing a cohomology theory (like K-theory for elliptic operators) in the framework of sub-elliptic operators directly. Moreover, whereas every manifold can be equipped with a Riemannian metric, there is no unified method to determine whether a given manifold carries a sub-Riemannian structure.

Many examples of sub-Riemannian manifolds are known. Among them we mention contact manifolds (for example, all odd dimensional spheres), CR-manifolds or Lie groups (even though it is not clear whether compact symmetric spaces have such a structure in general). In [6] we have classified the trivializable sub-Riemannian structures on n-dimensional Euclidean spheres $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ that are induced by a Clifford module structure on $\mathbb{R}^{n+1}$, (cf. [4]). As a result we could prove that (up to equivalence) only a few cases of bracket generating trivial sub-bundles of $T\mathbb{S}^n$ exist that determine such a structure. Moreover, it is known that there are sub-Riemannian structures induced by bracket generating distributions of co-dimenison 3 on each $4k+3$-dimensional sphere which in general (and similar to the contact structure) are not trivializable (cf. [18]). However, there are no bracket generating sub-bundles on even dimensional spheres.

The sub-Riemannian manifolds M we are going to treat here are the so-called nilmanifolds. More precisely, M is a compact quotient of a simply connected 2-step nilpotent Lie group by a lattice (= uniform discrete subgroup). Many of the properties we state in Section 2 are also valid for higher step cases. Based on arguments from Fourier analysis our approach relies on a heat kernel expression of the sub-Laplacian which is only known explicitly in the 2-step case (see [10, 11] and the references cited in these papers).

In some cases, sub-Laplacians can be thought as a countable sum of elliptic operators on a lower dimensional manifold. More precisely, let the sub-Laplacian Δ_{sub} act on the total space P of a principal bundle with structure group being a torus $\mathbb{T}$. Then, by decomposing the function space $C^{\infty}(P)$ into Fourier series, Δ_{sub} can be decomposed into a countable number of elliptic operators on the base space. All sub-Laplacians discussed here are of this type and we are using this structure together with an explicit expression of the heat kernel to calculate spectral zeta functions. As it turns out these spectral zeta functions are related to classical zeta functions such as the Riemann zeta function, the Hurwitz zeta function or the Epstein zeta function.

In our earlier papers [3, 8] and [7] we determined the spectral zeta function of the (sub)-Laplacian for low dimensional nilmanifolds. Here we deal with a class of nilmanifolds $L\backslash G$ associated to *pseudo H-type algebras* which will be defined in Section 6. We aim to explicitly calculate the heat trace and the spectrum of the (sub)-Laplacian on various of such type of low dimensional pseudo H-type nilmanifolds $L\backslash G$. In a forthcoming paper we plan to discuss a general pseudo H-type nilmanifold under these aspects and to decide whether there are isospectral but non-diffeomorphic cases. The structure of the paper is as follows:

In Section 2 we first introduce a geometric structure on manifolds called sub-Riemannian structure. In particular, all nilpotent Lie groups G can be equipped with such a structure and we define the corresponding sub-Laplacian as a "sum-of-squares operator". The heat kernel of the sub-Laplacian on G is described based on the work of Beals *et al*., (cf. [10, 11]).

We consider nilpotent Lie groups with lattices in Section 3 and calculate the heat kernel of the sub-Laplacian on the quotient space by a lattice. Such a quotient is called a (compact) nilmanifold.

In Section 4 the principal bundle structure of nilmanifolds is discussed. This structure induces a decomposition of the sub-Laplacian into a countable number of elliptic operators acting on line bundles defined on the base space (which always is a torus).

In order to have more concrete examples we provide heat traces and spectral zeta functions in an explicit form for low dimensional ($\dim \leq 6$) nilmanifolds in Section 5 following [3, 7, 8].

In Section 6 we introduce a class of two step nilpotent Lie groups, called "pseudo H-type Lie groups" by describing their Lie algebras [13, 14, 16]. For this purpose we begin with the general construction of Clifford algebras and exhibit their table in low dimensions. Then we present explicit formulas for the heat kernel of the sub-Laplacian of nilmanifolds corresponding to low dimensional pseudo H-type groups and with respect to a standard class of lattices.

In Section 7 we continue the determination of heat trace formulas for pseudo H-type nilmanifolds. As a result we can give an example of two nilmanifolds that are isospectral with respect to the sub-Laplacian (and the Laplacian). It would be an interesting problem to decide whether these nilmanifolds in fact are diffeomorphic.

Generalizing these examples we mention a family of isospectral pseudo H-type nilmanifolds (with respect to the sub-Laplacian) in Section 8.

We have added an appendix to the paper where we discuss a general formula for a kind of zeta function which (in the cases treated here) appears as a typical Mellin transform of the heat trace of the sub-Laplacian on nilmanifolds.

2. Sub-Riemannian Structure on Nilpotent Lie Groups and Heat Kernel

We introduce a sub-Riemannian structure on nilpotent Lie groups and describe the heat kernel for the sub-Laplacian following the paper [10] (see [15, 17] for a geometric aspect of this kernel construction).

2.1 *Sub-Riemannian structure on nilpotent Lie groups*

Let M be a manifold (smooth without boundary) and $\mathcal{H}$ a subbundle in the tangent bundle $T(M)$. We denote by $\Gamma(\mathcal{H})$ the space of vector fields taking values in $\mathcal{H}$.

Definition 2.1 — We call M with the subbundle $\mathcal{H}$ a sub-Riemannian manifold, if it satisfies the condition that evaluations of vector fields in a finite sum

$$\Gamma(\mathcal{H}) + [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})] + \big[\Gamma(\mathcal{H}), [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})]\big] + \Big[\Gamma(\mathcal{H}), \big[\Gamma(\mathcal{H}), [\Gamma(\mathcal{H}), \Gamma(\mathcal{H})]\big]\Big] + \cdots$$

span the tangent space $T_x(M)$ at any point $x \in M$. In this case we call the sub-bundle $\mathcal{H}$ *bracket generating*.

In most of the concrete cases we have a natural pointwise inner product in $\mathcal{H}$. A basic theorem on sub-Riemannian manifolds states the following:

***Theorem* 2.2** — *(Chow-Rashevskii Theorem, [12, 29]). Every two points on a connected sub-Riemannian manifold M can be connected by horizontal curves (in general they are only piecewise smooth).*

If we regard a manifold with such a sub-bundle as a configuration space of a physical system, then this result means that every two states can be transformed into each other.

Nilpotent Lie groups can be equipped with a sub-Riemannian structure in a natural way and such groups and their quotient spaces by discrete subgroups are our subject in this paper.

In the following we only deal with two step Lie algebras $\mathfrak{g}$ (and corresponding Lie groups G), that is

$$[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$$

and we assume that

$$[\mathfrak{g}, \mathfrak{g}] = \textit{ center of the Lie algebra } = \mathfrak{z}.$$

We fix bases $\{Z_k\}_{k=1}^d$ and $\{X_i\}_{i=1}^N$ of the center $\mathfrak{z}$ and its complement, respectively, and we assume they are "*orthonormal*". One obtains the orthogonal decomposition $\mathfrak{g} = \mathfrak{z}^\perp \oplus_\perp \mathfrak{z}$. Note that by the above assumption we have installed a left invariant sub-Riemannian metric on G.

Proposition 2.3 ([2, 27]) — The so called "Popp measure" with respect to this sub-Riemannian metric coincides with the Haar measure (modulo a constant factor).

Throughout the paper the corresponding structure constants of the Lie algebra will be denoted by $c_{i\,j}^k$, i.e.

$$[X_i, X_j] = \sum_k c_{i\,j}^k Z_k.$$

By making use of the *Campbell-Hausdorff formula* and the diffeomorphism, $\exp : \mathfrak{g} \xrightarrow{\sim} G$, the group law " $*$ " is expressed as

$$G \times G \cong \mathfrak{g} \times \mathfrak{g} \ni (X, Y) \longmapsto X * Y = X + Y + \frac{1}{2}[X, Y] \in \mathfrak{g} \cong G.$$

In the following, we work through the coordinates

$$G \cong \mathbb{R}^N \times \mathbb{R}^d \ni g = (x, z) = (x_1 \ldots, x_N, z_1, \ldots, z_d) \longleftrightarrow \sum x_i X_i + \sum z_k Z_k \in \mathfrak{g}.$$

We denote by $\tilde{X}_j$ the left invariant vector field on the group G. Then the vector fields

$$\left\{\tilde{X}_j, [\tilde{X}_i,\, \tilde{X}_j] \;:\; i, j = 1, \cdots N\right\}$$

span the tangent space at each point, so that the sub-bundle spanned by $\{\tilde{X}_i\}$ is bracket generating and defines a sub-Riemannian structure on the group G. We write Δ^G_{sub} for the second order differential operator

$$\Delta^G_{\text{sub}} := -\frac{1}{2}\sum_{i=1}^{N} \tilde{X}_i^2,$$

which is called a "sub-Laplacian". It is also defined as a composition of three operators, namely, the exterior derivative, the projection to the annihilator of the sub-bundle spanned by $\{\tilde{Z}_k\}$ in $T^*(G)$ (which is a subbundle in $T^*(G)$) and the adjoint of the exterior derivative with respect to the Riemannian metric introduced above. So this operator depends on the projection operator and does not depend on the particularly chosen orthonormal elements $\{X_j\}$.

We mention that the sub-Laplacian is also defined through the universal enveloping algebra $\mathcal{U}_\mathfrak{g}$, which can be regarded as the algebra of all left invariant differential operators. In fact, by the metric installed above the space $\text{Hom}(\mathfrak{g},\mathfrak{g}) \cong \mathfrak{g}^* \otimes \mathfrak{g}$ is identified with $\mathfrak{g} \otimes \mathfrak{g}$. Through this identification the identity map in $\text{Hom}(\mathfrak{g},\mathfrak{g})$ is mapped to the element $\sum X_i \otimes X_i + \sum Z_k \otimes Z_k$ in $\mathcal{U}_\mathfrak{g}$ and it defines a left invariant differential operator, which is the Laplacian. Similarly the operator corresponding to $-\frac{1}{2}\sum X_i \otimes X_i$ is the sub-Laplacian.

From an analytic point of view a fundamental property of a sub-Riemannian structure (which also corresponds to its geometric feature described by the Chow-Rashevskii theorem) is given by Hörmander's theorem.

***Theorem* 2.4** — *([Hörmander, [19]). The operator Δ^G_{sub} is hypo-elliptic or, more strongly, sub-elliptic, i.e., it satisfies a sub-elliptic estimate: for any bounded domain D in G (i.e. the closure $\overline{D}$ of D is compact), there exists a constant $C = C_D > 0$ and a number $0 < \delta < 2$ such that for any $u \in C_0^\infty(D)$*

$$||u||_{s-\delta} \le C_D \left(||\Delta^G_{\text{sub}}(u)||_s + ||u||_0\right),$$

where $||u||_s$ is a Sobolev norm of order s.

We fix a Haar measure on G (which coincides with the Euclidean measure) $dx_1 \cdots dx_N dz_1 \cdots dz_d$ and consider the space $L_2(G)$. The sub-Laplacian Δ^G_{sub} is positive and essentially selfadjoint in $L_2(G)$ when we consider it on the domain $C_0^\infty(G)$ (see [5] for an elementary proof of these facts including the case of higher step Grushin type operators descended from sub-Laplacians to a homogeneous space). Moreover, $\lambda = 0$ is an eigenvalue with eigenspace formed by the constant functions.

2.2 *Heat kernel for the sub-Laplacian*

We can consider the heat kernel for the sub-Laplacian and start from the spectral decomposition

$$\Delta^G_{\text{sub}} = \int_0^\infty \lambda\, dE_\lambda.$$

Here $\{ E_\lambda \}_{\lambda \geq 0}$ is the spectral measure of the unique selfadjoint extension of the sub-Laplacian Δ^G_{sub} in $L_2(G)$. The kernel function (distribution) $K(t, g, h)$ of the operators

$$e^{-t\Delta^G_{\text{sub}}} = \int_0^\infty e^{-t\lambda} dE_\lambda,$$

is called the *heat kernel* for the sub-Laplacian and it defines an element in $C^\infty(\mathbb{R}_+ \times G \times G)$. Since the operator Δ^G_{sub} is left invariant, K is of the form $K(t, g, h) = k^G(t,\, g^{-1} * h)$ with a smooth function $k^G \in C^\infty(\mathbb{R}_+ \times G)$.

2.3 *BGG formula*

In the paper [10] (see also [11]), an integral expression of the kernel function k^G (BGG formula) is given explicitly and will play a basic role in this paper.

Let $\sigma(\Delta^G_{\text{sub}}) \in C^\infty(T^*(G))$ be the principal symbol of the sub-Laplacian. We can express it through the coordinates

$$(x, z; \xi, \tau) \in T^*(G) \cong G \times \mathbb{R}^{N+d} = \mathbb{R}^N \times \mathbb{R}^d \times \mathbb{R}^N \times \mathbb{R}^d$$

as

$$\sigma\big(\Delta^G_{\text{sub}}\big)(x, z; \xi, \tau) = \frac{1}{2} \sum_{j=1}^N \Big(\xi_j + \frac{1}{2} \sum_{i,\,k} c^k_{i\,j} x_i \tau_k\Big)^2.$$

Then the space $\text{ch}(\Delta^G_{\text{sub}}) = \{ (x, z; \xi, \tau) \mid \sigma(\Delta^G_{\text{sub}})(x, z; \xi, \tau) = 0 \}$ in $T^*(G) \cong G \times \mathbb{R}^{N+d}$ is called the "*characteristic variety*" and can be seen as a sub-bundle in $T^*(G)$. Since $\sigma(\Delta^G_{\text{sub}})(x, z; \xi, \tau) = 0$ is equivalent to the conditions $\xi_j + \frac{1}{2} \sum_{i,\,k} c^k_{i\,j} x_i \tau_k = 0$ (for $j = 1, \ldots, N$) one may parametrize the characteristic variety through the variables (x, z, τ):

$$\text{ch}(\Delta^G_{\text{sub}}) = \Big\{ (x, z; -\frac{1}{2} \sum_{i,\,k} c^k_{i\,1} x_i \tau_k, \ldots, -\frac{1}{2} \sum_{i,\,k} c^k_{i\,N} x_i \tau_k, \tau) \Big\},$$

that is

$$G \times \mathbb{R}^d \longrightarrow T^*(G),\ (g, \tau) = (x, z, \tau) \mapsto \Big(x, z, -\frac{1}{2} \sum_{i,\,k} c^k_{i\,1} x_i \tau_k, \ldots, -\frac{1}{2} \sum_{i,\,k} c^k_{i\,N} x_i \tau_k, \tau\Big).$$

Putting $\omega_{i\,j}(\tau) := \sum_k c^k_{i\,j}\tau_k$ with $\tau \in \mathbb{R}^d$ we define a matrix function $\Omega(\tau)$ by

$$\mathbb{R}^d \ni \tau \longmapsto \Omega(\tau) = \big(\omega_{i\,j}(\tau)\big)_{i,j} \in \mathbb{R}^{N\times N}. \tag{2.1}$$

The kernel function $K(t,g,h) \in C^\infty(\mathbb{R}_+ \times G \times G)$ can be interpreted as a (fiber) integral on the characteristic variety of the sub-Laplacian.

***Theorem* 2.5** — (Beals *et al*., [10])

$$K(t,g,h) = k^G(t, g^{-1} * h) = \frac{1}{(2\pi t)^{N/2+d}} \int_{\mathbb{R}^d} e^{-\frac{f(\tau,\, g^{-1}*h)}{t}} \cdot W(\tau) d\tau,$$

where the functions $f = f(\tau, g) \in C^\infty(\mathbb{R}^d \times G)$ and $W(\tau)$ are given as follows: put $g = (x,z) \in \mathbb{R}^N \times \mathbb{R}^d$, then

$$f(\tau,g) = f(\tau,x,z) = \sqrt{-1} < \tau,\, z >_{\{d,0\}} + \frac{1}{2} < \Omega(\sqrt{-1}\tau)\coth\big(\Omega(\sqrt{-1}\tau)\big) \cdot x, x >_{\{N,0\}}$$

$$W(\tau) = \left\{ \det \frac{\Omega(\sqrt{-1}\tau)}{\sinh \Omega(\sqrt{-1}\tau)} \right\}^{1/2},$$

where $< z, z' >_{\{d,0\}} = \sum_{k=1}^{d} z_k z'_k$ is positive definite. Until Section 8 we shortly denote this inner product by $< \bullet, \bullet >$. However, in the last part of this paper we need to distinguish the signature of the quadratic form and therefore write $< \bullet, \bullet >_{\{r,s\}}$ (see the definition below).

We call the function $f = f(\tau, x, z)$ the "complex action function" and the function $W(\tau)$ the "volume function". The measure $W(\tau)d\tau$ is referred to as the "volume form".

Recall that the function f is constructed by the *complex Hamilton-Jacobi method*, and the volume function is also called *van Vleck determinant*. It is the Jacobian of the correspondence between the space of initial conditions and the space of boundary conditions when we solve the Hamilton equation which solution is the bicharacteristic flow. The volume function satisfies an equation called *transportation equation*.

3. Nilmanifolds

Based on the heat kernel expression of the sub-Laplacian explained in the last section we describe the heat kernel of the descended operator defined on the quotient space $L\backslash G$ (left coset space) by a lattice L. Such a space is called a nilmanifold.

3.1 *Uniform discrete subgroup*

In the following we assume that there exists a uniform discrete subgroup (lattice) L in G. In this connection we mention *Malćev's Theorem*:

***Theorem* 3.1** — (Malćev, [24, 28]). *A nilpotent Lie group has a lattice L (i.e., $L\backslash G$ is compact), if and only if, there exists a basis $\{X_i\}$ in the Lie algebra $\mathfrak{g}$ such that the structure constants $\{c^k_{i\,j}\}$ defined by*

$$[X_i,\, X_j] = \sum c^k_{i\,j} X_k$$

are all rational numbers. In this case we call $\{X_i\}$ a "rational basis".

Remark 3.2 : Even if $\{X_i\}$ is a rational basis the range $\exp(\{\sum n_i X_i \mid n_i \in \mathbb{Z}\})$ needs not to be a lattice in G, but it generates a lattice. Conversely, if L is a lattice in the group G, then $\log(L) := \exp^{-1}(L)$ contains such a basis.

3.2 *Heat kernel on nilmanifolds*

Let L be a lattice in a simply connected two step nilpotent Lie group $G \cong \mathbb{R}^N \times \mathbb{R}^d$. Then the quotient space $L\backslash G$ can be equipped with a sub-Riemannian structure naturally inherited from that of the group G and its sub-Laplacian, which we denote by $\Delta_{\text{sub}}^{L\backslash G}$, is the operator descended from the sub-Laplacian on G.

Given an element $g \in G$ we will denote by $[g] \in L\backslash G$ the corresponding class in the quotient space. Then, the heat kernel

$$K^{L\backslash G}(t, [g], [h]) \in C^\infty\Big(\mathbb{R}_+ \times L\backslash G \times L\backslash G\Big)$$

for the sub-Laplacian on the nilmanifold $L\backslash G$ is given by

$$\begin{aligned} K^{L\backslash G}(t, [g], [h]) &= K^{L\backslash G}(t, g, h) \\ &= \sum_{\gamma \in L} K(t, \gamma * g, h) = \sum_{\gamma \in L} k^G(t, g^{-1} * \gamma * h). \end{aligned}$$

This is guaranteed by the facts that

- $K(t, k * g, k * h) = K(t, g, h)$ for any $k \in G$ and the estimate
- $W(\tau) = O(\|\tau\|^{-j}),\ j > 0$ is arbitrary,
- the bilinear form $<\Big(\sqrt{-1}\Omega(\tau)\coth\big(\sqrt{-1}\Omega(\tau)\big)\Big) \cdot x,\, x>$

is *(strictly)* positive definite and $^\exists c > 0$,

$$\inf\Big\{ <\Big(\sqrt{-1}\Omega(\tau)\coth\big(\sqrt{-1}\Omega(\tau)\big)\Big)\cdot x,\, x> \;\Big|\; \tau\in\mathbb{R}^d\Big\} \geq c\|x\|^2$$
$$<\Big(\sqrt{-1}\Omega(\tau)\coth\big(\sqrt{-1}\Omega(\tau)\big)\Big)\cdot x,\, x> = O\big(\,\|\tau\|\|x\|^2\,\big).$$

We recall some crucial estimates:

Proposition 3.3 — (Small time asymptotic, [7]) Let $g=(x,z)\in G$ and $\delta>0$.

(i) There are constants $C, c>0$ such that:

$$|k^G(t,x,z)| \leq Ct^{-\frac{N}{2}-d}e^{-\frac{c\|x\|^2}{t}}.$$

(ii) Let $g=(x,z)$ with $\|z\|\geq\delta$. Then for any $\ell\in\mathbb{N}$ we can choose $C_\ell>0$ and $c>0$ such that

$$\big|k^G(t,x,z)\big| \leq C_\ell\frac{(1+\|x\|)^{2\ell}}{\|z\|^{2\ell}}t^{2\ell-\frac{N}{2}-d}e^{-\frac{c}{t}\|x\|^2}.$$

With these properties we have:

Proposition 3.4 ([7]) —

$$\begin{aligned}
\mathbf{tr}\left(e^{-t\Delta_{\text{sub}}^{L\backslash G}}\right) &= \sum_{\gamma\in L}\int_{\mathcal{F}_L} k^G\big(t,(x,z)^{-1}*\gamma*(x,z)\big)dxdz\\
&= \mathit{Vol}(L\backslash G)\cdot k^G(t,(0,0)) + \sum_{\gamma\in L\backslash\{0\}}\int_{\mathcal{F}_L} K\big(t,(x,z)^{-1}*\gamma*(x,z)\big)dxdz\\
&= (2\pi t)^{-N/2-d}\int_{\mathbb{R}^d} W(\tau)d\tau + O(t^\infty),
\end{aligned}$$

where $\mathcal{F}_L$ is a fundamental domain for the lattice L.

Let $\zeta_{L\backslash G}(s)$ be the spectral zeta function of the sub-Laplacian $\Delta_{\text{sub}}^{L\backslash G}$ defined as the Mellin transform of the heat trace:

$$\zeta_{L\backslash G}(s) = \frac{1}{\Gamma(s)}\int_0^\infty \left\{\mathbf{tr}\left(e^{-t\Delta_{\text{sub}}^{L\backslash G}}\right)-1\right\}t^{s-1}dt = \sum_{\substack{0<\lambda\,:\,\text{eigenvalues}\\ \text{of }\Delta_{\text{sub}}^{L\backslash G}}}\frac{1}{\lambda^s}. \tag{3.1}$$

Then as a corollary, we have:

Corollary 3.5 — The spectral zeta function (3.1) admits a meromorphic extension from $\mathrm{Re}(s)>s_p$ to the complex plane with only one simple pole at

$$s_p := N/2+d = \frac{1}{2}\dim\big(\mathfrak{g}/[\mathfrak{g},\mathfrak{g}]\big)+\dim[\mathfrak{g},\mathfrak{g}].$$

The residue of $\zeta_{L\backslash G}(s)$ in s_p is given by

$$\mathrm{Res}\big(\zeta_{L\backslash G}(s)\big)\big|_{s=s_p} = \frac{\mathrm{Vol}(L\backslash G)}{(2\pi)^{N/2+d}\Gamma(N/2+d)} \int_{\mathbb{R}^d} W(\tau)d\tau.$$

We mention some further properties of the spectral zeta function $\zeta_{L\backslash G}(s)$:

Proposition 3.6 — *(i)* $\zeta_{L\backslash G}(s)$ vanishes at the points $\{-1,-2,-3,\ldots\}$.

(ii) Independently of G and L, it holds $\zeta_{L\backslash G}(0) = -1$.

4. Nilpotent Lie Groups with Lattices

Assuming the existence of a lattice in the nilpotent Lie group G, we decompose the sub-Laplacian into differential operators acting on invariant subspaces according to a torus bundle structure. Then we provide the heat kernel expression for each component elliptic operator.

4.1 *Torus bundle and a family of elliptic operators*

Let $\mathbf{A} \cong \mathbb{R}^d$ (with an orthonormal basis $\{Z_k\}$) be the center of the simply connected two step nilpotent Lie group G and assume the existence of a lattice L in G. Then we have a principal bundle

$$L\backslash G \longrightarrow (L/L\cap\mathbf{A})\backslash(G/\mathbf{A}) \cong (L*\mathbf{A})\backslash G$$

with the structure group $\mathbf{A}/(\mathbf{A}\cap L) \cong \mathbb{T}^{\dim\mathbf{A}} = \mathbb{T}^d$. The base space $(L/L\cap\mathbf{A})\backslash(G/\mathbf{A}) \cong (L*\mathbf{A})\backslash G$ is also a torus of dimension $\dim G - \dim\mathbf{A} = N+d-d = N$. Since $\mathbf{A}$ is abelian, the subgroup $L*\mathbf{A}$ coincides with $L+\mathbf{A}$ (the sum in the Lie algebra).

Let $\mathbf{n}$ be an element in the "dual lattice" $(L\cap\mathbf{A})^*$ of $L\cap\mathbf{A}$, that is $\mathbf{n}$ is a linear function on $\mathbf{A}$ with the property that

$$\mathbf{n}(\gamma)\in\mathbb{Z} \text{ for all } \gamma\in L\cap\mathbf{A}.$$

We may express $\mathbf{n}$ in the form $\mathbf{n} = \sum_{k=1}^{d} n_k Z_k$ with integer coefficients $\{n_k \mid n_k\in\mathbb{Z}\}$ such that

$$\sum n_k < Z_k,\gamma > \in\mathbb{Z} \text{ for all } \gamma\in L\cap\mathbf{A}.$$

Each $f\in C^\infty(L\backslash G)$ is decomposed via a Fourier series expansion as

$$f(g) = \sum_{\mathbf{n}\in(L\cap\mathbf{A})^*} \int_{\mathbb{T}^d} f(g*\lambda)\overline{\chi_{\mathbf{n}}(\lambda)}d\lambda,$$

where $\chi_{\mathbf{n}}:\mathbb{T}^d\to U(1)$ with $\chi_{\mathbf{n}}(\lambda) = e^{2\pi\sqrt{-1}\mathbf{n}(\lambda)}$ is a unitary character corresponding to a dual element $\mathbf{n}\in(L\cap\mathbf{A})^*$. This induces a decomposition of the space of smooth functions on $L\backslash G$:

$$C^\infty(L\backslash G) = \sum_{\mathbf{n}\in(L\cap\mathbf{A})^*} \mathcal{F}^{(\mathbf{n})},$$

where

$$\mathcal{F}^{(\mathbf{n})} = \left\{ \int_{\mathbb{T}^d} f(g * \lambda)\, \overline{\chi_{\mathbf{n}}(\lambda)} d\lambda \;\middle|\; f \in C^\infty(L\backslash G) \right\}.$$

The subspace $\mathcal{F}^{(\mathbf{n})}$ can be seen as a space of smooth sections of a line bundle $E^{(\mathbf{n})}$ on the base space $(L + \mathbf{A})\backslash G = (L/L \cap \mathbf{A})\backslash(G/\mathbf{A})$ associated to the character $\chi_{\mathbf{n}}$. The sub-Laplacian leaves invariant each subspace $\mathcal{F}^{(\mathbf{n})}$ and so can be seen as a differential operator $\mathcal{D}^{(\mathbf{n})}$ acting on the line bundle $E^{(\mathbf{n})}$. Since the sub-bundle spanned by the (left) invariant vector fields $\{\tilde{X}_i\}$ defines a connection (i.e., its linear span is equivariant and transversal to the structure group action by $\mathbf{A}/(\mathbf{A} \cap L)$), each operator $\mathcal{D}^{(\mathbf{n})}$ is elliptic. So the sub-Laplacian

$$\Delta_{\text{sub}}^{L\backslash G} = -\frac{1}{2} \sum \tilde{X}_i^2$$

can be seen as an infinite sum of elliptic operators. Hence

$$\mathbf{tr}\left(e^{-\Delta_{\text{sub}}^{L\backslash G}}\right) = \sum_{\mathbf{n} \in (L \cap \mathbf{A})^*} \mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right)$$

and the spectral zeta function of $\Delta_{\text{sub}}^{L\backslash G}$ is expressed as

$$\begin{aligned} \zeta_{L\backslash G}(s) &= \text{sum of the spectral zeta function of } \mathcal{D}^{(\mathbf{n})} \text{ over } \mathbf{n} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \left\{ \mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{0})}}\right) - 1 \right\} t^{s-1} dt + \frac{1}{\Gamma(s)} \sum_{\mathbf{n} \neq 0} \int_0^\infty \mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) t^{s-1} dt. \end{aligned}$$

4.2 *Heat trace of the component operators I*

We give an expression of the heat trace of each operator $\mathcal{D}^{(\mathbf{n})}$.

By the invariance $K(t, g' * g, g' * h) = K(t, g, h)$ we know that

$$K^{L\backslash G}(t, [g], [h]) = \sum_{\gamma \in L} K(t, \gamma * g, h) \in C^\infty(\mathbb{R}_+ \times L\backslash G \times L\backslash G).$$

Let $\mathcal{F}_{L \cap \mathbf{A}}$ be a fundamental domain for the lattice $L \cap \mathbf{A}$ in the Euclidean space $\mathbf{A}$. Then the integral

$$k_{\mathcal{D}^{(\mathbf{n})}}(t, [g], [h]) = \int_{\mathcal{F}_{L \cap \mathbf{A}}} K^{L\backslash G}\big(t, [g], [h] * \lambda\big) \overline{\chi_{\mathbf{n}}(\lambda)}\, d\lambda$$

is the kernel function for the heat operator $e^{-t\mathcal{D}^{(\mathbf{n})}}$, that is it satisfies

$$\begin{aligned} k_{\mathcal{D}^{(\mathbf{n})}}(t, [g] * \theta, [h]) &= k_{\mathcal{D}^{(\mathbf{n})}}(t, [g * \theta], [h]) = \overline{\chi_{\mathbf{n}}(\theta)} k_{\mathcal{D}^{(\mathbf{n})}}(t, [g], [h]), \\ k_{\mathcal{D}^{(\mathbf{n})}}(t, [g], [h] * \theta) &= k_{\mathcal{D}^{(\mathbf{n})}}(t, [g], [h * \theta]) = \chi_{\mathbf{n}}(\theta) k_{\mathcal{D}^{(\mathbf{n})}}(t, [g], [h]), \end{aligned}$$

where $\theta \in \mathbf{A}$.

Let $\mathbb{M} = \{\mu_i\}$ be a set of complete representatives of the coset space $L/(L \cap \mathbf{A})$, then the trace of the heat operator $e^{-t\mathcal{D}^{(\mathbf{n})}}$ is given as follows:

Proposition 4.1 —

$$\mathrm{Vol}\big(\mathbf{A}/(L\cap\mathbf{A})\big)\cdot \mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \int_{\mathcal{F}_L}\left(\sum_{\gamma\in L}\int_{\mathcal{F}_{L\cap\mathbf{A}}} K(t,g,\,\gamma * g * \lambda)\overline{\chi_{\mathbf{n}}(\lambda)}d\lambda\right) dg$$

$$= \int_{\mathcal{F}_L}\left(\sum_{\mu_i\in\mathbb{M}}\sum_{\nu\in L\cap\mathbf{A}}\int_{\mathcal{F}_{L\cap\mathbf{A}}} k^G(t, g^{-1} * \mu_i * g * \nu * \lambda)\overline{\chi_{\mathbf{n}}(\lambda)}d\lambda\right) dg$$

$$= \int_{\mathcal{F}_L}\left(\sum_{\mu_i\in\mathbb{M}}\int_{\mathbb{R}^d} k^G(t, g^{-1} * \mu_i * g * \lambda)\overline{\chi_{\mathbf{n}}(\lambda)}d\lambda\right) dg$$

$$= \sum_{\mu_i\in\mathbb{M}}\int_{\mathcal{F}_L}\int_{\mathbb{R}^d} k^G(t, g^{-1} * \mu_i * g * \lambda)\overline{\chi_{\mathbf{n}}(\lambda)}d\lambda\, dg.$$

4.3 *Heat trace of the component operators II*

Based on the integral formula for the heat kernel stated in Theorem 2.5, we give a concrete expression of the formula in Proposition 4.1 for several concrete nilmanifolds. For this purpose we choose the structure constants $c_{i\,j}^k$ to be rational and for the sake of simplicity we assume that they are of the form

$$c_{i\,j}^k = \frac{2q_{i\,j}^k}{p_0}$$

with a common positive integer $p_0 \geq 1$ and integers $q_{i\,j}^k$. Then we fix the lattice

$$L = \Big\{\sum_{1\leq i\leq N} m_i X_i + \sum_{1\leq k\leq d}\frac{\ell_k}{p_0}Z_k \,\Big|\, m_i,\,\ell_k\,\in\mathbb{Z}\Big\},$$

and we choose the set

$$\mathbb{M} = \{\,\mu = \sum_{1\leq i\leq N} m_i X_i \mid m_i\in\mathbb{Z}\,\}$$

of complete representatives of the quotient group $(L\cap\mathbf{A})\backslash L = L/(L\cap\mathbf{A})$. Then, for each fixed $\mathbf{n} = p_0\sum_{k=1}^{d} n_k Z_k \in (L\cap\mathbf{A})^*$ ($n_k\in\mathbb{Z}$) we have

$$\mathrm{Vol}\big(\mathbf{A}/(L\cap\mathbf{A})\big)\cdot \mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) =$$

$$= \frac{1}{(2\pi t)^{N/2+d}}\int_{\mathcal{F}_L}\sum_{\mu\in\mathbb{M}}\int_{\mathbf{A}}\int_{\mathbb{R}^d} e^{-\sqrt{-1}\frac{<[\mu,x]+\lambda,\tau>}{t}}\cdot e^{-\frac{<\Omega(\sqrt{-1}\tau)\coth\Omega(\sqrt{-1}\tau)\cdot\mu,\,\mu>}{2t}} W(\tau)d\tau\overline{\chi_{\mathbf{n}}(\lambda)}\,d\lambda\,dx\,dz,$$

which we express in the form

$$\frac{1}{(2\pi t)^{N/2+d}} \int_{\mathcal{F}_L} \sum_{\mu \in \mathbb{M}} \int_{\mathbf{A}} \int_{\mathbb{R}^d} e^{-\sqrt{-1}\frac{<[\mu,x]+\lambda,\tau>}{t}} \cdot \varphi_t(\tau,\mu) d\tau \, \overline{\chi_{\mathbf{n}}(\lambda)} \, d\lambda \, dx \, dz = (*).$$

Here we write $\hat{\varphi}_t(\tau,\mu)$ for the Fourier transform of φ_t with respect to the τ-variable. Then

$$\begin{aligned}
(*) &= \frac{1}{t^{N/2+d} \cdot (2\pi)^{(N+d)/2}} \int_{\mathcal{F}_L} \sum_{\mu \in \mathbb{M}} \int_{\mathbf{A}} \hat{\varphi}_t \left(\frac{[\mu,x]+\lambda}{t}, \mu \right) \cdot e^{-2\pi\sqrt{-1}<\mathbf{n},\lambda>} \, d\lambda \, dx \, dz \\
&= \frac{1}{t^{N/2+d} \cdot (2\pi)^{(N+d)/2}} \int_{\mathcal{F}_L} \sum_{\mu \in \mathbb{M}} \int_{\mathbf{A}} \hat{\varphi}_t (u,\mu) \cdot e^{-2\pi\sqrt{-1}<\mathbf{n},\, tu+[x,\mu]>} t^d \, du \, dx \, dz \\
&= \frac{1}{(2\pi t)^{N/2}} \cdot {p_0}^d \cdot \sum_{\mu \in \mathbb{M}} \varphi_t(-2\pi t\mathbf{n},\mu) \cdot \int_{\underbrace{[0,1]\times\cdots\times[0,1]}_{N}} e^{-2\pi\sqrt{-1}<\mathbf{n},\,[x,\mu]>} \, dx \\
&= \frac{1}{(2\pi t)^{N/2}} \cdot {p_0}^d \cdot \sum_{\mu \in \mathbb{M}} \varphi_t(-2\pi t\mathbf{n},\mu) \cdot \int_{\underbrace{[0,1]\times\cdots\times[0,1]}_{N}} e^{-2\pi\sqrt{-1}<x,\,\Omega(\mathbf{n})(\mu)>} \, dx.
\end{aligned}$$

In the last line we have used the identification $\mathbf{n} \longleftrightarrow (n_1,\cdots,n_d) \in \mathbb{Z}^d \subset \mathbb{R}^d$ to define the matrix $\Omega(\mathbf{n})$ as in (2.1). With a basis $a_1(\mathbf{n}), \ldots, a_{b(\mathbf{n})}(\mathbf{n})$ in L the solution space $\mathbb{M}(\mathbf{n}) = \{\mu \in \mathbb{M} \mid \Omega(\mathbf{n})(\mu) = 0\}$ can be written as

$$\mathbb{M}(\mathbf{n}) = \left\{ \mu = \sum_{i=1}^{b(\mathbf{n})} m_i \, a_i(\mathbf{n}), \;\middle|\; m_i \in \mathbb{Z} \right\},$$

where $b(\mathbf{n}) \leq d$ and $b(\mathbf{n}) = d$ if and only if $\mathbf{n} = 0$. Hence

***Theorem* 4.2** — *The heat trace of the operator $\mathcal{D}^{(\mathbf{n})}$ is given by:*

$$\begin{aligned}
\mathbf{tr}\left(e^{-t\,\mathcal{D}^{(\mathbf{n})}}\right) &= \frac{1}{(2\pi t)^{N/2}} \sum_{\mu \in \mathbb{M}(\mathbf{n})} e^{-\frac{<\Omega(2\pi t\sqrt{-1}\mathbf{n})\,\coth\Omega(2\pi\sqrt{-1}t\mathbf{n})\mu,\,\mu>}{2t}} \sqrt{\det \frac{\Omega(2\pi\sqrt{-1}t\mathbf{n})}{\sinh\Omega(2\pi\sqrt{-1}t\mathbf{n})}} \\
&= \frac{1}{(2\pi t)^{N/2}} \sum_{\mu \in \mathbb{M}(\mathbf{n})} e^{-\frac{<\mu,\mu>}{2t}} \sqrt{\det \frac{\Omega(2\pi\sqrt{-1}t\mathbf{n})}{\sinh\Omega(2\pi\sqrt{-1}t\mathbf{n})}},
\end{aligned}$$

where, by $\Omega(\mathbf{n})(\mu) = 0$

$$< \Omega(2\pi t\sqrt{-1}\mathbf{n}) \, \coth\Omega(2\pi\sqrt{-1}t\mathbf{n})(\mu), \, \mu > = < \mu, \, \mu > = \sum m_i m_j < a_i(\mathbf{n}), a_j(\mathbf{n}) > .$$

5. Heat Trace Expression: Concrete Cases

To express the volume function $W(\tau)$ in Theorem 2.5 more explicitly we restrict ourselves to more specific classes of nilpotent Lie groups and lattices.

In this section we treat typical examples which were previously studied in [3, 7, 8]. In order to cover all low dimensional cases ($\dim \leq 6$) we list them here again.

5.1 *Three dimensional Heisenberg group $\mathcal{H}_3$*

Recall that the three dimensional Heisenberg algebra is determined by the non-trivial bracket relation $[X, Y] = 2Z$. We denote the elements of the corresponding Lie group $\mathcal{H}_3 \cong \mathbb{R}^3$ by $g = g(x, y, z) = xX + yY + zZ$. The multiplication $*$ in $\mathcal{H}_3$ is given by

$$g(x, y, z) * g(x', y', z') = (x + x')X + (y + y')Y + (z + z' + xy' - yx')Z.$$

We fix the lattice $L = \{ A(k, \ell, m) = kX + \ell Y + mZ \mid k, \ell, m \in \mathbb{Z} \}$ in $\mathcal{H}_3$. Let $\tilde{X}$ and $\tilde{Y}$ denote the left invariant vector fields:

$$\tilde{X} = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}, \qquad \text{and} \qquad \tilde{Y} = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}.$$

According to the calculation in Section 7 of [3] the heat trace for the sub-Laplacian

$$\Delta_{\text{sub}}^{L\backslash\mathcal{H}_3} = -\frac{1}{2}\big(\tilde{X}^2 + \tilde{Y}^2\big)$$

on the Heisenberg manifold $L\backslash\mathcal{H}_3$ is given by

$$\mathbf{tr}\Big(e^{-t\Delta_{\text{sub}}^{L\backslash\mathcal{H}_3}}\Big) = 4\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} k \cdot e^{-2\pi t k(2j+1)} + \sum_{(m,n)\in\mathbb{Z}^2} e^{-2\pi^2 t(m^2+n^2)}. \tag{5.1}$$

Proposition 5.1 — The spectral zeta function $\zeta_{L\backslash\mathcal{H}_3}(s)$ of the sub-Laplacian on the three dimensional Heisenberg manifold $L\backslash\mathcal{H}_3$ has the form:

$$\begin{aligned}\zeta_{L\backslash\mathcal{H}_3}(s) &= \frac{4}{(2\pi)^s}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{1}{k^{s-1}}\frac{1}{(2j+1)^s} + \frac{1}{(2\pi^2)^s}\sum_{\substack{n,m\in\mathbb{Z}\\ m^2+n^2\neq 0}}\frac{1}{(m^2+n^2)^s}\\ &= \frac{4}{(2\pi)^s}\zeta(\,s-1\,)\left(1-\frac{1}{2^s}\right)\zeta(\,s\,) + \frac{1}{(2\pi^2)^s}\,Ep^{(2)}(\,s\,),\end{aligned}$$

where the last term denotes the spectral zeta function of the two dimensional torus $\mathbb{T}^2 := \mathbb{R}^2/\{(m,n) \mid m, n \in \mathbb{Z}\}$ which is called Epstein zeta function (of two variables).

PROOF : Proposition 7.1 in [3]. □

As a consequence of Corollary 3.5 the function $\zeta_{L\backslash\mathcal{H}_3}(s)$ has only one simple pole at $s = 2$. Hence the residue at $s = 1$ of the first term of $\zeta_{L\backslash\mathcal{H}_3}(s)$ given by

$$\frac{4}{(2\pi)^s}\zeta(\, s-1\,)\left(1-\frac{1}{2^s}\right)\zeta(\, s\,)$$

and the residue of the Epstein zeta function, the second term of $\zeta_{L\backslash\mathcal{H}_3}(s)$ cancel. That is, we recover the following well known fact:

Corollary 5.2 — The residue A_{-1} of the Epstein zeta function

$$Ep^{(2)}(\, s\,) = \frac{A_{-1}}{s-1} + A_0 + \cdots\,,$$

at $s = 1$ is equal to π.

PROOF : This is calculated by using the values $\zeta(0) = -1/2$ and the residue of the Riemann zeta function $\zeta(s) = \dfrac{1}{s-1} + \gamma + \cdots$. □

Via the functional equations of the Riemann zeta function and the Epstein zeta function

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta(1-s),$$
$$\pi^{-s}\Gamma(s)\mathrm{Ep}^{(2)}(s) = \pi^{s-1}\Gamma(1-s)\mathrm{Ep}^{(2)}(1-s)$$

we can express the zeta function $\zeta_{L\backslash\mathcal{H}_3}(s)$ in the form

$$\zeta_{L\backslash\mathcal{H}_3}(s) = \frac{4\pi^{s-2}}{2^s}\frac{\Gamma((2-s)/2)\Gamma((1-s)/2)}{\Gamma(s/2)\Gamma((s-1)/2)}\zeta(1-s)\zeta(2-s)(1-2^{-s}) + \\ + \frac{1}{2^s\pi}\frac{\Gamma(1-s)}{\Gamma(s)}\,\mathrm{Ep}^{(2)}(1-s). \quad (5.2)$$

Using the well-known relations $\Gamma(s)\Gamma(1-s) = \dfrac{\pi}{\sin\pi s} = -s\Gamma(s)\Gamma(-s)$, and the multiplication formula $\sqrt{2\pi}\,\Gamma(s) = 2^{s-1/2}\Gamma(s/2)\Gamma(s/2+1/2)$, the function (5.2) can be rewritten as

$$\zeta_{L\backslash\mathcal{H}_3}(s) = \frac{\Gamma(1-s)^2\sin\pi s}{\pi}\Big\{\pi^{s-2}2^s(s-1)\zeta(2-s)\zeta(1-s)(1-2^{-s}) + \frac{1}{2^s\pi}\,\mathrm{Ep}^{(2)}(1-s)\Big\}.$$

Hence we recover the result of Proposition 3.6 in case of the Heisenberg manifold $L\backslash\mathcal{H}_3$:

Corollary 5.3 — At negative integers $s = -n$, the zeta function $\zeta_{L\backslash\mathcal{H}_3}(s)$ takes the value zero.

5.2 *Six-dimensional free nilmanifold*

Let $F_{(3+3)}$ be the free nilpotent Lie group of step 2 generated by 3 elements X_1, X_2, X_3. More precisely, $F_{(3+3)} \cong \mathbb{R}^6$ is a six dimensional Lie group with center spanned by Z_1, Z_2, Z_3 and one has the bracket relations

$$[X_1, X_2] = 2Z_1, \quad [X_1, X_3] = 2Z_2, \quad \text{and} \quad [X_2, X_3] = 2Z_3.$$

Further details on the discussion below can be found in Section 6 of [7] or Sections 13 and 14 of [8]. We consider a sub-Riemannian structure $\mathcal{H}$ on $F_{(3+3)}$ generated by the three left invariant vector fields $\{\tilde{X}_i\}_{i=1}^3$. The sub-Laplacian $\Delta_{\text{sub}}^{F_{(3+3)}}$ is given by:

$$\Delta_{\text{sub}}^{F_{(3+3)}} = -\frac{1}{2}\sum_{i=1}^{3} \tilde{X}_i^2.$$

According to the general formula in Theorem 2.5, the heat kernel

$$K_{\text{sub}}^{F_{(3+3)}}\big(t, (x,z), (\tilde{x}, \tilde{z})\big) \in C^\infty\big(\mathbb{R}_+ \times F_{(3+3)} \times F_{(3+3)}\big)$$

of $\Delta_{\text{sub}}^{F_{(3+3)}}$ has the integral form

$$K_{\text{sub}}^{F_{(3+3)}}\big(t, (x,z), (\tilde{x}, \tilde{z})\big) = \frac{1}{(2\pi t)^{9/2}} \int_{\mathbb{R}^3} \exp\left\{-\frac{A\big((\tilde{x}, \tilde{z})^{-1} * (x,z), \tau\big)}{t}\right\} W(\tau) d\tau,$$

where $(x, z) = \sum x_i X_i + \sum z_k Z_k \in F_{(3+3)}$. Here the function $A = A(x, z, \tau)$ can be expressed by the formula:

$$A(x, z, \tau) = \sqrt{-1}\sum_{i=1}^{3} \tau_i z_i + \frac{1}{2} < \Omega(\sqrt{-1}\tau) \coth \Omega(\sqrt{-1}\tau) \cdot x,\, x > .$$

Recall that $\Omega(\tau)$ denotes a general 3×3 anti-symmetric matrix:

$$\Omega(\tau) = \begin{pmatrix} 0 & \tau_1 & \tau_2 \\ -\tau_1 & 0 & \tau_3 \\ -\tau_2 & -\tau_3 & 0 \end{pmatrix}.$$

So, the volume function $W(\tau)$ is given by

$$W(\tau) = \sqrt{\det \frac{\sqrt{-1}\Omega(\tau)}{\sinh \sqrt{-1}\Omega(\tau)}} = \frac{\|\tau\|}{\sinh \|\tau\|}.$$

The function $A = A(x, z, \tau)$ can be written more explicitly in the form

$$< \Omega(\sqrt{-1}\tau) \coth \Omega(\sqrt{-1}\tau) \cdot x,\, x > = \left\langle \Big(\text{Id} + \frac{\|\tau\| \coth \|\tau\| \,-\, 1}{\|\tau\|^2}\Omega^2(\tau)\Big) \cdot x,\, x \right\rangle.$$

Let $L =\left\{ (m_1, m_2, m_3, k_1, k_2, k_3) \mid m_i, k_i \in \mathbb{Z} \right\}$ be a typical lattice in $F_{(3+3)}$. Then the heat kernel of the sub-Laplacian $\Delta_{\text{sub}}^{L\backslash F_{(3+3)}}$ on the nilmanifold $L\backslash F_{(3+3)}$ has the form:

$$\sum_{\gamma\in L} K_{\text{sub}}^{F_{(3+3)}}\big(t, \gamma * (x,z), (\tilde{x},\tilde{z})\big)$$

and its trace is obtained as the integral

$$\sum_{\gamma\in L} \int_{\mathcal{F}_L} K_{\text{sub}}^{F_{(3+3)}}\big(t, \gamma * (x,z), (x,z)\big) dxdz,$$

where $\mathcal{F}_L = [0,1] \times \ldots \times [0,1] = [0,1]^6$ is a fundamental domain of the lattice L.

The lattice $L \cap [\{Z_k\}]$ is selfdual and for $\mathbf{n} = \sum n_k Z_k$, with $n_k \in \mathbb{Z}$ the heat kernel of the operator $\mathcal{D}^{(\mathbf{n})}$ is expressed in an integral form

$$\textit{kernel function of } e^{-t\mathcal{D}^{(\mathbf{n})}} = \int_{[0,1]^3} K_{\text{sub}}^{L\backslash F^{(3+3)}}(t, g, \tilde{g} * \lambda)\, \chi_{(\mathbf{n})}^{-1}(\lambda)\, d\lambda_1 d\lambda_2 d\lambda_3,$$

where $g = (x,z),\ \tilde{g} = (\tilde{x},\tilde{z}) \in F_{(3+3)}$. Its trace is given by

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \frac{1}{(2\pi t)^{3/2}} \sum_{k\in\mathbb{Z}^3} \int_{[0,1]^3} e^{-\frac{1}{2t}\left\langle \ \mathrm{Id}+\frac{2\pi t|\mathbf{n}|\coth 2\pi t|\mathbf{n}|-1}{(2\pi t|\mathbf{n}|)^2}\Omega(2\pi t\sqrt{-1}\mathbf{n})^2 \ \cdot k, k\right\rangle} \times$$

$$\times \frac{2\pi t|\mathbf{n}|}{\sinh\big(2\pi t|\mathbf{n}|\big)} \exp\left\{4\pi\sqrt{-1}\,\langle x,\, \Omega(\mathbf{n})\cdot k\rangle\right\} dx,$$

and equals to

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \frac{1}{(2\pi t)^{3/2}} \sum_{\substack{k\in\mathbb{Z}^3\\ \Omega(\mathbf{n})\cdot k=0}} e^{-\frac{k_1^2+k_2^2+k_3^2}{2t}} \frac{|\mathbf{n}|}{\sinh\big(2\pi t|\mathbf{n}|\big)},$$

according to the choice of $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3$.

The spectral zeta function $\zeta_{\mathcal{D}^{(\mathbf{n})}}(s)$ of the elliptic operator $\mathcal{D}^{(\mathbf{n})}$ acting on the line bundle $E^{(\mathbf{n})}$ is meromorphic with only simple poles, the largest one is located at $s = 3/2$. The spectral zeta function of the sub-Laplacian $\Delta_{\text{sub}}^{L\backslash F_{(3+3)}}$ is given by

$$\zeta_{\text{sub}}^{L\backslash F_{(3+3)}}(s) = \zeta_{\mathcal{D}^{(0,0,0)}}(s) + \sum_{\mathbf{n}\in\mathbb{Z}^3\backslash\{0\}} \zeta_{\mathcal{D}^{(\mathbf{n})}}(s).$$

Let $(|n_1|, |n_2|)$ and $(|n_1|, |n_2|, |n_3|)$ denote the *greatest common divisor* of its entries where $\mathbf{n} = (n_1, n_2, n_3) \in \mathbb{Z}^3$.

In order to explicitly express the operator traces above we distinguish the cases $\mathbf{n} = (0,0,0) = 0$ and $\mathbf{n} \neq 0$ with the convention that $(|n_1|, |n_2|, 0) = (|n_1|, |n_2|)$, if $n_3 = 0$ and $(|n_1|, 0, 0) = |n_1|$ if $n_2 = n_3 = 0$, and so on.

(1) Let $\mathbf{n} = 0$, then

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{0})}}\right) = \frac{1}{(2\pi t)^{3/2}} \sum_{k\in\mathbb{Z}^3} e^{-\frac{k_1^2+k_2^2+k_3^2}{2t}} = \sum_{k\in\mathbb{Z}^3} e^{-2\pi^2 t(k_1^2+k_2^2+k_3^2)}.$$

(2) Let $\mathbf{n} \neq 0$, then

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \sum_{\ell\in\mathbb{Z}} e^{-2\pi^2 t \frac{(|n_1|,|n_2|,|n_3|)^2}{n_1^2+n_2^2+n_3^2}\ell^2} \frac{(|n_1|,|n_2|,|n_3|)}{\sinh\left(2\pi t\sqrt{n_1^2+n_2^2+n_3^2}\right)}.$$

***Theorem* 5.4** — *The heat kernel trace of* $\Delta_{\text{sub}}^{L\backslash F_{(3+3)}}$ *on* $L\backslash F_{(3+3)}$ *is given by:*

$$\sum_{\gamma\in L}\int_{\mathcal{F}_L} K_{\text{sub}}^{F_{(3+3)}}\big(t,\gamma*(x,z),(x,z)\big)dxdz$$
$$= \sum_{\ell_1,\ell_2,\ell_3\in\mathbb{Z}} e^{-2\pi^2 t\left(\ell_1^2+\ell_2^2+\ell_3^2\right)} + 3\cdot\sum_{\ell\in\mathbb{Z}}\sum_{k=1}^{\infty} e^{-2\pi^2\ell^2 t}\frac{2k}{\sinh(2\pi tk)}$$
$$+ 3\cdot \sum_{\substack{m_1,m_2\in\mathbb{Z},\, m_1\cdot m_2\neq 0\\ (|m_1|,|m_2|)=1}} \sum_{k=1}^{\infty}\sum_{\ell\in\mathbb{Z}} e^{-2\pi^2 t\frac{\ell^2}{m_1^2+m_2^2}} \frac{k}{\sinh\left(2\pi t\, k\sqrt{m_1^2+m_2^2}\right)}$$
$$+ \sum_{\substack{m_1,m_2,m_3\in\mathbb{Z},\, m_1\cdot m_2\cdot m_3\neq 0\\ (|m_1|,|m_2|,|m_3|)=1}} \sum_{k=1}^{\infty}\sum_{\ell\in\mathbb{Z}} e^{-2\pi^2 t\frac{\ell^2}{m_1^2+m_2^2+m_3^2}} \frac{k}{\sinh\left(2\pi tk\sqrt{m_1^2+m_2^2+m_3^2}\right)}.$$

PROOF : See Theorem 13.1 in [8]. □

5.3 *A* 5*-dimensional nilmanifold*

In dimension 5, there are only two nilpotent Lie groups of step 2 that do not decompose into lower dimensional groups. Let G_5 be the quotient group $F_{(3+3)}/\{zZ_3\}$. Alternatively, G_5 is defined as the Lie group with Lie algebra generated by 5 elements X_1, X_2, X_3 and Z_1, Z_2 and bracket relations

$$[X_1,\, X_2] = 2Z_1, \qquad [X_1,\, X_3] = 2Z_2, \qquad [X_2,\, X_3] = 0.$$

The second group among the indecomposable 5 dimensional nilpotent Lie groups of step two is the five dimensional Heisenberg group.

As in the case of $F_{(3+3)}$, we consider a sub-Laplacian

$$\Delta_{\text{sub}}^{G_5} = -\frac{1}{2}\sum_{i=1}^{3}\tilde{X}_i$$

on G_5 and fix a typical lattice $L = \{k_1X_1 + k_2X_2 + k_3X_3 + \ell_1Z_1 + \ell_2Z_2 \mid k_i, \ell_j \in \mathbb{Z}\}$. The heat trace $\mathbf{tr}\big(e^{-t\Delta_{\text{sub}}^{L\backslash G_5}}\big)$ on the nilmanifold $L\backslash G_5$ is calculated in the next proposition.

Proposition 5.5 —

$$\mathbf{tr}\left(e^{-t\Delta_{\text{sub}}^{L\backslash G_5}}\right) = \sum_{\ell_1,\ell_2,\ell_3\in\mathbb{Z}} e^{-2\pi^2 t(\ell_1^2+\ell_2^2+\ell_3^2)} + 2\sum_{\ell\in\mathbb{Z}}\sum_{k=1}^{\infty} e^{-2\pi^2 t\ell^2}\cdot\frac{2k}{\sinh\left(2\pi kt\right)}$$
$$+ \sum_{\substack{\ell_1,\ell_2\in\mathbb{Z}\\ \ell_1\cdot\ell_2\neq 0,(|\ell_1|,|\ell_2|)=1}} \sum_{k=1}^{\infty}\sum_{\ell\in\mathbb{Z}} e^{-2\pi^2 t\,\frac{\ell^2}{\ell_1^2+\ell_2^2}}\cdot\frac{k}{\sinh\left(2\pi t\,k\sqrt{\ell_1^2+\ell_2^2}\right)}. \quad (5.3)$$

PROOF : The precise derivation of this identity is given in [8], Theorem 12.1. □

The right hand side can be also expressed in the form:

Proposition 5.6 —

$$\mathbf{tr}\left(e^{-t\Delta_{\text{sub}}^{L\backslash G_5}}\right) = \sum_{\ell_1,\ell_2,\ell_3\in\mathbb{Z}} e^{-2\pi^2 t\left(\ell_1^2+\ell_2^2+\ell_3^2\right)} + \sum_{\ell\in\mathbb{Z}}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty} 8ke^{-t\left(2\pi^2\ell^2+2\pi k\,(2j+1)\right)}$$
$$+\,2\sum_{\substack{m_1,m_2\in\mathbb{Z}\\ m_1\cdot m_2\neq 0}}\sum_{\ell\in\mathbb{Z}}\sum_{j=0}^{\infty}(|m_1|,|m_2|)\cdot e^{-t\left(\frac{2\pi^2\ell^2\cdot(|m_1|,|m_2|)^2}{m_1^2+m_2^2}+2\pi\sqrt{m_1^2+m_2^2}(2j+1)\right)}.$$

PROOF : See Corollary 12.2 in [8]. □

As an immediate consequence, the spectral zeta function $\zeta_{L\backslash G_5}(s)$ is given as follows:

Proposition 5.7 —

$$\zeta_{L\backslash G_5}(s) = \frac{1}{(2\pi^2)^s}\sum_{\ell_i\in\mathbb{Z},\,|\ell_1|+|\ell_2|+|\ell_3|\neq 0}\frac{1}{\left({\ell_1}^2+{\ell_2}^2+{\ell_3}^2\right)^s}$$
$$+\sum_{\ell\in\mathbb{Z}}\sum_{k=1}^{\infty}\sum_{j=0}^{\infty}\frac{8k}{\left(2\pi^2\ell^2+2\pi k(2j+1)\right)^s}$$
$$+\sum_{m_1\cdot m_2\neq 0,\,m_i\in\mathbb{N}}\sum_{\ell\in\mathbb{Z}}\sum_{j=0}^{\infty}\frac{8(m_1,m_2)}{\left(\frac{2\pi^2\ell^2\cdot(m_1,m_2)^2}{m_1^2+m_2^2}+2\pi\sqrt{m_1^2+m_2^2}\,(2j+1)\right)^s}.$$

From this expression we cannot observe directly that the spectral zeta function $\zeta_{L\backslash G_5}(s)$ takes the value zero at each negative integer.

6. Spectral Zeta Function on Nilmanifolds of Pseudo H-Type Groups

According to the classification of low dimensional nilpotent Lie algebras $\mathfrak{g}$ (as for the cases $\dim \mathfrak{g} \leq 7$ see [25, 26]) and apart from $F_{(3+3)}$, there are two more such algebras of dimension six. Both are realized in a class of "pseudo H-type algebras" constructed from the minimal dimensional "admissible module" of the Clifford algebras $C\ell_{2,0}$, $C\ell_{1,1}$ and $C\ell_{0,2}$. We denote them by $\mathcal{N}_{2,0}$, $\mathcal{N}_{1,1}$ and $\mathcal{N}_{0,2}$, respectively. By $G_{2,0}$, $G_{1,1}$ and $G_{0,2}$ we mean the corresponding connected and simply connected nilpotent Lie groups. The algebras $\mathcal{N}_{2,0}$ and $\mathcal{N}_{0,2}$ are isomorphic and $\mathcal{N}_{1,1}$ is not isomorphic to $\mathcal{N}_{2,0}$. Hence we deal with $\mathcal{N}_{2,0}$ and $\mathcal{N}_{1,1}$. Both algebras contain a lattice which will be described below.

We start this section by introducing a class of 2-step nilpotent Lie algebras called *pseudo H-type algebras* (cf. [13, 16]).

6.1 *Pseudo H-type algebras*

Let $\mathbb{R}^{r,s} \cong \mathbb{R}^{r+s}$ be the $(r+s)$-dimensional Euclidean space with the non-degenerate symmetric bi-linear form

$$< x,\, y >_{\{r,\,s\}} =: \sum_{k=1}^{r} x_k y_k - \sum_{j=1}^{s} x_{r+j} y_{r+j}, \qquad x = (x_1, \ldots, x_{r+s}),\ y = (y_1, \ldots, y_{r+s}) \in \mathbb{R}^{r,s},$$

with signature (r, s). By $C\ell_{r,s}$ we denote the Clifford algebra generated by $\mathbb{R}^{r,s}$ (as for the definition of the Clifford algebra see [1, 23]). Table 1 shows some algebras of low dimension.

In this table, for example, $\mathbb{H}(2)$ denotes the 2×2 matrix algebra over the quaternion numbers and so on. By "Bott periodicity" the higher dimensional cases are obtained by taking tensor products with the matrix algebra $C\ell_{8,0} \cong \mathbb{R}(16) \cong C\ell_{0,8}$, i.e. one has $C\ell_{r,s} \otimes C\ell_{8,0} \cong C\ell_{r+8,\,s}$ or $C\ell_{r,s} \otimes C\ell_{0,8} \cong C\ell_{r,s+8}$.

Table 1: Table of Clifford algebras $C\ell_{r,s}$

8	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	$\mathbb{H}(32)$	$\mathbb{C}(64)$	$\mathbb{R}(128)$	$\mathbb{R}(128)\oplus\mathbb{R}(128)$	$\mathbb{R}(256)$
7	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$	$\mathbb{R}(64)$	$\mathbb{R}(64)\oplus\mathbb{R}(64)$	$\mathbb{R}(128)$	$\mathbb{C}(128)$
6	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$	$\mathbb{R}(32)\oplus\mathbb{R}(32)$	$\mathbb{R}(64)$	$\mathbb{C}(64)$	$\mathbb{H}(64)$
5	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$	$\mathbb{R}(32)$	$\mathbb{C}(32)$	$\mathbb{H}(32)$	$\mathbb{H}(32)\oplus\mathbb{H}(32)$
4	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$	$\mathbb{C}(16)$	$\mathbb{H}(16)$	$\mathbb{H}(16)\oplus\mathbb{H}(16)$	$\mathbb{H}(32)$
3	$\mathbb{C}(2)$	$\mathbb{R}(4)$	$\mathbb{R}(4)\oplus\mathbb{R}(4)$	$\mathbb{R}(8)$	$\mathbb{C}(8)$	$\mathbb{H}(8)$	$\mathbb{H}(8)\oplus\mathbb{H}(8)$	$\mathbb{H}(16)$	$\mathbb{C}(32)$
2	$\mathbb{R}(2)$	$\mathbb{R}(2)\oplus\mathbb{R}(2)$	$\mathbb{R}(4)$	$\mathbb{C}(4)$	$\mathbb{H}(4)$	$\mathbb{H}(4)\oplus\mathbb{H}(4)$	$\mathbb{H}(8)$	$\mathbb{C}(16)$	$\mathbb{R}(32)$
1	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}(2)$	$\mathbb{H}(2)\oplus\mathbb{H}(2)$	$\mathbb{H}(4)$	$\mathbb{C}(8)$	$\mathbb{R}(16)$	$\mathbb{R}(16)\oplus\mathbb{R}(16)$
s=0	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8)\oplus\mathbb{R}(8)$	$\mathbb{R}(16)$
s/r	r=0	1	2	3	4	5	6	7	8

Let V be a Clifford module, i.e. V is a real vector space with a module action denoted by $J : C\ell_{r,s} \times V \to V$. Moreover, we write $J_z : V \to V$ for each $z \in \mathbb{R}^{r,s}$. Then V is called *admissible* if it is equipped with a non-degenerate symmetric bi-linear form $< \bullet, \bullet >_V$ having the properties

$$< J_z(X),\, J_z(Y) >_V = < z, z >_{\{r,\,s\}} \cdot < X,\, Y >_V, \tag{6.1}$$

$$< J_z(X),\, Y >_V + < X,\, J_z(Y) >_V = 0. \tag{6.2}$$

In fact, the conditions (6.1) and (6.2) are equivalent to each other under the condition that ${J_z}^2 = - < z, z >_{\{r,s\}} \cdot \mathrm{Id}$. Irreducible modules need not to be admissible. If they are not admissible, then their double is admissible (a proof of this fact is found in [13]). We call such an admissible module *minimal*.

In this situation the non-degenerate symmetric bi-linear form must have positive definite and negative definite subspaces of the same dimension, if $s > 0$. Any admissible module can be decomposed into an orthogonal sum of minimal dimensional admissible modules.

Definition 6.1 — Let V be an admissible Clifford module of $C\ell_{r,s}$ with the non-degenerate symmetric bi-linear form $< \bullet, \bullet >_V$ satisfying the conditions (6.1) and (6.2). By defining an antisymmetric bi-linear map

$$[\bullet, \bullet] : V \ni\; X,\, Y \;\longmapsto\; [X,\, Y] \in \mathbb{R}^{r+s}$$

through the relation

$$< J_z(X),\, Y >_V = < z,\, [X, Y] >_{\{r,\,s\}},\; z \in \mathbb{R}^{r+s},\; X,\; Y \in V,$$

the direct sum $V \oplus_\perp \mathbb{R}^{r,s}$ has a Lie algebra structure of step two with the indefinite scalar product

$$< \bullet, \bullet >_V \oplus_\perp < \bullet, \bullet >_{\{r,\,s\}} .$$

If we start this construction from an admissible module V then we call the resulting nilpotent Lie algebra a *pseudo H-type algebra*. In the following it will be denoted by $\mathcal{N}_{r,s}$ and we write $G_{r,s}$ for the corresponding Lie group. Note that $\mathcal{N}_{r,s}$ and $G_{r,s}$ depend on the chosen module V, which in general is not unique. However, to keep the notation short we will not indicate this dependence.

Remark 6.2 : As a typical example we mention that the $(2k+1)$-dimensional Heisenberg algebra is of the form $\mathbb{R}^{2k,0} \oplus_\perp \mathbb{R}^{1,0}$.

The bi-linear map $V \times V \ni (X, Y) \mapsto [X, Y] \in \mathbb{R}^{r,s}$ is always surjective by the admissibility condition, so that the center $\mathfrak{z}$ of the algebra $\mathcal{N}_{r,s}$ coincides with $\mathbb{R}^{r,s} = [\mathcal{N}_{r,s}, \mathcal{N}_{r,s}] = [V, V]$.

Remark 6.3 : For the first time this type of algebra was defined in [21] (also see [22]) where the bi-linear form of the underlying Clifford algebra is positive definite. Therein it was called "generalized Heisenberg algebra (and group)". We will call such an algebra also a "classical H-type algebra". In these cases, irreducible $C\ell_{r,0}$-modules are always admissible with a positive definite bi-linear form.

Remark 6.4 : Even if there are two non-equivalent irreducible modules like in the cases of $C\ell_{3,0}$, $C\ell_{7,0}$, $C\ell_{0,1}$, $C\ell_{0,5}$ (and the tensor product algebras of these with $\overset{k}{\otimes}\, C\ell_{1,1}$ for any k and also with $C\ell_{8,0}$ or $C\ell_{0,8}$), the Lie algebra defined above with minimal admissible module is unique up to isomorphisms, see [16].

In [16] it was proved that the algebras $\mathcal{N}_{r,s} = V \oplus_{\perp} \mathbb{R}^{r,s}$ always admit lattices, or more strongly:

***Theorem* 6.5** — *For each orthonormal basis $\{Z_k\}$ in the center $\mathfrak{z} \cong \mathbb{R}^{r,s}$, there exists an orthonormal basis $\{X_i\}$ in V with respect to which not only the structure constants are ± 1 or 0, but also for each pair (X_i, X_j) of the basis the bracket $[X_i, X_j]$ is zero or, if not, there exists a unique element Z_k, $k = k(i, j)$ such that*

$$[X_i, X_j] = Z_{k(i,j)}, \text{ or } [X_i, X_j] = -Z_{k(i,j)}.$$

PROOF : See Theorem 1.2 and Theorem 1.3 of [16]. □

So the vectors $\{X_i, Z_k\}$ form a basis in the Lie algebra $\mathcal{N}_{r,s}$ and the range $\exp(\{X_i, Z_k\})$ $(= \{X_i, Z_k\}$, since our exponential map is the identity map) generates a lattice

$$L = \left\{ \sum m_i X_i + \frac{1}{2} \sum n_k Z_k \,\middle|\, m_i, n_k \in \mathbb{Z} \right\}$$

of the group $G_{r,s}$. We call such a basis an *integral basis* and the corresponding lattice an *integral lattice*. This is a generalization of a result on classical H-type algebras proved in [14].

Recall that the algebra $\mathcal{N}_{r,s}$ is equipped with a non-degenerate symmetric bi-linear form which is not positive definite in the case $s > 0$. We can as well install an inner product on $\mathcal{N}_{r,s}$ by assuming that the integral basis is orthonormal. Then we have two identifications

$$\mathrm{Hom}(\mathcal{N}_{r,s}, \mathcal{N}_{r,s}) \cong \mathcal{N}_{r,s}{}^* \otimes \mathcal{N}_{r,s} \cong \mathcal{N}_{r,s} \otimes \mathcal{N}_{r,s} \subset \mathcal{U}_{\mathcal{N}_{r,s}}$$

into the universal enveloping algebra $\mathcal{U}_{\mathcal{N}_{r,s}}$. Hence we have two operators corresponding to the identity map $Id \in \mathrm{Hom}(\mathcal{N}_{r,s}, \mathcal{N}_{r,s})$, namely

$$\sum_{i=1}^{N} \tilde{X}_i^2 + \sum_{k=1}^{r+s} \tilde{Z}_k^2 = -2\Delta_{\text{sub}} + \sum_{k=1}^{r+s} \tilde{Z}_k^2, \quad \text{and}$$
$$\sum_{i=1}^{N/2} \tilde{X}_i^2 - \sum_{j=1}^{N/2} \tilde{X}_{N/2+j}^2 + \sum_{k=1}^{r} \tilde{Z}_k^2 - \sum_{k=1}^{s} \tilde{Z}_{r+k}^2.$$

The second operator is called an *ultra-hyperbolic operator*.

In the sequel we determine the spectral zeta function for the sub-Laplacian on some low dimensional nilmanifolds ($\dim \mathcal{N}_{r,s} \leq 12$) with underlying Lie group of the above type and by fixing an integral lattice. The study of the ultra hyperbolic operators will be postponed to a forthcoming paper (see [20, 30] for the analysis of special cases of such operators).

6.2 *Algebra by $C\ell_{2,0}$-module*

Let Z_1 and Z_2 be the orthonormal generators of $C\ell_{2,0} \cong \mathbb{H}$ (= the quaternion number field), i.e. $Z_i \in \mathbb{R}^{2,0}$ with

$$Z_i{}^2 = -1, \ Z_1 Z_2 + Z_2 Z_1 = 0.$$

Take an element $v \in \mathbb{R}^{4,0}$ such that $< v, v >_{\{4,0\}} = 1$. Then the vectors

$$X_1 = v, \ X_2 = J_{Z_1} J_{Z_2}(v), \ X_3 = J_{Z_1}(v), \ X_4 = J_{Z_2}(v)$$

form a basis of $\mathbb{R}^{4,0}$. We have the following non-trivial commutation relations (all other brackets vanish):

$$[X_1, X_3] = Z_1, \ [X_1, X_4] = Z_2, \ [X_2, X_4] = -Z_1, \ [X_2, X_3] = Z_2.$$

The Lie algebra $\mathcal{N}_{2,0}$ is given as the orthogonal sum $\mathbb{R}^{4,0} \oplus_\perp \mathbb{R}^{2,0}$ with the integral basis $\{X_1, X_2, X_3, X_4, Z_1, Z_2\}$. Here $\{Z_1, Z_2\}$ forms a basis of the center $\mathfrak{z} = [\mathcal{N}_{2,0}, \mathcal{N}_{2,0}] = \mathbb{R}^{2,0}$. We take a lattice of the form

$$L = \left\{ \sum m_i X_i + \frac{1}{2} \sum k_j \, Z_j \;\middle|\; m_i, \ k_j \in \mathbb{Z} \right\},$$

which we call a *standard integral lattice*.

Let $\mathbf{A} \cong \mathbb{R}^{2,0}$ denote the center of the group $G_{2,0}$. The lattice dual to $L \cap \mathbf{A}$ is

$$\left\{ \mathbf{n} = 2 \sum n_k Z_k \mid n_k \in \mathbb{Z} \right\} \cong (L \cap \mathbf{A})^*,$$

and a set $\mathbb{M}$ of complete representatives of the quotient group $L/(L \cap \mathbf{A})$ can be chosen as

$$\mathbb{M} = \left\{ \ \sum m_i X_i \mid m_i \in \mathbb{Z} \ \right\}.$$

The matrix $\Omega_{2,0}(\tau) = \Omega(\tau), \tau = (\tau_1, \tau_2)$ which appears in the integral expression of the heat kernel (cf. Theorem 2.5) is given by

$$\Omega_{2,0}(\tau) = \begin{pmatrix} 0 & 0 & \tau_1 & \tau_2 \\ 0 & 0 & \tau_2 & -\tau_1 \\ -\tau_1 & -\tau_2 & 0 & 0 \\ -\tau_2 & \tau_1 & 0 & 0 \end{pmatrix}.$$

Let $\mathbf{n}$ be an element in the dual lattice $(L \cap \mathbf{A})^*$. Then for $\mathbf{n} \neq 0$ the matrix $\Omega(\mathbf{n})$ is non-singular. Hence in Theorem 4.2, it is enough to determine the quantity

$$W(2\pi\sqrt{-1}t\,\mathbf{n}) = \sqrt{\det \frac{\Omega(2\pi\sqrt{-1}\,t\,\mathbf{n})}{\sinh \Omega(2\pi\sqrt{-1}\,t\,\mathbf{n})}}$$

for each $\mathbf{n} = 2n_1 Z_1 + 2n_2 Z_2 \neq 0$. A direct calculation shows that

$$W(2\pi\sqrt{-1}t\,\mathbf{n}) = \frac{(4\pi t)^2 \cdot ({n_1}^2 + {n_2}^2)}{\sinh^2(4\pi t\sqrt{{n_1}^2 + {n_2}^2})}.$$

In conclusion, the heat trace of $\mathcal{D}^{(\mathbf{n})}$ for $\mathbf{n} \neq 0$ is given by:

Proposition 6.6 —

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = 4 \cdot \frac{{n_1}^2 + {n_2}^2}{\sinh^2(4\pi t\sqrt{{n_1}^2 + {n_2}^2})}.$$

From the last proposition we conclude that the spectral zeta function $\zeta^{(\mathbf{n})}_{\{2,0\}}(s)$ of $\mathcal{D}^{(\mathbf{n})}$ for $\mathbf{n} \neq 0$ can be expressed in the form:

$$\begin{aligned} \zeta^{(\mathbf{n})}_{\{2,0\}}(s) &= 4 \cdot \frac{{n_1}^2 + {n_2}^2}{\Gamma(s)} \int_0^\infty \frac{1}{\sinh^2\left(4\pi t\,\sqrt{{n_1}^2 + {n_2}^2}\,\right)}\, t^{s-1} dt \\ &= \frac{1}{2^{3s-4} \cdot \pi^s \cdot ({n_1}^2 + {n_2}^2)^{(s-2)/2}} \cdot \zeta(s-1), \end{aligned}$$

where $\zeta(s)$ denotes the Riemann zeta function.

6.3 *Algebra by $C\ell_{r,0}$-module*

Extending the result of the previous subsection we now treat the case $s = 0$ in general. Let $J : C\ell_{r,0} \times V \longrightarrow V$ be an admissible module for the Clifford algebra generated by the vector space $\mathbb{R}^{r,0}$ with positive definite inner product. Then the module space V can be equipped with a second positive definite inner product $< \bullet, \bullet >_V$ (cf. [21]). Hence in this case, by definition of the Lie bracket through the relation

$$< J_z(X),\, Y >_V = < z,\, [X, Y] >_{\{r,0\}},$$

we know that the matrix $\Omega(\tau)$ coincides with the matrix representation of the operation J_τ for $\tau = \sum \tau_k Z_k$ and we have

$$\Omega(\tau)^2 = -\sum_{k=1}^{r} {\tau_k}^2 \cdot \mathrm{Id}. \tag{6.3}$$

Hence the action function $f(\tau, x, z)$ and the volume element $W(\sqrt{-1}\tau)$ in the formula of the heat kernel in Theorem 2.5 are given as

$$\begin{aligned} f(\tau,\, x,\, z) &= \sqrt{-1} < \tau, z >_{\{r,0\}} + \frac{< \Omega(\sqrt{-1}\tau) \coth \Omega(\sqrt{-1}\tau) \cdot x,\, x >_V}{2} \\ &= \sqrt{-1} < \tau,\, z >_{\{r,0\}} + \frac{\|\tau\| \coth \|\tau\| \cdot < x,\, x >_V}{2}, \end{aligned} \tag{6.4}$$

$$W(\sqrt{-1}\tau) = \sqrt{\det \frac{\Omega(\sqrt{-1}\tau)}{\sinh \Omega(\sqrt{-1}\tau)}} = \left\{ \frac{\|\tau\|}{\sinh \|\tau\|} \right\}^{N/2}, \tag{6.5}$$

where N always is an even number. Now the heat kernel can be expressed in a more explicit form compared to the one in Theorem 2.5:

Proposition 6.7 — The heat kernel of the sub-Laplacian on $G_{r,0}$ is given by:

$$\begin{aligned} &K(t, g, h) = K(t, x, z, x', z') \\ &= \frac{1}{(4\pi\, t)^{N/2+r}} \int_{\mathbb{R}^r} e^{-\left\{ \frac{\sqrt{-1} < \tau, z - \frac{1}{2}[x, x'] >_{\{r,\,0\}}}{t} + \frac{\|\tau\| \coth \|\tau\| \cdot \|x - x'\|^2}{2t} \right\}} \left\{ \frac{\|\tau\|}{\sinh \|\tau\|} \right\}^{N/2} d\tau. \end{aligned} \tag{6.6}$$

PROOF : Formula (6.6) is obtained from Theorem 2.5 together with the identities (6.4), (6.5). □

We choose the integral basis $\{X_i\}$ for the given orthonormal basis $\{Z_k\}$ following the construction given in [16] and fix the lattice as above. From the previous expression of the heat kernel for the sub-Laplacian on the groups $G_{r,0}$ we obtain the heat trace of each component operator $\mathcal{D}^{(\mathbf{n})}$:

Proposition 6.8 — For $\mathbf{n} \neq 0$ in the dual lattice,

$$\mathbf{tr}\left(e^{-t\,\mathcal{D}^{(\mathbf{n})}}\right) = 2^{\frac{N}{2}} \cdot \frac{({n_1}^2 + \cdots + {n_r}^2)^{N/4}}{\sinh^{N/2}(4\pi t \sqrt{{n_1}^2 + \cdots + {n_r}^2})}.$$

Especially, for $r = 3,\ 4,\ 5,\ 6,\ 7,$ and 8 we know that $N = 4,\ 8,\ 8,\ 8,\ 8,$ and 16, respectively.

PROOF : According to (6.3) the matrix $\Omega(\mathbf{n})$ is invertible whenever $\mathbf{n} \neq 0$ and the assertion follows from Theorem 4.2. □

6.4 *Algebra* $\mathcal{N}_{1,1}$

The Clifford algebra $C\ell_{1,1}$ is generated by $\mathbb{R}^{1,1}$ with the basis $\{Z_1, Z_2\}$ satisfying the conditions

$$< Z_1, Z_1 >_{\{1,1\}} = 1, \;\; < Z_2, Z_2 >_{\{1,1\}} = -1, \;\; < Z_1, Z_2 >_{\{1,1\}} = 0.$$

Note that $C\ell_{1,1}$ is isomorphic to $\mathbb{R}(2)$, the algebra of real matrices of size 2. In this case the minimal admissible module V is of dimension 4 with the following integral basis: choose $v \in V$ with $< v, v >_V = 1$, and put

$$X_1 = v, X_2 = J_{Z_1} J_{Z_2}(v), X_3 = J_{Z_1}(v), X_4 = J_{Z_2}(v).$$

We have the following commutation relation table

Table 2: Commutation relation table of $\mathcal{N}_{1,1}$

[row,column]	X_1	X_2	X_3	X_4
X_1	0	0	Z_1	Z_2
X_2	0	0	Z_2	Z_1
X_3	$-Z_1$	$-Z_2$	0	0
X_4	$-Z_2$	$-Z_1$	0	0

The standard lattice L is given as

$$L = \{ \sum m_i X_i + \frac{1}{2} \sum k_j Z_j \mid m_i,\, k_j \in \mathbb{Z} \}.$$

Elements $\mathbf{n}$ in the dual lattice $(L \cap \mathbf{A})^*$ are expressed as $\mathbf{n} = 2n_1 Z_1 + 2n_2 Z_2$ with $n_i \in \mathbb{Z}$. In the case $|n_1| \neq |n_2|$, the matrix function in Theorem 4.2 is calculated as

$$\left\{ \det \left(\frac{\Omega(2\pi\sqrt{-1}\, t\, \mathbf{n})}{\sinh \Omega(2\pi\sqrt{-1}\, t\, \mathbf{n})} \right) \right\}^{1/2} = \frac{(4\pi t)^2 ({n_1}^2 - {n_2}^2)}{\sinh 4\pi t(n_1 + n_2) \cdot \sinh 4\pi t(n_1 - n_2)}. \tag{6.7}$$

So the heat trace of the operator $\mathcal{D}^{(\mathbf{n})}$ is obtained as follows:

Proposition 6.9 — (i) If $|n_1| \neq |n_2|$,

$$\mathbf{tr} \left(e^{-t\mathcal{D}^{(\mathbf{n})}} \right) = \frac{4({n_1}^2 - {n_2}^2)}{\sinh 4\pi t(n_1 + n_2) \cdot \sinh 4\pi t(n_1 - n_2)}.$$

(ii) If $n_1 = n_2 \neq 0$, then the solution space $\mathbb{M}(\mathbf{n})$ in the set $\mathbb{M} = \{ \sum m_i X_i \mid m_i \in \mathbb{Z} \}$ of complete representatives of the quotient group $L/(\mathbf{A} \cap L)$ is

$$\mathbb{M}(\mathbf{n}) = \{(m_1, -m_1, m_2, -m_2) \mid m_i \in \mathbb{Z}\}.$$

In case of $n_1 = -n_2$ we have $\mathbb{M}(\mathbf{n}) = \{(m_1, m_1, m_2, m_2) \mid m_i \in \mathbb{Z}\}$.

(iii) According to (ii) the heat trace of the operator $\mathcal{D}^{(\mathbf{n})}$ in case of $|n_1| = |n_2| \neq 0$ is given by

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \frac{2}{\pi t} \sum_{m_1,\, m_2 \in \mathbb{Z}} e^{-({m_1}^2 + {m_2}^2)/t} \frac{2n_1}{\sinh 8\pi\, t\, n_1}.$$

PROOF : Follows from (6.7) and Theorem 4.2. □

6.5 *Algebras $\mathcal{N}_{r,s}$ with $r + s = 3$*

In the case where $r + s = 3$ it is known that none of the groups $G_{3,0}$, $G_{2,1}$, $G_{1,2}$ and $G_{0,3}$ are isomorphic.

The minimal admissible module V of $C\ell_{2,2}$ is realized by $V = \mathbb{R}^{4,4}$. Moreover, the restriction of $C\ell_{2,1} \subset C\ell_{2,2}$ to V gives the minimal admissible module of $C\ell_{2,1}$. Similarly, we obtain an admissible module of the Clifford algebra $C\ell_{1,2}$ by a restriction of the action of $C\ell_{2,2}$ on its minimal admissible module $\mathbb{R}^{4,4}$. As before we denote the resulting pseudo H-type algebras by $\mathcal{N}_{2,1} \cong \mathbb{R}^{4,4} \oplus_\perp \mathbb{R}^{2,1}$ and $\mathcal{N}_{1,2} \cong \mathbb{R}^{4,4} \oplus_\perp \mathbb{R}^{1,2}$, respectively.

Proposition 6.10 — The algebras $\mathcal{N}_{2,1}$ and $\mathcal{N}_{1,2}$ are isomorphic.

PROOF : The Clifford algebra $C\ell_{2,2}$ is generated by $\mathbb{R}^{2,2}$ with the orthogonal basis $\{Z_1, Z_2, Z_3, Z_4\}$ and the inner products $< Z_1, Z_1 >_{\{2,2\}} = < Z_2, Z_2 >_{\{2,2\}} = 1$, $< Z_3, Z_3 >_{\{2,2\}} = < Z_4, Z_4 >_{\{2,2\}} = -1$ and $< Z_i, Z_j >_{\{2,2\}} = 0$ (for $i \neq j$). It is isomorphic to $\mathbb{R}(4)$, the 4 by 4 real matrix algebra. The minimal admissible module of $C\ell_{2,2}$ is eight dimensional and realized in the space $\mathbb{R}^{4,4}$. We can choose a vector $v \in \mathbb{R}^{4,4}$ satisfying

$$J_{Z_1} J_{Z_2} J_{Z_3} J_{Z_4}(v) = v \quad \text{and} \quad < v, v >_{\{4,4\}} = 1.$$

Put

$$X_1 = v, X_2 = J_{Z_1} J_{Z_2}(v), X_3 = J_{Z_1} J_{Z_3}(v), X_4 = J_{Z_1} J_{Z_4}(v),$$
$$X_5 = J_{Z_1}(v), X_6 = J_{Z_2}(v), X_7 = J_{Z_3}(v), X_8 = J_{Z_4}(v),$$

then the commutation relations with respect to this basis are given in the table below.

The inclusion $\mathbb{R}^{2,1} \subset \mathbb{R}^{2,2}$ induces the admissible module action of the Clifford algebra $C\ell_{2,1}$ by restricting the action of $C\ell_{2,2}$ to the same module $\mathbb{R}^{4,4}$ and it is also minimal. Thus we obtain the commutation relation table for the algebra $\mathcal{N}_{2,1}$ by putting $Z_4 = 0$.

Similarly, the commutation relation table for the algebra $\mathcal{N}_{1,2}$ is obtained by putting $Z_2 = 0$. More precisely, we have:

Table 3: Commutation relation table of $\mathcal{N}_{2,2}$

[row, column]	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	0	0	0	Z_1	Z_2	Z_3	Z_4
X_2	0	0	0	0	Z_2	$-Z_1$	Z_4	$-Z_3$
X_3	0	0	0	0	Z_3	$-Z_4$	Z_1	$-Z_2$
X_4	0	0	0	0	Z_4	Z_3	Z_2	Z_1
X_5	$-Z_1$	$-Z_2$	$-Z_3$	$-Z_4$	0	0	0	0
X_6	$-Z_2$	Z_1	Z_4	$-Z_3$	0	0	0	0
X_7	$-Z_3$	$-Z_4$	$-Z_1$	$-Z_2$	0	0	0	0
X_8	$-Z_4$	Z_3	Z_2	$-Z_1$	0	0	0	0

By comparing these tables, the transformation

$$\begin{aligned}
&X_1 \mapsto X_1, X_2 \mapsto X_2, X_3 \mapsto X_3, X_4 \mapsto X_4,\\
&X_5 \mapsto X_7, X_6 \mapsto X_8, X_7 \mapsto X_5, X_8 \mapsto X_6\\
&Z_1 \mapsto Z_3, Z_2 \mapsto Z_4, Z_3 \mapsto Z_1,
\end{aligned}$$

gives a Lie algebra isomorphism between $\mathcal{N}_{2,1}$ and $\mathcal{N}_{1,2}$. □

6.6 $\mathcal{N}_{2,1}$ *case*

As was mentioned before we can consider the Clifford algebra $C\ell_{2,1}$ as a subalgebra of $C\ell_{2,2}$. If the module action of $C\ell_{2,1}$ is defined as the restriction of the $C\ell_{2,2}$-action then both corresponding minimal admissible modules coincide as vector spaces.

We can fix the integral basis $\{X_i\}_{i=1}^{8} \cup \{Z_1, Z_2, Z_3\}$ as in the proof of Proposition 6.10. According to this basis we define the lattice

$$L_{2,1} := \Big\{ \sum m_i X_i + \frac{1}{2} \sum k_j Z_j \Big\}.$$

Proposition 6.11 —

$$\begin{aligned}
\det\big(\Omega(\tau) - \lambda\big) &= \big((\lambda^2 + {\tau_1}^2 + {\tau_2}^2 + {\tau_3}^2)^2 - 4{\tau_3}^2({\tau_1}^2 + {\tau_2}^2)\big)^2\\
&= \big(\lambda^4 + 2\lambda^2({\tau_1}^2 + {\tau_2}^2 + {\tau_3}^2) + ({\tau_1}^2 + {\tau_2}^2 - {\tau_3}^2)^2\big)^2,\\
W(\sqrt{-1}\,\tau) &= \left(\frac{{\tau_1}^2 + {\tau_2}^2 - {\tau_3}^2}{\sinh(\sqrt{{\tau_1}^2 + {\tau_2}^2} + \tau_3) \cdot \sinh(\sqrt{{\tau_1}^2 + {\tau_2}^2} - \tau_3)}\right)^2.
\end{aligned}$$

Table 4: Commutation relation table of $\mathcal{N}_{2,1}$

[row, column]	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	0	0	0	Z_1	Z_2	Z_3	0
X_2	0	0	0	0	Z_2	$-Z_1$	0	$-Z_3$
X_3	0	0	0	0	Z_3	0	Z_1	$-Z_2$
X_4	0	0	0	0	0	Z_3	Z_2	Z_1
X_5	$-Z_1$	$-Z_2$	$-Z_3$	0	0	0	0	0
X_6	$-Z_2$	Z_1	0	$-Z_3$	0	0	0	0
X_7	$-Z_3$	0	$-Z_1$	$-Z_2$	0	0	0	0
X_8	0	Z_3	Z_2	$-Z_1$	0	0	0	0

PROOF : As was mentioned earlier $\Omega(\tau)$ coincides with the matrix representation of J_τ and is obtained via Table 4. The identities in Proposition 6.11 then follow by a direct calculation. □

The heat trace of the operators $\mathcal{D}^{(\mathbf{n})}$ is given in the next proposition:

Proposition 6.12 — (1) If the dual element $\mathbf{n} = 2\sum n_k Z_k$ satisfies ${n_1}^2 + {n_2}^2 \neq {n_3}^2$, then

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = 16\cdot\left(\frac{{n_1}^2 + {n_2}^2 - {n_3}^2}{\sinh\ 4\pi t(\sqrt{{n_1}^2 + {n_2}^2} + n_3\,)\cdot\sinh 4\pi t\,(\sqrt{{n_1}^2 + {n_2}^2} - n_3)}\right)^2.$$

(2) If ${n_1}^2 + {n_2}^2 = {n_3}^2$, then

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \frac{1}{(\pi t)^2}\sum_{\mu\in\mathbb{M}(\mathbf{n})} e^{-\frac{<\mu,\mu>}{2t}}\cdot\left(\frac{2n_3}{\sinh 8\pi\, t\, n_3}\right)^2.$$

Moreover, for $\mu \in \mathbb{M}(\mathbf{n})$ we have

$$<\mu,\mu>= \frac{2{n_3}^2}{{d_0}^2}(\alpha^2 + \beta^2 + k^2 + \ell^2).$$

The set $\mathbb{M}(\mathbf{n})$ is given as follows: Let $d_0 > 0$ be the greatest common divisor of n_1, n_2, n_3. Defining $n_i := d_0 n_i'$ we can express the solution space $\mathbb{M}(\mathbf{n})$ as:

$$\mathbb{M}(\mathbf{n}) = \Big\{\mu = (\alpha n'_1 + \beta n'_2, \alpha n'_2 - \beta n'_1, -\alpha n'_3, -\beta n'_3,\\ k\, n'_1 + \ell\, n'_2, k\, n'_2 - \ell\, n'_1, -k\, n'_3, \ell\, n'_3) \;\Big|\; \alpha,\beta,k,\ell \in \mathbb{Z}\Big\}.$$

Table 5: Commutation relation table of $\mathcal{N}_{1,2}$

[row, column]	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	0	0	0	Z_1	0	Z_3	Z_4
X_2	0	0	0	0	0	$-Z_1$	Z_4	$-Z_3$
X_3	0	0	0	0	Z_3	$-Z_4$	Z_1	0
X_4	0	0	0	0	Z_4	Z_3	0	Z_1
X_5	$-Z_1$	0	$-Z_3$	$-Z_4$	0	0	0	0
X_6	0	Z_1	Z_4	$-Z_3$	0	0	0	0
X_7	$-Z_3$	$-Z_4$	$-Z_1$	0	0	0	0	0
X_8	$-Z_4$	Z_3	0	$-Z_1$	0	0	0	0

PROOF (1) : If $n_1^2+n_2^2 \neq n_3^2$, then it follows from Proposition 6.11 that $\Omega(\mathbf{n})$ is invertible and therefore $\mathbb{M}(\mathbf{n}) = \{0\}$. The sum in Theorem 4.2 reduces to the single term on the right of the above equation. (2): Follows by a direct calculation again using Theorem 4.2. □

6.7 $\mathcal{N}_{1,2}^{\min}$ *case*

The Clifford algebra $C\ell_{1,2}$ is identified with $\mathbb{R}(2) \oplus \mathbb{R}(2)$ and the minimal admissible module is realized in $\mathbb{R}^{2,2}$. The pseudo H-type algebra constructed by this module will be denoted by $\mathcal{N}_{1,2}^{\min}$. We choose a vector v in $\mathbb{R}^{2,2}$ such that

$$J_{Z_1}J_{Z_2}J_{Z_3}(v) = v, \text{ and } < v, v >_{\{2,2\}} = 1.$$

If we put

$$X_1 = v, X_2 = J_{Z_1}(v), X_3 = J_{Z_2}(v), X_4 = J_{Z_3}(v),$$

then $\{X_1, X_2, X_3, X_4, Z_1, Z_2, Z_3\}$ forms an integral basis of the Lie algebra $\mathcal{N}_{1,2}^{\min}$. The commutation relations are given in the following table:

We fix a standard lattice $L = L_{1,2} = \{\sum m_i X_i + \frac{1}{2}\sum k_j Z_j\}$. Then the function

$$W(\sqrt{-1}\tau) = \sqrt{\det\left(\frac{\Omega(\sqrt{-1}\tau)}{\sinh\Omega(\sqrt{-1}\tau)}\right)},$$

is calculated as

$$W(\sqrt{-1}\tau) = \frac{{\tau_1}^2 - {\tau_2}^2 - {\tau_3}^2}{\sinh\big(\tau_1 + \sqrt{{\tau_2}^2+{\tau_3}^2}\,\big)\cdot\sinh\big(\tau_1 - \sqrt{{\tau_2}^2+{\tau_3}^2}\,\big)}. \tag{6.8}$$

Table 6: Commutation relation of $\mathcal{N}^{\min}_{1,2}$

[row, column]	X_1	X_2	X_3	X_4
X_1	0	Z_1	Z_2	Z_3
X_2	$-Z_1$	0	Z_3	$-Z_2$
X_3	$-Z_2$	$-Z_3$	0	$-Z_1$
X_4	$-Z_3$	Z_2	Z_1	0

The heat trace of the operator $\mathcal{D}^{(\mathbf{n})}$ for each $\mathbf{n}$ in the dual of the lattice $\left\{\frac{1}{2}\sum k_j Z_j\right\}$ is given in the next proposition:

Proposition 6.13 — (1) For $\mathbf{n} = 0$,

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{0})}}\right) = \frac{1}{(2\pi t)^2}\sum_{\ell_i\in\mathbb{Z}} e^{-\frac{{\ell_1}^2+{\ell_2}^2+{\ell_3}^2+{\ell_4}^2}{2t}}.$$

(2) Let $\mathbf{n} = 2n_1Z_1 + 2n_2Z_2 + 2n_3Z_3$ with $n_1^2 \neq {n_2}^2 + {n_3}^2$ such that $\Omega(\sqrt{-1}\mathbf{n})$ is invertible. Then

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \frac{4\cdot({n_1}^2 - {n_2}^2 - {n_3}^2)}{\sinh 4\pi\, t\left(n_1 + \sqrt{{n_2}^2+{n_3}^2}\,\right)\cdot\sinh 4\pi\, t\left(n_1 - \sqrt{{n_2}^2+{n_3}^2}\,\right)}.$$

(3) Let $\mathbf{n} \neq 0$ with ${n_1}^2 = {n_2}^2 + {n_3}^2$. Then the solution space $\mathbb{M}(\mathbf{n})$ is given by

$$\mathbb{M}(\mathbf{n}) = \left\{\mu = \sum_{i=1}^{4}\ell_i X_i \in \mathbb{M} \;\middle|\; n_1\ell_3 = n_3\ell_1 - n_2\ell_2,\ n_1\ell_4 = -n_2\ell_1 - n_3\ell_2\right\}.$$

Put $n_i = d_0 n_i'$, where d_0 is the greatest common divisor of $\{n_i\}_{i=1}^3$. Then $\mathbb{M}(\mathbf{n})$ is characterized as the set

$$\mathbb{M}(\mathbf{n}) = \left\{ -\ell n_1' X_1 - k n_1' X_2 + (k n_2' - \ell n_3')X_3 + (k n_3' + \ell n_2')X_4 \mid k, \ell \in \mathbb{Z} \right\}.$$

Hence,

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \frac{1}{\pi t}\sum_{k,\ell\in\mathbb{Z}} e^{-\frac{{n_1}^2(k^2+\ell^2)}{{d_0}^2\, t}}\,\frac{2n_1}{\sinh 8\pi\, t\, n_1}.$$

PROOF : Apply Theorem 4.2 and (6.8). □

Remark 6.14 : We observe that for all $\mathbf{n}$ the heat trace of the operator $\mathcal{D}^{(\mathbf{n})}$ in Proposition 6.12 coincides with the square of the heat trace of the corresponding operator in Proposition 6.13. As a consequence, if $V \cong \mathbb{R}^{4,4}$ is any admissible module of $C\ell_{1,2}$ (there are four different types of module structure) and we fix a suitable lattice L in the pseudo H-type group $V \oplus_\perp \mathbb{R}^{1,2} \cong G_{1,2}$, then the nilmanifold $L\backslash G_{1,2}$ will be isospectral (with respect to the sub-Laplacian) with $L_{2,1}\backslash\mathcal{N}_{2,1}$.

7. The Algebras $\mathcal{N}_{r,s}$ with $r+s=4$

7.1 $\mathcal{N}_{4,0} \cong \mathcal{N}_{0,4} \cong \mathbb{H}(2)$ case : See Section 6.3.

7.2 $\mathcal{N}_{2,2} \cong \mathbb{R}(4)$ case : According to [16] we can choose a vector $v \in \mathbb{R}^{4,4}$ satisfying

$$J_{Z_1}J_{Z_2}J_{Z_3}J_{Z_4}(v) = v \text{ and } <v,v>_{\{4,4\}} = 1.$$

Fix an integral basis

$$X_1 = v,\ X_2 = J_{Z_1}J_{Z_2}(v),\ X_3 = J_{Z_1}J_{Z_3}(v),\ X_4 = J_{Z_1}J_{Z_4}(v),$$
$$X_5 = J_{Z_1}(v),\ X_6 = J_{Z_2}(v),\ X_7 = J_{Z_3}(v),\ X_8 = J_{Z_4}(v)$$

together with $\{Z_k\}_{k=1}^4$. The commutation relations of the Lie algebra $\mathcal{N}_{2,2}$ were given in Table 3. above. In order to calculate the volume function

$$W(\sqrt{-1}\tau) = \sqrt{\det\left(\frac{\Omega(\sqrt{-1}\tau)}{\sinh\Omega(\sqrt{-1}\tau)}\right)} \tag{7.1}$$

we need to distinguish the cases ${\tau_1}^2+{\tau_2}^2-{\tau_3}^2-{\tau_4}^2 \neq 0$ and ${\tau_1}^2+{\tau_2}^2-{\tau_3}^2-{\tau_4}^2 = 0$.

Proposition 7.1 — (1) If ${\tau_1}^2+{\tau_2}^2-{\tau_3}^2-{\tau_4}^2 \neq 0$, then

$$W(\sqrt{-1}\tau) = \left(\frac{{\tau_1}^2+{\tau_2}^2-{\tau_3}^2-{\tau_4}^2}{\sinh\left(\sqrt{{\tau_1}^2+{\tau_2}^2}+\sqrt{{\tau_3}^2+{\tau_4}^2}\right)\sinh\left(\sqrt{{\tau_1}^2+{\tau_2}^2}-\sqrt{{\tau_3}^2+{\tau_4}^2}\right)}\right)^2.$$

(2) If ${\tau_1}^2+{\tau_2}^2-{\tau_3}^2-{\tau_4}^2 = 0$, then we have

$$W(\sqrt{-1}\tau) = \frac{4({\tau_1}^2+{\tau_2}^2)}{\left(\sinh 2\sqrt{{\tau_1}^2+{\tau_2}^2}\right)^2}.$$

Proof : The matrix representation of $\Omega(\tau)$ is obtained through Table 3. A direct calculation gives the eigenvalues of $\Omega(\tau)$ and via (7.1) leads to the identities in either (1) or (2) depending on whether there are zero eigenvalues or $\Omega(\tau)$ is invertible. □

Let $\mathbf{n} = 2\sum_{i=1}^{4} n_k Z_k$ be in the dual of $\mathbf{A} \cap L$.

Proposition 7.2 — (1) If ${n_1}^2+{n_2}^2-{n_3}^2-{n_4}^2 \neq 0$, then

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \frac{16\cdot({n_1}^2+{n_2}^2-{n_3}^2-{n_4}^2)^2}{\sinh^2 4\pi t\left(\sqrt{{n_1}^2+{n_2}^2}+\sqrt{{n_3}^2+{n_4}^2}\right)\cdot\sinh^2 4\pi t\left(\sqrt{{n_1}^2+{n_2}^2}-\sqrt{{n_3}^2+{n_4}^2}\right)}.$$

(2) Assume that ${n_1}^2+{n_2}^2-{n_3}^2-{n_4}^2=0$ and put $n_i=d_0n_i'$, where d_0 is the greatest common divisor of $\{n_i\}_{i=1}^4$. Then the solution space $\mathbb{M}(\mathbf{n})$ is given as

$$\mathbb{M}(\mathbf{n})=\Big\{(-\alpha n_1'-\beta n_2',-\alpha n_2'+\beta n_1',\alpha n_3'-\beta n_4',\alpha n_4'+\beta n_3', kn_1'+\ell n_2',kn_2'-\ell n_1',-kn_3'-\ell n_4',-kn_4'+\ell n_3')\;\big|\;\alpha,\beta,k,\ell\in\mathbb{Z}\Big\}$$

and the heat trace of $\mathcal{D}^{(\mathbf{n})}$ takes the form

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right)=\frac{1}{(\pi t)^2}\sum_{\alpha,\beta,k,l\in\mathbb{Z}}e^{-\frac{(\alpha^2+\beta^2+k^2+\ell^2)({n_1}^2+{n_2}^2)}{{d_0}^2\,t}}\frac{4({n_1}^2+{n_2}^2)}{\sinh^2 8\pi t\sqrt{{n_1}^2+{n_2}^2}}.$$

PROOF (1) : In this case the matrix $\Omega(\mathbf{n})$ where $\mathbf{n}=(n_1,\cdots,n_4)$ is invertible and on the right hand side of the trace formula in Theorem 4.2 we have $\mathbb{M}(\mathbf{n})=\{0\}$. (2): The statements follow by a straightforward calculation from Theorem 4.2. □

7.3 $\mathcal{N}_{3,1}$ *case*

Let $\{Z_k\}$ be a basis of $\mathbb{R}^{3,1}$ with the properties

$$<Z_i,\,Z_i>_{\{3,1\}}=1\;(i=1,2,3),\;\;<Z_4,\,Z_4>_{\{3,1\}}=-1,\;\text{and}\;\;<Z_i,\,Z_j>_{\{3,1\}}=0\;(i\neq j).$$

We choose a vector $v\in\mathbb{R}^{4,4}$ with the properties

$$J_{Z_1}J_{Z_2}J_{Z_3}(v)=v,\;\;<v,\,v>_{\{4,4\}}=1$$

and fix an integral basis

$$X_1=v,\,X_2=J_{Z_1}(v),\,X_3=J_{Z_2}(v),\,X_4=J_{Z_3}(v),$$
$$X_5=J_{Z_4}(v),\,X_6=J_{Z_4}J_{Z_1}(v),\,X_7=J_{Z_4}J_{Z_2}(v),\,X_8=J_{Z_4}J_{Z_3}(v)$$

together with $\{Z_k\}_{k=1}^4$ of the Lie algebra $\mathcal{N}_{3,1}$. The commutation relations are given in the Table 7:

According to this table we consider the 8×8-matrix function

$$\Omega_{3,1}(\tau)=\begin{pmatrix}0&\tau_1&\tau_2&\tau_3&\tau_4&0&0&0\\-\tau_1&0&-\tau_3&\tau_2&0&\tau_4&0&0\\-\tau_2&\tau_3&0&-\tau_1&0&0&\tau_4&0\\-\tau_3&-\tau_2&\tau_1&0&0&0&0&\tau_4\\-\tau_4&0&0&0&0&\tau_1&\tau_2&\tau_3\\0&-\tau_4&0&0&-\tau_1&0&-\tau_3&\tau_2\\0&0&-\tau_4&0&-\tau_2&\tau_3&0&-\tau_1\\0&0&0&-\tau_4&-\tau_3&-\tau_2&\tau_1&0\end{pmatrix},\quad\text{where}\quad\tau=(\tau_1,\ldots,\tau_4)\in\mathbb{R}^4.$$

Table 7: Commutation relation of $\mathcal{N}_{3,1}$

[row, column]	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	Z_1	Z_2	Z_3	Z_4	0	0	0
X_2	$-Z_1$	0	$-Z_3$	Z_2	0	Z_4	0	0
X_3	$-Z_2$	Z_3	0	$-Z_1$	0	0	Z_4	0
X_4	$-Z_3$	$-Z_2$	Z_1	0	0	0	0	Z_4
X_5	$-Z_4$	0	0	0	0	Z_1	Z_2	Z_3
X_6	0	$-Z_4$	0	0	$-Z_1$	0	$-Z_3$	Z_2
X_7	0	0	$-Z_4$	0	$-Z_2$	Z_3	0	$-Z_1$
X_8	0	0	0	$-Z_4$	$-Z_3$	$-Z_2$	Z_1	0

We write $\Omega_{3,1}(\tau) = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$ with $A, B \in \mathbb{R}(4)$. Note that $A^2 = -({\tau_1}^2 + {\tau_2}^2 + {\tau_3}^2)\mathrm{Id}_{4\times 4}$ and

$$\begin{pmatrix} A + \lambda\,\mathrm{Id}_{4\times 4} & B \\ -B & A + \lambda\,\mathrm{Id}_{4\times 4} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Id}_{4\times 4} & -B \\ 0 & A + \lambda\ \mathrm{Id}_{4\times 4} \end{pmatrix} = \begin{pmatrix} A + \lambda\,\mathrm{Id}_{4\times 4} & 0 \\ -B & B^2 + (A + \lambda\,\mathrm{Id}_{4\times 4})^2 \end{pmatrix}.$$

Using these relations we have

$$\det\big(\Omega_{3,1}(\tau) - \lambda\big) = \det\big(\Omega_{3,1}(\tau) + \lambda\big) = \Big(\big(\lambda^2 - <\tau, \tau>_{\{3,1\}}\big)^2 + 4\lambda^2\left({\tau_1}^2 + {\tau_2}^2 + {\tau_3}^2\right)\Big)^2.$$

We conclude that the eigenvalues of the matrix $\Omega(\sqrt{-1}\tau)$ are given by

$$\pm(\|\tau'\| \pm \tau_4)$$

and these eigenvalues are of multiplicity two. Here we write $\|\tau'\| = \sqrt{{\tau_1}^2 + {\tau_2}^2 + {\tau_3}^2}$. Hence we have:

Proposition 7.3 —

$$W(\sqrt{-1}\tau) = \frac{\left(\|\tau'\|^2 - \tau_4^2\right)^2}{\sinh^2\left(\|\tau'\| + \tau_4\right)\sinh^2\left(\|\tau'\| - \tau_4\right)}.$$

PROOF : This directly follows from the spectral data of $\Omega(\sqrt{-1}\tau)$ given above. □

As before we fix a lattice

$$L = L_{3,1} = \Big\{\gamma = \sum m_i X_i + \frac{1}{2}\sum k_i Z_i \mid m_i, k_i \in \mathbb{Z}\Big\}$$

and take an element $\mathbf{n} = 2\sum_{i=1}^{4} n_i Z_i$ in the dual of $L \cap \mathbf{A}$, where $\mathbf{A} = \{\frac{1}{2}\sum k_i Z_i \mid k_i \in \mathbb{Z}\}$.

Proposition 7.4 — (1) If $\mathbf{n} = 0$, then

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \frac{1}{(2\pi t)^4}\sum_{m_i \in \mathbb{Z}} e^{-\frac{{m_1}^2+{m_2}^2+{m_3}^2+{m_4}^2}{2t}}.$$

(2) Let ${n_1}^2 + {n_2}^2 + {n_3}^2 - {n_4}^2 \neq 0$, then the matrix $\Omega(2\pi\sqrt{-1}\mathbf{n})$ is invertible. Hence, the trace of the operator $e^{-t\mathcal{D}^{(\mathbf{n})}}$ is given by

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \frac{16\cdot\left({n_1}^2+{n_2}^2+{n_3}^2-n_4^2\right)^2}{\sinh^2 4\pi t(\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}+n_4)\cdot\sinh^2 4\pi t(\sqrt{{n_1}^2+{n_2}^2+{n_3}^2}-n_4)}.$$

(3) Assume that $\mathbf{n} \neq 0$ and ${n_1}^2+{n_2}^2+{n_3}^2-{n_4}^2 = 0$. Let $d_0 > 0$ be the greatest common divisor of the four integers $\{n_1, n_2, n_3, n_4\}$ and define n_i' through the relation $n_i = d_0 n_i'$.

Then the solution space $\mathbb{M}(\mathbf{n})$ is given by

$$\mathbb{M}(\mathbf{n}) = \left\{\sum \ell_i X_i = (\ell_1,\ell_2,\ell_3,\ell_4,\ell_5,\ell_6,\ell_7,\ell_8) \mid \ell_i \in \mathbb{Z}\right\},$$

where each ℓ_i is is characterized as

$$\begin{aligned}
&\ell_1 = -n_4'k_1, \ell_2 = -n_4'k_2, \ell_3 = -n_4'k_3, \ell_4 = -n_4'k_4,\\
&\ell_5 = n_1'k_2 + n_2'k_3 + n_3'k_4,\\
&\ell_6 = -n_1'k_1 - n_3'k_3 + n_2'k_4,\\
&\ell_7 = -n_2'k_1 + n_3'k_2 - n_1'k_4,\\
&\ell_8 = -n_3'k_1 - n_2'k_2 + n_1'k_3.
\end{aligned}$$

Note that in this case the rank of the matrix $\Omega(\mathbf{n})$ is 4 and the relation $AB = BA$ gives the above characterization of the solution space $\mathbb{M}(\mathbf{n})$.

In particular, we have

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \frac{1}{(\pi t)^2}\sum_{\ell_i \in \mathbb{Z}} e^{-\frac{{n_4}^2\cdot\sum_{i=1}^{4}{\ell_i}^2}{{d_0}^2 t}} \cdot \frac{4{n_4}^2}{\sinh^2 8\pi t\, n_4}.$$

PROOF : Direct calculation using Theorem 4.2 and the spectral data above. □

7.4 $\mathcal{N}_{1,3}$ *case*

Let $\{Z_k\}_{k=1}^4$ be a basis of $\mathbb{R}^{1,3}$ with the properties

$$< Z_1,\, Z_1 >_{\{1,3\}} = 1, \;\; < Z_i,\, Z_i >_{\{1,3\}} = -1 \; (i = 2, 3, 4) \text{ and } < Z_i,\, Z_j >_{\{1,3\}} = 0 \; (i \neq j).$$

According to [16] we choose a vector $v \in \mathbb{R}^{4,4}$ such that

$$J_{Z_1} J_{Z_2} J_{Z_3}(v) = v, \quad \text{and} \quad < v,\, v >_{\{4,4\}} = 1.$$

Fix an integral basis

$$X_1 = v,\, X_2 = J_{Z_1}(v),\, X_3 = J_{Z_4} J_{Z_2}(v),\, X_4 = J_{Z_4} J_{Z_3}(v),$$
$$X_5 = J_{Z_2}(v),\, X_6 = J_{Z_3}(v),\, X_7 = J_{Z_4}(v),\, X_8 = J_{Z_4} J_{Z_1}(v)$$

together with $\{Z_k\}_{k=1}^4$ of the Lie algebra $\mathcal{N}_{1,3}$. The commutation relations are given as follows:

Table 8: Commutation relation of $\mathcal{N}_{1,3}$

[row, column]	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8
X_1	0	Z_1	0	0	Z_2	Z_3	Z_4	0
X_2	$-Z_1$	0	0	0	Z_3	$-Z_2$	0	Z_4
X_3	0	0	0	$-Z_1$	Z_4	0	$-Z_2$	$-Z_3$
X_4	0	0	Z_1	0	0	Z_4	$-Z_3$	Z_2
X_5	$-Z_2$	$-Z_3$	$-Z_4$	0	0	$-Z_1$	0	0
X_6	$-Z_3$	Z_2	0	$-Z_4$	Z_1	0	0	0
X_7	$-Z_4$	0	Z_2	Z_3	0	0	0	Z_1
X_8	0	$-Z_4$	Z_3	$-Z_2$	0	0	$-Z_1$	0

With $\tau = (\tau_1, \ldots, \tau_4) \in \mathbb{R}^4$ we define the 8×8 matrix

$$\Omega(\tau)_{1,3} = \begin{pmatrix} 0 & \tau_1 & 0 & 0 & \tau_2 & \tau_3 & \tau_4 & 0 \\ -\tau_1 & 0 & 0 & 0 & \tau_3 & -\tau_2 & 0 & \tau_4 \\ 0 & 0 & 0 & -\tau_1 & \tau_4 & 0 & -\tau_2 & -\tau_3 \\ 0 & 0 & \tau_1 & 0 & 0 & \tau_4 & -\tau_3 & \tau_2 \\ -\tau_2 & -\tau_3 & -\tau_4 & 0 & 0 & -\tau_1 & 0 & 0 \\ -\tau_3 & \tau_2 & 0 & -\tau_4 & \tau_1 & 0 & 0 & 0 \\ -\tau_4 & 0 & \tau_2 & \tau_3 & 0 & 0 & 0 & \tau_1 \\ 0 & -\tau_4 & \tau_3 & -\tau_2 & 0 & 0 & -\tau_1 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ -B & -A \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & \tau_1 & 0 & 0 \\ -\tau_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tau_1 \\ 0 & 0 & \tau_1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \tau_2 & \tau_3 & \tau_4 & 0 \\ \tau_3 & -\tau_2 & 0 & \tau_4 \\ \tau_4 & 0 & -\tau_2 & -\tau_3 \\ 0 & \tau_4 & -\tau_3 & \tau_2 \end{pmatrix}.$$

In this case it holds $AB + BA = 0$ and we observe the identity

$$\begin{pmatrix} A + \lambda \ \mathrm{Id}_{4\times 4} & B \\ -B & -A + \lambda \ \mathrm{Id}_{4\times 4} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{Id}_{4,4} & -(A + \lambda \ \mathrm{Id}_{4\times 4})^{-1} B \\ 0 & \mathrm{Id}_{4\times 4} \end{pmatrix} =$$

$$= \begin{pmatrix} A + \lambda \ \mathrm{Id}_{4\times 4} & 0 \\ -B & B(A + \lambda \ \mathrm{Id}_{4\times 4})^{-1} B - A + \lambda \ \mathrm{Id}_{4\times 4} \end{pmatrix}.$$

Then together with the relations $B^2 = ({\tau_2}^2 + {\tau_3}^2 + {\tau_4}^2) \cdot \ \mathrm{Id}_{4\times 4}$, $A^2 = -\tau_1^2 \cdot \ \mathrm{Id}_{4\times 4}$ and

$$(A + \lambda \ \mathrm{Id}_{4\times 4})^{-1} B = B(\lambda \ \mathrm{Id}_{4\times 4} - A)^{-1}$$

we have

$$\det\big(\Omega_{1,3}(\tau) - \lambda \ \mathrm{Id}_{8\times 8}\big) = \det\big(\Omega_{1,3}(\tau) + \lambda \ \mathrm{Id}_{8\times 8}\big) = \Big(\big(\lambda^2 - < \tau, \tau >_{\{1,3\}}\big)^2 + 4\lambda^2 \cdot {\tau_1}^2\Big)^2.$$

Hence the eigenvalues of the matrix $\Omega_{1,3}(\sqrt{-1}\tau)$ are given as

$$\pm(\|\tau'\| \pm \tau_1)$$

and each of them has multiplicity 2. Here we write $\|\tau'\| = \sqrt{{\tau_2}^2 + {\tau_3}^2 + {\tau_4}^2}$. The volume function $W(\sqrt{-1}\tau)$ is calculated as follows:

Proposition 7.5 —

$$W(\sqrt{-1}\tau) = \frac{\big(\|\tau'\|^2 - {\tau_1}^2\big)^2}{\sinh^2\big(\|\tau'\| + \tau_1\big) \cdot \sinh^2\big(\|\tau'\| - \tau_1\big)}.$$

PROOF : This follows from the spectral data of $\Omega(\tau)$ derived above. □

As before we fix a lattice

$$L = L_{1,3} = \Big\{ \sum m_i X_i + \frac{1}{2} \sum k_i Z_i \mid m_i, k_i \in \mathbb{Z} \Big\}.$$

Let $\mathbf{n} = 2 \sum n_i Z_i$ with $n_i \in \mathbb{Z}$ be an element in the lattice dual to $\{\frac{1}{2} \sum k_i Z_i\}$.

Proposition 7.6 — (1) If $\mathbf{n} = 0$, then

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \frac{1}{(2\pi t)^4} \sum_{m_i \in \mathbb{Z}} e^{-\frac{{m_1}^2 + {m_2}^2 + {m_3}^2 + {m_4}^2}{2t}}.$$

(2) Let ${n_1}^2 - {n_2}^2 - {n_3}^2 - {n_4}^2 \neq 0$, then the matrix $\Omega(2\pi\sqrt{-1}\mathbf{n})$ is invertible. Hence the trace of the operator $e^{-t\mathcal{D}^{(\mathbf{n})}}$ is given by

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \frac{16\cdot\left({n_1}^2 - {n_2}^2 - {n_3}^2 - n_4^2\right)^2}{\sinh^2 4\pi t\left(n_1 + \sqrt{{n_2}^2 + {n_3}^2 + {n_4}^2}\,\right) \cdot \sinh^2 4\pi t\,\left(n_1 - \sqrt{{n_2}^2 + {n_3}^2 + {n_4}^2}\right)}.$$

(3) Assume that $\mathbf{n} \neq 0$ and ${n_1}^2 - {n_2}^2 - {n_3}^2 - {n_4}^2 = 0$. Let $d_0 > 0$ be the greatest common divisor of the four integers $\{n_1, n_2, n_3, n_4\}$ and define n_i' by $n_i = d_0 n_i'$.

Then the solution space $\mathbb{M}(\mathbf{n})$ is given as

$$\mathbb{M}(\mathbf{n}) = \left\{\sum \ell_i X_i = (\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_6, \ell_7, \ell_8) \;\middle|\; \ell_i \in \mathbb{Z}\right\},$$

where each ℓ_i is characterized as

$$\begin{aligned}
\ell_1 &= n_2' k_1 + n_3' k_2 + n_4' k_3,\\
\ell_2 &= n_3' k_1 - n_2' k_2 + n_4' k_4,\\
\ell_3 &= n_4' k_1 - n_2' k_3 - n_3' k_4,\\
\ell_4 &= n_4' k_2 - n_3' k_3 + n_2' k_4,\\
\ell_5 &= n_1' k_2, \ell_6 = -n_1' k_1, \ell_7 = -n_1' k_4, \ell_8 = n_1' k_3.
\end{aligned}$$

This can be obtained by the relation

$$\begin{pmatrix} A & B \end{pmatrix} \cdot \begin{pmatrix} B \\ A \end{pmatrix} = 0.$$

Now we have

$$\mathbf{tr}\left(e^{-t\mathcal{D}^{(\mathbf{n})}}\right) = \frac{1}{(\pi t)^2} \sum_{\ell_i \in \mathbb{Z}} e^{-\frac{{n_1}^2 \cdot \sum_{i=1}^{4} {\ell_i}^2}{{d_0}^2\, t}} \cdot \frac{4{n_1}^2}{\sinh^2 8\pi t\, n_1}.$$

PROOF : Direct calculation using Theorem 4.2 and the spectral data above. □

As a corollary to Proposition 7.4 and Proposition 7.6 we mention:

Corollary 7.7 — The above nilmanifolds are isospectral with respect to the sub-Laplacians.

8. Final Remarks

In Proposition 6.10 we remarked that the algebras $\mathcal{N}_{2,1}$ and $\mathcal{N}_{1,2}$ are isomorphic. Both are constructed as pseudo H-type algebras by restricting the module structure from the minimal admissible module

$\mathbb{R}^{4,4}$ of the Clifford algebra $C\ell_{2,2}$. The algebras in Subsections 6.3 and 6.4 are constructed in the same way from the minimal admissible module of the Clifford algebra $C\ell_{3,3}$. In both cases the corresponding nilmanifolds are isospectral with respect to the sub-Laplacians. The lattices are obtained from the module structure for the Clifford algebras $C\ell_{2,2}$ or $C\ell_{3,3}$.

Let V be the minimal dimensional admissible module for a Clifford algebra $C\ell_{N,N}$ and fix an integral lattice L in the pseudo H-type algebra $\mathcal{N}_{N,N}$. Then by restricting the module structure to both subalgebras $C\ell_{N-s,N}$ and $C\ell_{N,N-s}$ we can construct two pseudo H-type algebras $\mathcal{N}_{N-s,N} \cong V \oplus_\perp \mathbb{R}^{N-s,N}$ and $\mathcal{N}_{N,N-s} \cong V \oplus_\perp \mathbb{R}^{N,N-s}$. Moreover, we have integral lattices $L_{N-s,N}$ and $L_{N,N-s}$ in each group $G_{N-s,N}$ and $G_{N,N-s}$, respectively, and corresponding sub-Laplacians. As a generalization of Corollary 7.7 we state here as a conjecture that the two nilmanifolds $L_{N-s,N}\backslash\mathcal{N}_{N-s,N}$ and $L_{N,N-s}\backslash\mathcal{N}_{N,N-s}$ are isospectral with respect to the sub-Laplacian.

This problem will be treated in a forthcoming paper [9] from the point of view of isospectrality.

9. Appendix : A General Formula

We did not provide expressions for the Mellin transform of the heat trace of the sub-Laplacian on the pseudo H-type nilmanifolds discussed in the last sections. As can be seen from the general formula in Theorem 4.2 and more concretely from earlier results in [3, 8] and Section 5 the spectral zeta functions of these sub-Laplacians have a common term. In this appendix we study its analytic behavior in the simplest case.

Let $A > 0$ and $B > 0$ be positive constants and consider the Mellin transform $Z(s, A, B)$ (divided by the Gamma function) of the function

$$F(t, A, B) = \sum_{\ell\in\mathbb{Z}} \sum_{k=1}^{\infty} e^{-At\ell^2} \frac{k}{\sinh Btk}.$$

More precisely, $Z(s, A, B)$ is given as the integral transform

$$\begin{aligned} Z(s, A, B) &= \frac{1}{\Gamma(s)} \int_0^\infty F(t, A, B) t^{s-1} dt \\ &= \frac{1}{\Gamma(s)} \int_0^\infty \left(\sum_{\ell\in\mathbb{Z}} \sum_{k=1}^{\infty} e^{-At\ell^2} \frac{k}{\sinh Btk} \right) t^{s-1}\, dt \\ &= \sum_{\ell\in\mathbb{Z}} \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \frac{2k}{\big(A\ell^2 + Bk(2n+1)\big)^s}. \end{aligned}$$

By the *Poisson summation formula* we have

$$F(t,A,B)=\sum_{\ell\in\mathbb{Z}}\sum_{k=1}^{\infty}e^{-At\ell^2}\,\frac{k}{\sinh\,Btk}=\sum_{\ell\in\mathbb{Z}}\sum_{k=1}^{\infty}\left(\frac{\pi}{tA}\right)^{\frac{1}{2}}e^{-\frac{\pi^2\ell^2}{tA}}\,\frac{k}{\sinh\,Btk},$$

and so

$$\begin{aligned}
Z(s,A,B)&=\frac{1}{\Gamma(s)}\int_0^\infty\sum_{\ell\in\mathbb{Z}}\sum_{k=1}^{\infty}\left(\frac{\pi}{tA}\right)^{\frac{1}{2}}e^{-\frac{\pi^2\ell^2}{tA}}\,\frac{k}{\sinh\,Btk}\,t^{s-1}\,dt\\
&=\frac{1}{\Gamma(s)}\int_0^\infty\sum_{0\neq\ell\in\mathbb{Z}}\sum_{k=1}^{\infty}\left(\frac{\pi}{tA}\right)^{\frac{1}{2}}e^{-\frac{\pi^2\ell^2}{tA}}\,\frac{k}{\sinh\,Btk}\,t^{s-1}\,dt\\
&\quad+\frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{\pi}{tA}\right)^{\frac{1}{2}}\sum_{k=1}^{\infty}\frac{k}{\sinh\,Btk}\,t^{s-1}\,dt\\
&=\textit{entire function}+\frac{1}{\Gamma(s)}\left(\frac{\pi}{A}\right)^{\frac{1}{2}}\int_0^\infty\sum_{k=1}^{\infty}\frac{k}{\sinh\,Btk}\,t^{s-3/2}\,dt\\
&=\mathcal{H}(s,A,B)\\
&\quad+2\sqrt{\frac{\pi}{A}}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{1}{B^{s-1/2}}\left(1-\frac{1}{2^{s-1/2}}\right)\zeta(s-3/2)\zeta(s-1/2)\\
&=\mathcal{H}(s,A,B)+\mathcal{K}(s,A,B).
\end{aligned}$$

Remark 9.1 : The product $\zeta(s-3/2)\zeta(s-1/2)$ is known to have the series expansion

$$\sum_{n=1}^{\infty}\frac{\sigma(n)}{n^{s-1/2}},$$

where $\sigma(n)=\sum_{d\,|\,n}d$ means the sum of all divisors of n.

The second term

$$\begin{aligned}
\mathcal{K}(s,A,B)&=\frac{1}{\Gamma(s)}\int_0^\infty\left(\frac{\pi}{tA}\right)^{\frac{1}{2}}\sum_{k=1}^{\infty}\frac{k}{\sinh\,Btk}\,t^{s-1}\,dt\\
&=2\sqrt{\frac{\pi}{A}}\frac{\Gamma(s-1/2)}{\Gamma(s)}\frac{1}{B^{s-1/2}}\left(1-\frac{1}{2^{s-1/2}}\right)\zeta(s-3/2)\zeta(s-1/2)
\end{aligned}$$

can also be expressed in the form

$$\begin{aligned}
\mathcal{K}(s,A,B) &= \frac{1}{\Gamma(s)}\sqrt{\frac{\pi}{A}}\frac{1}{2B}\int_0^\infty \left\{\frac{\pi^2}{2Bt}\sum_{k\in\mathbb{Z}}\frac{1}{\cosh^2\frac{\pi^2 k}{Bt}} - 1\right\} t^{s-5/2}\,dt \\
&= \frac{1}{\Gamma(s)}\sqrt{\frac{\pi}{A}}\frac{1}{2B}\int_0^1 \left\{\frac{\pi^2}{2Bt}\sum_{0\neq k\in\mathbb{Z}}\frac{1}{\cosh^2\frac{\pi^2 k}{Bt}} + \frac{\pi^2}{2Bt} - 1\right\} t^{s-5/2}\,dt \\
&\quad + \frac{1}{\Gamma(s)}\sqrt{\frac{\pi}{A}}\frac{1}{2B}\int_1^\infty \left\{\frac{\pi^2}{2Bt}\sum_{k\in\mathbb{Z}}\frac{1}{\cosh^2\frac{\pi^2 k}{Bt}} - 1\right\} t^{s-5/2}\,dt \\
&= \frac{1}{\Gamma(s)}\sqrt{\frac{\pi}{A}}\frac{\pi^2}{4B^2}\frac{1}{s-5/2} - \frac{1}{\Gamma(s)}\sqrt{\frac{\pi}{A}}\frac{1}{2B}\frac{1}{s-3/2} \\
&\quad + \frac{1}{\Gamma(s)}\sqrt{\frac{\pi}{A}}\frac{\pi^2}{4B^2}\int_0^1 \sum_{0\neq k\in\mathbb{Z}}\frac{1}{\cosh^2\frac{\pi^2 k}{Bt}} t^{s-7/2}\,dt \\
&\quad + \frac{1}{\Gamma(s)}\sqrt{\frac{\pi}{A}}\frac{1}{2B}\int_1^\infty \left\{\frac{\pi^2}{2Bt}\sum_{k\in\mathbb{Z}}\frac{1}{\cosh^2\frac{\pi^2 k}{Bt}} - 1\right\} t^{s-5/2}\,dt \\
&= \frac{1}{\Gamma(s)}\sqrt{\frac{\pi}{A}}\frac{\pi^2}{4B^2}\frac{1}{s-5/2} - \frac{1}{\Gamma(s)}\sqrt{\frac{\pi}{A}}\frac{1}{2B}\frac{1}{s-3/2} \\
&\quad + \mathcal{H}_1(s,A,B) + \mathcal{H}_\infty(s,A,B),
\end{aligned}$$

where the two functions $\mathcal{H}_1(s,A,B)$ and $\mathcal{H}_\infty(s,A,B)$ are entire.

Remark 9.2 : The Fourier transform of the function $\dfrac{x}{\sinh x}$ is given as

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{x}{\sinh x}e^{-\sqrt{-1}x\,\xi}\,dx = \left(\frac{\pi}{2}\right)^{3/2}\frac{1}{\cosh^2\frac{\pi\xi}{2}}$$

and the Poisson summation formula states that

$$\sum_{k\in\mathbb{Z}} f(2\pi k) = \frac{1}{\sqrt{2\pi}}\sum_{n\in\mathbb{Z}}\mathcal{F}(f)(n).$$

The following identity can be seen as a generalization of the *Jacobi identity* for the heat trace of the Laplacian on the torus, whose Mellin transform is the Epstein zeta function:

$$\begin{aligned}
F(t,A,B) &= \sum_{\ell\in\mathbb{Z}}\sum_{k=1}^\infty e^{-At\ell^2}\frac{k}{\sinh\, Btk} \\
&= \frac{\pi}{8t^2B^2}\sqrt{\frac{\pi}{tA}}\sum_{0\neq\ell\in\mathbb{Z}}\sum_{0\neq k\in\mathbb{Z}} e^{-\frac{\pi^2\ell^2}{tA}}\frac{1}{\cosh^2\left(\frac{\pi k}{2tB}\right)} \\
&\quad + \left(\frac{\pi}{8t^2B^2} - \frac{1}{2tB}\right)\sqrt{\frac{\pi}{tA}}\sum_{\ell\in\mathbb{Z}} e^{-\frac{\pi^2\ell^2}{tA}} + \frac{\pi}{8t^2B^2}\sqrt{\frac{\pi}{tA}}\sum_{0\neq k\in\mathbb{Z}}\frac{1}{\cosh^2\left(\frac{\pi k}{2tB}\right)}.
\end{aligned}$$

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