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# ON TREES WITH EQUAL 2-DOMINATION AND 2-OUTER-INDEPENDENT DOMINATION NUMBERS

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For a graph $G = (V, E)$, a subset $D \subseteq V(G)$ is a 2-dominating set if every vertex of $V(G) \setminus D$ has at least two neighbors in $D$, while it is a 2-outer-independent dominating set if additionally the set $V(G) \setminus D$ is independent. The 2-domination (2-outer-independent domination, respectively) number of $G$, is the minimum cardinality of a 2-dominating (2-outer-independent dominating, respectively) set of $G$. We characterize all trees with equal 2-domination and 2-outer-independent domination numbers.

**Key words** : 2-domination; 2-outer-independent domination; tree.

## 1. INTRODUCTION

Let $G = (V, E)$ be a graph. By the neighborhood of a vertex $v$ of $G$ we mean the set $N_G(v) = \{u \in V(G) \colon uv \in E(G)\}$. The degree of a vertex $v$, denoted by $d_G(v)$, is the cardinality of its neighborhood. By a leaf we mean a vertex of degree one, while a support vertex is a vertex adjacent to a leaf. We say that a subset of $V(G)$ is independent if there is no edge between any two vertices of this set. A path on $n$ vertices we denote by $P_n$. By a star we mean a connected graph in which exactly one vertex has degree greater than one. Let $uv$ be an edge of a graph $G$. By subdividing the edge $uv$ we mean removing it, and adding a new vertex, say $x$, along with two new edges $ux$ and $xv$.

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A subset $D \subseteq V(G)$ is a dominating set of $G$ if every vertex of $V(G) \setminus D$ has a neighbor in $D$, while it is a 2-dominating set, abbreviated 2DS, of $G$ if every vertex of $V(G) \setminus D$ has at least two neighbors in $D$. The domination (2-domination, respectively) number of $G$, denoted by $\gamma(G)$ ($\gamma_2(G)$, respectively), is the minimum cardinality of a dominating (2-dominating, respectively) set of $G$. A 2-dominating set of $G$ of minimum cardinality is called a $\gamma_2(G)$-set. Note that 2-domination is a type of multiple domination in which each vertex, which is not in the dominating set, is dominated at least $k$ times for a fixed positive integer $k$. Multiple domination in graphs was introduced by Fink and Jacobson [3], and was further studied for example in [1, 2, 4, 5, 8, 10]. For a comprehensive survey of domination in graphs, see [6, 7].

A subset $D \subseteq V(G)$ is a 2-outer-independent dominating set, abbreviated 2OIDS, of $G$ if every vertex of $V(G) \setminus D$ has at least two neighbors in $D$, and the set $V(G) \setminus D$ is independent. The 2-outer-independent domination number of $G$, denoted by $\gamma_2^{oi}(G)$, is the minimum cardinality of a 2-outer-independent dominating set of $G$. A 2-outer-independent dominating set of $G$ of minimum cardinality is called a $\gamma_2^{oi}(G)$-set. The study of 2-outer-independent domination in graphs was initiated in [9].

We characterize all trees with equal 2-domination and 2-outer-independent domination numbers.

## 2. Results

We begin with the following three straightforward observations.

*Observation* 1 — For every graph $G$ we have $\gamma_2^{oi}(G) \geq \gamma_2(G)$.

*Observation* 2 — Every leaf of a graph $G$ is in every $\gamma_2(G)$-set and in every $\gamma_2^{oi}(G)$-set.

*Observation* 3 — For every path there is a minimum 2-dominating set that contains all vertices that are at even distance from one of the leaves.

Let $T$ be a tree. We say that two vertices of $T$ of degree at least three are linked, if all interior vertices of the path joining them in $T$ have degree two. Then the path is called a link. Paths joining leaves of $T$ to the closest vertices of degree at least three we call chains. The length of a link or a chain is the number of its edges. A link or a chain is even (odd, respectively) if its length is even (odd, respectively). We say that a vertex of $T$ of degree at least three, say $x$, is within even range of a leaf, if there is a leaf, say $y$, such that all links and chains of the path joining $x$ and $y$ in $T$ are even.

Let $\mathcal{T}_0$ be a family of trees in which for every pair of adjacent vertices of degree at least three, at least one of them is within even range of a leaf.

*Lemma* 4 — If $T \in \mathcal{T}_0$, then $\gamma_2^{oi}(T) = \gamma_2(T)$.

PROOF : Observation 3 implies that for every tree there is a minimum 2-dominating set that contains all vertices of degree at least three that are within even range of a leaf. Let $D$ be such a set for the tree $T$. Suppose that some two adjacent vertices of $T$, say $x$ and $y$, do not belong to the set $D$. Since $T \in \mathcal{T}_0$, at least one of them has degree two. This is a contradiction as that vertex must have at least two neighbors in $D$. We now conclude that for every pair of adjacent vertices of $T$, the set $D$ contains at least one of them. Thus $V(T) \setminus D$ is an independent set. Consequently, $D$ is a 2OIDS of the tree $T$. Therefore $\gamma_2^{oi}(T) \leq \gamma_2(T)$. On the other hand, by Observation 1 we have $\gamma_2^{oi}(T) \geq \gamma_2(T)$. ■

We characterize all trees with equal 2-domination and 2-outer-independent domination numbers. For this purpose we introduce a family $\mathcal{T}$ of trees $T = T_k$ that can be obtained as follows. Let $T_1 \in \mathcal{T}_0$. If $k$ is a positive integer, then $T_{k+1}$ can be obtained recursively from $T_k$ by the following operation. Let $x$ be a vertex of $T_k$, which belongs to some $\gamma_2^{oi}(T)$-set. Let $y$ be the central vertex of a star, each edge of which can be subdivided any non-negative even number of times. Then join the vertices $x$ and $y$.

For checking whether a given vertex of a tree belongs to some of its minimum 2-outer-independent dominating sets, let us consider the following algorithm, which labels vertices of a tree $T$ as taken, omitted and undecided. Initialize by calling every leaf taken and every other vertex undecided. Root $T$ at a non-leaf vertex, say $r$. Let $u \neq r$ be a vertex of $T$, which has not already been decided, and such that all its children have been decided. If some child of $u$ has been omitted, then take $u$. Otherwise omit $u$ and take its parent.

*Proposition* 5 — Let $T$ be a tree, and let $v$ be a vertex of $T$. There exists a $\gamma_2^{oi}(T)$-set containing the vertex $v$ if and only if $v$ is a leaf or, rooting $T$ at $v$, the above algorithm labels at least one child of $v$ as omitted.

We now prove that for every tree of the family $\mathcal{T}$, the 2-domination and the 2-outer-independent domination numbers are equal.

*Lemma* 6 — If $T \in \mathcal{T}$, then $\gamma_2^{oi}(T) = \gamma_2(T)$.

PROOF : We use the induction on the number $k$ of operations performed to construct the tree $T$. If $T \in \mathcal{T}_0$, then by Lemma 4 we have $\gamma_2^{oi}(T) = \gamma_2(T)$. Let $k$ be a positive integer. Assume that the result is true for every $T' = T_k$ of the family $\mathcal{T}$ constructed by $k-1$ operations. Let $x$ be a vertex of $T'$ to which is attached the new tree $T_1$. It is easy to notice that $\gamma_2^{oi}(T_1) = \gamma_2(T_1)$. The vertices of $T_1$

at odd distance from the vertex of maximum degree, say $y$, form a $\gamma_2^{oi}(T_1)$-set. Let $D'$ be a $\gamma_2^{oi}(T')$-set that contains the vertex $x$. It is easy to observe that the elements of the set $D'$ together with the vertices of $T_1$ at odd distance from $y$, form a 2OIDS of the tree $T$. Thus $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + \gamma_2^{oi}(T_1)$. Now let us observe that there exists a $\gamma_2(T)$-set that does not contain the vertex $y$ and the vertices of $T_1$ at even distance from $y$. Let $D$ be such a set. Notice that all vertices of $T_1$ at odd distance from $y$ belong to the set $D$. Observe that $D \cap V(T')$ is a 2DS of the tree $T'$. Therefore $\gamma_2(T') \leq \gamma_2(T) - \gamma_2(T_1)$. We now get $\gamma_2^{oi}(T) \leq \gamma_2^{oi}(T') + \gamma_2^{oi}(T_1) = \gamma_2(T') + \gamma_2(T_1) \leq \gamma_2(T)$. This implies that $\gamma_2^{oi}(T) = \gamma_2(T)$. ■

We now prove that if the 2-domination and the 2-outer-independent domination numbers of a tree are equal, then the tree belongs to the family $\mathcal{T}$.

*Lemma* 7 — Let $T$ be a tree. If $\gamma_2^{oi}(T) = \gamma_2(T)$, then $T \in \mathcal{T}$.

PROOF : The result we obtain by the induction on the order $n$ of the tree $T$. Assume that the lemma is true for every tree $T'$ of order $n' < n$. If at most one vertex of $T$ has degree at least three, then it follows from the definition of the family $\mathcal{T}_0$ that $T \in \mathcal{T}_0 \subseteq \mathcal{T}$ as in the tree $T$ there is no pair of adjacent vertices of degree at least three. Now assume that at least two vertices of $T$ have degree at least three. Let $x$ be a vertex of $T$ of degree at least three, which is adjacent to exactly one link. Thus $x$ is adjacent to at least two chains. First assume that some of them is even. Let $T_x$ be the tree induced by the vertex $x$ and the chains adjacent to $x$. Let $S$ be the set of vertices of $V(T_x) \setminus \{x\}$ that are leaves or are at even distance from $x$. Let $T'$ be a tree obtained from $T$ by replacing $T_x$ with a path $P_3$, say $xyz$, where $z$ is the leaf. Let $D'$ be a $\gamma_2(T')$-set that contains the vertices $x$ and $z$. It is easy to observe that $D' \cup S \setminus \{z\}$ is a 2DS of the tree $T$. Thus $\gamma_2(T) \leq \gamma_2(T') + |S| - 1$. Now let us observe that there exists a $\gamma_2^{oi}(T)$-set that does not contain the vertices of $T_x$, which are not leaves and are at odd distance from $x$. Let $D$ be such a set. Observe that $\{z\} \cup D \cap V(T')$ is a 2OIDS of the tree $T'$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - |S| + 1$. We now get $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - |S| + 1 = \gamma_2(T) - |S| + 1 \leq \gamma_2(T')$. This implies that $\gamma_2^{oi}(T') = \gamma_2(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. It follows from the definition of the family $\mathcal{T}$ that $T \in \mathcal{T}$.

Now assume that all chains adjacent to $x$ are odd. Let $T_x$ be the tree induced by the vertex $x$ and the chains adjacent to $x$. The neighbor of $x$ that does not belong to $V(T_x)$ we denote by $k$. Let $S$ be the set of vertices of $T_x$ that are at odd distance from $x$. Let $T' = T - T_x$. Let $D'$ be any $\gamma_2(T')$-set. It is easy to observe that $D' \cup S$ is a 2DS of the tree $T$. Thus $\gamma_2(T) \leq \gamma_2(T') + |S|$. Now let us observe that there exists a $\gamma_2^{oi}(T)$-set that does not contain the vertex $x$ and the vertices of $T_x$ at even distance from $x$. Let $D$ be such a set. The set $V(T) \setminus D$ is independent, thus $k \in D$. Observe that $D \setminus S$ is a 2OIDS of the tree $T'$ of cardinality $\gamma_2^{oi}(T) - |S|$, and which contains the vertex $k$. Therefore $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - |S|$. We now get $\gamma_2^{oi}(T') \leq \gamma_2^{oi}(T) - |S| = \gamma_2(T) - |S| \leq \gamma_2(T')$. This implies that $\gamma_2^{oi}(T') = \gamma_2(T')$. By the inductive hypothesis we have $T' \in \mathcal{T}$. Moreover, there exists a $\gamma_2^{oi}(T')$-set that contains the vertex $k$. The tree $T_x$ is obtained from a star by subdividing each one of its edges a non-negative even number of times. The tree $T$ can be obtained from $T'$ by attaching the tree $T_x$ by joining the central vertex to the vertex $k$. Thus $T \in \mathcal{T}$. ■

As an immediate consequence of Lemmas 6 and 7, we have the following characterization of trees with equal 2-domination and 2-outer-independent domination numbers.

***Theorem* 8** — *Let $T$ be a tree. Then $\gamma_2^{oi}(T) = \gamma_2(T)$ if and only if $T \in \mathcal{T}$.*

## REFERENCES

1. M. Blidia, M. Chellali and L. Volkmann, Bounds of the 2-domination number of graphs, *Utilitas Mathematica*, **71** (2006), 209-216.

2. M. Blidia, O. Favaron and R. Lounes, Locating-domination, 2-domination and independence in stress, *Aust. J. Comb.*, **42** (2008), 309-316.

3. J. Fink and M. Jacobson, *n-domination in graphs,* Graph Theory with Applications to Algorithms and Computer Science, Wiley, New York, 1985, 282-300.

4. J. Fujisawa, A. Hansberg, T. Kubo, A. Saito, M. Sugita and L. Volkmann, *Independence and 2-domination in bipartite graphs,* Australasian Journal of Combinatorics 40 (2008), 265-268.

5. A. Hansberg and L. Volkmann, On graphs with equal domination and 2-domination numbers, *Discrete Mathematics*, **308** (2008), 2277-2281.

6. T. Haynes, S. Hedetniemi and P. Slater, *Fundamentals of Domination in Graphs,* Marcel Dekker, New York, 1998.

7. T. Haynes, S. Hedetniemi and P. Slater (eds), *Domination in Graphs: Advanced Topics*, Marcel Dekker, New York, 1998.

8. Y. Jiao and H. Yu, On graphs with equal 2-domination and connected 2-domination numbers, *Mathematica Applicata*, **17** (2004), suppl., 88-92.

9. N. Jafari Rad and M. Krzywkowski, *2-outer-independent domination in graphs,* to appear in the National Academy Science Letters.

10. R. Shaheen, Bounds for the 2-domination number of toroidal grid graphs, *Int. J. Comp. Math.*, **86** (2009), 584-588