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ON SUMS OF GENERALIZED RAMANUJAN SUMS

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Ramanujan sums have been studied and generalized by several authors. For example, Nowak [8] studied these sums over quadratic number fields, and Grytczuk [4] defined that on semigroups. In this note, we deduce some properties on sums of generalized Ramanujan sums and give examples on number fields. In particular, we have a relational expression between Ramanujan sums and residues of Dedekind zeta functions.

Key words : Ramanujan sums; arithmetical functions.

1. Introduction

For positive integers m and k the Ramanujan sum $c_k(m)$ is defined as

$$c_k(m) = \sum_{\substack{h \mod k \\ (h,k)=1}} \exp\left(2\pi i \frac{mh}{k}\right) = \sum_{d|m,k} d\mu\left(\frac{k}{d}\right)$$

where μ is the Möbius function. This sum was generalized by several authors. (For example, see [1, 5, 8, 4], and so on). In this paper, we define generalized Ramanujan sums in another way and show some properties on them.

Suppose that X is a non-empty set and F_X is the set of all mappings $A : X \to \mathbb{Z}$ such that there are only finitely many points $x \in X$ such that $A(x) \neq 0$. We see that F_X is an abelian group with respect to addition. For $A, B \in F_X$, we denote $A \leq B$ if $A(x) \leq B(x)$ for every $x \in X$. Let $I_X = \{A \in F_X : A \geq 0\}$. When X is the set of all prime ideals of some Dedekind domain O, we regard I_X as the set of all non-zero ideals of O. Now fix a real-valued function $\mathcal{N} : I_X \to \mathbb{Z}_{>0}$ such that $\mathcal{N}(0) = 1$, $\mathcal{N}(A) > 1$ if $A \neq 0$, and $\mathcal{N}(A + B) = \mathcal{N}(A)\mathcal{N}(B)$ for all $A, B \in I_X$. The

Möbius function μ on I_X is defined as $\mu(A) = (-1)^{\sum_{x \in X} A(x)}$ when $A(X) \subset \{0,1\}$ and $\mu(A) = 0$ otherwise. For $M, K \in I_X$, we put

$$C_K(M) = \sum_{\substack{D \in I_X \\ D \leq M, K}} \mathcal{N}(D)\mu(K - D).$$

There are many expressions on Ramanujan sums. For example,

$$\sum_{d|k} c_k(d) = k \prod_{p|k} (1 - 2/p),$$

where the product is over all prime divisors p of k, and

$$\sum_{d|n} c_d(m) = \begin{cases} n & \text{if } n|m, \\ 0 & \text{otherwise.} \end{cases}$$

It is also known that

$$\sum_{m=1}^{\infty} \frac{c_k(m)}{m} = -\Lambda(k) \quad \text{if } k \neq 1$$

where Λ is the von Mangoldt function. Firstly, we shall show these analogues. Put $[x] = \sharp\{A \in I_X : \mathcal{N}(A) \leq x\}$ for a real number $x > 0$ when X is at most countable. We shall show the next theorem.

***Theorem* 1** — *(1) For $K \in I_X$, we have*

$$\sum_{\substack{D \in I_X \\ D \leq K}} C_K(D) = \mathcal{N}(K) \prod_{p} \left(1 - \frac{2}{\mathcal{N}(A_p)}\right)$$

where the product is over points $p \in X$ such that $K(p) \neq 0$ and A_p is the map such that $A_p(p) = 1$ and $A_p(q) = 0$ if $p \neq q$.

(2) For $M, N \in I_X$, we have

$$\sum_{\substack{D \in I_X \\ D \leq N}} C_D(M) = \begin{cases} \mathcal{N}(N) & \textit{if } N \leq M, \\ 0 & \textit{otherwise.} \end{cases}$$

(3) Suppose that X is at most countable and $[x] = cx + O(x^{\alpha})$ for some $c > 0$ and $\alpha \in [0, 1)$. For $K \neq 0 \in I_X$, we have

$$\sum_{M \in I_X} \frac{C_K(M)}{N(M)} = -c\Lambda(K).$$

where $\Lambda(A) = \sum_{D \leq A} \mu(A - D) \log \mathcal{N}(D)$.

Chan and Kumchev [2] studied the sums

$$\sum_{m\le x}\left(\sum_{k\le y} c_k(m)\right)^n$$

where n is a positive integer, x and y are large real numbers. In particular, they obtain

$$\sum_{\substack{m\le x\\ k\le y}} c_k(m) = x + O(y^2).$$

We shall show an analogue of this expression.

Theorem 2 — *Suppose that X is at most countable and $[x] = cx + O(x^\alpha)$ for some $c > 0$ and $\alpha \in [0, 1)$.*

(1) If we fix $K \in I_X$, then

$$\sum_{\mathcal{N}(M)\le x} C_K(M) = \begin{cases} cx + O(x^\alpha) & \text{when } K = 0,\\ O(x^\alpha) & \text{otherwise.}\end{cases}$$

(2) Put $S(x, y) := \sum_{\substack{\mathcal{N}(M)\le x\\ \mathcal{N}(K)\le y}} C_K(M)$. For any $\lambda > \frac{2-\alpha}{1-\alpha}$, under the condition $y^\lambda \ll x$, considering $x \to \infty$,

$$S(x, y) = cx + o(x).$$

2. Preliminary

In this section, we review or construct some basic facts of arithmetical functions in a generalized situation. (See [1], [3], or [9]). Put $\mathcal{A} := \{f : I_X \to R\}$ where R is a commutative ring. When X is the set of prime numbers and $R \subset \mathbb{C}$, we may regard an elements of $\mathcal{A}$ as an arithmetical function in the usual case. Let f and $g \in \mathcal{A}$. The Dirichelet convolution $f * g$ is defined as

$$f * g(A) = \sum_{\substack{D\in I_X\\ D\le A}} f(D)g(A-D) = \sum_{\substack{B,C\in I_X\\ B+C=A}} f(B)g(C)$$

for $A \in I_X$. The operator $*$ on $\mathcal{A}$ is commutative, and associative. The identity element is the function δ such that $\delta(0) = 1$ and $\delta(A) = 0$ when $A \neq 0$. A function $f \in \mathcal{A}$ is invertible if and only if $f(0) \in R^\times$. For simplicity, we suppose that $R = \mathbb{R}$ or $\mathbb{C}$. The function μ is the inverse of the function 1 such that $1(A) = 1$ for all $A \in I_X$, that is, $\mu * 1 = \delta$. One can see that $f = g * 1$ if and only if $g = f * \mu$.

The partial summation formula is generalized as follows.

Lemma 3 — Suppose $[x] < \infty$ for all $x > 0$. Let $F : [1, \infty) \to \mathbb{C}$ be a C^1 function and $x \geq 1$. For $g \in \mathcal{A}$, put $S(x) = \sum_{\mathcal{N}(\mathcal{A}) \leq x} g(A)$. Then

$$\sum_{\mathcal{N}(A) \leq x} g(A) F(\mathcal{N}(A)) = S(x)F(x) - \int_1^x S(t)F'(t)dt.$$

This lemma is shown by the ordinary partial summation formula. So we omit the proof.

For a complex number $s = \sigma + it$, set

$$Z(s) = \sum_{A \in I_X} \frac{1}{\mathcal{N}(A)^s}.$$

Suppose $[x] = cx + R(x)$ and $R(x) = O(x^\alpha)$ for some $c > 0$ and $\alpha \in [0, 1)$. For $x > 0$ and $\sigma > 1$, using Lemma 3, we see that

$$\left| \sum_{\mathcal{N}(A) \leq x} \frac{1}{\mathcal{N}(A)^s} \right| = \sum_{\mathcal{N}(A) \leq x} \frac{1}{\mathcal{N}(A)^\sigma} = \frac{1}{x^\sigma}[x] + \sigma \int_1^x \frac{[t]}{t^{\sigma+1}} dt.$$

By our assumption, the above expression is

$$\frac{c}{x^{\sigma-1}} + O(x^{\alpha-\sigma}) + \frac{\sigma c x^{1-\sigma}}{1-\sigma} + \frac{\sigma c}{\sigma - 1} + O\left(\sigma \int_1^x t^{\alpha-\sigma-1} dt \right).$$

Considering $x \to \infty$, we have

$$Z(\sigma) = \frac{\sigma c}{\sigma - 1} + O\left(\sigma \int_1^\infty t^{\alpha-\sigma-1} dt \right).$$

Therefore, the series $Z(s)$ is absolutely convergent for $\sigma > 1$. Moreover, we can see $Z(s)$ has an analytic continuation to $\sigma > \alpha$, and the residue of $Z(s)$ at $s = 1$ is the constant c.

3. Proof of Theorem 1

For $f, g \in \mathcal{A}$ and $M, K \in I_X$, define the sum $S_{f,g}(M, K)$ as

$$S_{f,g}(M, K) = \sum_{D \leq M, K} f(D) g(K - D).$$

Note that $S_{\mathcal{N},\mu}(M,K) = C_K(M)$.

For $A \in I_X$, the function $\chi_A \in \mathcal{A}$ is defined as $\chi_A(B) = 1$ when $B \leq A$ and $\chi_A(B) = 0$ otherwise. Let $v(D) = \chi_K(D)f(D)g(K-D)$. When K is fixed, we see $S_{f,g}(M,K) = (v*1)(M)$ and

$$\begin{aligned}\sum_{D\leq N} S_{f,g}(D,K)h(N-D) &= \sum_{D\leq N}(v*1)(D)h(N-D)\\ &= (v*1*h)(N) = \sum_{D\leq N} v(D)(1*h)(N-D)\\ &= \sum_{D\leq N}\chi_K(D)f(D)g(K-D)(1*h)(N-D).\end{aligned}$$

Thus, we have that

$$\sum_{D\leq N} S_{f,g}(D,K)h(N-D) = \sum_{D\leq N,K} f(D)g(K-D)(1*h)(N-D)$$

which is an analogue of Theorem 1 and 2 in [1]. When $f(A) = \mathcal{N}(A)$, $g(A) = \mu(A)$, and $h(A) = 1$ for all $A \in I_X$, the above equation is

$$\sum_{D\leq N} C_K(D) = \sum_{D\leq N,K} \mathcal{N}(D)\mu(K-D)(1*1)(N-D).$$

Put $K = N$. We obtain

$$\begin{aligned}\sum_{D\leq K} C_K(D) &= \sum_{D\leq K}\mathcal{N}(D)\mu(K-D)(1*1)(K-D)\\ &= \sum_{D\leq K}\mathcal{N}(K-D)\mu(D)(1*1)(D)\\ &= \mathcal{N}(K)\sum_{D\leq K}\frac{\mu(D)(1*1)(D)}{N(D)}\\ &= \mathcal{N}(K)\prod_p\left(1+\frac{\mu(A_p)(1*1)(A_p)}{\mathcal{N}(A_p)}\right)\\ &= \mathcal{N}(K)\prod_p\left(1-\frac{2}{\mathcal{N}(A_p)}\right)\end{aligned}$$

where the product is over points $p \in X$ such that $K(p) \neq 0$ and A_p is the map $A_p(p) = 1$ and $A_p(q) = 0$ if $p \neq q$. Hence (1) of Theorem 1 is proved.

Next, in order to show (2), fix $M \in I_X$. Then we see $S_{f,g}(M,K) = (w * g)(K)$ where $w(A) = \chi_M(A)f(A)$ and

$$\begin{aligned}\sum_{D\le N} S_{f,g}(M,D)h(N-D) &= \sum_{D\le N}(w*g)(D)h(N-D)\\ &= (w*g*h)(N)\\ &= \sum_{D\le N} w(D)(g*h)(N-D)\\ &= \sum_{D\le N}\chi_M(D)f(D)(g*h)(N-D)\\ &= \sum_{D\le N,M} f(D)(g*h)(N-D).\end{aligned}$$

Thus we obtain

$$\sum_{D\le N} S_{f,g}(M,D)h(N-D) = \sum_{D\le N,M} f(D)(g*h)(N-D)$$

which is an analogue of Theorem 3 and 4 in [1]. When $f(A) = \mathcal{N}(A)$, $g(A) = \mu(A)$, and $h(A) = 1$ for all $A \in I_X$, the above equation is

$$\begin{aligned}\sum_{D\le N} C_D(M) &= \sum_{D\le N,M}\mathcal{N}(D)\delta(N-D)\\ &= \begin{cases}\mathcal{N}(N) & \text{if } N \le M,\\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

Hence (2) is proved.

To show (3) we use $Z(s)$ which is defined in the previous section. By an argument similar to that of Titchmarsh [9] p. 10, we obtain that

$$\begin{aligned}\sum_{M\in I_X}\frac{C_K(M)}{\mathcal{N}(M)^s} &= \sum_{M\in I_X}\frac{1}{N(M)^s}\sum_{D\le M,K}\mathcal{N}(D)\mu(K-D)\\ &= \sum_{D\le K}\mu(K-D)\mathcal{N}(D)\sum_{C\in I_X}\frac{1}{\mathcal{N}(C+D)^s}\\ &= Z(s)\phi_{1-s}(K)\end{aligned}$$

where $\phi_{1-s}(A) = \sum_{D \le A} \mu(A-D)\mathcal{N}(D)^{1-s}$. For $A \ne 0 \in I_X$, we see

$$\begin{aligned}
\phi_s(A) &= \sum_{D \le A} \mu(A-D)\mathcal{N}(D)^s \\
&= \sum_{D \le A} \mu(A-D)\exp(s \log \mathcal{N}(D)) \\
&= \sum_{D \le A} \mu(A-D)\left(\sum_{n=0}^{\infty} \frac{(s\log\mathcal{N}(D))^n}{n!}\right) \\
&= \sum_{D \le A} \mu(A-D)\left(\sum_{n=1}^{\infty} \frac{(s\log\mathcal{N}(D))^n}{n!}\right).
\end{aligned}$$

Thus, we have

$$\lim_{s\to 1} \frac{\phi_{1-s}(A)}{1-s} = \lim_{s\to 0} \frac{\phi_s(A)}{s} = \Lambda(A).$$

The expression (3) is proved from this and $\lim_{s\to 1}(s-1)Z(s) = c$.

4. Proof of Theorem 2

Firstly, we fix $K \in I_X$. Then,

$$\begin{aligned}
\sum_{\mathcal{N}(M)\le x} C_K(M) &= \sum_{\mathcal{N}(M)\le x} \sum_{\substack{D+E=K \\ D+A=M}} \mathcal{N}(D)\mu(E) \\
&= \sum_{D+E=K} \mathcal{N}(D)\mu(E)\left[\frac{x}{\mathcal{N}(D)}\right] \\
&= \sum_{D+E=K} \mathcal{N}(D)\mu(E)\left(c\frac{x}{\mathcal{N}(D)} + R\left(\frac{x}{\mathcal{N}(D)}\right)\right)
\end{aligned}$$

where $R(x) = O(x^\alpha)$. By the assumption, the above expression is

$$cx\sum_{D+E=K}\mu(E) + O\left(x^\alpha \sum_{D+E=K}\mathcal{N}(D)^{1-\alpha}\right).$$

Hence, (1) is shown.

Next, we shall show (2). We have

$$\begin{aligned}
S(x,y) &= \sum_{\substack{\mathcal{N}(M)\le x, \\ \mathcal{N}(K)\le y}} \sum_{D\le M,K} \mathcal{N}(D)\mu(K-D) \\
&= \sum_{\substack{\mathcal{N}(D+B)\le x \\ \mathcal{N}(D+A)\le y}} \mathcal{N}(D)\mu(A) \\
&= \sum_{\substack{D,A\in I_X \\ \mathcal{N}(D+A)\le y}} \mathcal{N}(D)\mu(A)\left[\frac{x}{\mathcal{N}(D)}\right].
\end{aligned}$$

By the assumption,

$$S(x,y) = \sum_{\substack{D,A\in I_X \\ \mathcal{N}(D+A)\le y}} \mathcal{N}(D)\mu(A)\left(c\frac{x}{\mathcal{N}(D)} + R\left(\frac{x}{\mathcal{N}(D)}\right)\right)$$
$$= cx \sum_{\substack{D,A\in I_X \\ \mathcal{N}(D+A)\le y}} \mu(A) + \sum_{\substack{D,A\in I_X \\ \mathcal{N}(D+A)\le y}} \mathcal{N}(D)\mu(A)R\left(\frac{x}{\mathcal{N}(D)}\right)$$

where $R(x) = O(x^\alpha)$.

Since

$$\sum_{\substack{D,A\in I_X \\ \mathcal{N}(D+A)\le y}} \mu(A) = \sum_{\substack{C\in I_X \\ \mathcal{N}(C)\le y}} \sum_{\substack{A\in I_X \\ A\le C}} \mu(A) = 1,$$

we obtain $S(x,y) = cx + T(x,y)$ where

$$T(x,y) = \sum_{\substack{D,A\in I_X \\ \mathcal{N}(D+A)\le y}} \mathcal{N}(D)\mu(A)R\left(\frac{x}{\mathcal{N}(D)}\right).$$

Note that

$$T(x,y) \ll \sum_{\substack{D,A\in I_X \\ \mathcal{N}(D+A)\le y}} \mathcal{N}(D)\left(\frac{x}{\mathcal{N}(D)}\right)^\alpha = \sum_{\mathcal{N}(A)\le y} \sum_{\mathcal{N}(D)\le y/\mathcal{N}(A)} x^\alpha \mathcal{N}(D)^{1-\alpha}$$
$$\ll x^\alpha \sum_{\mathcal{N}(A)\le y} \left(\frac{y}{\mathcal{N}(A)}\right)^{1-\alpha} \sum_{\mathcal{N}(D)\le y/\mathcal{N}(A)} 1 \ll x^\alpha y^{2-\alpha}.$$

Hence $S(x,y) = cx + O(x^\alpha y^{2-\alpha})$. If $\frac{2-\alpha}{1-\alpha} < \lambda$ and $y^\lambda \ll x$, then $T(x,y) = o(x)$. Therefore, Theorem 2 is proved.

5. Examples

Let F be a number field of degree d, O_F the integer ring of F, and $\mathcal{I}$ is the set of all non-zero ideals of O_F. The Möbius function $\mu : \mathcal{I} \to \mathbb{C}$ for O_F is defined as

$$\mu(\mathfrak{a}) = \begin{cases} (-1)^{\omega(\mathfrak{a})} & \mathfrak{a} \text{ is square free,} \\ 0 & \text{otherwise,} \end{cases}$$

where $\omega(\mathfrak{a})$ is the number of distinct prime factors of $\mathfrak{a}$ and one can define the Ramanujan sum as

$$C_{\mathrm{a}}(\mathfrak{b}) = \sum_{\mathrm{d}|\mathrm{a},\mathrm{b}} \mathcal{N}(\mathfrak{d})\mu\left(\frac{\mathfrak{a}}{\mathfrak{d}}\right)$$

where $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$ and $\mathcal{N}(\mathfrak{a}) = [O_F : \mathfrak{a}]$. By (1) of Theorem 1, one have

$$\sum_{\mathrm{d}|\mathrm{a}} C_{\mathrm{a}}(\mathfrak{d}) = \mathcal{N}(\mathfrak{a}) \prod_{\mathrm{p}|\mathrm{a}} \left(1 - \frac{2}{\mathcal{N}(\mathfrak{p})}\right)$$

for $\mathfrak{a} \in \mathcal{I}$. By (2) of Theorem 1, we have

$$\sum_{\mathrm{d}|\mathrm{a}} C_{\mathrm{d}}(\mathfrak{b}) = \begin{cases} \mathcal{N}(\mathfrak{a}) & \text{if } \mathfrak{a} \mid \mathfrak{b}, \\ 0 & \text{otherwise} \end{cases}$$

for $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$.

The following fact is well-known.

Lemma 4 (cf. Lang [6], Chap. VI Theorem 3, or Murty and Order [7]) — The number of ideals of O_F whose norms are less than or equal to x is

$$c_F x + R_F(x)$$

where c_F is the residue of the Dedekind zeta function $\zeta_F(s)$ of F at $s = 1$ and $R_F(x) = O(x^{1-\frac{1}{d}})$

It is well-known that the invariant c_F in the above lemma is given by

$$c_F = \frac{2^{r_1}(2\pi)^{r_2}\mathcal{R}h}{W\sqrt{D}}$$

where r_1 is the number of real primes, r_2 is the number of complex primes, $\mathcal{R}$ is the regulator, h is the class number, W is the number of roots of unity, and D is the absolute value of the discriminant of F. The von Mangoldt function Λ for F is the function such that $\Lambda(\mathfrak{a}) = \log \mathcal{N}(\mathfrak{a})$ if $\mathfrak{a}$ is a power of a prime ideal $\mathfrak{p}$, and $\Lambda(\mathfrak{a}) = 0$ otherwise. Using Lemma 4 and (3) of Theorem 1, we have

$$c_F = -\frac{1}{\Lambda(\mathfrak{a})} \sum_{\mathrm{b}} \frac{C_{\mathrm{a}}(\mathfrak{b})}{\mathcal{N}(\mathfrak{b})}$$

unless $\Lambda(\mathfrak{a}) = 0$.

In addition, if $\lambda > d + 1$ and $y^\lambda \ll x$, then

$$\sum_{\substack{\mathcal{N}(\mathrm{b}) \le x \\ \mathcal{N}(\mathrm{a}) \le y}} C_{\mathrm{a}}(\mathfrak{b}) = c_F x + o(x).$$

by Theorem 2.

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REFERENCES

1. T. M. Apostol, Arithmetical properties of generalized Ramanujan sums, *Pacific J. Math.*, **41**(2) (1972), 281-293.
2. T. H. Chan and A. V. Kumchev, On sums of Ramanujan sums, *Acta arithm.*, **152** (2012), 1-10.
3. H. Cohen, *Number theory. Vol. II. Analytic and modern tools*, Graduate Texts in Mathematics, 240. Springer, New York, 2007.
4. A. Grytczuk, On Ramanujan sums on arithmetical semigroup, *Tsukuba. J. Math.*, **16**(2) (1992), 315-319.
5. I. Kiuchi and Y. Tanigawa, On arithmetic functions related to the Ramanujan sum, *Period. Math. Hungar.*, **45**(1-2) (2002), 87-99.
6. S. Lang, *Algebraic number theory*, 2nd edition, Graduate Texts in Mathematics, 110. Springer-Verlag, New York, 1994.
7. R. Murty and J. V. Order, Counting integral ideals in a number field, *Expo. Math.*, **25** (2007), 53-66.
8. W. G. Nowak, The average size of Ramanujan sums over quadratic number fields, *Arch. Math.*, **99** (2012), 433-442.
9. E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd edition, revised by D. R. Heath-Brown, Oxford University Press, 1986.