Indian J. Pure Appl. Math., December 2014, 1017-1028

AN EFFICIENT METHOD FOR NONLINEAR FRACTIONAL DIFFERENTIAL EQUATIONS: COMBINATION OF THE ADOMIAN DECOMPOSITION METHOD AND SPECTRAL METHOD

E. Babolian*, A. R. Vahidi** and A. Shoja*

**Department of Mathematics, Science and Research Branch, Islamic Azad University, Tehran, Iran*

***Department of Mathematics, Shahr-e-Rey Branch, Islamic Azad University, Tehran, Iran*

e-mail: alrevahidi@yahoo.com

(*Received* 8 *August* 2013; *after final revision* 19 *November* 2013; *accepted* 26 *December* 2013)

In this paper, a novel iterative method is employed to give approximate solutions of nonlinear differential equations of fractional order. This approach is based on combination of two different methods which are the Adomian decomposition method (ADM) and the spectral Adomian decomposition method (SADM). The method reduces the nonlinear differential equations to systems of linear algebraic equations and then the resulting systems are solved by a numerical method. Investigating some illustrations, we demonstrate that the obtained numerical results are in a very good agreement with the exact solutions.

Key words : Adomian decomposition method; spectral method; fractional order; collocation method.

1. INTRODUCTION

The fractional calculus approach provides a powerful tool for the description of memory and hereditary properties of various materials and processes. Now a days, there is an increasing attention paid to fractional differential equations and their applications in different research areas. It is well known that these equations are concluded from many physical and chemical problems [1] such as the motion of a large thin plate in a Newtonian fluid, the process of cooling a semi-infinite body by radiation, the phenomena in electromagnetic acoustic viscoelasticity, electrochemistry and material science and so on.

Several methods have been previously proposed to solve fractional differential equations (FDEs), such as various integral transform methods including the Laplace, Fourier and Mellin transforms [1-10], symmetric group method [11, 12], Adomian decomposition method (ADM) [13-20], variational iteration method (VIM) [21], homotopy perturbation method (HPM) [22], homotopy analysis method (HAM) [23, 24], orthogonal polynomial method [1, 25], Oldham and Spanier LI method [2], Granwald-Letnikov method [1, 26], fractional Adams method [27], and several other methods [1, 4, 28, 29].

The Adomian decomposition method (ADM) [30-36] is a powerful technique to solve linear and nonlinear functional equations, which provides efficient algorithms for analytic approximate solutions and numeric simulations for real-world applications in the applied sciences and engineering. Using the ADM, we calculate a series solution, but in practice, we approximate the solution by a truncated series. The series sometimes coincides with the Taylor expansion of the true solution in the neighborhood about the point $x = 0$ for initial value problems. Although the series can be rapidly convergent in a small region, it has a slower convergence rate in the wider region (Examples [2, 3] in [37]). Also, in this paper, we will demonstrate that the ADM is not convergence in examples 1 and 2.

In this paper, in order to solve nonlinear fractional differential equations, we combine two methods: the Adomian decomposition method (ADM) and the spectral Adomian decomposition method (SADM) to overcome the difficulty of convergence. The proposed method reduces nonlinear differential equations to systems of linear algebraic equations and then the resulting systems are solved by a numerical method.

This paper is arranged as follows. In Section 2, we first recall some necessary definitions and mathematical preliminaries of the fractional calculus theory used throughout the paper. This is particularly important with fractional derivatives because there are several definitions available with some fundamental differences. Section 3 deals with spectral-Adomian method. In Section 4, we investigate four examples demonstrating the effectiveness of the new method. In Section 5, we summarize our findings.

2. Preliminaries and Notations

In order to proceed, we need the following definitions of fractional derivatives and integrals. We first introduce the Riemann-Liouville definition of fractional integral operator J_a^α.

Definition 2.1 — Let $\alpha \in R^+$. The operator J_a^α, defined on the usual Lebesgue space $L_1[a, b]$ by

$$\begin{aligned} J_a^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \\ J_a^0 f(x) &= f(x), \end{aligned} \tag{1}$$

for $a \leq x \leq b$, is called the Riemann-Liouville fractional integral operator of order α.

Properties of the operator J_a^α can be found in [1]. For $f \in L_1[a,b], \alpha, \beta \geq 0$ and $\gamma > -1$, we mention only the following:

(1) $J_a^\alpha f(x)$ exists for almost every $x \in [a,b]$,

(2) $J_a^\alpha J_a^\beta f(x) = J_a^{\alpha+\beta} f(x)$

(3) $J_a^\alpha J_a^\beta f(x) = J_a^\beta J_a^\alpha f(x)$

(4) $J_a^\alpha (x-a)^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)}(x-a)^{\alpha+\gamma}$.

Definition 2.2 — The fractional derivative of $f(x)$ in the Riemann-Liouville sense is defined as

$$D_a^\alpha f(x) = D^m J_a^{m-\alpha} f(x) = \frac{d^m}{dx^m} \frac{1}{\Gamma(m-\alpha)} \int_a^x (x-t)^{m-\alpha-1} f(t)dt, \tag{2}$$

where $m \in N$ and satisfies the relation $m-1 < \alpha \leq m$, and $f \in L_1[a,b]$.

Properties of the operator D_a^α can be found in [1, 4]. For $m-1 < \alpha \leq m$, $x > a$ and $\gamma > -1$ we mention only the following:

(1) $D_a^\alpha (X-a)^\gamma = \dfrac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)}(x-a)^{\gamma-\alpha}$,

(2) $D_a^\alpha J_a^\alpha f(x) = f(x)$.

3. The Spectral-Adomian Method

3.1 *The Adomian decomposition method*

Consider the following nonlinear differential equation:

$$L[y] + N[y] = f(x) \tag{3}$$

where L is a linear operator and N is a nonlinear operator from a Banach space E into E, f is a given function in E and we are looking for $y \in E$ satisfying (3).

The decomposition method suggests that the solution $y(x)$ be decomposed by the infinite series solution

$$y(x) = \sum_{k=0}^{\infty} y_k(x) \tag{4}$$

and the nonlinear operator N in eq. (3) is decomposed as follows:

$$N(y) = \sum_{i=0}^{\infty} A_i(y_0, y_1, \cdots, y_i), \tag{5}$$

where A_i are the so-called Adomian polynomials, obtained by

$$A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i}\left[N\left(\sum_{k=0}^{\infty}\lambda^k y_k\right)\right]_{\lambda=0}.$$

Substituting (4) and (5) into (3) gives the following recursive scheme:

$$\begin{cases} L[y_0] = f(x), \\ L[y_{i+1}] = A_i, \quad i = 0, 1, \cdots . \end{cases} \tag{6}$$

We define the M-term approximation solution as

$$\phi_M(x) = \sum_{i=0}^{M-1} y_i(x), \tag{7}$$

where we assume that

$$\lim_{M\to\infty} \phi_M(x) = y(x).$$

3.2 *Shifted Legendre polynomials*

The Legendre polynomials, denoted by $l_n(x)$, are orthogonal with respect to the weight function $w(x)$ = 1 over $I = [-1, 1]$, namely [38]

$$\int_{-1}^{1} l_n(x) l_m(x) dx = \frac{2}{2n+1}\delta_{nm},$$

where

$$\delta_{nm} = \begin{cases} 1, & n = m, \\ 0, & O.W. \end{cases}$$

In order to use these polynomials on the interval [0,1], we define the so-called shifted Legendre polynamials by introducing the change of variable x = $2t - 1$. Let the shifted Legendre polynomials $l_n(x)$ be denoted by $L_n(t)$. The shifted Legendre polynomials are orthogonal with respect to the weight function $w(t) = 1$ in the interval [0, 1] with the orthogonality property

$$\int_{0}^{1} L_n(t) L_m(t) dx = \frac{2}{2n+1}\delta_{nm}.$$

Then $L_i(t)$ can be obtained as follows:

$$\begin{aligned} L_{n+1}(t) &= \frac{(2n+1)(2t-1)}{n+1} L_n(t) - \frac{n}{n+1} L_{n-1}(t), n = 1, 2, \cdots , \\ & \quad L_0(t) = 1, \quad L_1(t) = 2t - 1. \end{aligned}$$

Note that $L_n(0) = (-1)^n$ and $L_n(1) = 1$.

3.3 *Collocation method*

Consider the linear fractional differential equation:

$$\sum_{k=0}^{n} D^{\alpha_k} y(x) = g(x), \tag{8}$$

where $\alpha_k \in (k, k+1]$. With initial conditions

$$y^{(i)}(0) = \beta_i \quad i = 0, 1, \cdots, n, \tag{9}$$

The unknown function $y(t)$ in problem (8), can be approximated by a truncated series of Legendre polynomials,

$$y_m(t) = \sum_{j=0}^{m} c_j L_j(t), \tag{10}$$

where c_j are unknowns. Here, the main purpose is to find c_j. In order to achieve this end, by putting (10) in (8) and (9), we have:

$$\sum_{j=0}^{m} c_j \sum_{k=0}^{n} D^{\alpha_k} L_j(t) = g(t), \tag{11}$$

$$\sum_{j=0}^{m} c_j L_j^{(i)}(0) = \beta_i, \quad i = 0, 1, \cdots, n. \tag{12}$$

Relation (12) forms a system with $n+1$ equations and $m+1$ unknowns. To construct the remaining $m-n$ equations, we substitute Legendre-Guass points $\{t_i\}_{i=1}^{m-n}$ in (11), to obtain $m-n$ equations. So, the method reduces the solution of eq. (8) to the solution of system $AC = b$, where A, C and b are

$$A = \begin{bmatrix} A1 \\ \hline A2 \end{bmatrix}, \quad C = [c_0, c_1, \cdots, c_m]^T, \quad b = \begin{bmatrix} b1 \\ \hline b2 \end{bmatrix},$$

and matrices $A1_{(m-n)\times(m+1)}$ and $A2_{(n+1)\times(m+1)}$ are defined by

$$\begin{aligned} A1[i,j] &= \sum_{k=0}^{n} D^{\alpha_k} L_j(t_i), \quad i = 1, 2, \cdots, m-n, \quad j = 0, 1, \cdots, m, \\ A2[i,j] &= L_j^{(i)}(0), \quad i = 0, 1, \cdots, n, \quad j = 0, 1, \cdots, m, \end{aligned}$$

and vectors $b1_{(m-n)\times 1} = g(t_i), i = 1, 2, \cdots, m-n$, $b2_{(n+1)\times 1} = \beta_i, i = 0, l, \cdots, n$.

3.4 *The methodology*

Consider the following differential equation

$$\begin{cases} L[y] + N[y] = g(x), \\ y^i(0) = \beta_i, \qquad\qquad i = 0, 1, \cdots, n. \end{cases} \tag{13}$$

Applying the Adomian decomposition method for the above problem, we have

$$L[y] = -N[y] + g(x) \tag{14}$$

$$\{y_0 = y(0) + xy'(0) + \frac{x^2}{2!}y''(0) + \ldots + \frac{x^n}{n!}y^{(n)}(0) + L^{-1}[g(x)], \tag{15}$$

$$y_{i+1} = L^{-1}[-A_i] \tag{16}$$

There are some problems since computation of L^{-1} is usually difficult or somtimes impossible. Consequently, to solve these equations, we propose the SADM. That is, we set

$$y_k(x) = \sum_{i=0}^{m} c_i^{(k)} L_i(x). \tag{17}$$

Considering Section 3.3, we have

$$c^{(0)} = A^{-1}b^{(0)}, \tag{18}$$

where

$$c^{(0)} = \left[c_0^{(0)}, c_1^{(0)}, \ldots, c_m^{(0)}\right]^T, \tag{19}$$

$$b^{(0)} = \left[g(x_1), g(x_2), \ldots, g(x_{m-n}), \beta_0, \beta_1, \ldots, \beta_n\right]^T, \tag{20}$$

where $x_1, x_2, \ldots, x_{m-n}$ are zeros of polynomial $L_{m-n}(2x-1)$ and

$$c^{(i)} = A^{-1}b^{(i)}, \tag{21}$$

where

$$c^{(i)} = \left[c_0^{(i)}, c_1^{(i)}, \ldots, c_m^{(i)}\right]^T, \tag{22}$$

$$b^{(i)} = \left[A_{i-1}(x_1), A_{i-1}(x_2), \ldots, A_{i-1}(x_{m-n}), 0, 0, \ldots, 0\right]^T, \;\; i = 1, 2, \ldots \tag{23}$$

4. TEST PROBLEMS

In this section, we demonstrate the effectiveness of the proposed method (SADM) by applying the method to three nonlinear FDEs. For each example, the maximum norm of the error between the n-term approximation of $u(x)$ and the exact solution is presented. All computations associated with the method have been performed by a personal computer having the Intel Pentium 4, 3 GHz processor, 4GB RAM and using Maple 13 with 32 digits precision.

Example 4.1 : Consider the following nonlinear initial value problem with the exact solution $y(x) = x^2$

$$\begin{aligned} &D^{\frac{3}{2}}y + D^{\frac{1}{2}}y + Dy + y + e^y = \frac{4\sqrt{x}}{\sqrt{\pi}} + \frac{8}{3}\,\frac{x^{\frac{3}{2}}}{\sqrt{\pi}} + 2x + x^2 + e^{x^2} \\ &y(0) = y'(0) = 0. \end{aligned} \tag{24}$$

We suppose that

$$\begin{cases} L[y] = D^{\frac{3}{2}}y + D^{\frac{1}{2}}y + Dy + y, \\ N[y] = e^y. \end{cases} \tag{25}$$

Hence, we have the following equation

$$L[y] + N[y] = g(x), \tag{26}$$

where $g(x) = \frac{4\sqrt{x}}{\sqrt{\pi}} + \frac{8}{3}\,\frac{x^{\frac{3}{2}}}{\sqrt{\pi}} + 2x + x^2 + e^{x^2}$. We have the approximate solution of equation (24) as

$$y_k(x) \simeq \sum_{j=0}^{7} c_j^{(k)}.L_j(x), \tag{27}$$

where $L_j(x)$ is shifted Legender in the interval [0, 1]. As we mentioned in Sec. 3.1, in the ADM, we consider the solution as infinite series of the form $y(x) = \sum_{k=0}^{\infty} y_k(x)$. We obtain the components of the solution by the following recurrence scheme

$$\begin{cases} D^{\frac{3}{2}}y_k + D^{\frac{1}{2}}y_k + Dy_k + y_k = g_k(x) \quad \text{for} \quad k = 0, 1, 2, \ldots \\ y_0'(0) = y'(0), y_0(0) = y(0) \;\; \text{and} \;\; y_k'(0) = y_k(0) = 0 \;\; \text{for} \;\; k = 1, 2, 3, \ldots, \end{cases} \tag{28}$$

where $g_0(x) = g(x), g_k(x) = A_{k-1}(x)$ for $k = 1, 2, 3, \ldots$ and A_i's are the Adomian polynomials. We can rewrite the system (28) as the compact form $AC^{(k)} = b^{(k)}$, where A is the coefficients of $C^{(k)}, C^{(k)}$ is the coefficients of L_j in (27) and components of $b^{(k)}$ are the values of $g_k(x)$ at roots of $L_6(2x - 1) = 0$.

In the following, we obtain $A = [a_{ij}]$

$$\begin{cases} a_{1j} = L_{j-1}(0), \\ a_{2j} = L'_{j-1}(0), \\ a_{ij} = D^{\frac{3}{2}}L_{j-1}(x_i) + D^{\frac{1}{2}}L_{j-1}(x_i) + DL_{j-1}(x_i) + L_{j-1}(x_i), \end{cases} \tag{29}$$
$$i = 3, 4, \ldots, 8, \qquad j = 1, 2, \ldots, 8,$$

where $x_i, i = 1, 2, \ldots, 6$, are the roots of $L_6(2x-1) = 0$.

To obtain $b^{(k)}$, we have

$$\begin{cases} b_1^{(k)} = y_k(0), \\ b_2^{(k)} = y_k'(0), \\ b_i^{(k)} = g_k(x_i), \\ i = 3, 4, \ldots, 8. \end{cases} \tag{30}$$

Consequently, we have $C^{(k)} = A^{-1}b^{(k)}$

The numerical results are presented in Table 1.

Table 1: The maximum norm of the error of approximate solutions $y_K(x)$

Method	ADM	SADM	N=7
K	$\|\|y(x) - y_K(x)\|\|_\infty$	$\|\|y(x) - y_K(x)\|\|_\infty$	CPU Times
10	failed	$1.0E-5$	0.313
20	failed	$1.6E-9$	1.031
30	failed	$3.0E-13$	11.766
40	failed	$7.0E-17$	142.672

Example 4.2 : Consider the following nonlinear initial value problem:

$$\begin{aligned} D^{\frac{5}{2}}y + y + y^2 &= (1 + er\ f(\sqrt{x}))e^x + e^{2x} \\ y(0) &= y'(0) = y''(0) = 1, \end{aligned} \tag{31}$$

with the exact solution e^x. Using the proposed method in this paper, we can calculate the approximate solution of this problem. Table 2 shows the obtained numerical results.

Table 2: The maximum norm of the error of approximate solutions $y_K(x)$

Method	ADM	SADM	N=20
K	$\|\|y(x) - y_K(x)\|\|_\infty$	$\|\|y(x) - y_K(x)\|\|_\infty$	CPU Times
5	failed	$5.0E-7$	0.531
10	failed	$1.5E-13$	0.640
15	failed	$3.0E-20$	0.719
20	failed	$2.0E-24$	0.938

Example 4.3 : Consider the following nonlinear initial value problem: [39]

$$D^3y + D^{\frac{5}{2}}y + y^2 = x^4 \tag{32}$$

$$y(0) = 0, y'(0) = 0, y''(0) = 2 \tag{33}$$

with the exact solution x^2. Applying the new proposed method here, we can evaluate the approximate solution of the above equation. Table 3 shows the efficiency of the method by the obtained numerical results. This problem was solved in [39] and obtained the exact solution. The $SADM$ received nearly zero error in 15th iteration with 32 Digits Computer.

Table 3: The maximum norm of the error of approximate solutions $y_K(x)$

Method	ADM		SADM N=7	
K	$\|\|y(x)-y_K(x)\|\|_\infty$	CPU Times	$\|\|y(x)-y_K(x)\|\|_\infty$	CPU Times
5	$2.0E-5$	2.156	$1.2E-17$	0.282
10	$5.0E-8$	46.212	$1.2E-31$	0.312
15	failed	failed	$8.0E-32$	0.313

Example 4.4 : Consider the following nonlinear initial value problem: [37]

$$D^\alpha y+(1+x^2)y^2 = \frac{x^{(1-\alpha)}}{(1-\alpha)\Gamma(1-\alpha)}+(1+x^2)(1+x)^2, \tag{34}$$

$$y(0) = 1, \qquad 0<\alpha\leq 1. \tag{35}$$

with the exact solution $y=1+x$. For solving the fractional differential equation, we set

$$L[y]=D^\alpha y+\mu y, \qquad N[y]=(1+x^2)y^2-\mu y. \tag{36}$$

A simple verification shows that for any $\alpha\in[0,1]$, there exists an interval $[a,b]$ that for every $\mu\in[a,b]$, the $(SADM)$ is convergent to the exact solution. The rate of convergence is dependent on the value of μ.

In the following table, we have obtained empirically the best value of μ, for some values of α.

α_i	0	0.2	0.4	0.6	0.8	1
μ_i	3.935	3.795	3.465	3.542	2.515	2

By applying spline interpolation, we have approximated the function $\mu=h(\alpha)$ with the following cubic spline,

$$\mu=\begin{cases} \frac{787}{200}-\frac{7587}{41800}\alpha-\frac{21673}{1672}\alpha^3, & 0\leq\alpha<0.2\\ \frac{732087}{209000}+\frac{13863}{2200}\alpha-\frac{3373}{1045}\alpha^2+\frac{68655}{1672}\alpha^3, & 0.2\leq\alpha<0.4\\ \frac{2306719}{209000}-\frac{2098551}{41800}\alpha+\frac{90999}{836}\alpha^2-\frac{3373}{44}\alpha^3, & 0.4\leq\alpha<0.6\\ -\frac{419683}{19000}+\frac{4824681}{41800}\alpha-\frac{36783}{220}\alpha^2+\frac{64121}{836}\alpha^3, & 0.6\leq\alpha<0.8\\ \frac{264087}{8360}-\frac{717867}{8360}\alpha+\frac{70575}{836}\alpha^2-\frac{23525}{836}\alpha^3, & 0.8\leq\alpha\leq 1 \end{cases} \tag{37}$$

Now, suppose that $\alpha \in [0,1]$ is given. By replacing it in (37), the nearly best value of μ, corresponding to α is obtained which $(SADM)$ is convergent to the exact solution.

Table 4 shows the residual error of $(SADM)$ for different values of α, by $Res.Error$ be mean $||L[y]+N[y]-g(x)||_{\infty}$.

Table 4: The Residual error of approximate solutions $y_K(x), N=7$

	$\alpha=0.25,$	$\mu=3.694$	$\alpha=0.7,$	$\mu=3.0898$	$\alpha=1,$	$\mu=2$
K	Res.Error	CPU Times	Res.Error	CPU Times	Res.Error	CPU Times
10	$1.0E-2$	0.438	$4.0E-3$	0.375	$2.5E-3$	0.344
20	$5.0E-4$	0.438	$1.0E-5$	0.390	$3.5E-6$	0.391
30	$6.0E-6$	0.500	$1.2E-7$	0.484	$1.2E-9$	0.485
40	$6.0E-8$	0.688	$7.0E-10$	0.672	$3.0E-12$	0.640

5. Conclusion

In this paper, we proposed a new method to solve nonlinear differential equations of fractional order. This method was based on the combination of the Adomian decomposition method and the spectral method, which reduced nonlinear differential equations to systems of linear algebraic equations. The obtained approximate solutions have shown the effectiveness of the new method.

References

1. I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, CA, 1999.

2. K. B. Oldham and J. Spanier, *The fractional calculus*, Academic Press, New York, London, 1974.

3. K. S. Miller and B. Ross, *An introduction to the fractional calculus and fractional differential equations*, Wiley, New York, 1993.

4. A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier, Amsterdam, 2006.

5. R. Gorenflo and F. Mainardi, *Fractional calculus: integral and differential equations of fractional order*, In: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, Wien/New York, 1997, pp. 223-276.

6. R. Groreflo and A. Y. Luchko, *The initial value problem for some fractional differential equations with the Caputo derivative*, Fachbreich Mathematik und Informatik, Freic Universitat Berlin, 1998.

7. F. Mainardi, *Fractional Calculus: some basic problems in continuum and statistical mechanics*, In: A. Carpinteri, F. Mainardi (Eds), Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, Wien/New York, 1997, pp. 291-348.

8. J. S. Duan, Time- and space-fractional partial differential equations, *J. Math. Phys.*, **46** (2005), 13504-13511.

9. J. S. Duan and M. Y. Xu, The problem for fractional diffusion-wave equations on finite interval and Laplace transform, *Appl. Meth. J. Chinese Univ. Ser. A*, **19** (2004), 165-171 (in Chinese with English abstract).

10. J. S. Duan, T. Chaolu and J. Sun, Solution for system of linear fractional differential equations with constant coefficients, *J. Math.*, (Wuhan), **29** (2009), 599-603.

11. J. S. Duan and T. Chaolu, Scale-invariant solution for fractional anomalous diffusion equation, *Ann. Differential Eqs.*, **22** (2006), 21-26.

12. J. S. Duan, A. P. Guo and W. Z. Yun, Similarity method to solve fractional diffusion model in fractal media, *J. Biomath.*, **25** (2010), 218-224.

13. V. Daftardar-Gejji and H. Jafari, Adomian decomposition: a tool for solving a system of fractional differential equations, *J. Math. Anal. Appl.*, **301** (2005), 508-518.

14. S. S. Ray and R. K. Bera, An approximate solution of a nonlinear fractional differential equation by Adomian decomposition method, *Appl. Math. Comput.*, **167** (2005), 561-571.

15. S. Momani and N. Shawagfeh, Decomposition method for solving fractional Riccati differential equations, *Appl. Math. Comput.*, **182** (2006), 1083-1092.

16. S. Momani, A numerical sheme for the solution of multi-order fractional differential equations, *Appl. Math. Comput.*, **182** (2006), 761-770.

17. J. S. Duan, J. Y. An and M. Y. Xu, Solution of system of fractional differential equations by Adomian decomposition method, *Appl. Math. J. Chinese Univ. Ser. B*, **22** (2007), 7-12.

18. A. M. A. El-Sayed, S. H. Behiry and W. E. Raslan, Adomian's decomposition method for solving an intermediate fractional advection-dispersion equation, *Comput. Math. Appl.*, **59** (2010), 1759-1765.

19. G. C. Wu, Adomian decomposition method for non-smooth initial value problems, *Math. Comput. Modelling*, **54** (2011), 2104-2108.

20. J. S. Duan, J. Sun and T. Chaolu, Nonlinear fractional differential equation combining Duffing equation and van der Pol equation, *J. Math. (Wuhan)*, **31** (2011), 7-10.

21. J. H. He, approximate analytical solution for seepage flow with fractional derivatives in porous media, *Comput. Methods Appl. Mech. Engrg.*, **167** (1998), 57-68.

22. Q. Wang, Homotopy perturbation method for fractional Kdv equation, *Appl. Math. Comput.*, **190** (2007), 1795-1802.

23. L. Song and H. Zhang, Application of homotopy analysis method to fractional Kdv-Burgers-Koramoto equation, *Phys. Lett. A*, **367** (2007), 88-94.

24. H. Xu, S. J. Liao and X. C. You, Analysis of nonlinear fractional partial differential equations with the homotopy analysis method, *Commun. Nonlinear Sci Numer. Simulat.*, **14** (2009), 1152-1156.

25. A. Saadatmandi and M. Dehghan, A new operational matrix for solving fractional-order differential equations, *Comput. Math. Appl.*, **59** (2010), 1326-1336.

26. J. S. Duan, Z. H. Liu, F. K. Zhang and T. Chaolu, Analytical solution and numerical solution to endolymph equation using fractional derivative, *Ann. Differential Eqs.*, **24** (2008), 9-12.

27. K. Dieth, N. J. Ford and A. D. Freed, Detailed error analysis for a fractional Adams method, *Numer. Algorithms*, **36** (2004), 31-52.

28. C. Li and Y. Wang, Numerical algorithm based on Adomian decomposition for fractional differential equations, *Comput. Math. Appl.*, **57** (2009), 1672-1681.

29. G. C. Wu, A fractional characteristic method for solving fractional partial differential equations, *Appl. Math. Lett.*, **24** (2011), 1046-1050.

30. G. Adomian, *Nonlinear stochastic operator equations*, Academic, Orlando, 1986.

31. G. Adomian, *Nonlinear Stochastic Systems Theory and Applications to Physics*, Kluwer, Academic, Dordrecht, 1989.

32. G. Adomian, R. Rach and R. Meyers, An efficient methodology for the physical sciences, *Kybernetes*, **20** (1991), 24-34.

33. G. Adomian, *Solving Frontier Problems of Physics: The Decomposition Method*, Kluwer, Academic, Dordrecht, 1994.

34. A. M. Wazwaz, *Partial Differential Equations: Methods and Applications*, Balkema, Lisse, 2002.

35. A. M. Wazwaz, *Partial Differential Equations and Solitary Waves Theory*, Higher Education, Beijing, Springer, Berlin, 2009.

36. A. M. Wazwaz, *Linear and Nonlinear Integral Equations: Methods and Applications*, Higher Education, Beijing, Springer, Berlin, 2011.

37. M. M. Hosseini and H. Nasabzadeh, On the convergence of Adomian decomposition method, *Appl. Math. Comput.*, **182** (2006), 536-543.

38. J. Shen and T. Tang, *High Order Numerical Methods and Algorithms*, Chinese Science Press, Beijing, 2005.

39. S. Kazem, S. Abbasbandy and Sunil Kumar, Fractional-order Legendre functions for solving fractional-order differential equations, *Appl. Math. Modelling*, **37** (2013), 5498-5510.