

***IFP*-INJECTIVE, *IFP*-FLAT MODULES AND LOCALIZATIONS¹**

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IFP-injective modules act in ways similar to injective modules. In this paper, we first investigate the existence of *IFP*-injective covers. It is shown that over any ring R , *IFP*-injective cover always exists. Secondly, we prove that $S^{-1}M$ is an *IFP*-injective $S^{-1}R$ -module for any *IFP*-injective R -module M over any ring R .

Key words : *IFP*-injective (flat) module; Pre(Cover); Localization; I -injective (flat) module; Pre(Envelope).

1. INTRODUCTION

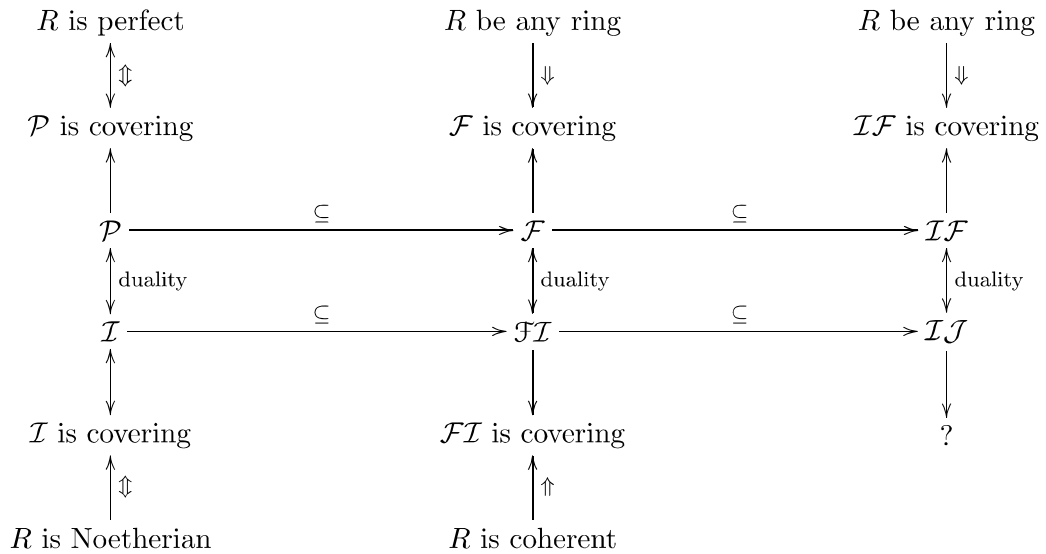
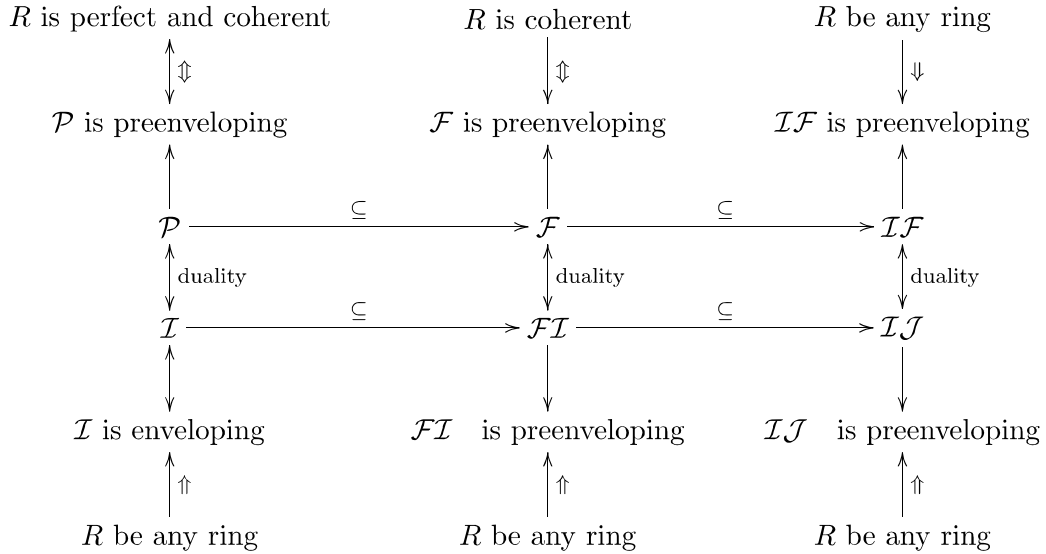
Throughout rings are associate with identity and all modules are unitary.

A ring R is left Noetherian if and only if the class of all injective left R -modules is covering (Enochs and Jenda, 2000, Theorem 5.4.1). In 2008, Pinzon proved that every left R -module has an absolutely pure cover for a left coherent ring R .

In what follows, ${}_R\mathcal{M}$, \mathcal{M}_R , \mathcal{P} , \mathcal{I} , \mathcal{F} , \mathcal{FI} , \mathcal{IF} , \mathcal{IJ} stand for the class of all left R -modules, right R -modules, projective modules, injective modules, flat modules, *FP*-injective modules, i.e., absolutely pure modules, see (Enochs, 1974), *IFP*-flat modules, *IFP*-injective modules, respectively. And every right R -module has an \mathcal{IF} -preenvelope over any ring (Lu and Liu, 2012).

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Now we have the following diagrams:



The first motivation of the present article is the “ ? ”. In section 2, we prove that every left R -module has an \mathcal{IJ} (i.e. IFP -injective) cover over any ring R . The result is used to study the following I -injective (I -flat) modules (Section 4).

Localizations of injective modules have been studied, see (Dade, 1981). In particular, we know that if R is commutative and noetherian, then for any injective R -module E , $S^{-1}E$ is an injective $S^{-1}R$ -module. And this result is false if R is not noetherian, see (Dade, 1981). In

section 3 of this paper, we prove that $S^{-1}M$ is an *IFP*-injective $S^{-1}R$ -module for any *IFP*-injective R -module M over any ring R (see Theorem 3.2). We also show that if $\varphi : D \rightarrow M$ is an *IFP*-injective cover of M as an $S^{-1}R$ -module (resp., as an R -module), then $\varphi : D \rightarrow M$ is also an *IFP*-injective cover of M as an R -module (resp., as an $S^{-1}R$ -module) (see Theorem 3.3).

In Section 4, *IFP*-injective modules are used to define the concepts of I -injective and I -flat modules. It is shown that a left R -module M is I -injective if and only if M is the kernel of an \mathcal{IJ} -precover $A \rightarrow B$ with A injective. For any ring R , we prove that a left R -module M is I -injective if and only if M is a direct sum of an injective left R -module and a reduced I -injective left R -module; a finitely presented right R -module L is I -flat if and only if L is the cokernel of an *IFP*-flat preenvelope $K \rightarrow F$ with F flat.

Let us recall some results and definitions we shall use below.

Let \mathcal{C} be a class of R -modules and M an R -module. Recall that a homomorphism $\phi : C \rightarrow M$ is a \mathcal{C} -precover of M (Enochs, 1981) if $C \in \mathcal{C}$ and the abelian group homomorphism $\text{Hom}(C', \phi) : \text{Hom}(C', C) \rightarrow \text{Hom}(C', M)$ is surjective for every $C' \in \mathcal{C}$. A \mathcal{C} -precover $\mathcal{C} : C \rightarrow M$ is said to be a \mathcal{C} -cover of M if every endomorphism $g : C \rightarrow C$ such that $\phi g = \phi$ is an isomorphism. Dually, we have the definitions of a \mathcal{C} -preenvelope and a \mathcal{C} -envelope. \mathcal{C} -covers (\mathcal{C} -envelopes) may not exist in general, but if they exist, they are unique up to isomorphisms.

Let M and N be R -modules. $M^+ = \text{Hom}_Z(M, Q/Z)$ denotes the character module of M . $\text{Hom}(M, N)$ (resp. $\text{Ext}^n(M, N)$) means $\text{Hom}_R(M, N)$ (resp. $\text{Ext}_R^n(M, N)$), and similarly $M \otimes N$ (resp. $\text{Tor}_n(M, N)$) denotes $M \otimes_R N$ (resp. $\text{Tor}_n^R(M, N)$) for an integer $n \geq 1$.

2. *IFP*-INJECTIVE COVERS

In this section, we will fix our attention on the existence of \mathcal{IJ} -covers. We will show that any module has an \mathcal{IJ} -cover over any ring. In the following, we first give some definitions and results needed in the sequel.

A right R -module M is called *IFP*-flat (Lu and Liu, 2012) if $\text{Tor}_1(M, R/I) = 0$ for all finitely presented left ideals I of R . Dually, A left R -module N is said to be *IFP*-injective (Lu and Liu, 2012) if $\text{Ext}^1(R/I, N) = 0$ for all finitely presented left ideals I of R .

Lemma 2.1 (Lu and Liu, 2012, Proposition 2.10) — Let R be a ring. Then \mathcal{IJ} and \mathcal{IF} are closed under pure submodules.

There are questions that arise naturally about classes of modules. Two of these are whether the direct sum of elements of the class remains in the class and whether the direct limit of elements in the class also remains in the class.

Lemma 2.2 (Lu and Liu, 2012, Theorem 2.8) — The following are true for any ring R .

- (1) Every direct limit of *IFP*-injective left R -modules is *IFP*-injective;
- (2) Every direct product of *IFP*-flat right R -modules is *IFP*-flat;
- (3) A left R -module M is *IFP*-injective if and only if M^{++} is *IFP*-injective;
- (4) A right R -module M is *IFP*-flat if and only if M^{++} is *IFP*-flat.

Proposition 2.3 — All injective modules are *IFP*-injective.

Proposition 2.4 — A ring R is left Noetherian if and only if every *IFP*-injective R -module is injective.

This follows from the fact that a ring is Noetherian if and only if the direct sum of injective modules is injective. If this were not the case in our ring, then we could form a direct sum of injective modules which is not injective but which is *IFP*-injective (Lu and Liu, 2012).

Lemma 2.5 (Lu and Liu, 2012, Theorem 2.6) — Let R be a ring. Then

- (1) A right R -module M is *IFP*-flat if and only if M^+ is *IFP*-injective.
- (2) A left R -module M is *IFP*-injective if and only if M^+ is *IFP*-flat.

It is known that a ring R is left Noetherian if and only if every left R -module M has an injective cover and if and only if every left R -module M has an injective precover (Enochs and Jenda, 2000, Theorem 5.4.1). If R is left coherent, then every left R -module M has an *FP*-injective cover (Pinzon, 2008). So the questions we ask are: For any ring R , whether there exists a class \mathcal{X} of R -module such that \mathcal{X} act in ways similar to the injective modules and is \mathcal{X} precovering or covering?

In order to prove the main result, we first recall some results which we need in the later.

Lemma 2.6 (Enochs and Jenda, 2000, Proposition 5.2.2) — If \mathcal{F} is a class of R -modules closed under direct sums, then an R -module M has an \mathcal{F} -precover if and only if there is a cardinal number \mathcal{N}_α such that any homomorphism $D \rightarrow M$ with $D \in \mathcal{F}$ has a factorization $D \rightarrow \mathcal{C} \rightarrow M$ with $\mathcal{C} \in \mathcal{F}$ and $\text{Card}(\mathcal{C}) \leq \mathcal{N}_\alpha$.

Lemma 2.7 (Bican, Bashir and Enochs, 2001, Theorem 5) — Let R be an arbitrary ring.

Then for each cardinal λ , there is a cardinal κ such that for any R -module M and any $L \leq M$ satisfying $\text{Card}(M) \geq \kappa$ and $\text{Card}(M/L) \leq \lambda$, the submodule L contains a nonzero submodule that is pure in M .

Lemma 2.8 (Enochs and Jenda, 2000, Corollary 5.2.7) — Let \mathcal{F} be a class of R -modules that is closed under well ordered inductive limits and M be an R -module. If M has an \mathcal{F} -precover, then it has an \mathcal{F} -cover.

We are now in a position to prove the following theorem.

Theorem 2.9 — *Let R be a ring. Then every left R -module has an \mathcal{IJ} -cover.*

PROOF : Let N be a left R -module with $\text{Card}(N) = \lambda$. We first prove that N has an \mathcal{IJ} -precover. Let κ be a cardinal as in Lemma 2.7. By Lemma 2.6, it suffices to show that any homomorphism $f : E \rightarrow N$ with E IFP-injective has a factorization $E \rightarrow F \rightarrow N$ with F IFP-injective and $\text{Card}(F) \leq \kappa$.

If $\text{Card}(E) \leq \kappa$, then we are done. Hence we may assume $\text{Card}(E) > \kappa$.

Let $K = \ker(f)$. Note that $\text{Card}(E/K) \leq \lambda$ since E/K embeds in N . Thus K contains a nonzero submodule E_0 which is pure in E by Lemma 2.7. The pure exact sequence $0 \rightarrow E_0 \rightarrow E \rightarrow E/E_0 \rightarrow 0$ induces the split exact sequence $0 \rightarrow (E/E_0)^+ \rightarrow E^+ \rightarrow E_0^+ \rightarrow 0$. Thus $(E/E_0)^+$ is IFP-flat since E^+ is IFP-flat by Lemma 2.5. So E/E_0 is IFP-injective by Lemma 2.5 again.

If $\text{Card}(E/E_0) \leq \kappa$, then we are done by Lemma 2.6 since f factors through E/E_0 .

Suppose $\text{Card}(E/E_0) > \kappa$. Put $\mathcal{X} = \{X : E_0 \leq X \leq K \text{ and } E/X \text{ is IFP-injective}\}$. Then \mathcal{X} is a nonempty set since $E_0 \in \mathcal{X}$. Let $\{X_i \in \mathcal{X} : i \in I\}$ be an ascending chain. Note that $E_0 \leq \bigcup X_i \leq K$ and $E/\bigcup X_i = E/\varinjlim X_i = \varinjlim (E/X_i)$ is IFP-injective by Lemma 2.2 since each E/X_i is IFP-injective. Thus $\bigcup X_i \in \mathcal{X}$, and so \mathcal{X} has a maximal element C by Zorn's Lemma.

We claim that $\text{Card}(E/C) \leq \kappa$. Suppose $\text{Card}(E/C) > \kappa$. Since $C \subseteq K$, there exists $g : E/C \rightarrow N$ with $\ker(g) = K/C$. Note that $\text{Card}((E/C)/(K/C)) = \text{Card}(E/K) \leq \lambda$, and so K/C contains a nonzero submodule K_1/C which is pure in E/C by Lemma 2.7. Therefore $E/K_1 \cong (E/C)/(K_1/C)$ is IFP-injective by the proof above, and hence $K_1 \in \mathcal{X}$, which contradicts the maximality of C . It is clear that E/C is IFP-injective and f factors through E/C . So N has an \mathcal{IJ} -precover by Lemma 2.6. Note that \mathcal{IJ} is closed under direct limits by Lemma 2.2. Thus N has an \mathcal{IJ} -cover by Lemma 2.8.

3. LOCALIZATIONS OF IFP-INJECTIVE AND IFP-FLAT MODULES

In this section, R will always be a commutative ring and $S \subset R$ will be a multiplicative set. We can form the ring of fractions $S^{-1}R$. There is a canonical ring homomorphism $R \rightarrow S^{-1}R$. For an R -module M , we also can construct the localization of M with respect to S , denoted by $S^{-1}M$, which is an $S^{-1}R$ -module, and hence an R -module. There is an obvious homomorphism $M \rightarrow S^{-1}M$.

Lemma 3.1 (Rotman, 1979, Theorem 3.84) — Let $S^{-1}R$ be multiplicative, and let N be a finitely presented R -module. For every R -module M , there is a natural isomorphism

$$\varphi : S^{-1}\mathrm{Hom}_R(N, M) \cong \mathrm{Hom}_{S^{-1}R}(S^{-1}N, S^{-1}M),$$

given by $g/1 \mapsto \bar{g}$, where $\bar{g}(n/1) = g(n) \otimes 1$ for all $n \in N$.

It is well known that if R is commutative and noetherian, then for any injective R -module E , $S^{-1}E$ is an injective $S^{-1}R$ -module. And this result is false if R is not noetherian, Dade (1981) showed, for every commutative ring k , that if $R = k[X]$, where X is an uncountable set of indeterminates, then there is a multiplicative subset $S \subseteq R$ and an injective R -module E such that $S^{-1}E$ is not an injective $S^{-1}R$ -module. Here we have

Theorem 3.2 — *Let S be a multiplicative subset of a ring R .*

- (1) *If M is an IFP-injective R -module, then $S^{-1}M$ is an IFP-injective $S^{-1}R$ -module.*
- (2) *If M is an IFP-injective $S^{-1}R$ -module, then M is an IFP-injective R -module.*
- (3) *If M is an IFP-flat R -module, then $S^{-1}M$ is an IFP-flat $S^{-1}R$ -module.*
- (4) *If M is an IFP-flat $S^{-1}R$ -module, then $S^{-1}M$ is an IFP-flat R -module.*

PROOF : (1) It suffices to extend any map $I \rightarrow S^{-1}E$ to a map $S^{-1}R \rightarrow S^{-1}E$, where I is a finitely presented ideal in $S^{-1}R$; that is, if $i : I \rightarrow S^{-1}R$ is the inclusion, then the induced map $i^* : \mathrm{Hom}_{S^{-1}R}(S^{-1}R, S^{-1}E) \rightarrow \mathrm{Hom}_{S^{-1}R}(I, S^{-1}E)$ is a surjection. Now there is a finitely presented ideal J in R with $I = S^{-1}J$. Naturality of the isomorphism in Lemma 3.1 gives a commutative diagram

$$\begin{array}{ccc} S^{-1}\mathrm{Hom}_R(R, M) & \longrightarrow & S^{-1}\mathrm{Hom}_R(J, M) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{S^{-1}R}(S^{-1}R, S^{-1}M) & \longrightarrow & \mathrm{Hom}_{S^{-1}R}(S^{-1}J, S^{-1}M). \end{array}$$

Since E is an injective R -module, $\text{Hom}_R(R, E) \rightarrow \text{Hom}_R(J, E)$ is a surjection, and note that $S^{-1} = S^{-1}R \otimes_R$ being right exact implies that the top arrow is also a surjection. But the vertical maps are isomorphisms, and so the bottom arrow is a surjection; that is, $S^{-1}M$ is an injective $S^{-1}R$ -module.

(2) Let I be a finitely presented ideal of R , $f \in \text{Hom}_R(I, M)$ and $\lambda : Ra \rightarrow R$ be the inclusion. By Pinzon (2005, Proposition 3.9), there is an $S^{-1}R$ -homomorphism $g : S^{-1}I \rightarrow M$ such that $g\theta = f$, where $\theta : I \rightarrow S^{-1}I$ is the canonical map. Note that $S^{-1}I$ is a finitely presented ideal of $S^{-1}R$. Since M is an IFP-injective $S^{-1}R$ -module, there exists $h : S^{-1}R \rightarrow M$ such that $h(S^{-1}\lambda) = g$. Let $\varphi : R \rightarrow S^{-1}R$ be the canonical map. Then, by Rotman (1979, Exercise 3.51, p. 102), we have the following commutative diagram:

$$\begin{array}{ccc} I & \xrightarrow{\theta} & S^{-1}I \\ \lambda \downarrow & & S^{-1}\lambda \downarrow \\ P & \xrightarrow{\varphi} & S^{-1}R \end{array}$$

Thus $(h\varphi)\lambda = h(\varphi\lambda) = h(S^{-1}\lambda)\theta = f$. So M is an IFP-injective R -module.

(3) Let I be a finitely presented ideal of R . Then there is a finitely presented ideal J of R such that $S^{-1}J = I$. By Rotman (1979, Theorem 9.49), we have

$$\text{Tor}_1^{S^{-1}R}(S^{-1}M, R/I) \cong \text{Tor}_1^{S^{-1}R}(S^{-1}M, S^{-1}R/J) \cong S^{-1}\text{Tor}_1^R(M, R/J) = 0.$$

So $S^{-1}M$ is an IFP-flat $S^{-1}R$ -module.

(4) If M is an IFP-flat $S^{-1}R$ -module, then M^+ is an IFP-injective $S^{-1}R$ -module. So M^+ is an IFP-injective R -module by (2) and hence M is an IFP-flat R -module.

Theorem 3.3 — *Let S be a multiplicative subset of a ring R and M an $S^{-1}R$ -module.*

(1) *If $\varphi : D \rightarrow M$ is an IFP-injective cover of M as an $S^{-1}R$ -module, then $\varphi : D \rightarrow M$ is also an IFP-injective cover of M as an R -module.*

(2) *If $\alpha : D \rightarrow M$ is an IFP-injective cover of M as an R -module, then $\alpha : D \rightarrow M$ is also an IFP-injective cover of M as an $S^{-1}R$ -module*

(3) *If $\psi : T \rightarrow M$ is an IFP-flat cover of M as an $S^{-1}R$ -module, then $\psi : T \rightarrow M$ is also an IFP-flat cover of M as an R -module.*

(4) *If $\beta : T \rightarrow M$ is an IFP-flat cover of M as an R -module, then $\beta : T \rightarrow M$ is also an IFP-flat cover of M as an $S^{-1}R$ -module.*

PROOF : (1) Let N be any *IFP*-injective R -module. Then $S^{-1}N$ is an *IFP*-injective $S^{-1}R$ -module by Theorem 3.2. For any R -homomorphism $f : N \rightarrow M$, there is an $S^{-1}R$ -homomorphism $g : S^{-1}N \rightarrow M$ such that $g\alpha = f$ by Pinzon (2005, Proposition 3.9), where $\alpha : N \rightarrow S^{-1}N$ is the canonical map. Thus there exists $\beta : S^{-1}N \rightarrow D$ such that $g = \varphi\beta$ since φ is an *IFP*-injective precover of M as an $S^{-1}R$ -module. Therefore, $\varphi(\beta\alpha) = (\varphi\beta)\alpha = g\alpha = f$, and so φ is an *IFP*-injective precover of M as an R -module. Let $h : D \rightarrow D$ be an R -homomorphism with $\varphi h = \varphi$. Note that h is also an $S^{-1}R$ -homomorphism by Rotman (1979, Exercise 3.50) since D is an $S^{-1}R$ -module. Thus h is an isomorphism, and so φ is an *IFP*-injective cover of M as an R -module.

(2) We first prove that D is an $S^{-1}R$ -module. Note that the canonical map $\varphi : M \rightarrow S^{-1}M$ is an isomorphism by Rotman (1979, Lemma 3.75) since M is an $S^{-1}R$ -module. Let $\theta : D \rightarrow S^{-1}D$ be the canonical map. Then $\varphi\alpha = (S^{-1}\alpha)\theta$. In addition, $S^{-1}D$ is an *IFP*-injective $S^{-1}R$ -module by Theorem 3.2 and so it is an *IFP*-injective R -module by Theorem 3.2. Since α is an *IFP*-injective cover of M as an R -module, there exists an R -homomorphism $\psi : S^{-1}D \rightarrow D$ such that $\alpha\psi = \varphi^{-1}(S^{-1}\alpha)$. So we have the commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{\alpha} & M \\ \theta \downarrow & & \varphi \downarrow \\ S^{-1}D & \xrightarrow{S^{-1}\alpha} & S^{-1}M \\ \psi \downarrow & & \varphi^{-1} \downarrow \\ D & \xrightarrow{\alpha} & M. \end{array}$$

Thus $\alpha(\psi\theta) = \varphi^{-1}(S^{-1}\alpha)\theta = \varphi^{-1}\varphi\alpha = \alpha$. Therefore, $\psi\theta$ is an isomorphism, and so there is an R -submodule H of $S^{-1}D$ such that $S^{-1}D = (D) \oplus H$. For any $x \in D$ and $s \in S$, there is a unique expression $\frac{x}{s} = \frac{a}{1} + b$ with $a \in D$ and $b \in H$. Thus $\frac{x}{1} = \frac{sx}{s} = \frac{sa}{1} + sb$, and hence $\frac{x - sa}{1} = sb = 0$. So $b = \frac{1}{s}(sb) = 0$. It follows that $H = 0$, and hence $D \cong \theta(D) = S^{-1}D$. Therefore D is an $S^{-1}R$ -module.

Let N be any *IFP*-injective $S^{-1}R$ -module and $f : N \rightarrow M$ any $S^{-1}R$ -homomorphism. Then N is also an *IFP*-injective R -module by Theorem 3.2. So there exists an R -homomorphism $g : N \rightarrow D$ such that $f = \alpha g$ since α is an *IFP*-injective precover of the R -module M . Note that g is also an $S^{-1}R$ -homomorphism by Rotman (1979, Exercise 3.50) since N and D are both $S^{-1}R$ -modules. Thus α is an *IFP*-injective cover of M as an $S^{-1}R$ -module.

(3) and (4) are similar to the proof of (1) and (2). □

4. *I*-INJECTIVE MODULES AND *I*-FLAT MODULES

Definition 4.1 — A left R -module M is called *I*-injective if $\text{Ext}^1(G, M) = 0$ for every *IFP*-injective left R -module G .

A right R -module N is said to be *I*-flat if $\text{Tor}_1(N, G) = 0$ for every *IFP*-injective left R -module G .

Remark 4.2 : (1) By Wakamutsu's Lemma (see Xu, 1996, Lemma 2.1.1), any kernel of an \mathcal{IJ} -cover is *I*-injective.

(2) Recall that a left R -module M is called *FP*-injective (Stenström, 1970) if $\text{Ext}^1(N, M) = 0$ for every finitely presented left R -module N . M is called copure injective (Enochs and Jenda, 1993) if $\text{Ext}^1(G, M) = 0$ for every injective left R -module G . A right R -module N is said to be copure flat (Enochs and Jenda, 1993) if $\text{Tor}_1(N, G) = 0$ for every injective left R -module G . Obviously, we have the following implications:

injective modules \Rightarrow *I*-*injective modules* \Rightarrow *copure injective modules*;

flat modules \Rightarrow *I*-*flat modules* \Rightarrow *copure flat modules*.

However, the converse is not true in general. See Proposition 4.5.

Proposition 4.3 — Let R be a ring. Then a right R -module M is *I*-flat if and only if M^+ is *I* injective.

PROOF : It holds by the standard isomorphism $\text{Ext}^1(N, M^+) \cong \text{Tor}_1(M, N)^+$ for every *IFP*-injective left R -module N . \square

Proposition 4.4 — The following are equivalent for a left R -module M :

- (1) M is *I*-injective;
- (2) For every exact sequence $0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$ with E *IFP*-injective, $E \rightarrow L$ is an \mathcal{IJ} -precover of L ;
- (3) M is the kernel of an \mathcal{IJ} -precover $f : A \rightarrow B$ with A injective;
- (4) M is injective with respect to every exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with C *IFP*-injective.

PROOF : (1) \Rightarrow (2) and (1) \Rightarrow (4) are clear by definitions.

(2) \Rightarrow (3). It is obvious since there exists a short exact sequence $0 \rightarrow M \rightarrow E(M) \rightarrow$

$E(M)/M \rightarrow 0$.

(3) \Rightarrow (1). Let M be the kernel of an \mathcal{IJ} -precover $f : A \rightarrow B$ with A injective. Then we have an exact sequence $0 \rightarrow M \rightarrow A \rightarrow A/M \rightarrow 0$. So, for any IFP -injective left R -module N , the sequence $\text{Hom}(N, A) \rightarrow \text{Hom}(N, A/M) \rightarrow \text{Ext}^1(N, M) \rightarrow 0$ is exact. It is easy to verify that $\text{Hom}(N, A) \rightarrow \text{Hom}(N, A/M) \rightarrow 0$ is exact by (3). Thus $\text{Ext}^1(N, M) = 0$, and so (1) follows.

(4) \Rightarrow (1). For every IFP -injective left R -module N , there exists a short exact sequence $0 \rightarrow K \rightarrow P \rightarrow N \rightarrow 0$ with P projective, which induces an exact sequence $\text{Hom}(P, M) \rightarrow \text{Hom}(K, M) \rightarrow \text{Ext}^1(N, M) \rightarrow 0$. Note that $\text{Hom}(P, M) \rightarrow \text{Hom}(K, M) \rightarrow 0$ is exact by (4). Hence $\text{Ext}^1(N, M) = 0$, as desired. \square

Proposition 4.5 — The following hold for any ring R :

- (1) A left R -module M is injective if and only if M is I -injective and $IFP\text{-id}(M) \leq 1$.
- (2) A right R -module N is flat if and only if N is I -flat and $IFP\text{-fd}(N) \leq 1$.

PROOF : (1)(\Rightarrow). It is trivial.

(\Leftarrow). Let M be an I -injective left R -module and $IFP\text{-id}(M) \leq 1$. Then there is an exact sequence $0 \rightarrow M \rightarrow E \rightarrow L \rightarrow 0$ with E injective. Note that L is IFP -injective. So the exact sequence is split, and hence M is injective.

(2)(\Rightarrow). It is trivial.

(\Leftarrow). For any I -flat right R -module N with $IFP\text{-fd}(N) \leq 1$, we have N^+ is I -injective by Proposition 4.3. Thus N^+ is injective by (1) since $IFP\text{-id}(N^+) \leq 1$. So N is flat. \square

Recall that a left R -module M is called reduced (Enochs and Jenda, 2000) if M has no nonzero injective submodules.

Proposition 4.6 — Let R be a ring. Then the following are equivalent for a left R -module M :

- (1) M is a reduced I -injective left R -module;
- (2) M is the kernel of an \mathcal{IJ} -cover $f : A \rightarrow B$ with A injective.

PROOF : (1) \Rightarrow (2). By Proposition 4.5, the natural map $\pi : E(M) \rightarrow E(M)/M$ is an \mathcal{IJ} -precover. Note that $E(M)/M$ has an \mathcal{IJ} -cover, and $E(M)$ has no nonzero direct summand K contained in M since M is reduced. It follows that $\pi : E(M) \rightarrow E(M)/M$ is an \mathcal{IJ} -cover by Xu (1996, Corollary 1.2.8), and hence (2) follows.

(2) \Rightarrow (1). Let M be the kernel of an \mathcal{IJ} -cover $\alpha : A \rightarrow B$ with A injective. By Remark 4.2(1), M is I -injective. Now let K be an injective submodule of M . Suppose $A = K \oplus L$, $p : A \rightarrow L$ is the projection and $i : L \rightarrow A$ is the inclusion. It is easy to see that $\alpha(ip) = \alpha$ since $\alpha(K) = 0$. Therefore ip is an isomorphism since α is a cover. Thus i is epic, and hence $A = L, K = 0$. So M is reduced. \square

Theorem 4.7 — *Let R be a ring. Then a left R -module M is I -injective if and only if M is a direct sum of an injective left R -module and a reduced I -injective left R -module.*

PROOF : (\Leftarrow). It is clear.

(\Rightarrow). Let M be an I -injective left R -module. Consider the exact sequence $0 \rightarrow M \rightarrow E(M) \rightarrow E(M)/M \rightarrow 0$. Note that $E(M) \rightarrow E(M)/M$ is an \mathcal{IJ} -injective precover of $E(M)/M$ by Proposition 4.4. But $E(M)/M$ has an \mathcal{IJ} -injective cover $L \rightarrow E(M)/M$, so we have the commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
 0 & \longrightarrow & K & \xrightarrow{f} & L & \longrightarrow & E(M)/M & \longrightarrow & 0 \\
 & & \downarrow \phi & & \downarrow \gamma & & \parallel & & \\
 0 & \longrightarrow & M & \longrightarrow & E(M) & \longrightarrow & E(M)/M & \longrightarrow & 0 \\
 & & \downarrow \sigma & & \downarrow \beta & & \parallel & & \\
 0 & \longrightarrow & K & \longrightarrow & L & \longrightarrow & E(M)/M & \longrightarrow & 0
 \end{array}$$

Note that $\beta\gamma$ is an isomorphism, and so $E(M) = \ker(\beta) \oplus \text{im}(\gamma)$. Thus L and $\ker(\beta)$ are injective (for $\text{im}(\gamma) \cong L$). Therefore K is a reduced I -injective module by Proposition 4.6. Since $\sigma\phi$ is an isomorphism by the Five Lemma, we have $M = \ker(\sigma) \oplus \text{im}(\phi)$, where $\text{im}(\phi) \cong K$. In addition, we get the commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & \ker(\sigma) & \longrightarrow & \ker(\beta) & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 0 & \longrightarrow & M & \xrightarrow{\alpha} & E(M) & \longrightarrow & E(M)/M & \longrightarrow & 0 \\
 & & \downarrow \sigma & & \downarrow \beta & & \parallel & & \\
 0 & \longrightarrow & K & \xrightarrow{f} & L & \longrightarrow & E(M)/M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
 \end{array}$$

Hence $\ker(\sigma) \cong \ker(\beta)$. This completes the proof. \square

Proposition 4.8 — Let R be a ring.

(1) If M is a finitely presented I -flat right R -module, then M is the cokernel of an \mathcal{IF} -preenvelope $f : K \rightarrow F$ with F flat.

(2) If L is the cokernel of an \mathcal{IF} -preenvelope $f : K \rightarrow F$ with F flat, then L is I -flat.

PROOF : (1) Let M be a finitely presented I -flat right R -module. There is an exact sequence $0 \rightarrow K \rightarrow F \rightarrow M \rightarrow 0$ with F projective and both F and K finitely generated. We claim that $K \rightarrow F$ is an \mathcal{IF} -preenvelope. In fact, for any IFP -flat right R -module Q , we have $\text{Tor}_1(M, Q^+) = 0$, and so we get the following commutative diagram with the first row exact:

$$\begin{array}{ccccc} 0 & \longrightarrow & K \otimes Q & \xrightarrow{\alpha} & F \otimes Q^+ \\ & & \tau_1 \downarrow & & \tau_2 \downarrow \\ & & \text{Hom}(K, Q)^+ & \xrightarrow{\theta} & \text{Hom}(F, Q)^+ \end{array}$$

By Colby (1975, Lemma 2), τ_1 is an epimorphism and τ_2 is an isomorphism. Thus θ is a monomorphism, and hence $\text{Hom}(F, Q) \rightarrow \text{Hom}(K, Q)$ is epic, as required.

(2) There is an exact sequence $0 \rightarrow \text{im}(f) \xrightarrow{i} F \rightarrow L \rightarrow 0$. It is clear that $i : \text{im}(f) \rightarrow F$ is an \mathcal{IF} -preenvelope. For any IFP -injective left R -module N , N^+ is IFP -flat by Lemma 2.5. Thus we obtain an exact sequence $\text{Hom}(F, N^+) \rightarrow \text{Hom}(\text{im}(f), N^+) \rightarrow 0$, which yields the exactness of $(F \otimes N)^+ \rightarrow (\text{im}(f) \otimes N)^+ \rightarrow 0$. So the sequence $0 \rightarrow \text{im}(f) \otimes N \rightarrow F \otimes N$ is exact. Thus the exactness of $0 \rightarrow \text{Tor}_1(L, N) \rightarrow \text{im}(f) \otimes N \rightarrow F \otimes N$ implies $\text{Tor}_1(L, N) = 0$. \square

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