Indian J. Pure Appl. Math., **44**(5): 673-681, October 2013

DIFFERENTIAL POLYNOMIAL RINGS WHICH ARE GENERALIZED ASANO PRIME RINGS

M. R. Helmi*, H. Marubayashi**[1] and A. Ueda***

**Departement of Mathematics, Andalas University,*
Padang, West Sumatera, 25163, *Indonesia*
***Faculty of Sciences and Engineering, Tokushima Bunri University*,
Sanuki, Kagawa, 769-2193, *Japan*
****Department of Mathematics, Shimane University*,
Matsue, Shimane, 690-8504 *Japan*
e-mails: monika@fmipa.unand.ac.id; marubaya@kagawa.bunri-u.ac.jp
ueda@riko.shimane-u.ac.jp

(*Received* 18 *July* 2012; *accepted* 16 *January* 2013)

Let $R[x;\delta]$ be a differential polynomial ring over a prime Goldie ring R in an indeterminate x, where δ is a derivation of R. In this paper, we describe explicitly the group of δ-stable v-R-ideals and using this results, we show that $R[x;\delta]$ is a generalized Asano prime ring if and only if R is a δ-generalized Asano prime ring.

Key words : Differential polynomial ring; generalized Asano prime ring; prime Goldie ring; δ-stable v-ideal.

[1]The second author was supported by Grant-in-Aid for Scientific Research (No. 21540056) of Japan Society for the Promotion of Science.

1. INTRODUCTION

Throughout this paper, R denotes a prime Goldie ring with simple Artinian quotient ring Q (in other word, R is an order in a simple Artinian ring Q) and $R[x;\delta]$ is a differential polynomial ring over R in an indeterminate x with multiplication $xa = ax + \delta(a)$, where δ is a derivation of R.

We define the concept of δ-Krull prime rings and prove that $R[x;\delta]$ is a Krull prime ring if and only if R is a δ-Krull prime ring. We also determine the set of all maximal v-ideals of $R[x;\delta]$ when it is a Krull prime ring.

In [8], we defined a notion of a σ-generalized Asano prime ring motivated by [1] and [2] and proved that a skew polynomial ring $R[x;\sigma]$ over R in an indeterminate x is a generalized Asano prime ring if and only if R is a σ-generalized Asano prime ring, where σ is an automorphism of R.

In this paper, we define a notion of δ-generalized Asano prime rings that are δ-Krull prime rings whose δ-v-ideals are invertible. We obtain that $R[x;\delta]$ is a generalized Asano prime ring if and only if R is a δ-generalized Asano prime ring, which is proved by using the complete description of maximal v-ideals of $R[x;\delta]$.

We refer readers to [9] or [10] for details of maximal orders and R-ideals.

2. δ-GENERALIZED ASANO PRIME RINGS

First we introduce some notation. For any right (left) R-ideals I (J), let

$$(R:I)_l = \{q \in Q \mid qI \subseteq R\} \quad ((R:J)_r = \{q \in Q \mid Jq \subseteq R\})$$

which is a left (right) R-ideal and

$$I_v = (R:(R:I)_l)_r \quad ({}_vJ = (R:(R:J)_r)_l)$$

which is a right (left) R-ideal containing I (J). I (J) is called a right (left) v-ideal if $I_v = I$ $({}_vJ = J)$. In case I is a two-sided R-ideal, it is said to be a v-ideal if $I_v = I = {}_vI$.

Let $E(Q/R)$ be the injective hull of a right R-module Q/R and let $F(\tau)$ be a right Gabriel topology corresponding to the torsion theory cogenerated by $E(Q/R)$. Then

$$F(\tau) = \{H : \text{right ideal of } R \mid (R : r^{-1}H)_l = R \text{ for any } r \in R\}$$

where $r^{-1}H = \{a \in R \mid ra \in H\}$ by [11, Proposition 5.5, p.147]. For a right ideal I of R, let

$$\text{cl}(I) = \{r \in R \mid rH \subseteq I \text{ for some } H \in F(\tau)\}.$$

If $\text{cl}(I) = I$, then I is said to be τ-closed. We note that $I \subseteq \text{cl}(I) \subseteq I_v$ for any right R-ideal of R and so right v-ideals are τ-closed. R is called a right τ-Noetherian ring if R satisfies the a.c.c. on τ-closed right ideals. Similarly we define τ-closed left ideals and left τ-Noetherian rings and R is said to be τ-Noetherian if it is right and left τ-Noetherian. We note that if R is right or left τ-Noetherian, then R satisfies the a.c.c. on right or left v-ideals of R.

Let δ be a derivation of R, that is, $\delta(ab) = \delta(a)b + a\delta(b)$ for $a, b \in R$. Then δ is extended to a derivation of Q by $\delta(ac^{-1}) = \delta(a)c^{-1} - ac^{-1}\delta(c)c^{-1}$, where $a \in R$ and c is a regular element of R.

An R-ideal I is called a δ-stable ideal (or a δ-ideal for short) if $\delta(I) \subseteq I$. R is called a δ-Krull prime ring if it is a τ-Noetherian and a δ-maximal order in Q, that is, $O_l(A) = R = O_r(A)$ for any δ-ideal A of R, where

$$O_l(A) = \{q \in Q \mid qA \subseteq A\} \text{ and } O_r(A) = \{q \in Q \mid Aq \subseteq A\}.$$

In case $\delta = 0$, R is said to be a Krull prime ring. Then we have the following.

Proposition 2.1 — R is a δ-Krull prime ring if and only if $S = R[x; \delta]$ is a Krull prime ring.

PROOF : By [9, Proposition 2.3.15], R is τ-Noetherian if and only if $R[x; \delta]$ is τ-Noetherian. Suppose R is a δ-Krull prime ring and let A be a non-zero ideal of

S. Then $A' = AT$, where $T = Q[x;\delta]$, is an ideal of T by [6, Theorem 9.20 (a)]. Let $q \in O_l(A)$, that is $qA \subseteq A$. Then $qA' \subseteq A'$ and $q \in O_l(A') = T$ since T is a principal ideal ring. Write $q = q_n x^n + \cdots + q_1 x + q_0$, where $q_i \in Q$ and let

$$L(A) = \{a_l \in R \mid \exists f(x) = a_l x^l + \cdots + a_1 x + a_0 \in A\},$$

which is a δ-ideal of R. For any $a \in L(A)$, there is an $a(x) = ax^l + \cdots + a_1 x + a_0 \in A$ and $qa(x) \in A$ implies $q_n a \in L(A)$, that is $q_n L(A) \subseteq L(A)$. Hence $q_n \in R$. Continuing this method, we have $q \in S$ and similarly $O_r(A) = S$. Hence S is a maximal order and so it is a Krull prime ring.

Conversely suppose that S is a Krull prime ring and let $\mathfrak{a}$ be a δ-ideal of R. Then $A = \mathfrak{a}[x;\delta]$ is an ideal of S. Let $q \in O_l(\mathfrak{a})$. Then $qA \subseteq A$ and so $q \in S \cap Q = R$. Similarly $O_r(\mathfrak{a}) = R$ since $S = R[x;-\delta] = \{x^n a_n + \cdots + a_1 x + a_0 \mid a_i \in R\}$. Hence R is a δ-maximal order and so it is a δ-Krull prime ring. □

If R is a δ-Krull prime ring, then for any δ-ideal A,

$$(R : A)_l = A^{-1} = \{q \in Q \mid AqA \subseteq A\} = (R : A)_r$$

and so $A_v = A^{-1-1} = {}_vA$ follows.

Let $D_\delta(R)$ be the set of all δ-v-ideals. For $A,\ B \in D_\delta(R)$, we define $A \circ B = (AB)_v$. The following lemma is proved in the standard way ([9, Theorem 2.1.2]). However we give a complete proof for reader's convenience.

Lemma 2.2 — If R is a δ-Krull prime ring, then $D_\delta(R)$ is a free abelian group generated by maximal δ-v-ideals of R with multiplication $\circ$.

PROOF : Let $\mathfrak{m}$ be any maximal δ-v-ideal of R. Then $\mathfrak{m}$ is v-invertible, that is $\mathfrak{m}^{-1} \circ \mathfrak{m} = R = \mathfrak{m} \circ \mathfrak{m}^{-1}$ because $R \supseteq \mathfrak{m}^{-1}\mathfrak{m} \supset \mathfrak{m}$ and $R \supseteq \mathfrak{m}\mathfrak{m}^{-1} \supset \mathfrak{m}$.

For any different maximal δ-v-ideals $\mathfrak{m}$ and $\mathfrak{n}$ of R, we claim that $\mathfrak{m} \circ \mathfrak{n} = \mathfrak{m} \cap \mathfrak{n} = \mathfrak{n} \circ \mathfrak{m}$. From $\mathfrak{m} \supset \mathfrak{a} = \mathfrak{m} \cap \mathfrak{n} \supseteq \mathfrak{m}\mathfrak{n}$, we have $R \supset \mathfrak{m}^{-1} \circ \mathfrak{a} \supseteq {}_v(\mathfrak{m}^{-1}\mathfrak{m}\mathfrak{n}) = \mathfrak{n}$ and so $\mathfrak{m}^{-1} \circ \mathfrak{a} = \mathfrak{n}$. Thus $\mathfrak{a} = \mathfrak{m} \circ \mathfrak{n}$ and similarly $\mathfrak{a} = \mathfrak{n} \circ \mathfrak{m}$.

Next we will show that any δ-v-ideal of R is a finite product of maximal δ-v-ideals of R. Assume, on the contrary, that there is a δ-v-ideal $\mathfrak{a}$ of R which is not a finite product of maximal δ-v-ideals of R. Choose $\mathfrak{a}$ maximal with this property. Then $\mathfrak{a}$ is not a maximal δ-v-ideal of R. Let $\mathfrak{m}$ be a maximal δ-v-ideal of R with $\mathfrak{m} \supset \mathfrak{a}$. Then $R \supseteq \mathfrak{m}^{-1} \circ \mathfrak{a} \supset \mathfrak{a}$ and so $\mathfrak{m}^{-1} \circ \mathfrak{a}$ is a finite product of maximal δ-v-ideals of R. Thus so is $\mathfrak{a}$, which is a contradicion.

Finally let $\mathfrak{a}$ be any δ-v-ideal and $\mathfrak{b} = \{r \in R \mid r\mathfrak{a} \subseteq R\}$. Then $\mathfrak{b}$ is a δ-v-ideal of R and so $\mathfrak{b}$ and $\mathfrak{b} \circ \mathfrak{a}$ are both finite product of maximal δ-v-ideals of R. Hence $\mathfrak{a} = \mathfrak{b}^{-1} \circ (\mathfrak{b} \circ \mathfrak{a})$ is a finite product of maximal δ-v-ideals $\mathfrak{m}$ of R or its inverse $\mathfrak{m}^{-1}$. Thus $D_\delta(R)$ is a free abelian group generated by maximal δ-v-ideals.

A δ-ideal P of R is called a δ-prime ideal if $IJ \subseteq P$ for δ-ideals I and J of R, then $I \subseteq P$ or $J \subseteq P$.

Proposition 2.3 — Suppose $S = R[x; \delta]$ is a Krull prime ring. Then $\{\mathfrak{m}[x; \delta], M \mid \mathfrak{m}$ is a maximal $\delta - v - ideal$ of R and $M = M' \cap S$, where M' is a maximal ideal of $T = Q[x; \delta]\}$ is the set of all maximal v-ideals of S.

PROOF : Let M be any maximal v-ideal of S and $\mathfrak{m} = M \cap R$. It is easy to see that M is a prime ideal and $\mathfrak{m}$ is a δ-prime v-ideal.

(i) In case $\mathfrak{m} \neq (0)$. By [7, Lemma 1.3], $\mathfrak{m}[x; \delta]$ is a prime ideal and it is a v-ideal (see the proof of [12, Lemma 3]). Hence $M = \mathfrak{m}[x; \delta]$ and $\mathfrak{m}$ is a maximal δ-v-ideal since M is a minimal prime ideal of S by [10, Proposition 5.1.9]. Conversely suppose $\mathfrak{m}$ is a maximal δ-v-ideal. Then it is a δ-prime ideal and so $\mathfrak{m}[x; \delta]$ is a prime v-ideal. Hence it is a maximal v-ideal.

(ii) In case $\mathfrak{m} = (0)$. Put $M' = MT$, a proper ideal of T. To prove that M' is a maximal ideal of T, let N' be a maximal ideal of T with $N' \supseteq M'$, Then $N' = NT$, where $N = N' \cap S$. Since S is τ-Noetherian and T is a principal ideal ring, it follows that

$$\begin{aligned} N' &= N'_v = (T : (T : N')_l)_r = (T : T(S : N)_l)_r \\ &= (S : (S : N)_l)_r T = N_v T. \end{aligned}$$

Thus N is a v-ideal and so $N = M$. Hence M' is a maximal ideal of T with $M = M' \cap S$.

Conversely if M' is a maximal ideal of T, then $M = M' \cap S$ is a maximal v-ideal which is clear from the proof above. $\square$

Lemma 2.4 — Suppose $S = R[x;\delta]$ is a Krull prime ring. If A is a v-S-ideal contained in $T = Q[x;\delta]$ with $\mathfrak{a} = A \cap Q \neq (0)$, then $A = \mathfrak{a}[x;\delta]$.

PROOF : By Lemma 2.2 and Proposition 2.3, $A = \mathfrak{b}[x;\delta] \circ B$ where $\mathfrak{b}$ is a δ-v-ideal and $B = (M_1^{e_1} \cdots M_k^{e_k})_v$ such that $M_i' = M_i T$ is a maximal ideal of T, $1 \leq i \leq k$. Thus $T = AT = (\mathfrak{b}[x;\delta]B)_v T = BT$ and so $e_1 = \cdots = e_k = 0$. Hence $A = \mathfrak{b}[x;\delta]$ with $\mathfrak{b} = A \cap Q = \mathfrak{a}$. $\square$

An R-ideal A is said to be invertible if $A^{-1}A = R = AA^{-1}$. Then we call R a δ-generalized Asano prime ring if it is a δ-Krull prime ring whose δ-v-ideals are invertible. In case $\delta = 0$, R is said to be a generalized Asano prime ring. Let R be a δ-Krull prime ring. Then, by Lemma 2.2, R is a δ-generalized Asano prime ring if and only if any δ-v-ideal of R is invertible, which is equivalent to that any maximal δ-v-ideal of R is invertible.

Lemma 2.5 — Suppose R is a δ-generalized Asano prime ring. If $B = (M_1^{e_1} \cdots M_k^{e_k})_v$ is a v-ideal of $S = R[x;\delta]$, where $M_i' = M_i T$ is a maximal ideal of T, then B is invertible.

PROOF : Since BT is an ideal, $BT = wT$ for some central element w in T by [3, Remark 1, p.95] and so $w^{-1}B$ is a v-S-ideal such that $w^{-1}B \subseteq T$ with $\mathfrak{a} = w^{-1}B \cap Q \neq (0)$. It follows that from Lemma 2.4 that $w^{-1}B = \mathfrak{a}[x;\delta]$. Hence $B = w\mathfrak{a}[x;\delta]$ is invertible. $\square$

Now we obtain the main result of this paper.

***Theorem* 2.6** — *Let R be a prime Goldie ring in a simple Artinian ring Q. Then R is a δ-generalized Asano prime ring if and only if $S = R[x;\delta]$ is a generalized Asano prime ring.*

PROOF : Suppose R is a δ-generalized Asano prime ring. Let A be a v-ideal of S. Then $A = \mathfrak{a}[x;\delta] \circ B = \mathfrak{a}[x;\delta]B$, where $\mathfrak{a}$ is a δ-v-ideal and B is as in Lemma 2.5. Hence A is invertible and so S is a generalized Asano prime ring.

Conversely suppose S is a generalized Asano prime ring. Let $\mathfrak{a}$ be a δ-v-ideal of R. Then $A = \mathfrak{a}[x;\delta]$ is invertible with $A^{-1} = \mathfrak{a}^{-1}[x;\delta]$ and so $\mathfrak{a}$ is invertible. Hence R is a δ-generalized Asano prime ring. □

In [1] and [2], she studied Noetherian generalized Dedekind prime rings, which are generalized Asano prime rings in our sense. We give an example of non-Noetherian generalized Asano prime rings.

Lemma 2.7 — Let D be a commutative domain with quotient field K and $R = D[\mathbf{x}]$ be a polynomial ring over D in indeterminates $\mathbf{x} = \{x_i\}$ (we do not assume that $\{x_i\}$ is finite). Further let $\mathbf{x}'$ be a finite subset of $\mathbf{x}$ and I be a $D[\mathbf{x}']$-ideal. Then $(R : I[\mathbf{x}-\mathbf{x}'])_l = (D[\mathbf{x}'] : I)_l[\mathbf{x}-\mathbf{x}']$ and in particular $(I[\mathbf{x}-\mathbf{x}'])_v = I_v[\mathbf{x}-\mathbf{x}']$.

PROOF : It is clear $(D[\mathbf{x}'] : I)_l[\mathbf{x}-\mathbf{x}'] \subseteq (R : I[\mathbf{x}-\mathbf{x}'])_l$. Let α be an element in $(R : I[\mathbf{x}-\mathbf{x}'])_l$ and $K(\mathbf{x}')$ be the quotient field of $K[\mathbf{x}']$. Then $IK(\mathbf{x}') = K(\mathbf{x}')$ and $\alpha I \subseteq R$ imply $\alpha \in \alpha K(\mathbf{x}') = \alpha I K(\mathbf{x}') \subseteq RK(\mathbf{x}') \subseteq K(\mathbf{x}')[\mathbf{x}-\mathbf{x}']$. So we can write $\alpha = q_l\mathbf{x_l} + \cdots + q_i\mathbf{x_i} + \cdots + q_0$, where $q_i \in K(\mathbf{x}')$ and $\mathbf{x_i}$'s are subsets of $\mathbf{x}-\mathbf{x}'$. Hence $q_i I \subseteq D[\mathbf{x}']$ for each i and $q_i \in (D[\mathbf{x}'] : I)_l$. Therefore $\alpha \in (D[\mathbf{x}'] : I)_l[\mathbf{x}-\mathbf{x}']$ follows. □

Lemma 2.8 — Let D be a commutative Krull domain and $R = D[\mathbf{x}]$ be a polynomial ring over D in indeterminates $\mathbf{x}$, where $\mathbf{x}$ is infinite or finite. Then

(1) R is a Krull domain and it is not Noetherian if $\mathbf{x}$ is an infinite set.

(2) For any maximal v-ideal P of R, there exists a finite subset $\mathbf{x}'$ of $\mathbf{x}$ such that $P = P_0[\mathbf{x}-\mathbf{x}']$, where $P_0 = P \cap D[\mathbf{x}']$ and P_0 is a maximal v-ideal of $D[\mathbf{x}']$.

PROOF : (1) See for example [4, Theorem 4.2.9].

(2) Let P be a maximal v-ideal of $R = D[\mathbf{x}]$. Then there exists a finite subset of $\mathbf{x}'$ of $\mathbf{x}$ with $P_0 = D[\mathbf{x}'] \cap P \neq (0)$. If $AB \subseteq P_0$ for ideals A and B of $D[\mathbf{x}']$, then $A[\mathbf{x}-\mathbf{x}']B[\mathbf{x}-\mathbf{x}'] = AB[\mathbf{x}-\mathbf{x}'] \subseteq P_0[\mathbf{x}-\mathbf{x}'] \subseteq P$ and so we have $A[\mathbf{x}-\mathbf{x}'] \subseteq P$ or $B[\mathbf{x}-\mathbf{x}'] \subseteq P$. Hence $A \subseteq P_0$ or $B \subseteq P_0$ holds and P_0 is a prime ideal of $D[\mathbf{x}']$. Now $P_0[\mathbf{x}-\mathbf{x}'] \subseteq P$ and so $(P_0)_v[\mathbf{x}-\mathbf{x}'] = (P_0[\mathbf{x}-\mathbf{x}'])_v \subseteq P_v = P$ by Lemma 2.7. Hence $(P_0)_v = P_0$ and P_0 is a prime v-ideal. Since P is a minimal

prime ideal, we have $P = P_0[\mathbf{x} - \mathbf{x}']$. □

Example : Let D be a generalized Dedekind domain. Then $R = D[\mathbf{x}]$ is also a generalized Dedekind domain. If $\mathbf{x}$ is infinite, then R is not Noetherian.

PROOF : Since R is a Krull domain, it is sufficient to prove that any maximal v-ideal is invertible. But this follows from the relation $P = P_0[\mathbf{x} - \mathbf{x}']$ because $D[\mathbf{x}']$ is a generalized Dedekind domain. □

A generalized Asano prime ring is a Krull prime ring. But the converse does not necessarily hold (see [5, p.8, Example 1.10]).

REFERENCES

1. E. Akalan, On Generalized Dedekind prime rings, *J. of Algebra*, **320** (2008), 2907-2916.

2. E. Akalan, On rings whose reflexive ideals are principal, *Comm. in Algebra* **38** (2010), 3174-3180.

3. S. Amitsur, Derivations in simple rings, *Proc. London Math. Soc. (3)* **7** (1957), 87-112.

4. S. Balcerzyk and T. Józefiak, *Commutative Noetherian and Krull Rings*, Ellis Horwood, Warszawa (1989).

5. R. M. Fossum, *The Divisor Class Group of a Krull Domaian*, Springer-Verlag, Berlin (1973).

6. K. R. Goodearl and R. B. Warfield, JR., *An Intoroduction to Noncommutative Noetherian Rings*, London Mathematical Society, Cambridge (1989).

7. K. Kishimoto, H. Marubayashi and A. Ueda, An Ore extension over a v-HC order, *J. of Okayama Unv.*, **27** (1985), 107-120.

8. H. Marubayashi, Intan Muchtadi-Alamsyah and A. Ueda, Skew polynomial rings which are generalized Asano prime rings, *J. Algebra Appl.*, **12** (2013), 7.

9. H. Marubayashi and F. Van Oystaeyen, *Prime Divisors and Non-commutative Valuation Rings*, Lecture Notes in Math. 2059, Springer (2012).

10. J. C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, Wiley, Chichester (1987).

11. B. Stenström, *Rings of Quotients*, Grundlehren Math. **217**, Springer-Verlag, Berlin, (1975).

12. Y. Wang, H. Marubayashi, E. Suwastika and A. Ueda, The group of divisors of an Ore extension over a Noetherian integrally closed domain, *The Aligarh Bulletin of Math.*, **29 (2)** (2010), 149-152.