



Algorithms for solving the optimization problems using fuzzy and intuitionistic fuzzy set

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Abstract In this article, the crisp, fuzzy and intuitionistic fuzzy optimization problem is formulated. The basic definitions and notations related to optimization problems are given in the preliminaries section. Algorithms for solving the optimization problems using fuzzy and intuitionistic fuzzy set is presented in this article. Then, with the help of the proposed algorithm the optimal solution of the crisp, fuzzy and intuitionistic fuzzy optimization problems are determined. A new theorem related to type-2 fuzzy/type-2 intuitionistic fuzzy optimization problems is proposed and proved. Some new and concrete results related to type-2 fuzzy/type-2 intuitionistic fuzzy optimization problems are presented. To illustrate the proposed method, some real-life numerical examples are presented. The proposed article provides seven fully worked examples with screenshots of output summaries from the software used in the computations for better understanding. The advantages of the proposed approach as compared to other existing work are also specified. Detail analyses of the comparative study as well the discussion are given. To show the advantages of the proposed approach, superiority analysis is discussed. Comparison analysis and the advantages of the proposed operators are also discussed. Some managerial applications and the advantages of the proposed approach are given.

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Finally, conclusion and future research directions are also given.

Keywords Fuzzy set · Fuzzy number · TFN · TrFN · Intuitionistic fuzzy set · Intuitionistic fuzzy number · TIFN · TrIFN · Type-2 FTP · Type-2 IFTP · Type-2 IFSTP · FAP · IFAP · IFSAP · Linear programming method · Integer programming problem · Optimal solution · Optimal assignment · PSK theorem

1 Introduction

The transportation problem (i.e., TP) is a special class of the LPP (i.e., linear programming problem), widely used in the areas of a communication network, employee scheduling, personnel management, aggregate planning, inventory control and so forth. In several real life situations, there is a need for shipping the manufactured goods from different origins (Factories) to different destinations (Warehouses). The TP deals with shipping commodities from different origins to various destinations. The objective of the crisp transportation problem (CTP) is to determine the optimum amount of a commodity to be transported from various supply points (origins) to different demand points (destinations) so that the total transportation cost is minimum or total transportation profit is maximum. A minimization TP involves cost data, in this case, the objective of the solution is to minimize the total cost. On the other hand, a maximization TP involves sales, revenue or profit data, in this case, the objective of the solution is to maximize the total profit. The unit costs/profits, that is, the cost/profit of transporting one unit from a particular origin (supply point) to a particular destination (demand point),

the amounts available at the origins ($O_i; i = 1, 2, 3, \dots, m$) and the amounts required at the destinations ($D_j; j = 1, 2, 3, \dots, n$) are the parameters of the TP.

Let us consider m origins (plant locations) and n destinations (distribution centres). The production capacity of the i th plant/origin is a_i and the number of units required at the j th destination is b_j . The transportation cost/profit of one unit from the i th origin to j th destination is c_{ij}/p_{ij} and let x_{ij} be the number of units shipped from the i th origin to j th destination. Our aim is to determine the transportation schedule to minimize/maximize the total transportation cost/profit satisfying supply and demand constraints.

Now, the mathematical model of the above TP (i.e., crisp TP) is given by

$$\begin{aligned}
 \text{(Model 1) Minimize } Z &= \sum_{i=1}^m \sum_{j=1}^n c_{ij} \times x_{ij} \text{ or} \\
 \text{Maximize } Z &= \sum_{i=1}^m \sum_{j=1}^n p_{ij} \times x_{ij}
 \end{aligned}
 \tag{1}$$

subject to

$$\sum_{j=1}^n x_{ij} = a_i, \quad \text{for } i = 1, 2, \dots, m \text{ (Row restriction)} \tag{2}$$

$$\sum_{i=1}^m x_{ij} = b_j, \quad \text{for } j = 1, 2, \dots, n \text{ (Column restriction)} \tag{3}$$

$$x_{ij} \geq 0, \quad \text{for } i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n \tag{4}$$

Hitchcock (1941) developed a crisp TP.

Finding a best way for assigning several finite number of objects to each other is indeed a difficult task and many problems (assignment problems (AP)) are to be faced while doing so. A special case of AP arises when our main focus is to allot some equal number of destination to a number of origins keeping the cost minimum or profit maximum. Some examples of such phenomenon are:

- *People* → *projects*.
- *Jobs* → *machines*
- *Workers* → *jobs*
- *Teachers* → *classes* etc.

AP has a significant feature of establishing a unique way for assigning finite number of objects or tasks to each other and by uniqueness we mean that an object can be assigned to only one job or task. Such type of relation makes the number of sources and destination equal and the values of necessities and capacity is unity that is one. There are basically two approaches for solving an AP which are

- *Linear programming*
- *Transportation method*

D. Konig’s (Hungarian mathematician) assignment method is relatively much faster and reliable in applicability. Such method is also referred as ‘Hungarian Method of Assignment Problem’.

Due to its effectiveness and reliability, AP is applied across the world in various real-life circumstances. The sole application of an AP lies in industry but such tools can be applied in some other areas too. As discussed, an AP has exactly n jobs and corresponds to n persons and a person may avail at most one job. The task is to find an optimum assignment in order to keep the cost minimum or profit maximum. In doing so, a well-known Hungarian Method is developed by Kuhn (1955) which was recognised as first method practically applicable for solving typical APs.

Now, the mathematical model of $n \times n$ balanced AP is presented as follows:

$$\begin{aligned}
 \text{(Model 2) Minimize } Z &= \sum_{i=1}^n \sum_{j=1}^n c_{ij} \times x_{ij} \text{ or} \\
 \text{Maximize } Z &= \sum_{i=1}^n \sum_{j=1}^n p_{ij} \times x_{ij}
 \end{aligned}
 \tag{5}$$

subject to

$$\sum_{j=1}^n x_{ij} = 1, \quad \text{for } i = 1, 2, \dots, n \text{ (Row restriction)} \tag{6}$$

$$\sum_{i=1}^n x_{ij} = 1, \quad \text{for } j = 1, 2, \dots, n \text{ (Column restriction)} \tag{7}$$

$$x_{ij} = 0 \text{ or } 1, \quad \text{for all } i, j \tag{8}$$

where c_{ij} is the cost of assigning the job j to the machine i .

$x_{ij} = 1$ if the job j is assigned to the machine i , and $x_{ij} = 0$, otherwise.

In case of maximization AP, our objective is to determine the assignment of machines to jobs so that the total profit of completing all the jobs is maximum.

In real-life applications, many allocation/optimization problems are uncertain in nature. In that, the CTPs and crisp assignment problems (CAPs) are not an exception, they can also involve uncertainties. So, in this article, the author considers the optimization problems that having uncertainty and hesitation in its parameter. The author formulates the optimization problems (e.g., TPs and APs) and utilizes TIFNs to deal with uncertainty and hesitation. The formulated optimization problems have been transformed into crisp one and solved by linear programming method or integer programming method. Since the existing methods (e.g., fuzzy modified distribution method (FMODI), intuitionistic fuzzy modified distribution method (IFMODI), intuitionistic fuzzy min-zero min-cost method (IFMZMCM), fuzzy Hungarian method (FHM), intuitionistic fuzzy Hungarian method (IFHM), intuitionistic fuzzy

reduction method (IFRM)) has many steps to solve the fuzzy and intuitionistic fuzzy optimization problems but so many times it will be very complicated. In general, fuzzy and intuitionistic fuzzy optimization problems are first converted into equivalent crisp optimization problems. Second, the obtained crisp optimization problems are converted into equivalent linear programming or integer linear programming problem (ILPP), which are then solved by TORA (Temporary-Ordered Routing Algorithm) software. It solves the problems by using the branch and bound methodology.

The TORA software especially helpful when the problem is complex. Such complexity may arise when the problem involves a large number of decision variables. However, use of TORA software necessary requires database or data availability about the environment surrounding the problem also. Hence, intuitionistic fuzzy optimization technique with the use of TORA software has evolved special techniques which can measure, compare, control and predict the probable behavior of the variables of the system in a scientific method. It can handle both the maximization as well as the minimization problems. Further, it uses scientific approach to arrive at the solution. So the author has been illustrated a very easy and scientific method to solve different types of optimization problems under crisp, fuzzy and intuitionistic fuzzy environment (IFE). The crisp, fuzzy and intuitionistic fuzzy optimization problems have wide applications in industry, organization, telecommunication, coal transportation, satellite launching, timetabling problems, capital investment, multi-passive-sensor, dynamic facility location and so on. Hence the proposed algorithm is illustrated by real-life numerical examples. To the best of my knowledge, the problem of solving optimization problems under crisp, fuzzy and IFE using single algorithm is new in literature.

The concept of considering the optimization problems by using fuzzy and intuitionistic fuzzy set is new in literature. The proposed approach in this article is computationally simple and efficient. To check the reliability and efficiency of the proposed algorithm, seven numerical examples are presented. Some real-life examples are stated and also it is solved by using the proposed algorithm. The proposed real-life examples are strengthening both the quality and quantity of this article. Results and discussions are given. Due to solving the real-life problems, this article is useful to the variety of researchers and the public. Due to this, it would be more attracted to the various kinds of researchers, practitioners and advanced students in the future. By using the proposed algorithm a DM has the following advantages:

- There is no need to find out the basic feasible solution.
- There is no need to apply the optimality test. Because the solution obtained by the proposed approach is always optimal.
- The proposed algorithm is a single stage approach. So, the use of the IFMODI is not required.

The rest of the paper is structured as follows. The historical aspect of different kinds of optimization problems under crisp, fuzzy and IFE is discussed in Sect. 2. Section 3 presents some basic definitions such as fuzzy set, fuzzy number, TFN, TrFN, intuitionistic fuzzy set, intuitionistic fuzzy number, TIFN and TrIFN, and also theorems related to accuracy function of TrIFN. Arithmetic operations on TIFNs and TrIFNs are discussed in Sect. 4. Section 5 represents the Varghese and Kuriakose (2012) centroid formula for triangular intuitionistic fuzzy numbers (TIFNs). Moreover, this section describes the score function and accuracy function of TIFNs, and also theorems related to accuracy function of TIFN. Section 6, explicitly presents a formulation for the different kinds of optimization problems under crisp, fuzzy and IFE. Section 7 presents the linear and integer programming method based approach to solving various kinds of optimization problems under crisp, fuzzy and IFE. Real-life examples are presented in Sect. 8. Section 9 is devoted to comparison of the suggested approach with the existing methods. The paper is ended by conclusion in Sect. 10.

2 Historical aspects

Everyone knows that the optimization problem is one of the most important problems of management science. Overall, the CAPs and CTPs both are called optimization problems or allocation problems. The CAP deals with assigning jobs $(J_i, i = 1, 2, 3, \dots, n)$ to machines $(M_j, j = 1, 2, 3, \dots, n)$ and the CTP deals with assigning sources $(S_i$ or $O_i, i = 1, 2, 3, \dots, n)$ to destinations $(D_j, j = 1, 2, 3, \dots, n)$. The CAP is a particular case of the CTP where the destinations are tasks/jobs and the sources are assignees.

In the history of mathematics, Hitchcock (1941) presented the fundamental concepts of crisp TPs. The transportation algorithm for solving TPs with equality constraints was given by Dantzig (1963). Taha (2008), Ghazali et al. (2012), Ahmed et al. (2016), Ficker et al. (2017), Xie et al. (2017), Gupta and Arora (2018), Sadeghi (2018), Roy and Mahapatra (2018), Das and Jana (2018) and many authors have solved classical/STPs under crisp environment. Hungarian algorithm for solving APs was given by Kuhn (1955). Thompson (1981), Balinski (1986), Singh (2012), Hossen and Akther (2017) and many authors have solved crisp assignment problems (CAPs).

Many of the allocation problems are imprecise (vague) in nature in today's world such as in corporate or in the industry, due to variations (the variations may be either big or small) in the parameters. In this case, the use of ordinary (crisp) set theory is not possible. So, to deal quantitatively with imprecise (vague) information in making the decision, Zadeh (1965) introduced the fuzzy set (FS) theory and has applied it successfully to the different kinds of fields. Crisp sets are the sets, that we have used most of our life. In a crisp set, an element is either a member of the set or not. Fuzzy sets (FSs), on the other hand, allow elements to be partially in a set. Each element is given a degree of membership (i.e., membership value) in a set. This membership value can range from the numerical value 0 (i.e., zero) to 1 (i.e., one). Here, the numerical values 0 and 1 refer to "not an element of the set" and "a member of the set" respectively. Clearly, if it is one only allowed the extreme membership values of 0 and 1, that this would actually be equivalent to crisp sets. A membership function is a relationship between "the values of an element" and "its degree of membership" in a set. In a fuzzy set, the membership value lies between 0 and 1. Generally, the membership value is also called the level of acceptance or level of satisfaction. In a crisp set, the numerical values 0 and 1 respectively, represent "the element belongs to the set" and "the element not belong to the set". Due to the special features of fuzzy set theory, it has been used in numerous fields such as power engineering, consumer electronics, industrial automation, optimization, robotics and control systems engineering.

After the invention of fuzzy set theory, many authors (for example, Mohideen and Kumar 2010; Sharma et al. 2015; Baykasoğlu and Subulan 2017; Kumar 2016a, b, 2017a, 2018a; Sujatha et al. 2018; Vidhya and Ganesan 2018; Habib 2018; Bharati 2018; Gupta et al. 2018; Mishra et al. 2018; Saini et al. 2018; Ngastiti et al. 2018; Maheswari and Ganesan 2018; Sam'an et al. 2018; Bisht and Srivastava 2018; Ebrahimnejad and Verdegay 2018a) have solved TPs under fuzzy environment successfully. Similarly, several authors have solved fuzzy assignment problems (FAPs) successfully. However, the fuzzy set considered the membership value (or degree of membership) of an element in the set but it did not consider the non-membership value (or degree of non-membership) and hesitation index of an element in that particular set.

In conventional TP, i.e., in crisp TP supply, demand and costs are fixed crisp numbers. Hence in this situation, the decision maker can predict transportation cost precisely. On the contrary, in real-world TPs, the supplies, i.e., the availability of the goods and demands of the goods both are not known exactly. These are uncertain quantities with hesitation due to numerous factors like unexpected situations, lack of good communications, error in data, seasonal

changes, understanding of markets, rare materials in the market (it depends on the nature and quality of the material), unawareness of customers and many more. Also, the costs are in uncertain quantities with hesitation due to numerous factors like variation in rates of fuels (e.g., petrol, diesel, coal, and gas), traffic jams, weather etc. In such situations, the DM cannot predict transportation cost precisely. Hence, the DM may hesitate. To address this issue, Atanassov (1983) proposed the intuitionistic fuzzy set (IFS) which is more reliable than the fuzzy set proposed by Zadeh (1965). The major advantage of the intuitionistic fuzzy set over fuzzy set is that intuitionistic fuzzy set separates "the degree of membership" and "degree of non-membership" of an element in the set. Hence, with the help of IFS theory, the decision maker can decide the following:

- the degree of acceptance,
- the degree of non-acceptance and
- the degree of hesitation for some quantity.

In case the DM consider the IFS theory in CTP, the DM can decide about the level of acceptance and non-acceptance for the transportation cost or profit.

Similarly, in conventional assignment problem, the performing time (or cost/profit) of each job of the workers (or machines) is not known exactly. This may be due to lack of experience, capacity, interest, situations on that particular day, knowledge, the agility of persons, physical ability, understanding and so on. In such situation, the DM cannot predict performing (the processing time of particular machine) time exactly. Hence the decision maker may hesitate. In this situation, by using the intuitionistic fuzzy set theory the DM can decide about the level of acceptance and non-acceptance for the assignment cost/profit/time. Due to this, the application of IFS theory becomes very popular in TP, assignment problem, decision making, planning, manufacturing, scheduling, medical diagnosis, image processing, washing machines, facial pattern recognition, knowledge-based systems for multiobjective optimization of power systems, control of subway systems and unmanned helicopters, vacuum cleaners and so on.

After the invention of intuitionistic fuzzy set theory many authors (for instance, Hussain and Kumar 2012a, b, c, 2013; Kumar and Hussain 2014a; Singh and Yadav 2015; Kumar and Hussain 2015, 2016a; Ebrahimnejad and Verdegay 2016; Gupta and Anupum 2017; Kumar 2018b, c, d; Mahmoodirad et al. 2018; Roy et al. 2018; Bharati and Singh 2018a; Ebrahimnejad and Verdegay 2018b; Hunwisai et al. 2018; Pathade and Ghadle 2018; Abhishekh and Nishad 2018; Sidhu and Kumar 2019; Pratihari et al. 2020; Smarandache and Broumi 2020) have solved transportation/linear programming problems under neutrosophic/IFE successfully. Similarly, a lot of works based on AP in IFE has been done by several researchers such as Mukherjee and

Basu (2012), Kumar and Hussain (2014b, c, d, 2016b, c), Kumar (2017b), Pothiraj and Rajaram (2017), Kumar (2018e, 2020a, b, c, d) and many others.

The solid transportation problem (i.e., STP) is a generalization of the classical TP (i.e., crisp TP) in which three-dimensional properties are taken into account in the objective and constraint set instead of the source (origin) and destination. Shell (1955) stated an extension of well-known transportation problem, that is, crisp TP is called a STP in which bounds are given on three items, namely, supply, demand, and conveyance. In many industrial problems, a homogeneous product is transported from an origin to a destination by means of different modes of transport called conveyances, such as trucks, cargo flights, goods trains, ships and so forth. Study on various kinds of STPs has been done by several researchers (e.g., Haley 1962; Patel and Tripathy 1989; Basu et al. 1994; Jimenez and Verdegay 1996; Li et al. 1997a and many others). An exact method for solving the four index transportation problem and industrial application was presented by Pham and Dott (2013). In literature, Bit et al. (1993), Gen et al. (1995), Li et al. (1997b), Jimenez and Verdegay (1998, 1999), Liu (2006), Ojha et al. (2009), Baidya et al. (2014) and many others have solved fuzzy solid transportation problems (FSTPs). Similarly, a lot of works based on STP under IFE has been done by several researchers such as Aggarwal and Gupta (2016, 2017), Das et al. (2017), Kumar (2018f, g, 2019a, b, c, d) and many others.

Pierskalla (1967) introduced the three-index assignment problem as a straightforward extension of the classical two-dimensional AP. The multidimensional AP was proposed by Pierskalla (1968). Other than Pierskalla, a number of researchers (Frieze and Yadegar 1981; Balas and Saltzman 1991; Crama and Spieksma 1992; Magos and Miliotis 1994; Magos 1996; Storms and Spieksma 2003; Anuradha and Pandian 2012; Kavitha and Pandian 2012) have also studied the concept of SAPs under crisp environment. Recently, Kadhem (2017) discussed heuristic solution approaches to the SAP. Thus, many authors have proposed different approaches to solve the different types of assignment problems when its parameter was in well known crisp numbers. That is, efficient algorithms have been developed for solving different types of optimization problems when the parameters values are known precisely. Anuradha (2015) presented fuzzy solid assignment problems (FSAPs). Therefore, several authors have solved IFTP and IFAP. However, IFSTP and IFSAP have not been investigated by the other studies. To address these issues, two important notions have to be discussed: Linear programming method (LP) and integer programming method (IP). To the best of my knowledge, the problem of solving optimization problems under crisp, fuzzy and IFE using single algorithm is new in literature.

3 Preliminaries

Before studying the proposed algorithm and model, some key terms are defined which is as follows:

Linear programming is concerned with the maximization or minimization of a linear objective function in many variables subject to linear equality and inequality constraints. An objective function represents some principal objective criterion or goal that measures the effectiveness of the system (such as maximizing profits or productivity, or minimizing cost or consumption). Decision variables describe the quantities that the DMs would like to determine. They are the unknowns of a mathematical programming model. Typically we will determine their optimum values with an optimization method.

Optimization problem is a problem in which a objective function is to be minimize or maximize subject to the certain conditions. In general, the solution means the final answer to a problem, but the convention in optimization problem (e.g., linear programming (LP), integer programming (IP), TP, AP and so on) is quite different. Here, any specification of values for the decision variables (e.g. x_1, x_2, \dots, x_n or $x_{11}, x_{12}, \dots, x_{nn}$) is called a solution, regardless of whether it is a desirable or even an allowable choice. Based on the nature of the solution, the solution can be defined in different ways. So, the different types of solution can be defined as follows.

A feasible solution is a solution for which all the constraints are satisfied. An infeasible solution is a solution for which at least one constraint is violated. An optimal solution is a feasible solution that has the most favorable value of the objective function. The most favorable value is the smallest value if the objective function is to be minimized, whereas it is the largest value if the objective function is to be maximized. Optimal value is a minimum or maximum value of the objective function to be calculated in optimization problem (e.g., IP, LP, AP, TP and so on). Constraints are linear inequalities or equations involved in an optimization problem. The restrictions normally are referred to as constraints. In most practical problems the variables are required to be non-negative. This special kind of constraint is called a non-negativity restriction. The supply (a_i), demand (b_j), cost (c_{ij} , \tilde{c}_{ij} , \tilde{c}_{ijk}^l), time (t_{ij} , \tilde{t}_{ij} , \tilde{t}_{ijk}^l) and capacity (e_k) are also referred to as the parameters of the model.

Definition 3.1 (Zadeh 1965) Let A be a classical (ordinary) set and $\mu_A(x): A \rightarrow [0, 1]$. A fuzzy set A^* with the membership function (membership grade) $\mu_A(x)$ is defined by, $A^* = \{(x, \mu_A(x)): x \in A \text{ and } \mu_A(x) \in [0, 1]\}$.

Definition 3.2 (Klir and Yuan 2005) The fuzzy number (FN) \tilde{A} is an extension of a regular (ordinary) number in

the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible values have its own weight (membership value) between the numerical values 0 and 1. The weight (membership value/function) denoted by $\mu_A(x)$ that satisfies the following conditions.

(1) The membership function $\mu_A(x)$ is piecewise continuous. (2) The membership function $\mu_A(x)$ is a convex fuzzy subset. (3) The membership function $\mu_A(x)$ is the normality of a fuzzy subset (this statement represent that for at least one element x_0 the membership grade should be 1 (i.e., $\mu_A(x_0) = 1$)).

Definition 3.3 (Nayagam et al. 2008) A fuzzy number (FN) A is defined to be a triangular fuzzy number (TFN) if its membership functions $\mu_A: \mathbb{R} \rightarrow [0, 1]$ is equal to

$$\mu_A(x) = \begin{cases} \frac{x - \omega_1}{\omega_2 - \omega_1}, & x \in [\omega_1, \omega_2], \\ \frac{\omega_3 - x}{\omega_3 - \omega_2}, & x \in [\omega_2, \omega_3], \\ 0, & \text{Otherwise.} \end{cases}$$

Definition 3.4 (Klir and Yuan 2005) A trapezoidal fuzzy number (TrFN) is denoted by $\tilde{\omega}$ and is defined by $\tilde{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$, where $\omega_1, \omega_2, \omega_3$ and ω_4 are real numbers, then its membership function $\mu_{\tilde{\omega}}(x)$ is as follows:

$$\mu_{\tilde{\omega}}(x) = \begin{cases} 0, & \text{for } x < \omega_1, \\ \frac{x - \omega_1}{\omega_2 - \omega_1}, & \text{for } \omega_1 \leq x \leq \omega_2, \\ 1, & \text{for } \omega_2 \leq x \leq \omega_3, \\ \frac{\omega_4 - x}{\omega_4 - \omega_3}, & \text{for } \omega_3 \leq x \leq \omega_4, \\ 0, & \text{for } x > \omega_4, \end{cases}$$

The graphical representation of a membership function $\mu_{\tilde{\omega}}(x) = (\omega_1, \omega_2, \omega_3, \omega_4)$ is given in the following figure (Fig. 1).

Definition 3.5 (Liou and Wang 1992) We define a ranking function R from F(R) to R, which maps each fuzzy number into the real line, F(R) represents the set of all

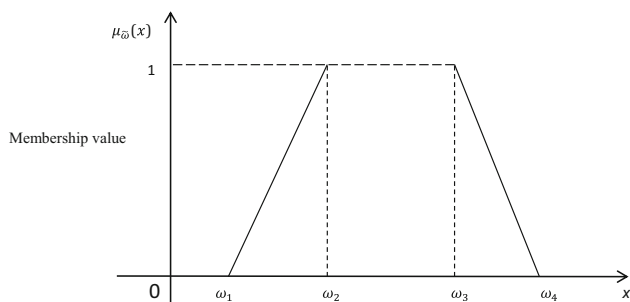


Fig. 1 Graphical representation of TrFN

trapezoidal fuzzy numbers (TrFNs). If R be any ranking function, then the ranking function of the given trapezoidal fuzzy number (TrFN) $\tilde{\omega} = (\omega_1, \omega_2, \omega_3, \omega_4)$ can be defined as follows.

$$R(\tilde{\omega}) = \frac{\omega_1 + \omega_2 + \omega_3 + \omega_4}{4} \tag{9}$$

Definition 3.6 (Atanassov 1986) Let X be the universal set and the membership function, non-membership functions defined on X by $\mu_A(x) : X \rightarrow [0, 1]$, $\vartheta_A(x) : X \rightarrow [0, 1]$.

Degree of membership function and non-membership functions are $\mu_A(x), \vartheta_A(x)$ which always satisfies the conditions $\mu_A(x) + \vartheta_A(x) \geq 0$ and $\mu_A(x) + \vartheta_A(x) \leq 1$ i.e., $0 \leq \mu_A(x) + \vartheta_A(x) \leq 1$ for all $x \in X$ then the set $A = \{ \langle x, \mu_A(x), \vartheta_A(x) \rangle : x \in X \}$ is an Intuitionistic Fuzzy Set (IFS).

In addition, the intuitionistic fuzzy set index or hesitation margin of x in A is denoted by $\pi_A(x)$ and is defined by $\pi_A(x) = 1 - (\mu_A(x) + \vartheta_A(x))$.

$\pi_A(x)$ is the degree of indeterminacy of $x \in X$ to the IFS A and $\pi_A(x) \in [0, 1]$ that is, $\pi_A(x) : X \rightarrow [0, 1]$ and $0 \leq \pi_A(x) \leq 1$ for every $x \in X$.

Where, the notation $\pi_A(x)$ expresses the lack of knowledge of whether x belongs to IFS A or not.

Definition 3.7 (Burillo et al. 1994) If the following holds then an intuitionistic fuzzy subset $A = \{ \langle x, \mu_A(x), \vartheta_A(x) \rangle : x \in X \}$ of the real line R is called an intuitionistic fuzzy number (IFN):

- (i) $\exists m \in R, \mu_A(m) = 1$ and $\vartheta_A(m) = 0$, where m denotes the mean value of A.
- (ii) μ_A is a continuous mapping from $R \rightarrow [0, 1]$ and for all $x \in R$, the relation $0 \leq \mu_A(x) + \vartheta_A(x) \leq 1$ (i.e., $\mu_A(x) + \vartheta_A(x) \geq 0$ and $\mu_A(x) + \vartheta_A(x) \leq 1$) holds.

The membership function and non-membership function of the IFS ‘A’ is of the following form:

$$\mu_A(x) = \begin{cases} 0, & \text{for } -\infty < x \leq m - \alpha, \\ p_1 = f_1(x), & \text{for } x \in [m - \alpha, m], \\ 1, & \text{for } x = m, \\ q_1 = h_1(x), & \text{for } x \in [m, m + \beta], \\ 0, & \text{for } m + \beta \leq x < \infty. \end{cases}$$

Here, $p_1 = f_1(x)$ is strictly increasing function in $[m - \alpha, m]$ and $q_1 = h_1(x)$ is strictly decreasing function in $[m, m + \beta]$.

$$\vartheta_A(x) = \begin{cases} 1, & \text{for } -\infty < x \leq m - \alpha', \\ p_2 = f_2(x), & \text{for } x \in [m - \alpha', m]; 0 \leq p_1 + p_2 \leq 1, \\ 0, & \text{for } x = m, \\ q_2 = h_2(x), & \text{for } x \in [m, m + \beta']; 0 \leq q_1 + q_2 \leq 1, \\ 1, & \text{for } m + \beta' \leq x < \infty. \end{cases}$$

Here,

- ‘ m ’ denotes the mean value of A .
 - ‘ α ’ denotes the left spread of membership function $\mu_A(x)$.
 - ‘ β ’ denotes the right spread of membership function $\mu_A(x)$.
 - ‘ α' ’ denotes the left spread of non-membership function $\vartheta_A(x)$.
 - ‘ β' ’ denotes the right spreads of non-membership function $\vartheta_A(x)$.
- Symbolically, the intuitionistic fuzzy number \tilde{A}^I is represented as $A_{IFN} = (m; \alpha, \beta; \alpha', \beta')$.

Definition 3.8 (Varghese and Kuriakose 2012) A triangular intuitionistic fuzzy number (TIFN) \tilde{A}^I is an intuitionistic fuzzy set in R with the following membership function $\mu_\omega(x)$ and non-membership function $\vartheta_\omega(x)$:

$$\mu_\omega(x) = \begin{cases} 0, & \text{for } x < \omega_1, \\ \frac{x - \omega_1}{\omega_2 - \omega_1}, & \text{for } \omega_1 \leq x \leq \omega_2, \\ 1, & \text{for } x = \omega_2, \\ \frac{\omega_3 - x}{\omega_3 - \omega_2}, & \text{for } \omega_2 \leq x \leq \omega_3, \\ 0, & \text{for } x > \omega_3, \end{cases}$$

$$\vartheta_\omega(x) = \begin{cases} 1, & \text{for } x < \omega'_1, \\ \frac{\omega_2 - x}{\omega_2 - \omega'_1}, & \text{for } \omega'_1 \leq x \leq \omega_2, \\ 0, & \text{for } x = \omega_2, \\ \frac{x - \omega_2}{\omega'_3 - \omega_2}, & \text{for } \omega_2 \leq x \leq \omega'_3, \\ 1, & \text{for } x > \omega'_3. \end{cases}$$

where $\omega'_1 \leq \omega_1 \leq \omega_2 \leq \omega_3 \leq \omega'_3$ and $\mu_\omega(x), \vartheta_\omega(x) \leq 0.5$ for $\mu_\omega(x) = \vartheta_\omega(x)$, for all $x \in R$. This TIFN can be written as $\tilde{\omega}^I = (\omega_1, \omega_2, \omega_3)(\omega'_1, \omega_2, \omega'_3)$.

3.1 Particular cases (Kumar 2019e)

Let $\tilde{J}^I = (j_1, j_2, j_3)(j'_1, j_2, j'_3)$ be an arbitrary TIFN. Then, we have the following two cases:

- Case 1* If $j'_1 = j_1, j'_3 = j_3$ then \tilde{J}^I represent Triangular Fuzzy Number (TFN). Symbolically, it can be written as $\tilde{J} = (j_1, j_2, j_3)$.
- Case 2* If $j'_1 = j_1 = j_2 = j_3 = j'_3 = s$ then \tilde{J}^I represent a real number s .

Definition 3.9 (Singh and Yadav 2016) A Trapezoidal Intuitionistic Fuzzy Number (TrIFN) \tilde{A}^I is an ‘Intuitionistic Fuzzy Number (IFN)’ with the following ‘membership function $(\mu_{\tilde{A}^I})'$ and ‘non-membership function $(\vartheta_{\tilde{A}^I})'$:

$$\mu_{\tilde{A}^I}(x) = \begin{cases} \frac{x - a_1}{b_1 - a_1} = \mu_{\tilde{A}^I}^L(x), & \text{for } a_1 < x \leq b_1, \\ 1, & \text{for } b_1 \leq x \leq c_1, \\ \frac{d_1 - x}{d_1 - c_1} = \mu_{\tilde{A}^I}^R(x), & \text{for } c_1 \leq x < d_1, \\ 0, & \text{otherwise} \end{cases} \quad \text{and}$$

$$\vartheta_{\tilde{A}^I}(x) = \begin{cases} \frac{b'_1 - x}{b'_1 - a'_1} = \vartheta_{\tilde{A}^I}^L(x), & \text{for } a'_1 < x \leq b'_1, \\ 0, & \text{for } b'_1 \leq x \leq c'_1, \\ \frac{x - c'_1}{d'_1 - c'_1} = \vartheta_{\tilde{A}^I}^R(x), & \text{for } c'_1 \leq x < d'_1, \\ 1, & \text{otherwise} \end{cases}$$

where $a'_1 \leq a_1 \leq b'_1 \leq b_1 \leq c_1 \leq c'_1 \leq d_1 \leq d'_1$ (see Fig. 2).

This TrIFN is denoted by $\tilde{A}^I = (a_1, b_1, c_1, d_1; a'_1, b'_1, c'_1, d'_1)$ (refer to Fig. 2)

3.2 Particular cases

- Case 1* If $b'_1 = b_1, c'_1 = c_1$, then \tilde{A}^I also represents a TrIFN. It is denoted by $\tilde{A}^I = (a_1, b_1, c_1, d_1; a'_1, b_1, c_1, d'_1)$.
- Case 2* If $b'_1 = b_1 = c_1 = c'_1$, then \tilde{A}^I also represents a Triangular Intuitionistic Fuzzy Number (TIFN). It is denoted by $\tilde{A}^I = (a_1, b_1, d_1; a'_1, b_1, d'_1)$.
- Case 3* If $a'_1 = a_1, b'_1 = b_1, c'_1 = c_1, d'_1 = d_1$, then \tilde{A}^I represents a Trapezoidal Fuzzy Number (TrFN). It is denoted by $\tilde{A}^I = (a_1, b_1, c_1, d_1)$.
- Case 4* If $a'_1 = a_1, b'_1 = b_1 = c_1 = c'_1, d'_1 = d_1$, then \tilde{A}^I represents a Triangular Fuzzy Number (TFN). It is denoted by $\tilde{A}^I = (a_1, b_1, d_1)$.
- Case 5* If $a'_1 = a_1 = b'_1 = b_1, c_1 = c'_1 = d_1 = d'_1$, then \tilde{A}^I represents the crisp interval $[b_1, c_1]$.
- Case 6* If $a'_1 = a_1 = b'_1 = b_1 = c_1 = c'_1 = d_1 = d'_1 = m$, then \tilde{A}^I represents a real number m .

3.3 Score function and accuracy function of a TrIFN (Singh and Yadav 2016)

Let $\tilde{A}^I = (a_1, b_1, c_1, d_1; a'_1, b'_1, c'_1, d'_1)$ be an arbitrary TrIFN.

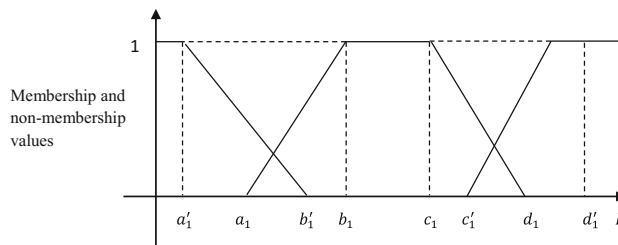


Fig. 2 Graphical representation of TrIFN

Definition 3.10 The left integral and right integral values of \tilde{A}^I for the membership function $\mu_{\tilde{A}^I}$ are denoted by $I_L(\mu_{\tilde{A}^I})$ and $I_R(\mu_{\tilde{A}^I})$ and defined by $I_L(\mu_{\tilde{A}^I}) = \int_0^1 g_{\tilde{A}^I}^L(y)dy$ and $I_R(\mu_{\tilde{A}^I}) = \int_0^1 g_{\tilde{A}^I}^R(y)dy$ respectively.

Definition 3.11 The left integral and right integral values of \tilde{A}^I for the non-membership function $\vartheta_{\tilde{A}^I}$ are denoted by $I_L(\vartheta_{\tilde{A}^I})$ and $I_R(\vartheta_{\tilde{A}^I})$ and defined by $I_L(\vartheta_{\tilde{A}^I}) = \int_0^1 h_{\tilde{A}^I}^L(y)dy$ and $I_R(\vartheta_{\tilde{A}^I}) = \int_0^1 h_{\tilde{A}^I}^R(y)dy$ respectively.

Definition 3.12 The generalized score function for the membership function $\mu_{\tilde{A}^I}$ is denoted by $S^\alpha(\mu_{\tilde{A}^I})$ with index of optimism $\alpha \in [0, 1]$ and is defined by $S^\alpha(\mu_{\tilde{A}^I}) = \alpha\{I_R(\mu_{\tilde{A}^I}) + I_L(\mu_{\tilde{A}^I})\}$.

Definition 3.13 The generalized score function for the non-membership function $\vartheta_{\tilde{A}^I}$ is denoted by $S^\alpha(\vartheta_{\tilde{A}^I})$ with index of optimism $\alpha \in [0, 1]$ and is defined by $S^\alpha(\vartheta_{\tilde{A}^I}) = (1 - \alpha)\{I_L(\vartheta_{\tilde{A}^I}) + I_R(\vartheta_{\tilde{A}^I})\}$.

Hence, $f^\alpha(\tilde{A}^I) = \frac{\alpha\{(c_1 + d_1) + (a_1 + b_1)\} + (1 - \alpha)\{(a'_1 + b'_1) + (c'_1 + d'_1)\}}{4}$ (10)

This gives $f^0(\tilde{A}^I) = \frac{a'_1 + b'_1 + c'_1 + d'_1}{4}$ (10a)

$f^{0.5}(\tilde{A}^I) = \frac{\{(c_1 + d_1) + (a_1 + b_1)\} + \{(a'_1 + b'_1) + (c'_1 + d'_1)\}}{8}$ (10b)

$f^1(\tilde{A}^I) = \frac{a_1 + b_1 + c_1 + d_1}{4}$ (10c)

Theorem 3.1 The generalized accuracy function f^α is a linear function for all $\alpha = 0, 0.5$ and 1

Proof Let $\tilde{\varphi}^I = (\eta_1, \eta_2, \eta_3, \eta_4; \eta'_1, \eta'_2, \eta'_3, \eta'_4)$ and $\tilde{\psi}^I = (\zeta_1, \zeta_2, \zeta_3, \zeta_4; \zeta'_1, \zeta'_2, \zeta'_3, \zeta'_4)$ be two TrIFNs. Then \square

$$\begin{aligned} f^\alpha(\tilde{\varphi}^I \oplus \tilde{\psi}^I) &= f^\alpha[(\eta_1, \eta_2, \eta_3, \eta_4; \eta'_1, \eta'_2, \eta'_3, \eta'_4) + (\zeta_1, \zeta_2, \zeta_3, \zeta_4; \zeta'_1, \zeta'_2, \zeta'_3, \zeta'_4)] \\ &= f^\alpha(\eta_1 + \zeta_1, \eta_2 + \zeta_2, \eta_3 + \zeta_3, \eta_4 + \zeta_4; \eta'_1 + \zeta'_1, \eta'_2 + \zeta'_2, \eta'_3 + \zeta'_3, \eta'_4 + \zeta'_4) \\ &= \frac{\alpha\{(\eta_1 + \zeta_1) + (\eta_2 + \zeta_2) + (\eta_3 + \zeta_3) + (\eta_4 + \zeta_4)\} + (1 - \alpha)\{(\eta'_1 + \zeta'_1) + (\eta'_2 + \zeta'_2) + (\eta'_3 + \zeta'_3) + (\eta'_4 + \zeta'_4)\}}{4} \\ &= \frac{\alpha(\eta_1 + \eta_2 + \eta_3 + \eta_4) + (1 - \alpha)(\eta'_1 + \eta'_2 + \eta'_3 + \eta'_4)}{4} + \frac{\alpha(\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4) + (1 - \alpha)(\zeta'_1 + \zeta'_2 + \zeta'_3 + \zeta'_4)}{4} \\ &= f^\alpha(\tilde{\varphi}^I) + f^\alpha(\tilde{\psi}^I). \text{ Hence proved.} \end{aligned}$$

Definition 3.14 The generalized accuracy function of \tilde{A}^I is denoted by $f^\alpha(\tilde{A}^I)$ and is defined by

$$f^\alpha(\tilde{A}^I) = \frac{S^\alpha(\mu_{\tilde{A}^I}) + S^\alpha(\vartheta_{\tilde{A}^I})}{2} = \frac{\alpha\{I_R(\mu_{\tilde{A}^I}) + I_L(\mu_{\tilde{A}^I})\} + (1 - \alpha)\{I_L(\vartheta_{\tilde{A}^I}) + I_R(\vartheta_{\tilde{A}^I})\}}{2}$$

In case of TrIFN \tilde{A}^I we have, $\frac{x-a_1}{b_1-a_1} = \mu_{\tilde{A}^I}^L(x) \Rightarrow g_{\tilde{A}^I}^L(y) = [a_1 + (b_1 - a_1)y]$.

So, $\int_0^1 g_{\tilde{A}^I}^L(y)dy = \int_0^1 [a_1 + (b_1 - a_1)y]dy = \frac{(a_1+b_1)}{2} = I_L(\mu_{\tilde{A}^I})$.

Similarly, $\frac{(c_1+d_1)}{2} = I_R(\mu_{\tilde{A}^I})$, $\frac{(a'_1+b'_1)}{2} = I_L(\vartheta_{\tilde{A}^I})$ and $\frac{(c'_1+d'_1)}{2} = I_R(\vartheta_{\tilde{A}^I})$.

Remark 3.1 α represents the degree of optimism of a decision-maker (DM) (Liou and Wang (1992)). A large α indicates a higher degree of optimism because when α increases we see that the weight of the acceptance level of the assignment/transportation cost is also increases. So, we can observe the following.

1. $\alpha = 0$ gives non-membership value which represents a pessimistic decision maker's viewpoint as DM is curious about the non-acceptance level of the assignment/transportation cost.
2. $\alpha = 1$ gives membership value of the assignment/transportation cost which represents an optimistic decision maker's viewpoint as DM is curious about the acceptance level of the assignment/transportation cost.
3. For, $\alpha = 0.5$, the score functional value represents a moderate decision maker's viewpoint because the DM gives equal importance to the acceptance and non-acceptance level of the assignment/transportation cost.

Let $\tilde{\varphi}^I = (\zeta_1, \eta_1, \theta_1, \delta_1; \zeta'_1, \eta'_1, \theta'_1, \delta'_1)$ be TrIFN. Now, we define zero, non-negative, positive, negative TrIFNs and ordering of TrIFNs based on accuracy function is as follows (Singh and Yadav 2016):

Definition 3.15 If $f^\alpha(\tilde{\varphi}^I) = 0$ then the TrIFN $\tilde{\varphi}^I = (\zeta_1, \eta_1, \theta_1, \delta_1; \zeta'_1, \eta'_1, \theta'_1, \delta'_1)$ is said to be zero TrIFN.

Definition 3.16 If $f^\alpha(\tilde{\varphi}^I) \geq 0$ then the TrIFN $\tilde{\varphi}^I = (\zeta_1, \eta_1, \theta_1, \delta_1; \zeta'_1, \eta'_1, \theta'_1, \delta'_1)$ is said to be non-negative TrIFN.

Definition 3.17 If $f^\alpha(\tilde{\varphi}^I) > 0$ then the TrIFN $\tilde{\varphi}^I = (\zeta_1, \eta_1, \theta_1, \delta_1; \zeta'_1, \eta'_1, \theta'_1, \delta'_1)$ is said to be positive TrIFN.

Definition 3.18 If $f^\alpha(\tilde{\varphi}^I) < 0$ then the TrIFN $\tilde{\varphi}^I = (\zeta_1, \eta_1, \theta_1, \delta_1; \zeta'_1, \eta'_1, \theta'_1, \delta'_1)$ is said to be negative TrIFN.

Definition 3.19 Let $\tilde{\sigma}^I = (\sigma_1, \sigma_2, \sigma_3, \sigma_4; \sigma'_1, \sigma'_2, \sigma'_3, \sigma'_4)$ and $\tilde{\rho}^I = (\rho_1, \rho_2, \rho_3, \rho_4; \rho'_1, \rho'_2, \rho'_3, \rho'_4)$ be two different TrIFNs. Then

- (i) $\tilde{\sigma}^I \geq \tilde{\rho}^I$ if $f^\alpha(\tilde{\sigma}^I) \geq f^\alpha(\tilde{\rho}^I)$
- (ii) $\tilde{\sigma}^I \leq \tilde{\rho}^I$ if $f^\alpha(\tilde{\sigma}^I) \leq f^\alpha(\tilde{\rho}^I)$
- (iii) $\tilde{\sigma}^I = \tilde{\rho}^I$ if $f^\alpha(\tilde{\sigma}^I) = f^\alpha(\tilde{\rho}^I)$
- (iv) $\text{Min}(\tilde{\sigma}^I, \tilde{\rho}^I) = \tilde{\sigma}^I$ if $\tilde{\sigma}^I \leq \tilde{\rho}^I$ or $\tilde{\rho}^I \geq \tilde{\sigma}^I$
- (v) $\text{Max}(\tilde{\sigma}^I, \tilde{\rho}^I) = \tilde{\sigma}^I$ if $\tilde{\sigma}^I \geq \tilde{\rho}^I$ or $\tilde{\rho}^I \leq \tilde{\sigma}^I$

4 Basic arithmetic operations on TIFNs and TrIFNs

Mahapatra and Roy (2009), Shaw and Roy (2012), Mahapatra and Roy (2013), Kumar and Hussain (2016a) have proposed some arithmetic operations on triangular intuitionistic fuzzy number. The advantages of the proposed operators are as follows:

- (i) Addition, subtraction, and multiplication of two fuzzy numbers is always a fuzzy number.
- (ii) Use of the proposed operators is very simple and easy.
- (iii) Understanding the proposed operators is very easy.
- (iv) The proposed operators are reliable and efficient.
- (v) In case of triangular intuitionistic fuzzy number, if we add or subtract two triangular intuitionistic fuzzy number then the resulting triangular intuitionistic fuzzy number is always depending on

membership into membership and non-membership into non-membership. Due to this, we need not mix both the membership and non-membership values. We can able to deal independently both the membership and non-membership values.

The arithmetic operations of TIFNs and TrIFNs may be defined as follows:

Let $\tilde{\varphi}^I = (\eta_1, \eta_2, \eta_3)(\eta'_1, \eta'_2, \eta'_3)$ and $\tilde{\psi}^I = (\zeta_1, \zeta_2, \zeta_3)(\zeta'_1, \zeta'_2, \zeta'_3)$ be any two arbitrary TIFNs. The arithmetic operations of TIFNs are defined as follows (Kumar and Hussain 2016a):

Addition/sum of $\tilde{\varphi}^I$ and $\tilde{\psi}^I$: $\tilde{\varphi}^I \oplus \tilde{\psi}^I = (\eta_1 + \zeta_1, \eta_2 + \zeta_2, \eta_3 + \zeta_3)(\eta'_1 + \zeta'_1, \eta'_2 + \zeta'_2, \eta'_3 + \zeta'_3)$

Subtraction/difference of $\tilde{\varphi}^I$ and $\tilde{\psi}^I$: $\tilde{\varphi}^I \ominus \tilde{\psi}^I = (\eta_1 - \zeta_3, \eta_2 - \zeta_2, \eta_3 - \zeta_1)(\eta'_1 - \zeta'_3, \eta'_2 - \zeta'_2, \eta'_3 - \zeta'_1)$

Multiplication/Product of $\tilde{\varphi}^I$ and $\tilde{\psi}^I$:
 $\tilde{\varphi}^I \otimes \tilde{\psi}^I = (\eta_1 \Re(\tilde{\psi}^I), \eta_2 \Re(\tilde{\psi}^I), \eta_3 \Re(\tilde{\psi}^I))(\eta'_1 \Re(\tilde{\psi}^I), \eta'_2 \Re(\tilde{\psi}^I), \eta'_3 \Re(\tilde{\psi}^I))$

if $\Re(\tilde{\varphi}^I), \Re(\tilde{\psi}^I) \geq 0$

Scalar product or scalar multiplication:

- (i) $k\tilde{\varphi}^I = (k\eta_1, k\eta_2, k\eta_3)(k\eta'_1, k\eta'_2, k\eta'_3)$, for $k \geq 0$
- (ii) $k\tilde{\varphi}^I = (k\eta_3, k\eta_2, k\eta_1)(k\eta'_3, k\eta'_2, k\eta'_1)$, for $k < 0$

Let $\tilde{\varphi}^I = (\eta_1, \eta_2, \eta_3, \eta_4; \eta'_1, \eta'_2, \eta'_3, \eta'_4)$ and $\tilde{\psi}^I = (\zeta_1, \zeta_2, \zeta_3, \zeta_4; \zeta'_1, \zeta'_2, \zeta'_3, \zeta'_4)$ be any two arbitrary TrIFNs. The arithmetic operations of TrIFNs are defined as follows (Singh and Yadav 2016):

Addition or sum of $\tilde{\varphi}^I$ and $\tilde{\psi}^I$:

$$\tilde{\varphi}^I \oplus \tilde{\psi}^I = (\eta_1 + \zeta_1, \eta_2 + \zeta_2, \eta_3 + \zeta_3, \eta_4 + \zeta_4; \eta'_1 + \zeta'_1, \eta'_2 + \zeta'_2, \eta'_3 + \zeta'_3, \eta'_4 + \zeta'_4)$$

Subtraction or difference of $\tilde{\varphi}^I$ and $\tilde{\psi}^I$:

$$\tilde{\varphi}^I \ominus \tilde{\psi}^I = (\eta_1 - \zeta_4, \eta_2 - \zeta_2, \eta_3 - \zeta_3, \eta_4 - \zeta_1; \eta'_1 - \zeta'_4, \eta'_2 - \zeta'_2, \eta'_3 - \zeta'_3, \eta'_4 - \zeta'_1)$$

Multiplication or product of $\tilde{\varphi}^I$ and $\tilde{\psi}^I$: $\tilde{\varphi}^I \otimes \tilde{\psi}^I = (\eta_1 f(\tilde{\psi}^I), \eta_2 f(\tilde{\psi}^I), \eta_3 f(\tilde{\psi}^I), \eta_4 f(\tilde{\psi}^I); \eta'_1 f(\tilde{\psi}^I), \eta'_2 f(\tilde{\psi}^I), \eta'_3 f(\tilde{\psi}^I), \eta'_4 f(\tilde{\psi}^I))$

if $f(\tilde{\varphi}^I), f(\tilde{\psi}^I) \geq 0$

Scalar product or scalar multiplication:

- (i) $c\tilde{\varphi}^I = (c\eta_1, c\eta_2, c\eta_3, c\eta_4; c\eta'_1, c\eta'_2, c\eta'_3, c\eta'_4)$,
for $c \geq 0$
- (ii) $c\tilde{\varphi}^I = (c\eta_4, c\eta_3, c\eta_2, c\eta_1; c\eta'_4, c\eta'_3, c\eta'_2, c\eta'_1)$,
for $c < 0$

Remark 4.1 All the parameters of the crisp, fuzzy and intuitionistic fuzzy optimization problems such as supply, demand, cost, revenue or profit, conveyance capacity, time and production are in positive. Since, in TPs and APs, negative parameters have no physical meaning. Hence, in the proposed algorithm all the parameters may be assumed as non-negative crisp, fuzzy and intuitionistic fuzzy numbers. On the basis of this idea, we are not necessary to further investigate on multiplication operation involving negative values.

5 Centroid of the TIFN and its ordering principles

Ranking of alternatives in IFE plays a major role in decision-making and optimization problems. Burillo et al. (1994) introduced the definition of intuitionistic fuzzy number (IFN) and studied its important properties. Several authors like Grzegorzewski (2003), Mitchell (2004), Nehi and Maleki (2005), Ban (2008), Nayagam et al. (2008), Wei and Tang (2010), Guha and Chakraborty (2010), Deng Feng Li et al. (2010), Nehi (2010), De et al. (2012), Das and Guha (2013), Rezvani (2013a, b, c), Kumar and Kaur (2013), Zhang and Nan (2013), Jafarian and Rezvani (2013), Shabani and Jamkhaneh (2014), Nishad et al. (2014), De and Das (2014), Zeng et al. (2014), Li and Yang (2015), Wan et al. (2016), Das and Guha (2016), Bharati and Singh (2018b), Chutia and Saikia (2018) have studied intuitionistic fuzzy numbers (IFNs) and analyzed its properties. Varghese and Kuriakose (2012), have proposed centroid (ranking) of TIFN with the single score using its non-membership and membership function.

Definition 5.1 (Varghese and Kuriakose 2012) Let $\tilde{\varphi}^I = (\eta_1, \eta_2, \eta_3)(\eta'_1, \eta'_2, \eta'_3)$ and $\tilde{\psi}^I = (\zeta_1, \zeta_2, \zeta_3)(\zeta'_1, \zeta'_2, \zeta'_3)$ be two TIFNs. Then the set of TIFNs is defined as follows:

- (i) $\Re(\tilde{\varphi}^I) > \Re(\tilde{\psi}^I) \Leftrightarrow \tilde{\varphi}^I \succ \tilde{\psi}^I$
- (ii) $\Re(\tilde{\varphi}^I) < \Re(\tilde{\psi}^I) \Leftrightarrow \tilde{\varphi}^I \prec \tilde{\psi}^I$
- (iii) $\Re(\tilde{\varphi}^I) = \Re(\tilde{\psi}^I) \Leftrightarrow \tilde{\varphi}^I \approx \tilde{\psi}^I$

where

$$\Re(\tilde{\varphi}^I) = \frac{1}{3} \left[\frac{(\eta'_3 - \eta'_1)(\eta_2 - 2\eta'_3 - 2\eta'_1) + (\eta_3 - \eta_1)(\eta_1 + \eta_2 + \eta_3) + 3(\eta_3^2 - \eta_1^2)}{\eta'_3 - \eta'_1 + \eta_3 - \eta_1} \right] \tag{11}$$

$$\Re(\tilde{\psi}^I) = \frac{1}{3} \left[\frac{(\zeta'_3 - \zeta'_1)(\zeta_2 - 2\zeta'_3 - 2\zeta'_1) + (\zeta_3 - \zeta_1)(\zeta_1 + \zeta_2 + \zeta_3) + 3(\zeta_3^2 - \zeta_1^2)}{\zeta'_3 - \zeta'_1 + \zeta_3 - \zeta_1} \right]$$

Whenever the above formulae doesn't provide finite value then we can make use of the following formulae.

The score function for the membership function $\mu_\varphi(x)$ is denoted by $S(\mu_\varphi(x))$ and is defined by $S(\mu_\varphi(x)) = \frac{\eta_1 + 2\eta_2 + \eta_3}{4}$.

The score function for the non-membership function $\vartheta_\varphi(x)$ is denoted by $S(\vartheta_\varphi(x))$ and is defined by $S(\vartheta_\varphi(x)) = \frac{\eta'_1 + 2\eta'_2 + \eta'_3}{4}$.

The accuracy function of $\tilde{\varphi}^I$ is denoted by $f(\tilde{\varphi}^I)$ and is defined by

$$f(\tilde{\varphi}^I) = \frac{S(\mu_\varphi(x)) + S(\vartheta_\varphi(x))}{2}$$

$$f(\tilde{\varphi}^I) = \frac{(\eta_1 + 2\eta_2 + \eta_3) + (\eta'_1 + 2\eta'_2 + \eta'_3)}{8}$$

Similarly we can write, $f^\alpha(\tilde{\varphi}^I)$

$$= \frac{\alpha(\eta_1 + 2\eta_2 + \eta_3) + (1 - \alpha)(\eta'_1 + 2\eta'_2 + \eta'_3)}{4} \tag{12}$$

This gives $f^0(\tilde{\varphi}^I) = \frac{\eta'_1 + 2\eta_2 + \eta'_3}{4}$. (12a)

$f^{0.5}(\tilde{\varphi}^I) = \frac{(\eta_1 + 2\eta_2 + \eta_3) + (\eta'_1 + 2\eta'_2 + \eta'_3)}{8}$ (12b)

$f^1(\tilde{\varphi}^I) = \frac{\eta_1 + 2\eta_2 + \eta_3}{4}$ (12c)

Theorem 5.1 The generalized accuracy function f^α is a linear function for all $\alpha = 0, 0.5$ and 1

Proof Let $\tilde{\varphi}^I = (\eta_1, \eta_2, \eta_3)(\eta'_1, \eta'_2, \eta'_3)$ and $\tilde{\psi}^I = (\zeta_1, \zeta_2, \zeta_3)(\zeta'_1, \zeta'_2, \zeta'_3)$ be two TIFNs. Then for all $k \geq 0$, we have

$$f^\alpha(k\tilde{\varphi}^I \oplus \tilde{\psi}^I) = f^\alpha \left[k \left((\eta_1, \eta_2, \eta_3)(\eta'_1, \eta'_2, \eta'_3) \right) \oplus (\zeta_1, \zeta_2, \zeta_3)(\zeta'_1, \zeta'_2, \zeta'_3) \right]$$

We Know That (Wkt) $k\tilde{\varphi}^I = k[(\eta_1, \eta_2, \eta_3)(\eta'_1, \eta'_2, \eta'_3)]$
 $= [(k\eta_1, k\eta_2, k\eta_3)(k\eta'_1, k\eta'_2, k\eta'_3)],$
 for all $k \geq 0$

$$f^\alpha(k\tilde{\varphi}^I \oplus \tilde{\psi}^I) = f^\alpha[(k\eta_1, k\eta_2, k\eta_3)(k\eta'_1, k\eta'_2, k\eta'_3) \oplus (\zeta_1, \zeta_2, \zeta_3)(\zeta'_1, \zeta'_2, \zeta'_3)]$$

$$k\tilde{\varphi}^I \oplus \tilde{\psi}^I = [(k\eta_1 + \zeta_1, k\eta_2 + \zeta_2, k\eta_3 + \zeta_3)(k\eta'_1 + \zeta'_1, k\eta'_2 + \zeta'_2, k\eta'_3 + \zeta'_3)]$$

$$\therefore \tilde{\varphi}^I \oplus \tilde{\psi}^I = (\eta_1 + \zeta_1, \eta_2 + \zeta_2, \eta_3 + \zeta_3)$$

$$(\eta'_1 + \zeta'_1, \eta'_2 + \zeta'_2, \eta'_3 + \zeta'_3)$$

$$f^\alpha(k\tilde{\varphi}^I + \tilde{\psi}^I) = \frac{\alpha[k\eta_1 + \zeta_1 + 2(k\eta_2 + \zeta_2) + k\eta_3 + \zeta_3] + (1 - \alpha)[k\eta'_1 + \zeta'_1 + 2(k\eta_2 + \zeta_2) + k\eta'_3 + \zeta'_3]}{4}$$

$$\therefore f^\alpha(\tilde{\varphi}^I) = \left(\frac{\alpha(\eta_1 + 2\eta_2 + \eta_3) + (1 - \alpha)(\eta'_1 + 2\eta_2 + \eta'_3)}{4} \right)$$

$$= \frac{\alpha[k\eta_1 + \zeta_1 + 2k\eta_2 + 2\zeta_2 + k\eta_3 + \zeta_3] + (1 - \alpha)[k\eta'_1 + \zeta'_1 + 2k\eta_2 + 2\zeta_2 + k\eta'_3 + \zeta'_3]}{4}$$

$$= \left(\frac{[\alpha(k\eta_1 + 2k\eta_2 + k\eta_3) + (1 - \alpha)(k\eta'_1 + 2k\eta_2 + k\eta'_3)] + \alpha(\zeta_1 + 2\zeta_2 + \zeta_3) + (1 - \alpha)(\zeta'_1 + 2\zeta_2 + \zeta'_3)}{4} \right)$$

$$= k \left(\frac{\alpha(\eta_1 + 2\eta_2 + \eta_3) + (1 - \alpha)(\eta'_1 + 2\eta_2 + \eta'_3)}{4} \right) + \left(\frac{\alpha(\zeta_1 + 2\zeta_2 + \zeta_3) + (1 - \alpha)(\zeta'_1 + 2\zeta_2 + \zeta'_3)}{4} \right)$$

$$f^\alpha(k\tilde{\varphi}^I \oplus \tilde{\psi}^I) = kf^\alpha(\tilde{\varphi}^I) + f^\alpha(\tilde{\psi}^I)$$

negative (i.e., either positive or equal to zero), positive (i.e., greater than zero), negative (i.e., less than zero) TIFNs and ordering (i.e., \succcurlyeq and \preccurlyeq between any two TIFNs) of TIFNs based on accuracy function is as follows (Kumar and Hussain 2016a):

Definition 5.2 If $f^\alpha(\tilde{\varphi}^I) = 0$ then the TIFN $\tilde{\varphi}^I = (\eta_1, \eta_2, \eta_3)(\eta'_1, \eta'_2, \eta'_3)$ is said to be zero TIFN.

Definition 5.3 If $f^\alpha(\tilde{\varphi}^I) \geq 0$ then the TIFN $\tilde{\varphi}^I = (\eta_1, \eta_2, \eta_3)(\eta'_1, \eta'_2, \eta'_3)$ is said to be non-negative TIFN.

Definition 5.4 If $f^\alpha(\tilde{\varphi}^I) > 0$ then the TIFN $\tilde{\varphi}^I = (\eta_1, \eta_2, \eta_3)(\eta'_1, \eta'_2, \eta'_3)$ is said to be positive TIFN.

Definition 5.5 If $f^\alpha(\tilde{\varphi}^I) < 0$ then the TIFN $\tilde{\varphi}^I = (\eta_1, \eta_2, \eta_3)(\eta'_1, \eta'_2, \eta'_3)$ is said to be negative TIFN.

□

Similarly it can be proved for $k < 0$. This implies that the accuracy function f^α is a linear function for all $\alpha = 1, 0.5$ and 0 .

From the accuracy function, we have

- (i) $f^\alpha(\tilde{\varphi}^I) > f^\alpha(\tilde{\psi}^I) \Leftrightarrow \tilde{\varphi}^I \succ \tilde{\psi}^I$
- (ii) $f^\alpha(\tilde{\varphi}^I) < f^\alpha(\tilde{\psi}^I) \Leftrightarrow \tilde{\varphi}^I \prec \tilde{\psi}^I$
- (iii) $f^\alpha(\tilde{\varphi}^I) = f^\alpha(\tilde{\psi}^I) \Leftrightarrow \tilde{\varphi}^I \approx \tilde{\psi}^I$

Let us consider $\tilde{\varphi}^I = (\eta_1, \eta_2, \eta_3)(\eta'_1, \eta'_2, \eta'_3)$ be an arbitrary TIFN. Next, we define zero (0, zero means it is neither positive nor negative, that is, equal to zero), non-

Definition 5.6 The ordering \succcurlyeq and \preccurlyeq between any two TIFNs $\tilde{\varphi}^I$ and $\tilde{\psi}^I$ are defined as follows:

- (i) $\tilde{\varphi}^I \succcurlyeq \tilde{\psi}^I$ iff $\tilde{\varphi}^I \succ \tilde{\psi}^I$ or $\tilde{\varphi}^I \approx \tilde{\psi}^I$ and
- (ii) $\tilde{\varphi}^I \preccurlyeq \tilde{\psi}^I$ iff $\tilde{\varphi}^I \prec \tilde{\psi}^I$ or $\tilde{\varphi}^I \approx \tilde{\psi}^I$

Definition 5.7 Let $\{\tilde{\varphi}_i^I, i = 1, 2, \dots, n\}$ be a set of TIFNs. If $f^\alpha(\tilde{\varphi}_k^I) \leq f^\alpha(\tilde{\varphi}_i^I) \forall i$, then the TIFN $\tilde{\varphi}_k^I$ is the minimum of $\{\tilde{\varphi}_i^I, i = 1, 2, \dots, n\}$.

Definition 5.8 Let $\{\tilde{\varphi}_i^I, i = 1, 2, \dots, n\}$ be a set of TIFNs. If $f^\alpha(\tilde{\varphi}_i^I) \geq f^\alpha(\tilde{\varphi}_k^I) \forall i$, then the TIFN $\tilde{\varphi}_i^I$ is the maximum of $\{\tilde{\varphi}_i^I, i = 1, 2, \dots, n\}$.

5.1 Example Let $\tilde{\varphi}^I = (5, 8, 10)(1, 8, 11)$ and $\tilde{\psi}^I = (3, 8, 16)(0, 8, 19)$, then

$$\text{(Model 4) Minimize } \tilde{Z}^I = \sum_{i=1}^m \sum_{j=1}^n \tilde{c}_{ij}^I \otimes x_{ij} \tag{14}$$

$$\Re(\tilde{\varphi}^I) = \frac{1}{3} \left[\frac{(11-1)(8-2(11)-2(1)) + (10-5)(5+8+10) + 3(11^2-1^2)}{11-1+10-5} \right] = \frac{1}{3} \left[\frac{315}{15} \right] = 7 \text{ and}$$

$$\Re(\tilde{\psi}^I) = \frac{1}{3} \left[\frac{(19-0)(8-2(19)-2(0)) + (16-3)(3+8+16) + 3(19^2-0^2)}{19-0+16-3} \right] = \frac{1}{3} \left[\frac{864}{32} \right] = 9$$

$$\Rightarrow \Re(\tilde{\varphi}^I) < \Re(\tilde{\psi}^I)$$

Similarly, from the accuracy function we can prove

$$f(\tilde{\varphi}^I) < f(\tilde{\psi}^I)$$

Thus in each case we have, $\tilde{\varphi}^I \prec \tilde{\psi}^I$.

6 Mathematical formulations of various kinds of optimization problems

This section consists of two subsections. Both sections are present the mathematical model of the problem.

6.1 Mathematical formulation of type-2 fuzzy, type-2 intuitionistic fuzzy, solid and type-2 intuitionistic fuzzy solid transportation problems

Let us consider the TP with m origins (rows) and n destinations (columns). Let $\tilde{c}_{ij} = [c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4]$ and $\tilde{c}_{ij}^I = (c_{ij}^1, c_{ij}^2, c_{ij}^3)(c_{ij}^1, c_{ij}^2, c_{ij}^3)$ be the cost of transporting one unit of the product (manufactured goods) from i th origin to j th destination, $b_j(j = 1, 2, 3, \dots, n)$ be the quantity of commodity needed at destination j , $a_i(i = 1, 2, 3, \dots, m)$ denotes the quantity of commodity available at origin i , x_{ij} is the quantity of manufactured goods transported from i th origin to j th destination, so as to minimize the total fuzzy and intuitionistic fuzzy transportation costs.

Now, the mathematical model of the above TP is given by

$$\text{(Model 3) Minimize } \tilde{Z} = \sum_{i=1}^m \sum_{j=1}^n \tilde{c}_{ij} \otimes x_{ij} \tag{13}$$

subject to the constraints (2) to (4).

subject to the constraints (2) to (4).

Consider transportation with m origins, n destinations, and l conveyances. Let c_{ijk} be the unit cost of transporting one unit of the product from i th origin to j th destination by means of the k th conveyance. Let $a_i(i = 1, 2, \dots, m)$ be the quantity of commodity available at origin i . Let $b_j(j = 1, 2, \dots, n)$ be the amount of the quantity of commodity needed at destination j . Let $e_k(k = 1, 2, \dots, l)$ be the amount of the material transported by k th conveyance. Let $x_{ijk}(i = 1, 2, \dots, m; j = 1, 2, \dots, n; \text{ and } k = 1, 2, \dots, l)$ be the number of units of quantity transported from i th origin to j th destination by means of the k th conveyance. Our aim is to determine transportation schedule to minimize the transportation cost satisfying supply, demand and conveyance constraints.

$$\text{(Model 5) Minimize } Z = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l c_{ijk} \times x_{ijk} \tag{15}$$

subject to

$$\sum_{j=1}^n \sum_{k=1}^l x_{ijk} = a_i, \text{ for } i = 1, 2, \dots, m \tag{16}$$

$$\sum_{i=1}^m \sum_{k=1}^l x_{ijk} = b_j, \text{ for } j = 1, 2, \dots, n \tag{17}$$

$$\sum_{i=1}^m \sum_{j=1}^n x_{ijk} = e_k, \text{ for } k = 1, 2, \dots, l \tag{18}$$

$$x_{ijk} \geq 0, \text{ for } i = 1, 2, \dots, m; j = 1, 2, \dots, n \text{ and } k = 1, 2, \dots, l \tag{19}$$

Consider transportation with m origins, n destinations, and l conveyances. Let us take $\tilde{c}_{ijk} = [c_{ijk}^1, c_{ijk}^2, c_{ijk}^3, c_{ijk}^4]$ and $\tilde{c}_{ijk}^I = (c_{ijk}^1, c_{ijk}^2, c_{ijk}^3)(c_{ijk}^1, c_{ijk}^2, c_{ijk}^3)$ be the unit cost of transporting one unit of the manufactured goods from i th origin to j th destination by means of the k th conveyance. Let us take $a_i(i = 1, 2, \dots, m)$ be the quantity of commodity available at origin i . Let us take $b_j(j = 1, 2, \dots, n)$

be the amount of the quantity of commodity needed at destination j . Let $e_k(k = 1, 2, \dots, l)$ be the amount of the material transported by k th conveyance. Let us take $x_{ijk}(i = 1, 2, \dots, m; j = 1, 2, \dots, n; \text{ and } k = 1, 2, \dots, l)$ be the number of units of quantity transported from i th origin to j th destination by means of the k th conveyance. Our aim is to determine the transportation schedule that minimizes the total fuzzy and intuitionistic fuzzy transportation costs while satisfying supply, demand and conveyance limits.

Now, the mathematical model of the above TP is given by

$$\text{(Model 6) Minimize } \tilde{Z} = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l \tilde{c}_{ijk} \otimes x_{ijk} \quad (20)$$

subject to the constraints (16) to (19).

$$\text{(Model 7) Minimize } \tilde{Z}^l = \sum_{i=1}^m \sum_{j=1}^n \sum_{k=1}^l \tilde{c}_{ijk}^l \otimes x_{ijk} \quad (21)$$

subject to the constraints (16) to (19).

6.2 Mathematical formulation of fuzzy, intuitionistic fuzzy, solid and intuitionistic fuzzy solid assignment problems

Let us consider the situation of assigning n machines to n jobs (one machine per job) and each machine is capable of doing any job at different costs. Let $\tilde{c}_{ij} = [c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4]$ and $\tilde{c}_{ij}^l = (c_{ij}^1, c_{ij}^2, c_{ij}^3)(c_{ij}^l, c_{ij}^2, c_{ij}^3)$ be the cost of assigning the j th job to the i th machine. Let $x_{ij}(i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, n)$ be the decision variable that denotes the assignment of the machine i to the job j . Our aim is to determine the assignment of machines to jobs so that the total fuzzy cost and the total intuitionistic fuzzy cost of completing all the jobs are minimum. This situation is known as balanced fuzzy assignment problem and balanced intuitionistic fuzzy assignment problem.

Now, the mathematical model of the above AP is given by

$$\text{(Model 8) Minimize } \tilde{Z} = \sum_{i=1}^n \sum_{j=1}^n \tilde{c}_{ij} x_{ij} \quad (22)$$

Subject to the constraints (6) to (8).

$$\text{(Model 9) Minimize } \tilde{Z}^l = \sum_{i=1}^n \sum_{j=1}^n \tilde{c}_{ij}^l x_{ij} \quad (23)$$

Subject to the constraints (6) to (8).

Consider the situation of n jobs in n factory and the factory has n machines to process the jobs. Each job in a factory has to be associated with only one machine. Let us take $c_{ijk}(i = 1, 2, \dots, n; j = 1, 2, \dots, n \text{ and } k = 1, 2, \dots, n)$

be the cost of job j ($j = 1, 2, \dots, n$) is processed by the machine i ($i = 1, 2, \dots, n$) in the factory k ($k = 1, 2, \dots, n$). Let x_{ijk} be the decision variable that denotes the assignment of j th job to i th machine in the k th factory. Our objective is to determine the assignment of jobs to machines at minimum assignment cost. This situation is known as balanced SAP (BSAP).

$$\text{(Model 10) Minimize } Z = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n c_{ijk} \otimes x_{ijk} \quad (24)$$

subject to the constraints

$$\sum_{j=1}^n \sum_{k=1}^n x_{ijk} = 1, \quad \text{for } i = 1, 2, \dots, n \quad (25)$$

$$\sum_{i=1}^n \sum_{k=1}^n x_{ijk} = 1, \quad \text{for } j = 1, 2, \dots, n \quad (26)$$

$$\sum_{i=1}^n \sum_{j=1}^n x_{ijk} = 1, \quad \text{for } k = 1, 2, \dots, n \quad (27)$$

$$x_{ijk} = 0 \text{ or } 1, \text{ for all } i, j \text{ and } k \quad (28)$$

where c_{ijk} is the cost of assigning the j th job to the i th machine in the k th factory. $x_{ijk} = 1$ if the job j is assigned to the machine i in the factory k , and $x_{ijk} = 0$, otherwise.

Consider the situation of n jobs in n factory and the factory has n machines to process the jobs. Each job in a factory has to be associated with only one machine. Let us take \tilde{c}_{ijk} and \tilde{c}_{ijk}^l be the fuzzy and intuitionistic fuzzy costs of the job j ($j = 1, 2, \dots, n$) is processed by the machine i ($i = 1, 2, \dots, n$) in the factory k ($k = 1, 2, \dots, n$) respectively. Let $x_{ijk}(i = 1, 2, \dots, n; j = 1, 2, \dots, n; \text{ and } k = 1, 2, \dots, n)$ be the decision variable that denotes the assignment of j th job to i th machine in the k th factory. Our objective is to determine the assignment of jobs to machines at minimum intuitionistic fuzzy assignment cost. This situation is known as BIFSAP (balanced intuitionistic fuzzy solid assignment problem).

$$\text{(Model 11) Minimize } \tilde{Z} = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \tilde{c}_{ijk} \otimes x_{ijk} \quad (29)$$

subject to the constraints (25) to (28).

Now, the mathematical model of the type-2 IFSTP is given by

$$\text{(Model 12) Minimize } \tilde{Z}^l = \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n \tilde{c}_{ijk}^l \otimes x_{ijk} \quad (30)$$

subject to Eqs. (25)–(28).

7 Algorithm

The proposed algorithm for determining optimal solution consists of the following steps:

Step 1 Construct the optimization problems under crisp, fuzzy and IFE. i.e., there are three different optimization problems which are

- (i) crisp optimization problem
 - (ii) fuzzy optimization problem
 - (iii) intuitionistic fuzzy optimization problem
- Hence formulate the optimization problems under these three different situations.

Step 2

- (i) If the formulated optimization problems, which is only having the crisp parameters, then go to step 5 (i).
- (ii) If the formulated optimization problems having both the crisp and fuzzy parameters, then go to step 3 (i).
- (iii) If the formulated optimization problems having both the crisp and intuitionistic fuzzy parameters, then go to step 3 (ii).

Step 3

- (i) By using the ranking method proposed by Liou and Wang (1992), calculate ranking index for each fuzzy number of the given fuzzy optimization problem. The formula for ranking of TrFNs is shown in Eq. (9).
- (ii) By using the ranking method proposed by Varghese and Kuriakose (2012), calculate ranking index for each intuitionistic fuzzy number of the given intuitionistic fuzzy optimization problem. The formula for ranking of TIFNs is shown in Eq. (11). If the ranking method is not suitable. i.e., if it does not yield finite values, then use the accuracy function to calculate the ranking index. The accuracy function for TrIFN and TIFN is given in Eqs. (10b) and (12b).

Step 4 By using steps 3 (i) and 3 (ii) of the proposed algorithm, in a given fuzzy and intuitionistic fuzzy optimization problem, replace every fuzzy and intuitionistic fuzzy number by their respective ranking indices. This step yields the crisp optimization problem corresponding to every fuzzy and intuitionistic fuzzy optimization problems

Step 5 After using step 2 (i) or/and step 4 of the proposed algorithm, now

- (i) Solve the crisp optimization problem by using TORA software. This step yields the optimal solution and optimal objective value for crisp optimization problems as mentioned in steps 2 (i) or 4.
- (ii) From the obtained solution in step 5(i), the minimum/maximum objective value of fuzzy optimization problem can be determined by using the Eqs. (13), (20), (22) and (29).
- (iii) From the obtained solution in step 5(i), the minimum/maximum objective value of intuitionistic fuzzy optimization problem can be determined by using the Eqs. (14), (21), (23) and (30).

Note 7.1 Crisp optimization problem is nothing but it is having a crisp parameters. Similarly, fuzzy optimization problem and intuitionistic fuzzy optimization problem both are having fuzzy and intuitionistic fuzzy parameters respectively.

Now, we are going to prove the new theorem called PSK theorem in optimization problems under fuzzy and IFE. Subsequently, we are going to present some new and important remarks.

Theorem 7.1 (PSK theorem in fuzzy and intuitionistic fuzzy optimization problems) *If some or all of the elements (i.e., fuzzy numbers or IFNs or its mixture of both) in the type-2 fuzzy/type-2 intuitionistic fuzzy optimization problems (optimization costs or profits) are replaced by equivalent fuzzy numbers or IFNs or both (their ranking values, for example), the new type-2 fuzzy/type-2 intuitionistic fuzzy optimization problems has the same set of optimal solutions.*

Proof Type-2 fuzzy/type-2 intuitionistic fuzzy optimization problem is an optimization problem, in which only the optimization costs (or profits) are represented in terms of fuzzy number/IFN or the mixture of both. We know that in a type-2 fuzzy/type-2 intuitionistic fuzzy optimization problems the decision variables are all crisp numbers. In type-2 fuzzy/type-2 intuitionistic fuzzy optimization problems, the decision variables depend on its constraints, as well as, feasible, and non-negative restrictions. But the constraints, as well as feasible and non-negative restrictions, are all having the crisp decision variables. From the above discussion, the optimal solution of type-2 fuzzy/type-2 intuitionistic fuzzy optimization problem is always crisp numbers. The optimal solution is always unchanged if

all the costs or profits of the type-2 fuzzy/type-2 intuitionistic fuzzy optimization problem is unchanged. Further, our assumption is to replace equivalent fuzzy numbers or IFNs or mixture of both in the type-2 fuzzy/type-2 intuitionistic fuzzy optimization problems. So, the costs or profits are unchanged in a new type-2 fuzzy/type-2 intuitionistic fuzzy optimization problem and only the existing type-2 fuzzy/type-2 intuitionistic fuzzy optimization costs or profits are replaced by its equivalent fuzzy numbers or IFNs or mixture of both. This implies that, if some or all of the elements (i.e., fuzzy numbers or IFNs or its mixture of both) in the type-2 fuzzy/type-2 intuitionistic fuzzy optimization problems (optimization costs or profits) are replaced by equivalent fuzzy numbers or IFNs or its mixture of both (their ranking values, for example), the new type-2 fuzzy/type-2 intuitionistic fuzzy optimization problems has the same set of optimal solutions. Hence proved the theorem. \square

Remark 7.1 The type-2 fuzzy/type-2 intuitionistic fuzzy optimization problem and its equivalent crisp optimization problem both are having the same occupied/assigned cells. That is, in both the case, the occupied/assigned cells are unchanged.

Remark 7.2 Every feasible solution/assignment to the type-2 fuzzy/type-2 intuitionistic fuzzy optimization problem is also a feasible solution/assignment to its equivalent crisp optimization problem. Similarly, every optimal solution/assignment to the type-2 fuzzy/type-2 intuitionistic fuzzy optimization problem is also an optimal solution/assignment to its equivalent crisp optimization problem.

Remark 7.3 If every optimization problem is possible to be solved using TORA software, then intuitionistic fuzzy optimization problem is also possible to be solved by TORA software.

Now, the proposed algorithm can be illustrated with the help of following real life numerical examples.

8 Illustrative examples

For each proposed model, the real life numerical examples have been illustrated in this section.

8.1 Example: 1 type-2 FTP with TrFNs

A company has four plants $S_i (i = 1, 2, 3, 4)$ and four warehouses $D_j (j = 1, 2, 3, 4)$. The number of units, available at the plants are 25, 18, 37 and 20 respectively. Demands at D_1, D_2, D_3 and D_4 are 16, 28, 34 and 22 respectively. Units cost of transportation (given in terms of TrFNs) are given in Table 1.

Find the allocation so that total FTC (fuzzy transportation cost) is minimum.

Solution The given problem having the crisp and fuzzy parameters. Here, the supply and demand are crisp numbers but the cost is TrFN. Therefore, the given problem is a Type-2 FTP.

From steps 3 (i) and 4, we get the following CTP (crisp transportation problem) (see Table 2).

From model (1), we can write this crisp TP (see Table 2) as follows:

Table 2 CTP corresponding to example 1

	Warehouses				Supply
	D ₁	D ₂	D ₃	D ₄	
Plants					
S ₁	4	10	15	20	25
S ₂	12	6	16	8	18
S ₃	9	5	14	22	37
S ₄	25	20	7	18	20
Demand	16	28	34	22	

Table 1 Type-2 FTP with TrFNs

	Warehouses				Supply
	D ₁	D ₂	D ₃	D ₄	
Plants					
S ₁	(1, 3, 5, 7)	(7, 9, 11, 13)	(12, 14, 16, 18)	(17, 19, 21, 23)	25
S ₂	(9, 11, 13, 15)	(3, 5, 7, 9)	(13, 15, 17, 19)	(5, 7, 9, 11)	18
S ₃	(6, 8, 10, 12)	(2, 4, 6, 8)	(11, 13, 15, 17)	(19, 21, 23, 25)	37
S ₄	(22, 24, 26, 28)	(17, 19, 21, 23)	(4, 6, 8, 10)	(15, 17, 19, 21)	20
Demand	16	28	34	22	

$$\begin{aligned} \text{Minimize } Z &= 4x_{11} + 10x_{12} + 15x_{13} + 20x_{14} + 12x_{21} \\ &+ 6x_{22} + 16x_{23} + 8x_{24} + 9x_{31} + 5x_{32} \\ &+ 14x_{33} + 22x_{34} + 25x_{41} + 20x_{42} + 7x_{43} \\ &+ 18x_{44} \end{aligned}$$

subject to the constraints

$$x_{11} + x_{12} + x_{13} + x_{14} = 25 \text{ (Row 1 restriction)}$$

$$x_{21} + x_{22} + x_{23} + x_{24} = 18 \text{ (Row 2 restriction)}$$

$$x_{31} + x_{32} + x_{33} + x_{34} = 37 \text{ (Row 3 restriction)}$$

$$x_{41} + x_{42} + x_{43} + x_{44} = 20 \text{ (Row 4 restriction)}$$

$$x_{11} + x_{21} + x_{31} + x_{41} = 16 \text{ (Column 1 restriction)}$$

$$x_{12} + x_{22} + x_{32} + x_{42} = 28 \text{ (Column 2 restriction)}$$

$$x_{13} + x_{23} + x_{33} + x_{43} = 34 \text{ (Column 3 restriction)}$$

$$x_{14} + x_{24} + x_{34} + x_{44} = 22 \text{ (Column 4 restriction)}$$

$$x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34}, x_{41}, x_{42}, x_{43}, x_{44} \geq 0 \text{ (non-negative restriction)}$$

By applying TORA software to this problem, we get the following optimal solution, which is given in screenshot 1.

The output image 1 (or screenshot 1) shows the optimal solution and optimal objective value of the crisp TP corresponding to type-2 FTP (refer to Fig. 3).

From step 5 (i), we get the following optimal solution and optimal objective value for the crisp TP.

$$x_{11} = 16, x_{13} = 5, x_{14} = 4, x_{24} = 18, x_{32} = 28, x_{33} = 9, x_{43} = 20$$

$$\begin{aligned} \text{Min } Z &= (4 \times 16) + (15 \times 5) + (20 \times 4) + (8 \times 18) \\ &+ (5 \times 28) + (14 \times 9) + (7 \times 20) \\ &= \text{Rs. } 769 \end{aligned}$$

(31)

The optimal solution of type-2 FTP and its CTP both are the same. Since the parameters supply and demand are crisp numbers.

From step 5 (ii), we can write the optimum objective value of the given type-2 FTP is as follows:

$$\begin{aligned} \text{Min } \tilde{Z} &= (1, 3, 5, 7) \times 16 + (12, 14, 16, 18) \times 5 \\ &+ (17, 19, 21, 23) \times 4 + (5, 7, 9, 11) \times 18 \\ &+ (2, 4, 6, 8) \times 28 + (11, 13, 15, 17) \times 9 \\ &+ (4, 6, 8, 10) \times 20 \end{aligned}$$

$$\begin{aligned} \text{Min } \tilde{Z} &= (16, 48, 80, 112) + (60, 70, 80, 90) \\ &+ (68, 76, 84, 92) + (90, 126, 162, 198) \\ &+ (56, 112, 168, 224) + (99, 117, 135, 153) \\ &+ (80, 120, 160, 200) \end{aligned}$$

$$\text{Min } \tilde{Z} = (469, 669, 869, 1069) \tag{32}$$

8.2 Example: 2 type-2 IFTP with TrIFNs

A firm has three factories $F_i (i = 1, 2, 3)$ and three warehouses $D_j (j = 1, 2, 3)$. The number of units, available at the factories is 20, 15 and 25 respectively. Demands at D_1, D_2 and D_3 are 27, 19 and 14 respectively. Units cost of transportation (given in terms of TrIFNs) are given in the following table (refer to Table 3).



Fig. 3 Output summary corresponding to the type-2 FTP and its crisp TP

Table 3 Type-2 IFTP with TrIFNs

	Warehouses			Supply
	D ₁	D ₂	D ₃	
Factories				
F ₁	(2, 4, 5, 7; 1, 3, 6, 8)	(7, 9, 10, 12; 6, 8, 11, 13)	(4, 6, 7, 9; 3, 5, 8, 10)	20
F ₂	(15, 17, 18, 20; 14, 16, 19, 21)	(3, 5, 6, 8; 2, 4, 7, 9)	(11, 13, 14, 16; 10, 12, 15, 17)	15
F ₃	(12, 14, 15, 17; 11, 13, 16, 18)	(5, 7, 8, 10; 4, 6, 9, 11)	(6, 8, 9, 11; 5, 7, 10, 12)	25
Demand	27	19	14	

Find the allocation so that total intuitionistic fuzzy transportation cost is minimum.

Solution The given problem having the crisp and intuitionistic fuzzy parameters. Here, the supply and demand are crisp numbers but the cost is TrIFN. Therefore, the given problem is a type-2 IFTP.

From steps 3 (ii) and 4, we get the following CTP (see Table 4).

From model (1), we can write this CTP (see Table 4) as follows:

$$\text{Minimize } Z = 4.5x_{11} + 9.5x_{12} + 6.5x_{13} + 17.5x_{21} + 5.5x_{22} + 13.5x_{23} + 14.5x_{31} + 7.5x_{32} + 8.5x_{33}$$

subject to the constraints

- $x_{11} + x_{12} + x_{13} = 20$ (Factory 1 (F₁) or Row 1 restriction)
- $x_{21} + x_{22} + x_{23} = 15$ (Factory 2 (F₂) or Row 2 restriction)
- $x_{31} + x_{32} + x_{33} = 25$ (Factory 3 (F₃) or Row 3 restriction)
- $x_{11} + x_{21} + x_{31} = 27$ (Warehouse 1 (D₁) or Column 1 restriction)
- $x_{12} + x_{22} + x_{32} = 19$ (Warehouse 2 (D₂) or Column 2 restriction)
- $x_{13} + x_{23} + x_{33} = 14$ (Warehouse 3 (D₃) or Column 3 restriction)

Table 4 CTP corresponding to example 2

	Warehouses			Supply
	D ₁	D ₂	D ₃	
Factories				
F ₁	4.5	9.5	6.5	20
F ₂	17.5	5.5	13.5	15
F ₃	14.5	7.5	8.5	25
Demand	27	19	14	

$$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} \geq 0 \text{ (non-negative restriction)}$$

By applying TORA software to this problem, we get the following optimal solution, which is given in screenshot 2.

The output image 2 (or screenshot 2) shows the optimal solution and optimal objective value of the crisp TP corresponding to type-2 IFTP (refer to Fig. 4).

From step 5 (i), we get the following optimal solution and optimal objective value for the crisp TP.

$$\begin{aligned} x_{11} = 20, x_{22} = 15, x_{31} = 7, x_{32} = 4, x_{33} = 14 \\ \text{Min } Z = 4.5 \times 20 + 5.5 \times 15 + 14.5 \times 7 + 7.5 \times 4 + 8.5 \times 14 \\ = 423 \end{aligned} \tag{33}$$

The optimal solution of type-2 IFTP and its CTP both are the same. Since the parameters supply and demand are crisp numbers.

From step 5 (iii), we can write the optimum objective value of the given type-2 IFTP is as follows:

$$\begin{aligned} \text{Min } \tilde{Z}^I &= (2, 4, 5, 7; 1, 3, 6, 8) \times 20 + (3, 5, 6, 8; 2, 4, 7, 9) \\ &\times 15 + (12, 14, 15, 17; 11, 13, 16, 18) \times 7 \\ &+ (5, 7, 8, 10; 4, 6, 9, 11) \times 4 \\ &+ (6, 8, 9, 11; 5, 7, 10, 12) \times 14 \\ \text{Min } \tilde{Z}^I &= (40, 80, 100, 140; 20, 60, 120, 160) \\ &+ (45, 75, 90, 120; 30, 60, 105, 135) \\ &+ (84, 98, 105, 119; 77, 91, 112, 126) \\ &+ (20, 28, 32, 40; 16, 24, 36, 44) \\ &+ (84, 112, 126, 154; 70, 98, 140, 168) \end{aligned}$$

$$\text{Min } \tilde{Z} = (273, 393, 453, 573; 213, 333, 513, 633) \tag{34}$$

8.3 Example: 3 real-life type-2 IFSTP

Coal is a kind of crucial energy source in the development of economy and society in India. But, how to transport the coal from different mines to the different areas economically is also an important issue in the coal transportation in



Fig. 4 Output summary corresponding to the type-2 IFTP and its crisp TP

India. For example, consider the following real-life example.

There are three different coal mines (e.g., Neyveli, Khammam and Belpahar) to supply coal to three different cities (e.g., Chennai, Nellore and Kakinada) in India. There are three kinds of transportation in India. So, three kinds of conveyances are available to be selected, that is, train, truck, and cargo ship. Now, the task of the decision-maker/contractor is to make the transportation plan for next month. At the beginning of this task, the contractor needs to obtain the basic data, such as origin/source availability, conveyance capacity, destination demand/requirement, and transportation cost of the coal (per ton). In fact, since the transportation plan is made in advance, he/she generally cannot get these data exactly. According to the past experience of the contractor, the transportation costs (\tilde{c}_{ijk} , $i = 1, 2, 3$; $j = 1, 2, 3$ and $k = 1, 2, 3$) per ton (in rupees) are estimated as triangular intuitionistic fuzzy numbers. The contractor is well known about the availabilities (a_i , $i = 1, 2, 3$), demands (b_j , $j = 1, 2, 3$) and conveyance capacities (e_k , $k = 1, 2, 3$) of the coal (per ton), so these parameters are represented by real numbers. Therefore, the crisp supply, crisp demand, crisp conveyance capacity and uncertain transportation cost are listed in the following table (see Table 5). Now, the objective of the decision-maker/contractor is to make the transportation schedule which minimizes the total transportation cost.

Solution The given problem having the crisp and intuitionistic fuzzy parameters. Here, the supply, demand, and

conveyance capacity are crisp numbers but the cost is TIFN. Therefore, the given problem is a type-2 IFSTP.

From steps 3 (ii) and 4, we get the following CSTP (crisp solid transportation problem) (see Table 6).

From model (5), we can write this CSTP (see Table 6) as follows:

$$\text{Minimize } Z = 4 x_{111} + 6 x_{112} + 5 x_{113} + 8 x_{121} + 1 x_{122} + 3 x_{123} + 4 x_{131} + 7 x_{132} + 3 x_{133} + 5x_{211} + 4 x_{212} + 8 x_{213} + 4 x_{221} + 2 x_{222} + 6 x_{223} + 1 x_{231} + 3 x_{232} + 8 x_{233} + 2 x_{311} + 7 x_{312} + 6 x_{313} + 8 x_{321} + 7 x_{322} + 4 x_{323} + 3 x_{331} + 9 x_{332} + 7 x_{333}$$

subject to the constraints

$$\begin{aligned} &x_{111} + x_{112} + x_{113} + x_{121} + x_{122} + x_{123} + x_{131} + x_{132} + x_{133} = 10, \text{ (Factory 1 constraint)} \\ &x_{211} + x_{212} + x_{213} + x_{221} + x_{222} + x_{223} + x_{231} + x_{232} + x_{233} = 13, \text{ (Factory 2 constraint)} \\ &x_{311} + x_{312} + x_{313} + x_{321} + x_{322} + x_{323} + x_{331} + x_{332} + x_{333} = 11, \text{ (Factory 3 constraint)} \\ &x_{111} + x_{112} + x_{113} + x_{211} + x_{212} + x_{213} + x_{311} + x_{312} + x_{313} = 12, \text{ (Store 1 constraint)} \\ &x_{121} + x_{122} + x_{123} + x_{221} + x_{222} + x_{223} + x_{321} + x_{322} + x_{323} = 7, \text{ (Store 2 constraint)} \\ &x_{131} + x_{132} + x_{133} + x_{231} + x_{232} + x_{233} + x_{331} + x_{332} + x_{333} = 15, \text{ (Store 3 constraint)} \\ &x_{111} + x_{121} + x_{131} + x_{211} + x_{221} + x_{231} + x_{311} + x_{321} + x_{331} = 11, \text{ (Conveyance 1 constraint)} \\ &x_{112} + x_{122} + x_{132} + x_{212} + x_{222} + x_{232} + x_{312} + x_{322} + x_{332} = 14, \text{ (Conveyance 2 constraint)} \end{aligned}$$

Table 5 Tabular representation of real-life type-2 IFSTP

	E1			E2			E3			Capacity e_k
Conveyance	E1			E2			E3			e_1
	E2			E3			E3			e_2
	E3			E3			E3			e_3
Retail stores →	W1			W2			W3			Supply a_i
Factories ↓										
F1	\tilde{c}'_{111}	\tilde{c}'_{112}	\tilde{c}'_{113}	\tilde{c}'_{121}	\tilde{c}'_{122}	\tilde{c}'_{123}	\tilde{c}'_{131}	\tilde{c}'_{132}	\tilde{c}'_{133}	a_1
F2	\tilde{c}'_{211}	\tilde{c}'_{212}	\tilde{c}'_{213}	\tilde{c}'_{221}	\tilde{c}'_{222}	\tilde{c}'_{223}	\tilde{c}'_{231}	\tilde{c}'_{232}	\tilde{c}'_{233}	a_2
F3	\tilde{c}'_{311}	\tilde{c}'_{312}	\tilde{c}'_{313}	\tilde{c}'_{321}	\tilde{c}'_{322}	\tilde{c}'_{323}	\tilde{c}'_{331}	\tilde{c}'_{332}	\tilde{c}'_{333}	a_3
Demand b_j	b_1			b_2			b_3			

W1: $\tilde{c}'_{111} = (3, 4, 5) (2, 4, 6)$; $\tilde{c}'_{112} = (4, 6, 8) (3, 6, 9)$; $\tilde{c}'_{113} = (2, 5, 8) (1, 5, 9)$; $\tilde{c}'_{211} = (2, 5, 8) (1, 5, 9)$; $\tilde{c}'_{212} = (3, 4, 5) (2, 4, 6)$; $\tilde{c}'_{213} = (7, 8, 9) (6, 8, 10)$; $\tilde{c}'_{311} = (1, 2, 3) (0, 2, 4)$; $\tilde{c}'_{312} = (5, 8, 10) (1, 8, 11)$; $\tilde{c}'_{313} = (4, 6, 8) (3, 6, 9)$

W2: $\tilde{c}'_{121} = (4, 8, 12) (3, 8, 13)$; $\tilde{c}'_{122} = (0.5, 1, 1.5) (0, 1, 2)$; $\tilde{c}'_{123} = (1, 3, 5) (0, 3, 6)$; $\tilde{c}'_{221} = (3, 4, 5) (2, 4, 6)$; $\tilde{c}'_{222} = (1, 2, 3) (0, 2, 4)$; $\tilde{c}'_{223} = (2, 6, 10) (1, 6, 11)$; $\tilde{c}'_{321} = (7, 8, 9) (6, 8, 10)$; $\tilde{c}'_{322} = (5, 8, 10) (1, 8, 11)$; $\tilde{c}'_{323} = (3, 4, 5) (2, 4, 6)$

W3: $\tilde{c}'_{131} = (3, 4, 5) (2, 4, 6)$; $\tilde{c}'_{132} = (5, 8, 10) (1, 8, 11)$; $\tilde{c}'_{133} = (1, 3, 5) (0, 3, 6)$; $\tilde{c}'_{231} = (0.5, 1, 1.5) (0, 1, 2)$; $\tilde{c}'_{232} = (1, 3, 5) (0, 3, 6)$; $\tilde{c}'_{233} = (4, 8, 12) (3, 8, 13)$; $\tilde{c}'_{331} = (1, 3, 5) (0, 3, 6)$; $\tilde{c}'_{332} = (3, 8, 16) (0, 8, 19)$; $\tilde{c}'_{333} = (5, 8, 10) (1, 8, 11)$

Demand: $b_1 = 12, b_2 = 7, b_3 = 15$; Supply: $a_1 = 10, a_2 = 13, a_3 = 11$; Conveyance: $e_1 = 11, e_2 = 14, e_3 = 9$

Table 6 CSTP corresponding to example 3

	E1			E2			E3			Capacity e_k
Conveyance	E1			E2			E3			11
	E2			E3			E3			14
	E3			E3			E3			9
Retail stores →	W1			W2			W3			Supply a_i
Factories ↓										
F1	4	6	5	8	1	3	4	7	3	10
F2	5	4	8	4	2	6	1	3	8	13
F3	2	7	6	8	7	4	3	9	7	11
Demand b_j	12			7			15			

$$x_{113} + x_{123} + x_{133} + x_{213} + x_{223} + x_{233} + x_{313} + x_{323} + x_{333} = 9, \text{ (Conveyance 3 constraint)}$$

$$x_{ijk} \geq 0, \text{ for } i = 1, 2, 3; j = 1, 2, 3 \text{ and } k = 1, 2, 3$$

By applying TORA software to this problem, we get the following optimal solution, which is given in screenshots 3 and 4.

The output images 3 and 4 (or screenshots 3 and 4) show the optimal solution and optimal objective value of the crisp STP corresponding to type-2 IFSTP (refer to Fig. 5 and Fig. 6).

From step 5 (i), we get the following optimal solution and optimal objective value for the crisp STP.

$$x_{212} = 1, x_{311} = 11, x_{122} = 1, x_{222} = 6, x_{133} = 9, x_{231} = 0, x_{232} = 6$$

The minimum objective value Z

$$= (2 \times 11) + (4 \times 1) + (1 \times 1) + (2 \times 6) + (1 \times 0) + (3 \times 6) + (3 \times 9) = 84$$

(35)

The optimal solution of type-2 IFSTP and its CSTP both are the same. Because the parameters supply, demand and conveyance capacity are crisp numbers.

From step 5 (iii), we can write the optimum objective value of the given type-2 IFSTP is as follows:



Fig. 5 Output summary-1 corresponding to the type-2 IFSTP and its crisp STP



Fig. 6 Output summary-2 corresponding to the type-2 IFSTP and its crisp STP

$$\begin{aligned} \text{Min } \tilde{Z}^1 &= (3, 4, 5) (2, 4, 6) \times 1 + (1, 2, 3) (0, 2, 4) \times 11 + (0.5, 1, 1.5) (0, 1, 2) \times 1 + (1, 2, 3) (0, 2, 4) \times 6 + (1, 3, 5) (0, 3, 6) \times 9 + (0.5, 1, 1.5) (0, 1, 2) \times 0 + (1, 3, 5) (0, 3, 6) \times 6 \\ \text{Min } \tilde{Z}^1 &= (3, 4, 5) (2, 4, 6) + (11, 22, 33) (0, 22, 44) + (0.5, 1, 1.5) (0, 1, 2) + (6, 12, 18) (0, 12, 24) + (9, 27, 45) (0, 27, 54) + (0, 0, 0) (0, 0, 0) + (6, 18, 30) (0, 18, 36) \end{aligned}$$

$$\begin{aligned} \text{Min } \tilde{Z}^1 &= (35.5, 84, 132.5) (2, 84, 166) \\ \Re(\tilde{Z}^1) &= 84 \end{aligned} \tag{36}$$

8.4 Example: 4 FAP with TrFNs

Let us consider the 4 × 4 fuzzy assignment problem (FAP). Further, in that FAP, let us consider the following: (i) the

Table 7 FAP with TrFNs

Jobs	Machines			
	M ₁	M ₂	M ₃	M ₄
J ₁	(1, 3, 5, 7)	(7, 9, 11, 13)	(12, 14, 16, 18)	(17, 19, 21, 23)
J ₂	(9, 11, 13, 15)	(3, 5, 7, 9)	(13, 15, 17, 19)	(5, 7, 9, 11)
J ₃	(6, 8, 10, 12)	(2, 4, 6, 8)	(11, 13, 15, 17)	(19, 21, 23, 25)
J ₄	(22, 24, 26, 28)	(17, 19, 21, 23)	(4, 6, 8, 10)	(15, 17, 19, 21)

rows representing four jobs— J_1, J_2, J_3 and J_4 and (ii) the columns representing four machines— M_1, M_2, M_3 and M_4 .

The cost matrix $[\tilde{c}_{ij}]_{4 \times 4}$ is given whose elements are TrFN (see Table 7).

The aim of the problem is to find the optimal assignment (optimal allocation) so that the total cost of job assignment becomes a minimum. Table 7 represents the 4×4 FAP with TrFNs.

Solution The given problem having the crisp and fuzzy parameters. Here, the cost of the given AP is TrFN and the value of the decision variables is the crisp number. Therefore, the given problem is a FAP.

From steps 3 (i) and 4, we get the following CAP (crisp assignment problem) (see Table 8).

From model (2), we can write this CAP (see Table 8) as follows:

$$\text{Minimize } Z = 4 x_{11} + 10 x_{12} + 15 x_{13} + 20 x_{14} + 12 x_{21} + 6 x_{22} + 16 x_{23} + 8 x_{24} + 9 x_{31} + 5 x_{32} + 14 x_{33} + 22 x_{34} + 25 x_{41} + 20 x_{42} + 7 x_{43} + 18 x_{44}$$

subject to the constraints

$$\begin{aligned} x_{11} + x_{12} + x_{13} + x_{14} &= 1 \text{ (Row 1 restriction)} \\ x_{21} + x_{22} + x_{23} + x_{24} &= 1 \text{ (Row 2 restriction)} \\ x_{31} + x_{32} + x_{33} + x_{34} &= 1 \text{ (Row 3 restriction)} \\ x_{41} + x_{42} + x_{43} + x_{44} &= 1 \text{ (Row 4 restriction)} \\ x_{11} + x_{21} + x_{31} + x_{41} &= 1 \text{ (Column 1 restriction)} \\ x_{12} + x_{22} + x_{32} + x_{42} &= 1 \text{ (Column 2 restriction)} \\ x_{13} + x_{23} + x_{33} + x_{43} &= 1 \text{ (Column 3 restriction)} \\ x_{14} + x_{24} + x_{34} + x_{44} &= 1 \text{ (Column 4 restriction)} \\ x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34}, x_{41}, x_{42}, x_{43}, x_{44} &\geq 0 \text{ and integer (non-negative restriction)} \end{aligned}$$

By applying TORA software to this problem, we get the following optimal solution, which is given in screenshot 5.

Table 8 CAP corresponding to example 4

Jobs	Machines			
	M ₁	M ₂	M ₃	M ₄
J ₁	4	10	15	20
J ₂	12	6	16	8
J ₃	9	5	14	22
J ₄	25	20	7	18

The output image 5 (or screenshot 5) shows the optimal solution and optimal objective value of the crisp AP corresponding to FAP (refer to Fig. 7).

From step 5 (i), we get the following optimal solution and optimal objective value for the crisp AP.

$$x_{11} = 1, x_{24} = 1, x_{32} = 1, x_{43} = 1$$

All other variables are at zero level.

The optimal solution to the given FAP and its equivalent CAP both are the same. Because, the values of decision variables in each of the problems are crisp numbers.

$$\begin{aligned} \text{Min } Z &= 4 \times 1 + 8 \times 1 + 5 \times 1 + 7 \times 1 \\ &= 24 \end{aligned} \tag{37}$$

From step 5 (ii), we can write the optimum objective value for the given FAP is as follows:

$$\begin{aligned} \text{Min } \tilde{Z} &= (1, 3, 5, 7) \times 1 + (5, 7, 9, 11) \times 1 + (2, 4, 6, 8) \\ &\quad \times 1 + (4, 6, 8, 10) \times 1 \\ &= (12, 20, 28, 36) \end{aligned} \tag{38}$$

$$\mathfrak{R}(\tilde{Z}) = 24$$

8.5 Example: 5 IFAP with TIFNs

A Radio/TV/computer manufacturing company has three different jobs. The three different jobs are assembling, testing, and packing respectively. These three different jobs can be done by the three different persons/workers. That is, persons A, B and C. All the three different persons are capable of doing all the three different jobs, but a person can undertake only one job/task at a time. The performing time (in hrs) of each job is not known precisely because of the lack of experience, capacity, interest, situations on that particular day, knowledge, the agility of persons, physical ability, understanding and so forth. Therefore, the time consumption of all the jobs is not known exactly, which is given in the form of triangular intuitionistic fuzzy numbers (TIFNs). From the past experience, the time (in hrs) that each person takes to do each job is given in the following table (refer to Table 9).



Fig. 7 Output summary corresponding to the FAP and its crisp AP

Table 9 IFAP with TIFNs

Jobs→	Assembling	Testing	Packing
Persons ↓			
A	(7, 21, 29) (2, 21, 34)	(7, 20, 57) (3, 20, 61)	(12, 25, 56) (8, 25, 60)
B	(8, 9, 16) (2, 9, 22)	(4, 12, 35) (1, 12, 38)	(6, 14, 28) (3, 14, 31)
C	(5, 9, 22) (2, 9, 25)	(10, 15, 20) (5, 15, 25)	(4, 16, 19) (1, 16, 22)

Table 10 CAP corresponding to example 5

Jobs→	Assembling	Testing	Packing
Persons ↓			
A	19	28	31
B	11	17	16
C	12	15	13

Find the assignment of persons/workers to jobs that will minimize the total time taken.

Solution The given problem having the crisp and intuitionistic fuzzy parameters. Here, the cost of the given AP is TIFN and the value of the decision variables is the crisp number. Therefore, the given problem is an IFAP.

From steps 3 (ii) and 4, we get the following CAP (see Table 10).

From model (2), we can write this CAP (see Table 10) as follows:

$$\text{Minimize } Z = 19 x_{11} + 28 x_{12} + 31 x_{13} + 11 x_{21} + 17 x_{22} + 16 x_{23} + 12 x_{31} + 15 x_{32} + 13 x_{33}$$

subject to the constraints

- $x_{11} + x_{12} + x_{13} = 1$ (Row 1 restriction)
- $x_{21} + x_{22} + x_{23} = 1$ (Row 2 restriction)
- $x_{31} + x_{32} + x_{33} = 1$ (Row 3 restriction)
- $x_{11} + x_{21} + x_{31} = 1$ (Column 1 restriction)
- $x_{12} + x_{22} + x_{32} = 1$ (Column 2 restriction)
- $x_{13} + x_{23} + x_{33} = 1$ (Column 3 restriction)
- $x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} \geq 0$ and integer (non-negative restriction).

By applying TORA software to this problem, we get the following optimal solution, which is given in screenshot 6.

The output image 6 (or screenshot 6) shows the optimal solution and optimal objective value of the crisp AP corresponding to IFAP (refer to Fig. 8).

From step 5 (i), we get the following optimal solution and optimal objective value for the crisp AP.

$$x_{11} = 1, x_{22} = 1, x_{33} = 1$$

All other variables are at zero level. In other words, the optimal assignment is

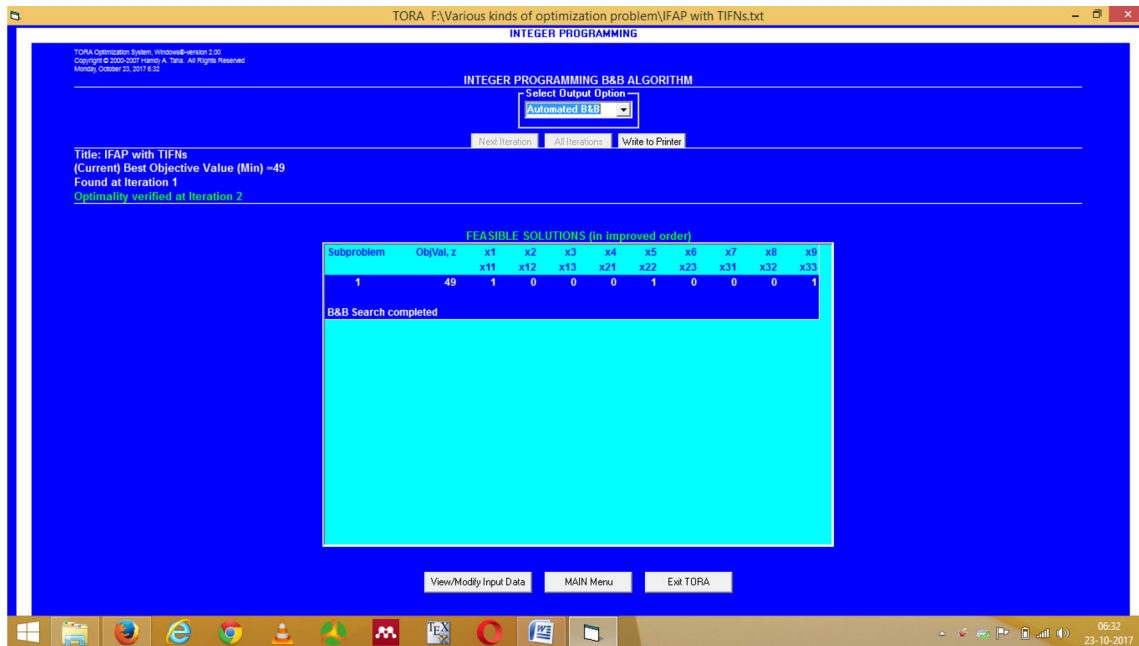


Fig. 8 Output summary corresponding to the IFAP and its crisp AP

Person A → Assembling, Person B → Testing, Person C
→ Packing

The optimal solution to the given IFAP and its equivalent CAP both are the same. Because, the values of decision variables in each of the problems are crisp numbers.

$$\begin{aligned} \text{Min } Z &= 19 \times 1 + 17 \times 1 + 13 \times 1 \\ &= 49 \end{aligned} \tag{39}$$

From step 5 (iii), we can write the optimum objective value for the given IFAP is as follows:

$$\begin{aligned} \text{Min } \tilde{Z}^I &= (7, 21, 29)(2, 21, 34) \times 1 + (4, 12, 35)(1, 12, 38) \\ &\quad \times 1 + (4, 16, 19)(1, 16, 22) \times 1 \\ &= (15, 49, 83)(4, 49, 94) \end{aligned} \tag{40}$$

$$\Re(\tilde{Z}^I) = 49 \text{ (hours)}$$

8.6 Example: 6 maximization type IFAP with TIFNs

Let us consider the 4×4 intuitionistic fuzzy assignment problem (IFAP). Further, in that IFAP, let us consider the following:

- (i) the rows representing four machines (namely, M_1, M_2, M_3 and M_4) and
- (ii) the columns representing four jobs (namely, J_1, J_2, J_3 and J_4).

The profit matrix $[c_{ij}^I]$ is given and its elements are in the form of TIFN (see Table 11).

The objective of the problem is to find the optimal assignment (optimal allocation) so that the total profit of job assignment becomes a maximum. Table 11 shows the 4×4 IFAP with TIFNs.

Solution The given problem having the crisp and intuitionistic fuzzy parameters. Here, the profit of the given AP

Table 11 IFAP with TIFNs

Jobs →	J_1	J_2	J_3	J_4
Machines ↓				
M_1	(27, 50, 109)	(56, 67, 111)	(8, 22, 120)	(75, 100, 128)
M_2	(13, 50, 123)	(40, 67, 127)	(4, 22, 124)	(62, 100, 141)
M_3	(52, 68, 93)	(43, 90, 119)	(34, 56, 93)	(60, 70, 89)
M_4	(44, 68, 101)	(35, 90, 127)	(18, 56, 109)	(50, 70, 99)
	(72, 80, 109)	(78, 90, 108)	(85, 98, 150)	(52, 68, 93)
	(58, 80, 123)	(65, 90, 121)	(76, 98, 159)	(44, 68, 101)
	(23, 40, 81)	(44, 58, 90)	(63, 89, 109)	(64, 72, 95)
	(17, 40, 87)	(38, 58, 96)	(49, 89, 123)	(51, 72, 108)

Table 12 CAP corresponding to example 6

Jobs→	J_1	J_2	J_3	J_4
Machines ↓				
M_1	62	78	50	101
M_2	71	85	61	73
M_3	87	92	111	71
M_4	48	64	87	77

is TIFN and the value of the decision variables is the crisp number. Therefore, the given problem is an IFAP.

From steps 3 (ii) and 4, we get the following CAP (see Table 12).

From model (2), we can write this CAP (see Table 12) as follows:

$$\text{Max } Z = 62x_{11} + 78x_{12} + 50x_{13} + 101x_{14} + 71x_{21} + 84x_{22} + 61x_{23} + 73x_{24} + 87x_{31} + 92x_{32} + 111x_{33} + 71x_{34} + 48x_{41} + 64x_{42} + 87x_{43} + 77x_{44}$$

subject to the constraints

- $x_{11} + x_{12} + x_{13} + x_{14} = 1$ (Row 1 restriction)
- $x_{21} + x_{22} + x_{23} + x_{24} = 1$ (Row 2 restriction)
- $x_{31} + x_{32} + x_{33} + x_{34} = 1$ (Row 3 restriction)
- $x_{41} + x_{42} + x_{43} + x_{44} = 1$ (Row 4 restriction)
- $x_{11} + x_{21} + x_{31} + x_{41} = 1$ (Column 1 restriction)
- $x_{12} + x_{22} + x_{32} + x_{42} = 1$ (Column 2 restriction)
- $x_{13} + x_{23} + x_{33} + x_{43} = 1$ (Column 3 restriction)
- $x_{14} + x_{24} + x_{34} + x_{44} = 1$ (Column 4 restriction)
- $x_{11}, x_{12}, x_{13}, x_{14}, x_{21}, x_{22}, x_{23}, x_{24}, x_{31}, x_{32}, x_{33}, x_{34}, x_{41}, x_{42}, x_{43}, x_{44} \geq 0$ and integer (non-negative restriction)

By applying TORA software to this problem, we get the following optimal solution, which is given in screenshot 7.

The output image 7 (or screenshot 7) shows the optimal solution and optimal objective value of the crisp AP corresponding to IFAP (refer to Fig. 9).

From step 5 (i), we get the following optimal solution and optimal objective value for the crisp AP.

$$x_{14} = 1, x_{22} = 1, x_{31} = 1, x_{43} = 1$$

All other variables are at zero level.

In other words, the optimal assignment is

$$M_1 \rightarrow J_4, M_2 \rightarrow J_2, M_3 \rightarrow J_1, M_4 \rightarrow J_3$$

The optimal solution to the given IFAP and its equivalent CAP both are the same. Because, the values of decision variables in each of the problems are crisp numbers.

$$\text{Max } Z = 101 \times 1 + 85 \times 1 + 87 \times 1 + 87 \times 1 = 360 \tag{41}$$

From step 5 (iii), we can write the optimum objective value for the given IFAP is as follows:

$$\begin{aligned} \text{Max } \tilde{Z}^I &= (75, 100, 128)(62, 100, 141) \times 1 \\ &\quad + (43, 90, 119)(35, 90, 127) \times 1 \\ &\quad + (72, 80, 109)(58, 80, 123) \times 1 \\ &\quad + (63, 89, 109)(49, 89, 123) \times 1 \end{aligned}$$

$$\text{Max } \tilde{Z}^I = (253, 359, 465)(204, 359, 514) \tag{42}$$

$$\Re(\tilde{Z}^I) = \text{Rs. } 360$$

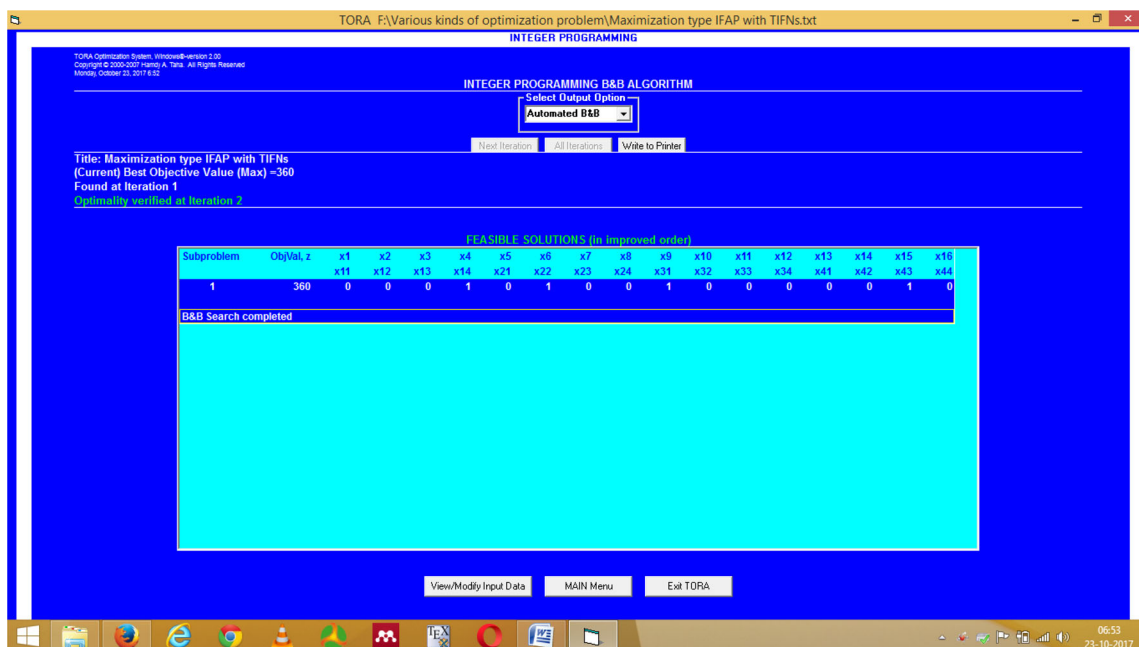


Fig. 9 Output summary corresponding to the IFAP and its crisp AP

8.7 Example: 7 real-life IFSAP

A computer/TV manufacturing company has three jobs. That is, assembling (J_1), testing (J_2) and packing (J_3). These three jobs can be done by three employees’ (workers) $M_1, M_2,$ and M_3 respectively. Let us suppose that there are three factories denoted by $F_1, F_2,$ and F_3 . All the three employees’ are capable of doing all the three jobs with different factories and all the three factories are capable of doing all the three jobs with different employees’ but a single employee can undertake only one job with only one factory at a time.

The performing time of each job is not known exactly due to some of the common factors like lack of experience, understanding, capacity (i.e., person’s ability), interest.

So, the time consumption for all the jobs with different factories are not known exactly, which are given in the form of TIFNs.

From the past experience, it is known that $\tilde{t}_{ijk} (i = 1, 2, 3; j = 1, 2, 3; \text{ and } k = 1, 2, 3)$ is the assignment time of the j th job in the k th factory to be performed by the i th employee.

The time (in hours) that each employee takes to do each job with different factories is given in the following table (refer to Table 13).

Therefore, three employees’, three factories and three jobs can be associated with only one of the others (i.e., only one employee is associated with only one factory with only one job).

Find the assignment of employees’ to jobs with different factories that will minimize the total time taken.

Solution The given problem having the crisp and intuitionistic fuzzy parameters. Here, the time of the given AP is TIFN and the value of the decision variables is the crisp number. Therefore, the given problem is an IFSAP.

From steps 3 (ii) and 4, we get the following CSAP (crisp solid assignment problem) (see Table 14).

From model (10), we can write this CSAP (see Table 14) as follows:

$$\begin{aligned} \text{Minimize } Z = & 10x_{111} + 11x_{112} + 6x_{113} + 8x_{121} + \\ & 6x_{122} + 7x_{123} + 10x_{131} + 6x_{132} + 7x_{133} + 13x_{211} + \\ & 12x_{212} + 8x_{213} + 12x_{221} + 7x_{222} + 6x_{223} + 27x_{231} \\ & + 12x_{232} + 12x_{233} + 15x_{311} + 7x_{312} + 8x_{313} + 10x_{321} \\ & + 8x_{322} + 9x_{323} + 9x_{331} + 9x_{332} + 10x_{333} \end{aligned}$$

subject to the constraints

$$\begin{aligned} & x_{111} + x_{112} + x_{113} + x_{121} + x_{122} + x_{123} + x_{131} \\ & + x_{132} + x_{133} = 1, \text{ (Employee 1 constraint)} \\ & x_{211} + x_{212} + x_{213} + x_{221} + x_{222} + x_{223} + x_{231} \\ & + x_{232} + x_{233} = 1, \text{ (Employee 2 constraint)} \\ & x_{311} + x_{312} + x_{313} + x_{321} + x_{322} + x_{323} + x_{331} \\ & + x_{332} + x_{333} = 1, \text{ (Employee 3 constraint)} \\ & x_{111} + x_{112} + x_{113} + x_{211} + x_{212} + x_{213} + x_{311} \\ & + x_{312} + x_{313} = 1, \text{ (Job 1 constraint)} \\ & x_{121} + x_{122} + x_{123} + x_{221} + x_{222} + x_{223} + x_{321} \\ & + x_{322} + x_{323} = 1, \text{ (Job 2 constraint)} \\ & x_{131} + x_{132} + x_{133} + x_{231} + x_{232} + x_{233} + x_{331} \\ & + x_{332} + x_{333} = 1, \text{ (Job 3 constraint)} \\ & x_{111} + x_{121} + x_{131} + x_{211} + x_{221} + x_{231} + x_{311} \\ & + x_{321} + x_{331} = 1, \text{ (Factory 1 constraint)} \\ & x_{112} + x_{122} + x_{132} + x_{212} + x_{222} + x_{232} + x_{312} \\ & + x_{322} + x_{332} = 1, \text{ (Factory 2 constraint)} \end{aligned}$$

Table 13 IFSAP with TIFNs

									Capacity e_k	
Factories	F1			F2			F3			1
	F2			F3			F3			1
Jobs →	Assembling (J_1)			Testing (J_2)			Packing (J_3)			Supply a_i
Employees ↓										
E_1	\tilde{t}_{111}	\tilde{t}_{112}	\tilde{t}_{113}	\tilde{t}_{121}	\tilde{t}_{122}	\tilde{t}_{123}	\tilde{t}_{131}	\tilde{t}_{132}	\tilde{t}_{133}	1
E_2	\tilde{t}_{211}	\tilde{t}_{212}	\tilde{t}_{213}	\tilde{t}_{221}	\tilde{t}_{222}	\tilde{t}_{223}	\tilde{t}_{231}	\tilde{t}_{232}	\tilde{t}_{233}	1
E_3	\tilde{t}_{311}	\tilde{t}_{312}	\tilde{t}_{313}	\tilde{t}_{321}	\tilde{t}_{322}	\tilde{t}_{323}	\tilde{t}_{331}	\tilde{t}_{332}	\tilde{t}_{333}	1
Demand b_j	1			1			1			

$J_1: \tilde{t}_{111} = (3, 13, 14) (2, 13, 15); \tilde{t}_{112} = (8, 12, 15) (3, 12, 17); \tilde{t}_{113} = (4, 6, 8) (3, 6, 9); \tilde{t}_{211} = (4, 16, 19) (2, 16, 21); \tilde{t}_{212} = (3, 13, 14) (2, 13, 24); \tilde{t}_{213} = (4, 8, 12) (3, 8, 13); \tilde{t}_{311} = (2, 17, 18) (0, 17, 32); \tilde{t}_{312} = (5, 8, 10) (1, 8, 11); \tilde{t}_{313} = (4, 8, 12) (3, 8, 13)$

$J_2: \tilde{t}_{121} = (4, 8, 12) (3, 8, 13); \tilde{t}_{122} = (4, 6, 8) (3, 6, 9); \tilde{t}_{123} = (5, 8, 10) (1, 8, 11); \tilde{t}_{221} = (3, 13, 14) (2, 13, 24); \tilde{t}_{222} = (5, 8, 10) (1, 8, 11); \tilde{t}_{223} = (4, 6, 8) (3, 6, 9); \tilde{t}_{321} = (3, 13, 14) (2, 13, 15); \tilde{t}_{322} = (4, 8, 12) (3, 8, 13); \tilde{t}_{323} = (3, 8, 16) (0, 8, 19)$

$J_3: \tilde{t}_{131} = (3, 13, 14) (2, 13, 15); \tilde{t}_{132} = (4, 6, 8) (3, 6, 9); \tilde{t}_{133} = (5, 8, 10) (1, 8, 11); \tilde{t}_{231} = (10, 34, 37) (6, 34, 41); \tilde{t}_{232} = (3, 13, 14) (2, 13, 24); \tilde{t}_{233} = (3, 13, 14) (2, 13, 24); \tilde{t}_{331} = (3, 8, 16) (0, 8, 19); \tilde{t}_{332} = (3, 8, 16) (0, 8, 19); \tilde{t}_{333} = (3, 13, 14) (2, 13, 15)$

Table 14 CSAP corresponding to example 7

										Capacity e_k		
Factories	F1	F2			F1	F2			F1	F2	F3	1
				F3			F3				1	
											1	
Jobs →	Assembling (J_1)			Testing (J_2)			Packing (J_3)			Supply a_i		
Employees ↓												
E_1	10	11	6	8	6	7	10	6	7	1		
E_2	13	12	8	12	7	6	27	12	12	1		
E_3	15	7	8	10	8	9	9	9	10	1		
Demand b_j	1			1			1					

$$x_{113} + x_{123} + x_{133} + x_{213} + x_{223} + x_{233} + x_{313} + x_{323} + x_{333} = 1, \text{ (Factory 3 constraint)}$$

$$x_{ijk} \geq 0 \text{ and integer, } \forall i, j, k = 1, 2, 3.$$

By applying TORA software to this problem, we get the following optimal solution, which is given in screenshots 8 and 9.

The output images 8 and 9 (or screenshots 8 and 9) show the optimal solution and optimal objective value of the crisp SAP corresponding to IFSAP (refer to Figs. 10 and 11).

From step 5 (i), we get the following optimal solution and optimal objective value for the crisp SAP.

$$x_{113} = 1, x_{222} = 1, x_{331} = 1$$

All other variables are at zero level.

Therefore, the optimal assignment to the given real-life IFSAP is

- Employee 1 Job 1 (Assembling) and Factory 3
- Employee 2 Job 2 (Testing) and Factory 2
- Employee 3 Job 3 (Packing) and Factory 1

The optimal solution to the given IFSAP and its equivalent CSAP both are same. Because, the values of decision variables in each of the problems are crisp numbers.

$$\begin{aligned} \text{Min } Z &= 6 \times 1 + 7 \times 1 + 9 \times 1 \\ &= 22 \text{ hours} \end{aligned} \tag{43}$$

From step 5 (iii), we can write the optimum objective value for the given IFSAP is as follows:

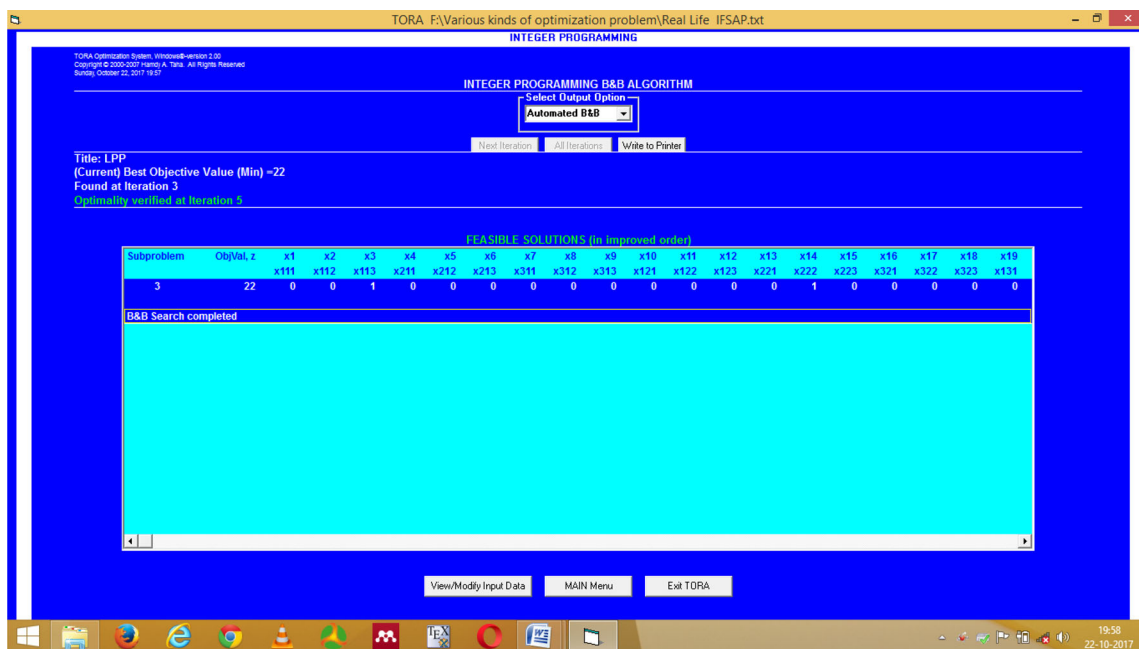


Fig. 10 Output summary-1 corresponding to the IFSAP and its crisp SAP



Fig. 11 Output summary-2 corresponding to the IFSAP and its crisp SAP

$$\begin{aligned} \text{Min } \tilde{Z}^I &= (4, 6, 8)(3, 6, 9) + (5, 8, 10)(1, 8, 11) \\ &\quad + (3, 8, 16)(0, 8, 19) \\ &= (12, 22, 34)(4, 22, 39) \end{aligned}$$

$$\Re(\tilde{Z}^I) = 22 \text{ Hours} \tag{44}$$

9 Results and discussion

Comparison between the proposed method and the existing method is given in Tables 15 and 16 respectively.

The minimum total intuitionistic fuzzy transportation cost of problem (3) is

$$\text{Min } \tilde{Z}^I = (35.5, 84, 132.5) (2, 84, 166) \tag{29}$$

The result in (29) can be explained (refer to Fig. 12) as follows:

Let $\mu_{\tilde{z}^I}(c)$ be the membership and $\vartheta_{\tilde{z}^I}(c)$ be the non-membership value at c . Then

1. The degree of acceptance of the transportation cost c for the DM is $100\mu_{\tilde{z}^I}(c)\%$.
2. The degree of non-acceptance of the transportation cost c for the DM is $100\vartheta_{\tilde{z}^I}(c)\%$.
3. The degree of hesitation for the acceptance of c is given by $100(1 - \mu_{\tilde{z}^I}(c) - \vartheta_{\tilde{z}^I}(c))\%$.
4. The optimal transportation cost i.e., the minimum total transportation cost lies in $[35.5, 132.5]$ (cost in rupees).
5. 100% expect are in favour that the total transportation cost is 84 rupees as $\mu_{\tilde{z}^I}(c) = 1, c = 84$ rupees.

6. Beyond (2, 166), the transportation cost is totally unacceptable (refer to Fig. 12).

Values of $\mu_{\tilde{z}^I}(c)$ and $\vartheta_{\tilde{z}^I}(c)$ at different values of c can be determined using equations given below.

$$\mu_{\tilde{z}^I}(c) = \begin{cases} 0, & \text{for } c < 35.5, \\ \frac{c - 35.5}{48.5}, & \text{for } 35.5 \leq c \leq 84, \\ 1, & \text{for } c = 84, \\ \frac{132.5 - c}{48.5}, & \text{for } 84 \leq c \leq 132.5, \\ 0, & \text{for } c > 132.5, \end{cases}$$

$$\vartheta_{\tilde{z}^I}(c) = \begin{cases} 1, & \text{for } c < 2, \\ \frac{84 - c}{82}, & \text{for } 2 \leq c \leq 84, \\ 0, & \text{for } c = 84, \\ \frac{c - 84}{82}, & \text{for } 84 \leq c \leq 166, \\ 1, & \text{for } c > 166. \end{cases}$$

If we consider the result in (29) as an intuitionistic fuzzy assignment time (time in hours) then we can describe the results as follows:

Let $\mu_{\tilde{z}^I}(t)$ be the membership and $\vartheta_{\tilde{z}^I}(t)$ be the non-membership value at t . Then

1. The degree of acceptance of the assignment time t for the DM is $100\mu_{\tilde{z}^I}(t)\%$.
2. The degree of non-acceptance of the assignment time t for the DM is $100\vartheta_{\tilde{z}^I}(t)\%$.
3. The degree of hesitation for the acceptance of t is given by $100(1 - (\mu_{\tilde{z}^I}(t) + \vartheta_{\tilde{z}^I}(t)))\%$.
4. The optimal assignment time i.e., the minimum total performing time lies in $[35.5, 132.5]$ (time in hours).

Table 15 Comparative analysis

Ranking method	Example number	Crisp solution methods				Fuzzy/intuitionistic fuzzy solution methods				
		VAM	MODI	Zero point method	TORA	Fuzzy/intuitionistic fuzzy zero point method	Improved fuzzy/improved intuitionistic fuzzy zero point method	Fuzzy modified/intuitionistic fuzzy modified distribution method	PSK method	Proposed method
Liou and Wang (1992), Chen and Hsieh (1999)	8.1	Rs. 769	Rs. 769	Rs. 769	Rs. 769	Min $\tilde{Z} = (469, 669, 869, 1069)$	Min $\tilde{Z} = (469, 669, 869, 1069)$	Min $\tilde{Z} = (469, 669, 869, 1069)$	Min $\tilde{Z} = (469, 669, 869, 1069)$	Min $\tilde{Z} = (469, 669, 869, 1069)$
Kumar (2018d)	8.2	Rs. 423	Rs. 423	Rs. 423	Rs. 423	Min $\tilde{Z}^I = (273, 393, 453, 573; 213, 333, 513, 633)$	Min $\tilde{Z}^I = (273, 393, 453, 573; 213, 333, 513, 633)$	Min $\tilde{Z}^I = (273, 393, 453, 573; 213, 333, 513, 633)$	Min $\tilde{Z}^I = (273, 393, 453, 573; 213, 333, 513, 633)$	Min $\tilde{Z}^I = (273, 393, 453, 573; 213, 333, 513, 633)$
Varghese and Kuriakose (2012)	8.3	–	–	–	Rs. 84	–	–	–	Min $\tilde{Z}^I = (35.5, 84, 132.5, 166)$	Min $\tilde{Z}^I = (35.5, 84, 132.5, 166)$

Table 16 Comparative analysis

Ranking method	Example number	Solution method		Solution method		
		Hungarian method	Reduction method	Fuzzy/IFHM	Fuzzy/IFRM	Proposed method
Liou and Wang (1992), Chen and Hsieh (1999)	8.4	$x_{11} = 1, x_{24} = 1, x_{32} = 1, x_{43} = 1$ and Min $Z = \text{Rs. } 24$	Not applicable	$x_{11} = 1, x_{24} = 1, x_{32} = 1, x_{43} = 1$ and Min $\tilde{Z} = (12, 20, 28, 36)$	Not applicable	$x_{11} = 1, x_{24} = 1, x_{32} = 1, x_{43} = 1$ and Min $\tilde{Z} = (12, 20, 28, 36)$
Varghese and Kuriakose (2012)	8.5	$x_{11} = 1, x_{22} = 1, x_{33} = 1$ and Min $Z = 49$ (hours)	Not applicable	$x_{11} = 1, x_{22} = 1, x_{33} = 1$ and Min $\tilde{Z}^I = (15, 49, 83)(4, 49, 94)$	Not applicable	$x_{11} = 1, x_{22} = 1, x_{33} = 1$ and Min $\tilde{Z}^I = (15, 49, 83)(4, 49, 94)$
Varghese and Kuriakose (2012)	8.6	$x_{14} = 1, x_{22} = 1, x_{31} = 1, x_{43} = 1$ and Rs. 360	Not applicable	$x_{14} = 1, x_{22} = 1, x_{31} = 1, x_{43} = 1$ and Max $\tilde{Z}^I = (253, 359, 465)(204, 359, 514)$	Not applicable	$x_{14} = 1, x_{22} = 1, x_{31} = 1, x_{43} = 1$ and Max $\tilde{Z}^I = (253, 359, 465)(204, 359, 514)$
Varghese and Kuriakose (2012)	8.7	Not applicable	$x_{113} = 1, x_{222} = 1, x_{331} = 1$ and Min $Z = 22$ (hours)	Not applicable	$x_{113} = 1, x_{222} = 1, x_{331} = 1$ and Min $\tilde{Z}^I = (12, 22, 34)(4, 22, 39)$	$x_{113} = 1, x_{222} = 1, x_{331} = 1$ and Min $\tilde{Z}^I = (12, 22, 34)(4, 22, 39)$

- 100% expect are in favour that the total assignment time is 84 h as $\mu_{z^I}(t) = 1, t = 84$ hours.
- Beyond (2, 166), the assignment time (time in hours) is totally un-acceptable (refer to Fig. 12).

Values of $\mu_{z^I}(t)$ and $\vartheta_{z^I}(t)$ at different values of t are evaluated using Eqs. (45) and (46), respectively,

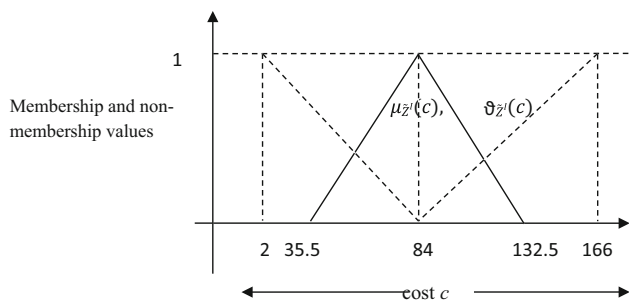


Fig. 12 Graphical representation of type-2 IFSTC

$$\mu_{z'}(t) = \begin{cases} 0, & \text{for } t < 35.5, \\ \frac{t - 35.5}{48.5}, & \text{for } 35.5 \leq t \leq 84, \\ 1, & \text{for } t = 84, \\ \frac{132.5 - t}{48.5}, & \text{for } 84 \leq t \leq 132.5, \\ 0, & \text{for } t > 132.5, \end{cases} \quad (45)$$

$$\vartheta_{z'}(t) = \begin{cases} 1, & \text{for } t < 2, \\ \frac{84 - t}{82}, & \text{for } 2 \leq t \leq 84, \\ 0, & \text{for } t = 84, \\ \frac{t - 84}{82}, & \text{for } 84 \leq t \leq 166, \\ 1, & \text{for } t > 166. \end{cases} \quad (46)$$

The advantages of the proposed approach as compared to other existing work are given in Table 17.

10 Conclusion and future research

A new and computationally simple algorithm for solving different types of optimization problems under crisp, fuzzy and IFE by using single algorithm is presented in this paper. The proposed algorithm is very easy to understand and to apply in solving different types of optimization problems under crisp, fuzzy and IFE because this method is

based on crisp linear programming problem or integer linear programming problem. The proposed algorithm provides the optimal solution and optimal objective value for different types of optimization problems under crisp, fuzzy and IFE simultaneously with one LP model. Due to this, this proposed algorithm can be served as an important tool for a decision-maker when he/she handles various types of optimization problems under crisp, fuzzy and IFE. In addition to prove this, for each proposed model the relative real-life numerical example is presented. The crisp, fuzzy and intuitionistic fuzzy optimization problems have wide applications in industry, organization, telecommunication, coal transportation, satellite launching, time-tabling problems, capital investment, multi-passive-sensor, dynamic facility location and so on.

If the manager/decision maker uses the proposed ideas, to solve the management related issues then they will get feasible and optimal solutions.

If the person who has less knowledge but the aware of operates the computer, then that person also can solve the problems by using the proposed method.

To solve the optimization problems under crisp, fuzzy and IFE, the manager/decision maker is not necessary to study the different algorithms. Because, the proposed algorithm gives the optimal solution under all situation.

The proposed algorithm yields an optimal solution for the crisp, fuzzy and intuitionistic fuzzy optimization problems directly. In this algorithm, we have the following advantages:

1. Use of the existing methods, namely, modified distribution method, FMODI, IFMODI, min zero–min cost method, fuzzy min zero–min cost method, intuitionistic fuzzy min zero–min cost method, Hungarian method, fuzzy Hungarian method, IFHM, reduction method, fuzzy reduction method and IFRM which are not required.

Table 17 Differences between existing methods and the proposed method

Serial number	Existing method	Proposed method
1	Computationally it is not a simple	Computationally it is very simple
2	It takes more time to solve the problem	It takes very less time to solve the problem
3	Sometimes it gives a feasible solution but not an optimal solution	Always it gives an optimal solution
4	It does not use to solve crisp, fuzzy and IFTPs at a time. This statement is also applicable to the AP	It is useful to solve crisp, fuzzy and IFTPs at a time. This statement is also applicable to the AP
5	It does not use to do the sensitivity analysis	It is useful to do the sensitivity analysis
6	The solution obtained by the existing method is not reliable and also, sometimes it is not efficient	The solution obtained by the proposed method is always reliable and efficient
7	The existing method is not helpful to solve the complex problem	A complex problem is also solved by the proposed approach

2. This proposed algorithm can help the decision makers/managers to reduce their computational time. Apart from that, it always provides optimal solutions.
3. It is applicable to solve the different types of optimization problems under crisp, fuzzy and IFE.
4. For crisp TP and IFSTPs, we are not required to find a basic feasible solution. Since the proposed algorithm gives the optimal solution directly. So, we need not apply the optimality test to the obtained solution. Therefore, the proposed algorithm is called a single stage algorithm.
5. The optimal solution of various kinds of optimization problems has physical meaning as it never yields the negative solutions.
6. The proposed algorithm especially helpful for the DM if the problems have large number of decision variables. i.e., if the problem is too complex.
7. From the proposed algorithm, we can do the sensitivity analysis in optimization problems under crisp, fuzzy and IFE.
8. From the proposed algorithm, the decision maker/manager can handle the balanced, unbalanced, maximization and minimization problems at a time. Unbalanced fuzzy optimization problems and unbalanced intuitionistic fuzzy optimization problems can be solved by the proposed algorithm in an easy manner because it is based on a crisp linear programming method.
9. We need not balance the problem when it is unbalanced (it is fit for both the transportation and APs under crisp, fuzzy and IFE).

Our future research will concentrate to solve the multi-objective optimization problems under crisp, fuzzy and IFE by using a single algorithm.

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