



Parameter estimation of the inverted exponentiated Rayleigh distribution based on progressively first-failure censored samples

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Abstract We study statistical inference of the inverted exponentiated Rayleigh distribution under progressively first-failure censoring samples in our paper. Specifically, we deal with Maximum likelihood and Bayes estimators of parameters. The observed Fisher matrix is conducive to obtain asymptotic confidence interval. Parametric bootstrap methods are applied to provide the confidence intervals. Bayes estimators in terms of squared error loss function are derived with Metropolis–Hastings technique, which are helpful to construct highest posterior density credible intervals. We compare the behavior of various estimators by conducting Monte Carlo simulations. A set of actual data is analyzed to introduce the proposed methods.

Keywords Maximum likelihood estimation · Parametric bootstrap · Metropolis–Hastings algorithm · Progressively first-failure censoring

1 Introduction

The definition of Rayleigh distribution was originally given by Rayleigh who concerned a problem about the area of acoustics, which was a particular form of the Weibull distribution. Its hazard rate function would grow linearly over time. Due to this important property, Rayleigh distribution is indispensable in numerous application fields. Some applications of Rayleigh distribution could be found

in construction of physics model fields, for instance, sound radiation, ray radiation, wave heights and wind speed. Considering that Rayleigh distribution could exactly represent Instantaneous state, one would use it to describe instantaneous peak values in communication theory, see Gómez-Déniz and Gómez-Déniz (2013). Polovko (1968) and Dyer and Whisenand (1973) pointed out that Rayleigh distribution played a significant role in Electronic Vacuum Devices and Communication Engineering such as the swing of radio noise and envelopes of certain stochastic processes, which had probability density function of Rayleigh. In addition, the Rayleigh distribution fits very well with the model that grew rapidly over time. So it is very popular in probability and statistics. It is widely used as a useful life model in the field of reliability, while it applies in other sciences, including operations, statistics and biology. In the recent past, the distributions associated with the Rayleigh distribution have attracted the attention of many authors. Raqab and Madi (2009) considered informative and non-informative priors to get estimation and prediction of the exponentiated Rayleigh model. Mahmoud and Ghazal (2016) proposed estimations of exponentiated Rayleigh distribution with type-II censoring model. Also, Kundu and Raqabb (2005) introduced different estimation procedures to estimate the unknown parameter(s) of generalized Rayleigh distribution. Wu (2006) discussed the problem of accepting the sampling scheme when the samples came from the generalized Rayleigh distribution. Apart from that, Ali (2015) derived the Bayesian estimation for inverted Rayleigh distribution. Soliman et al. (2010) obtained estimation and prediction under lower record values for the inverted Rayleigh distribution. In a recent study, Rastogi and Tripathi (2014) showed that the estimation of unknown parameters using type-II

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progressive censoring, meanwhile the distribution was inverted exponentiated Rayleigh distribution.

We note that the hazard rate of inverted exponentiated Rayleigh distribution is not monotonous. In view of numerous practical situations, it is universal that hazard rate is nonmonotone. Therefore, we are interested in the inverted exponentiated Rayleigh distribution when the observed data is progressive first-failure censored. To the best of our knowledge, there is no research about this case. We consider the estimation of the parameters for the inverted exponentiated Rayleigh distribution by using the progressive first failure censored sample. The failure rate function of the inverted exponentiated Rayleigh distribution is unimodal. With the help of the advantage, the inverted exponentiated Rayleigh distribution is favored by many statisticians. In fact, the lifetime of electrical and mechanical components, the treatment of hospital patients will change with time, and is not only decreasing with the growth of time. Therefore, inverted exponentiated Rayleigh distribution can be used to fit the data, because its failure rate function is not monotonous. When we estimate the parameters of the inverted exponentiated Rayleigh distribution, the progressive first-failure censoring is used to obtain samples. The progressive first-failure censoring sample is the most time saving and cost efficient sampling method in the known censored modes. This method of obtaining samples can help us save experimental costs and carry out experiments at a fast speed.

In consideration of cost and time, many experimenters pay more attention on censoring in life testing of reliability analysis, such as Aslam et al. (2018), Aslam et al. (2019) and Sajid Ali and Butt (2019). More importantly, deletion is reasonable. Aslam et al. (2018), Muhammad (2018), Aslam (2019) and Aslam and Arif (2018) have studied it. The most common censoring schemes have been used widely by many authors with all kinds of lifetime models, which are type-I and type-II censoring, one can refer to Kay (1976), Aslam et al. (2018) and Sinha (1986). However, there is a more popular censoring scheme, progressive censoring that could removal test units at time points not final terminal point by Balakrishnan and Aggarwala (2000), Nasrullah Khan (2018) applied this censoring method. And more, Balasooriya (1995) proposed that experimenter could divide the test items into several groups in order to cut short the time and cost of experiment. However, the more attractive censoring is uniting the characteristics of progressive censoring and grouping censoring into a better censoring, namely, progressive first-failure censoring scheme, Wu and Kus (2009) explained it in detail. Some recent studies, one can refer to Soliman et al. (2011) and Soliman et al. (2012).

The cumulative distribution function(cdf) of inverted exponentiated Rayleigh distribution(IERD), denotes by IERD (α, β) , is written as

$$F(x; \alpha, \beta) = 1 - \left(1 - e^{-(\beta/x^2)}\right)^\alpha, \quad x, \alpha, \beta > 0 \quad (1.1)$$

α is a shape parameter, the scale parameter as β . And corresponding probability density function (pdf) can be given by

$$f(x; \alpha, \beta) = 2\alpha\beta x^{-3} e^{-(\beta/x^2)} \left(1 - e^{-(\beta/x^2)}\right)^{\alpha-1}, \quad x, \alpha, \beta > 0 \quad (1.2)$$

Next, we give the IERD's reliability and failure rate functions, respectively,

$$r(t) = \left(1 - e^{-(\beta/t^2)}\right)^\alpha, \quad t > 0, \quad (1.3)$$

$$h(t) = 2\alpha\beta t^{-3} e^{-(\beta/t^2)} \left(1 - e^{-(\beta/t^2)}\right)^{-1}, \quad t > 0 \quad (1.4)$$

Figure 1 shows the failure rate of inverted exponentiated Rayleigh distribution for $\alpha = 0.5, 0.8, 1$ and $\beta = 1$. It is not hard to find that failure rate of IERD is nonmonotone.

We should firstly define this progressively first-failure censored scheme as below: we assume there are n independent groups in a life experiment, each of which has k units. Once the first failed group is discovered, we will remove the group and R_1 groups from the experiment, and the second failed group has taken place while R_2 groups and the group are got rid of samples at random, which is discovered the second failure, and so on. The procedure would work m times, then we can get $x_{1:m:n:k} < x_{2:m:n:k} < \dots < x_{m:m:n:k}$ with the progressively censoring scheme $R = (R_1, R_2, \dots, R_m)$ as progressively first-failure censoring order statistics, which comes from a population with pdf $f(\cdot)$ and cdf $F(\cdot)$. For simplicity, we shall utilize x_i to take place of $x_{i:m:n:k}$. It is obvious that $m + R_1 + R_2 + \dots + R_m = n$. The joint pdf with progressively first-failure censoring dataset was derived by Wu and Kus (2009):

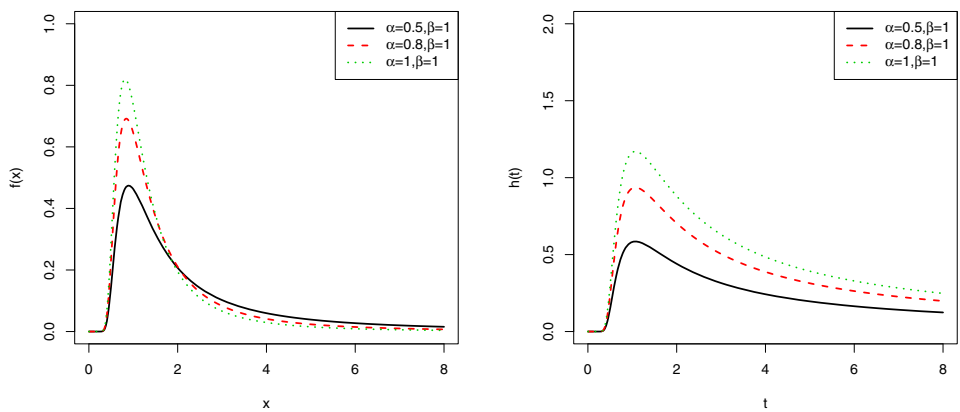
$$f(x_1, x_2, \dots, x_m) = Ck^m \prod_{i=1}^m f(x_i) (1 - F(x_i))^{k(R_i+1)-1}; \quad (1.5)$$

$$0 < x_1 < x_2 < \dots < x_m < \infty.$$

$C = n(n - R_1 - 1)(n - R_1 - R_2 - 1) \dots (n - R_1 - R_2 - \dots - R_{m-1} - m + 1)$. Note that the censored scheme $R = (R_1, R_2, \dots, R_m)$ could be determined in advance.

Obviously, if $k = 1$ and $R_1 = R_2 = \dots = R_m = 0$, the progressively first-failure censored scheme becomes the complete sample. With group size $k = 1$, the scheme develops into the progressively type-II censoring. Also, when

Fig. 1 Plots of $f(t)$ and $h(t)$ of IERD when $\beta = 1$



(a) Graph of pdf of IERD when $\beta = 1$

(b) Graph of $h(t)$ of IERD when $\beta = 1$

$R = (R_1 = R_2 = \dots = R_{m-1} = 0)$ and $R_m = n - m$, the progressively first-failure censoring scheme is changed into first failure type-II sample, and it corresponds to first failure censored sample and let $R = (0, \dots, 0)$. And more interesting, this censoring plan can be regarded as traditional type-II censoring plan while $k = 1$ and $R = (0, \dots, n - m)$. Hence, progressively first-failure censored is a generalization of censoring whose advantages is considering test cost and time.

The structure of article is as below. Section 2 deals with MLEs, which uses the observed Fisher information matrix to obtain parameters' asymptotic confidence intervals (CI) and coverage probabilities (CP). We provide the parametric bootstrap CIs of the parameters with constructing the censoring sample in Sect. 3. Section 4 discusses Bayesian estimators of parameters, and we cover the Bayes estimations using Metropolis–Hastings (M–H) technique to gain highest posterior density (HPD) credible intervals of parameters in the same section. Section 5 adopts a method of simulation study by Monte Carlo. It is applied in order to evaluate the performance of different censored schemes, group number and group sizes. Finally, we illustrate all the methods of this article by a real dataset in the last Section.

2 Maximum likelihood estimation

2.1 Point estimation

x_1, x_2, \dots, x_m can be seen as a censoring sample, which is from the inverted exponentiated Rayleigh distribution (α, β) whose censoring scheme is \mathbf{R} . Using Eqs. (1.1), (1.2) and (1.5), the likelihood function can be obtained

$$L(\alpha, \beta) = c k^m 2^m \alpha^m \beta^m \prod_{i=1}^m x_i^{-3} e^{\sum_{i=1}^m \beta/x_i^2} e^{\sum_{i=1}^m \{ \alpha k(R_i+1) - 1 \} \ln(1 - e^{-\beta/x_i^2})}. \tag{2.1}$$

We can also write the log-likelihood function as

$$\begin{aligned} \log L(\alpha, \beta) &\propto m \ln(\alpha) + m \ln(\beta) - \beta \sum_{i=1}^m \left(\frac{1}{x_i^2} \right) \\ &+ \sum_{i=1}^m \{ \alpha k(R_i + 1) - 1 \} \ln(1 - e^{-\beta/x_i^2}) \end{aligned} \tag{2.2}$$

By taking derivatives from Eq. (2.2) with respect to α and β , then equating to zero in order to get the equations:

$$\frac{\partial \log L(\alpha, \beta)}{\partial \alpha} = \frac{m}{\alpha} + \sum_{i=1}^m k(R_i + 1) \ln(1 - e^{-\beta/x_i^2}) = 0, \tag{2.3}$$

$$\begin{aligned} \frac{\partial \log L(\alpha, \beta)}{\partial \beta} &= \frac{m}{\beta} - \sum_{i=1}^m \frac{1}{x_i^2} \\ &+ \sum_{i=1}^m \frac{1}{x_i^2} \frac{\alpha k(R_i + 1) - 1}{1 - e^{-\beta/x_i^2}} e^{-\beta/x_i^2} = 0. \end{aligned} \tag{2.4}$$

From Eq. (2.3) obtain that

$$\hat{\alpha} = - \frac{m}{\sum_{i=1}^m k(R_i + 1) \ln(1 - e^{-\beta/x_i^2})}, \tag{2.5}$$

Then replacing the Eq. (2.4) with Eq. (2.5)

$$\begin{aligned} \frac{m}{\beta} - \sum_{i=1}^m \left(\frac{1}{x_i^2} \right) \\ + \sum_{i=1}^m \frac{1}{x_i^2} \frac{-m \{ k(R_i + 1) - 1 \}}{\{ \sum_{i=1}^m k(R_i + 1) \ln(1 - e^{-\beta/x_i^2}) \} (1 - e^{-\beta/x_i^2})} e^{-\beta/x_i^2} = 0. \end{aligned} \tag{2.6}$$

The $\hat{\beta}$ can be got with the help of Newtown-Raphson iteration method. At the same time, with the invariance character of MLEs, $r(t)$ and $h(t)$ can be derived at a pre-determined time t

$$r(t) = \left(1 - e^{-(\hat{\beta}/t^2)}\right)^{\hat{\alpha}} \tag{2.7}$$

$$h(t) = 2\hat{\alpha}\hat{\beta}t^{-3}e^{-(\hat{\beta}/t^2)}\left(1 - e^{-(\hat{\beta}/t^2)}\right)^{-1} \tag{2.8}$$

Here, $t > 0$.

2.2 Confidence interval estimation

On the base of likelihood equation, this section infers observed Fisher information (OFI). We rename $\theta = (\alpha, \beta)$, we can obtain θ 's Fisher information matrix

$$I(\theta) = E \begin{bmatrix} -\frac{\partial^2 \log L(\alpha, \beta)}{\partial \alpha^2} & -\frac{\partial^2 \log L(\alpha, \beta)}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \log L(\alpha, \beta)}{\partial \beta \partial \alpha} & -\frac{\partial^2 \log L(\alpha, \beta)}{\partial \beta^2} \end{bmatrix}$$

Here,

$$\begin{aligned} \frac{\partial^2 \log L(\alpha, \beta)}{\partial \alpha^2} &= -\frac{m}{\alpha^2} \\ \frac{\partial^2 \log L(\alpha, \beta)}{\partial \beta^2} &= -\frac{m}{\beta^2} - \sum_{i=1}^m \frac{\{k\alpha(R_i + 1) - 1\}e^{-\beta/x_i^2}}{x_i^4(1 - e^{-\beta/x_i^2})^2} \\ \frac{\partial^2 \log L(\alpha, \beta)}{\partial \alpha \partial \beta} &= \frac{\partial^2 \log L(\alpha, \beta)}{\partial \beta \partial \alpha} = \sum_{i=1}^m \frac{k(R_i + 1)e^{-\beta/x_i^2}}{x_i^2(1 - e^{-\beta/x_i^2})^2} \end{aligned}$$

We use OFI matrix in our calculations, not the Fisher information matrix, because the expectation of the above expressions are hard to solve. The OFI matrix is obtained as $I(\hat{\theta})$ with $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$

$$I(\hat{\theta}) = \begin{bmatrix} -\frac{\partial^2 \log L(\alpha, \beta)}{\partial \alpha^2} & -\frac{\partial^2 \log L(\alpha, \beta)}{\partial \alpha \partial \beta} \\ -\frac{\partial^2 \log L(\alpha, \beta)}{\partial \beta \partial \alpha} & -\frac{\partial^2 \log L(\alpha, \beta)}{\partial \beta^2} \end{bmatrix}_{\theta=\hat{\theta}}$$

Next, the inverted of OFI matrix is the observed variance-covariance matrix of MLEs we need,

$$I^{-1}(\hat{\theta}) = \begin{bmatrix} \hat{V}ar(\hat{\alpha}) & \hat{C}ov(\hat{\alpha}, \hat{\beta}) \\ \hat{C}ov(\hat{\beta}, \hat{\alpha}) & \hat{V}ar(\hat{\beta}) \end{bmatrix}$$

The asymptotic distribution of $\hat{\theta}$ is a normal distribution as $\hat{\theta} \sim N(\theta, I^{-1}(\hat{\theta}))$ by Lawless (2011) given, which estimation method is MLE. Therefore, bivariate normal distribution as $100(1 - \varepsilon)\%$ CI for θ is $\hat{\theta} \pm z_{\varepsilon/2}\sqrt{\hat{V}ar(\hat{\theta})}$. And the CP can be gained by Monte Carlo simulation as

$$CP_{\theta} = P \left[\left| \frac{(\hat{\theta} - \theta)}{\sqrt{\hat{V}ar(\hat{\theta})}} \right| \leq z_{\varepsilon/2} \right]$$

where the right $(\varepsilon/2)$ th percentile of standard normal distribution as $z_{\varepsilon/2}$, and θ could be α or β .

3 Bootstrap estimation for confidence intervals

Obviously, the CIs with asymptotic normal is executed well in view of the large effective sample m . Nevertheless, the sample m in the actual situation is not always large. The bootstrap method would be a wise choice. In this section, we propose two parametric bootstrap algorithms: (I) percentile bootstrap (Boot-p) method utilizes theory of Efron (1982) and (II) percentile bootstrap-t (Boot-t) method in view of Hall (1988). We show the bootstrap procedures below.

3.1 Boot-p method

- (1) Step 1: Compute MLEs $(\hat{\alpha}, \hat{\beta})$ with the original sample x_1, x_2, \dots, x_m , where n , k and \mathbf{R} are the amount of groups, the size of each group and censored schemes, respectively.
- (2) Step 2: Use $(\hat{\alpha}, \hat{\beta})$ to create an independent sample, which in order to compute the MLEs $(\hat{\alpha}^*, \hat{\beta}^*)$.
- (3) Step 3: Repeat N times of Step 2, we can acquire a series of bootstrap estimates $(\hat{\alpha}_i^*, \hat{\beta}_i^*)$, $i = 1, 2, \dots, N$.
- (4) Step 4: The $100(1 - 2\varepsilon)\%$ Boot-p CI of $\hat{\alpha}$ is provided as

$$(\hat{\alpha}_{Boot-p(\varepsilon)}^*, \hat{\alpha}_{Boot-p(1-\varepsilon)}^*)$$

after ordering $\hat{\alpha}_{(1)}^* \leq \hat{\alpha}_{(2)}^* \leq \dots \leq \hat{\alpha}_{(m)}^*$. As for β , we can use the same method to gain the similar result.

3.2 Boot-t method

The initial two steps of Boot-t method are alike as Boot-p method.

- (3) Step 3: Do last step N times to compute a series of statistics:

$$T_{zi}^* = \frac{(\hat{\alpha}^*) - \hat{\alpha}}{\sqrt{I^{-1}(\hat{\alpha}^*)}}$$

$i = 1, 2, \dots, N$.

- (4) Step 4: The circa $100(1 - 2\varepsilon)\%$ Boot-t CI of $\hat{\alpha}$ is given by

$$(\hat{\alpha} - T_{\alpha Boot-t(\varepsilon)}^* \sqrt{I^{-1}(\hat{\alpha}^*)}, \hat{\alpha} + T_{\alpha Boot-t(1-\varepsilon)}^* \sqrt{I^{-1}(\hat{\alpha}^*)})$$

after ordering $T_{(\alpha 1)}^* \leq T_{(\alpha 2)}^* \leq \dots \leq T_{(\alpha m)}^*$.

Similarly, the same approach is used to β would achieve the confidence interval of β .

4 Bayesian estimation

Although the traditional methods of estimation are always effective, it is possible to appear hard in mathematical. In the other hand, the simulation method would acquire more attention, for example, Markov Chain Monte Carlo (MCMC). And mroe, the simulation can get point estimation, at the same time, provide the interval estimation. For parameter estimation we have considered squared error loss function. We suppose that gamma priors for both α, β :

$$\pi_1(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-b\alpha}, \alpha > 0, a > 0, b > 0, \tag{4.1}$$

$$\pi_2(\beta) = \frac{d^c}{\Gamma(c)} \beta^{c-1} e^{-d\beta}, \beta > 0, c > 0, d > 0. \tag{4.2}$$

here a, b, c, d reflect prior knowledge about parameters. The model shall more flexible because of hyper-parameters, and when the super-parameters are zero, which is the non-informative prior. In addition, the prior distribution of gamma distribution is very reasonable, many authors have applied this prior, see Gupta and Singh (2013) and Almutairi et al. (2015). The posterior distribution of parameters can be given with combining the likelihood function and the prior distributions

$$h_{\alpha,\beta}(\alpha, \beta|data) = \frac{l(data|\alpha, \beta)\pi_1(\alpha)\pi_2(\beta)}{\int_0^\infty \int_0^\infty l(data|\alpha, \beta)\pi_1(\alpha)\pi_2(\beta)d\alpha d\beta} \tag{4.3}$$

the joint posterior can be written as

$$h_{\alpha,\beta}(\alpha, \beta|data) \propto \alpha^{m+a-1} \beta^{m+c-1} \prod_{i=1}^m \frac{e^{-\beta_i/x_i^2}}{1 - e^{-\beta_i/x_i^2}} e^{\{-\alpha[b - \sum_{i=1}^m k(R_i+1) \ln(1 - e^{-\beta_i/x_i^2})] - d\beta\}} \tag{4.4}$$

Because we don't have enough prior information, so consider $a = b = c = d = 0$. Next, we should give the marginal posterior distributions of parameters α, β , respectively:

$$h_\alpha(\alpha, \beta|data) \propto \alpha^{m+a-1} e^{\{-\alpha[b - \sum_{i=1}^m k(R_i+1) \ln(1 - e^{-\beta_i/x_i^2})]\}} \tag{4.5}$$

$$h_\beta(\alpha, \beta|data) \propto \beta^{m+c-1} \prod_{i=1}^m \frac{e^{-\beta_i/x_i^2}}{1 - e^{-\beta_i/x_i^2}} e^{\{\sum_{i=1}^m \alpha k(R_i+1) \ln(1 - e^{-\beta_i/x_i^2}) - d\beta\}} \tag{4.6}$$

We can find that $h_\alpha(\alpha, \beta|data)$ is a gamma density whose parameters are $(m + a, b - \sum_{i=1}^m k(R_i + 1) \ln(1 - e^{-\beta_i/x_i^2}))$, it's not difficult to use any gamma generating algorithm to obtain samples of α . However, the samples of β are not directly to get by standard approaches. For this reason, we consider to apply Metropolis–Hastings algorithm.

4.1 Point estimation of Metropolis–Hastings algorithm

There are many methods to solve Bayesian estimation, for example, Lindley's method, importance sampling procedure and Metropolis–Hastings algorithm (M–H). However, M–H is more attractive than other two methods, because: (I) the numerical procedures and computational are relative straightforward, (II) it provides point estimations, also gives HPD intervals for parameters, and (III) it allows a full analysis of the data. M–H approach is more flexible, simple and effective. For applications of the algorithm, one can refer to Dey and Pradhan (2014).

M–H approach is developed by Metropolis et al. (1952) and later extended by Hastings (1970), and now it is a more popular MCMC method. The M–H algorithm works as below:

- (1) Step 1: give an initial surmise of β^0 and make $t=1$.
- (2) Step 2: Crate α^t by using Gamma $(m + a, b - \sum_{i=1}^m k(R_i + 1) \ln(1 - e^{-\beta_i/x_i^2}))$.
- (3) Step 3: Crate β^* from $h_\beta(\alpha, \beta|data)$ with the proposal distribution $N(\beta^{t-1}, \sigma^2)$, where σ^2 is the variance of β .
- (4) Step 4: Calculate $\rho(\beta^*|\beta^{t-1}) = \min \left\{ \frac{f(\beta^*|x)q(\beta^{t-1}|\beta^*)}{f(\beta^{t-1}|x)q(\beta^*|\beta^{t-1})}, 1 \right\}$.
- (5) Step 5: Take v from Uniform(0,1).
- (6) Step 6: If $v \leq \rho(\beta^*|\beta^{t-1})$ then $\beta^t = \beta^*$, otherwise $\beta^t = \beta^{t-1}$.
- (7) Step 7: let $t = t + 1$.
- (8) Step 8: Repeat Step 2–7 N times.

We discard M initial samples in order to get an independent sample. Thus, the approximate Bayes estimation of $\theta = (\alpha, \beta)$ is obtained as

$$\hat{\theta}_{MH} = \frac{1}{N - M} \sum_{j=M+1}^N \theta_j$$

4.2 Highest posterior density credible interval

The point estimation does not take the sampling error into consideration. It is difficult to make a judgement by simply

Table 1 Bias and MSE of MLE and Bayes estimate of the α assume $\alpha = 0.5, \beta = 1$

(k, n, m)	CS	$\hat{\alpha}_{ML}$		$\hat{\alpha}_{Bayes}$		$\hat{\alpha}_{Bayes}$	
				Prior 1		Prior 2	
		Bias	MSE	Bias	MSE	Bias	MSE
(2, 50, 25)	(25, 0 * 24)	0.0587	0.0306	0.0313	0.0273	0.0499	0.0271
	(1 * 25)	0.0871	0.0552	0.0521	0.0403	0.0685	0.0408
	(0 * 24, 25)	0.0872	0.0905	0.0713	0.0722	0.0910	0.0623
(2, 50, 30)	(20, 0 * 29)	0.0457	0.0224	0.0259	0.0212	0.0369	0.0203
	((2, 0, 0) * 10)	0.0687	0.0372	0.0355	0.0307	0.0428	0.0260
	(0 * 29, 20)	0.0692	0.0492	0.0534	0.0484	0.0656	0.0356
(3, 50, 25)	(25, 0 * 24)	0.0989	0.0359	0.0272	0.0283	0.0530	0.0307
	(1 * 25)	0.1056	0.1158	0.0654	0.0609	0.0830	0.0537
	(0 * 24, 25)	0.1305	0.1429	0.1116	0.1716	0.1027	0.0738
(3, 50, 30)	(1 * 25)	0.0522	0.0249	0.0201	0.0218	0.0333	0.0196
	((2, 0, 0) * 10)	0.0558	0.0434	0.0343	0.0321	0.0554	0.0328
	(0 * 29, 20)	0.0596	0.0577	0.0634	0.0611	0.0850	0.0462

Table 2 Bias and MSE of MLE and Bayes estimate of the β assume $\alpha = 0.5, \beta = 1$

(k, n, m)	CS	$\hat{\beta}_{ML}$		$\hat{\beta}_{Bayes}$		$\hat{\beta}_{Bayes}$	
				Prior 1		Prior 2	
		Bias	MSE	Bias	MSE	Bias	MSE
(2, 50, 25)	(25, 0 * 24)	0.0638	0.0672	- 0.0013	0.0616	0.0263	0.0557
	(1 * 25)	0.0915	0.0781	0.0160	0.0571	0.0425	0.0525
	(0 * 24, 25)	0.0790	0.0872	0.0169	0.0719	0.0412	0.0598
(2, 50, 30)	(20, 0 * 29)	0.0680	0.0584	0.0112	0.0548	0.0135	0.0464
	((2, 0, 0) * 10)	0.0799	0.0664	0.0136	0.0501	0.0135	0.0454
	(0 * 29, 20)	0.0747	0.0756	0.0071	0.0580	0.0425	0.0530
(3, 50, 25)	(25, 0 * 24)	0.0529	0.0573	- 0.0004	0.0466	0.0291	0.0460
	(1 * 25)	0.0697	0.0693	0.0123	0.0518	0.0310	0.0456
	(0 * 24, 25)	0.0911	0.0783	0.0291	0.0674	0.0343	0.0545
(3, 50, 30)	(1 * 25)	0.0614	0.0503	- 0.0047	0.0413	0.0158	0.0366
	((2, 0, 0) * 10)	0.0520	0.0517	0.0011	0.0421	0.0183	0.0370
	(0 * 29, 20)	0.0600	0.0571	0.0059	0.0500	0.0404	0.0462

Table 3 Bias and MSE of MLE and Bayes estimate of $r(t)$ assume $t=2$

(k, n, m)	CS	$r(\hat{t})_{ML}$		$r(\hat{t})_{Bayes}$		$r(\hat{t})_{Bayes}$	
				Prior 1		Prior 2	
		Bias	MSE	Bias	MSE	Bias	MSE
(2, 50, 25)	(25, 0 * 24)	- 0.0139	0.0058	- 0.0107	0.0056	- 0.0134	0.0056
	(1 * 25)	- 0.0192	0.0109	- 0.0161	0.0071	- 0.0250	0.0071
	(0 * 24, 25)	- 0.0080	0.0187	- 0.0241	0.0091	- 0.0296	0.0077
(2, 50, 30)	(20, 0 * 29)	- 0.0122	0.0050	- 0.0114	0.0047	- 0.0108	0.0043
	((2, 0, 0) * 10)	- 0.0181	0.0060	- 0.0086	0.0051	- 0.0178	0.0051
	(0 * 29, 20)	- 0.0143	0.0099	- 0.0113	0.0064	- 0.0224	0.0060
(3, 50, 25)	(25, 0 * 24)	- 0.0184	0.0068	- 0.0110	0.0059	- 0.0200	0.0058
	(1 * 25)	- 0.0091	0.0163	- 0.0196	0.0095	- 0.0313	0.0086
	(0 * 24, 25)	- 0.0267	0.0195	- 0.0317	0.0126	- 0.0434	0.0115
(3, 50, 30)	(1 * 25)	- 0.0173	0.0059	- 0.0103	0.0054	- 0.0133	0.0050
	((2, 0, 0) * 10)	- 0.0128	0.0093	- 0.0126	0.0066	- 0.0270	0.0069
	(0 * 29, 20)	- 0.0023	0.0171	- 0.0243	0.0087	- 0.0250	0.0078

Table 4 Bias and MSE of MLE and Bayes estimate of $h(t)$ assume $t=2$

(k, n, m)	CS	$h(\hat{t})_{ML}$		$h(\hat{t})_{Bayes}$		$h(\hat{t})_{Bayes}$	
				Prior 1		Prior 2	
		Bias	MSE	Bias	MSE	Bias	MSE
(2, 50, 25)	(25, 0 * 24)	0.0068	0.0004	0.0033	0.0003	0.0042	0.0002
	(1 * 25)	0.0108	0.0008	0.0056	0.0005	0.0065	0.0005
	(0 * 24, 25)	0.0145	0.0016	0.0079	0.0008	0.0078	0.0005
(2, 50, 30)	(20, 0 * 29)	0.0061	0.0003	0.0027	0.0003	0.0037	0.0002
	((2, 0, 0) * 10)	0.0080	0.0004	0.0036	0.0004	0.0042	0.0003
	(0 * 29, 20)	0.0102	0.0007	0.0051	0.0005	0.0064	0.0004
(3, 50, 25)	(25, 0 * 24)	0.0075	0.0005	0.0037	0.0003	0.0049	0.0003
	(1 * 25)	0.0108	0.0014	0.0062	0.0008	0.0073	0.0005
	(0 * 24, 25)	0.0161	0.0019	0.0013	0.0147	0.0109	0.0008
(3, 50, 30)	(1 * 25)	0.0075	0.0004	0.0034	0.0002	0.0038	0.0002
	((2, 0, 0) * 10)	0.0082	0.0005	0.0047	0.0003	0.0060	0.0003
	(0 * 29, 20)	0.0098	0.0008	0.0075	0.0008	0.0073	0.0005

Table 5 AL and associated CP of the parameter $\alpha = 0.5$ with 95% asymptotic CI, bootstrap CI, HPD credible interval

(k, n, m)	CS	$\hat{\alpha}_{ML}$		$\hat{\alpha}_{Boot-p}$		$\hat{\alpha}_{Boot-t}$		$\hat{\alpha}_{Bayes}$		$\hat{\alpha}_{Bayes}$	
								Prior 1		Prior 2	
		AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
(2, 50, 25)	(25, 0 * 24)	0.5738	0.96	0.6998	0.93	0.6545	0.71	0.5557	0.95	0.5334	0.96
	(1 * 25)	0.7256	0.96	1.0055	0.87	1.0523	0.68	0.6979	0.94	0.6541	0.96
	(0 * 24, 25)	0.8584	0.93	1.3734	0.88	1.3591	0.64	0.8796	0.93	0.7812	0.94
(2, 50, 30)	(20, 0 * 29)	0.5104	0.96	0.6536	0.89	0.7406	0.68	0.5000	0.94	0.4982	0.95
	((2, 0, 0) * 10)	0.5899	0.97	0.7560	0.89	0.9881	0.68	0.5660	0.95	0.5453	0.96
	(0 * 29, 20)	0.6896	0.96	0.9260	0.93	1.3020	0.60	0.6584	0.95	0.6265	0.95
(3, 50, 25)	(25, 0 * 24)	0.6247	0.97	0.8397	0.89	0.8694	0.70	0.6063	0.95	0.5940	0.94
	(1 * 25)	0.7686	0.92	1.2403	0.90	1.1614	0.80	0.7665	0.94	0.7325	0.95
	(0 * 24, 25)	0.9960	0.94	1.6846	0.89	1.8679	0.68	1.0489	0.94	0.9204	0.96
(3, 50, 30)	(1 * 25)	0.5592	0.98	0.7424	0.89	0.7279	0.72	0.5481	0.94	0.5391	0.96
	((2, 0, 0) * 10)	0.6668	0.96	0.8764	0.92	0.9143	0.64	0.6413	0.95	0.6230	0.95
	(0 * 29, 20)	0.7328	0.88	1.0421	0.93	1.1710	0.65	0.7658	0.96	0.7416	0.96

$$f(x; \alpha, \beta) = \alpha\beta x^{-2} e^{-(\beta/x)} \left(1 - e^{-(\beta/x)}\right)^{\alpha-1}, \quad x > 0, \alpha > 0, \beta > 0$$

$$f(x; \alpha, \beta) = \alpha\beta \frac{1}{x(1+x)} e^{-\beta \ln(1+1/x)} \left(1 - e^{-\beta \ln(1+1/x)}\right)^{\alpha-1},$$

$$x > 0, \alpha > 0, \beta > 0$$

We present different criterions to evaluate the optimum degree for fitting these distributions, which applying MLE to compute subtractive log-likelihood function ($-\ln L$), Akaike information criterion (AIC), Bayesian information criterion (BIC) and Kolmogorov–Smirnov ($K-S$) statistic, we just to know p values of $K-S$. Here, the maximum value for likelihood function of an estimated model is named L .

And AIC is defined that $AIC = 2 \times (p - \ln(L))$, and the amount of distribution’s parameters as p . We also give definition of $BIC : BIC = p \times \ln(n) - 2 \times \ln(L)$, and the sum of observations in sample as n . Thus the minimum $-\ln L, AIC, BIC$ and highest p values of $K-S$ statistics values would be the criterions for best distribution. Apart from that, we propose a graphic method for the optimum degree of fitting different distributions at Fig. 2.

Table 8 lists that $-\ln L, AIC, BIC$ and p values.

Next, we divide the dataset into $n = 31$ groups, which has $k = 2$ units in each group, to generate first-failure censored sample. Then, we consider three progressive

Table 6 AL and associated CP of the parameter $\beta = 1$ with 95% asymptotic CI, bootstrap CI, HPD credible interval

(k, n, m)	CS	$\hat{\beta}_{ML}$		$\hat{\beta}_{Boot-p}$		$\hat{\beta}_{Boot-t}$		$\hat{\beta}_{Bayes}$		$\hat{\beta}_{Bayes}$	
								Prior 1		Prior 2	
		AL	CP	AL	CP	AL	CP	AL	CP	AL	CP
(2, 50, 25)	(25, 0 * 24)	0.9249	0.96	1.0417	0.92	0.9785	0.81	0.9098	0.93	0.8784	0.95
	(1 * 25)	0.9274	0.94	1.0416	0.90	1.0279	0.68	0.9068	0.94	0.8627	0.97
	(0 * 24, 25)	1.0173	0.93	1.0983	0.93	1.1624	0.65	0.9820	0.93	0.9395	0.96
(2, 50, 30)	(20, 0 * 29)	0.8902	0.95	0.9637	0.95	1.0291	0.73	0.8657	0.94	0.8446	0.93
	((2, 0, 0) * 10)	0.8601	0.94	0.9711	0.88	1.0653	0.74	0.8401	0.96	0.8194	0.95
	(0 * 29, 20)	0.9369	0.94	1.0505	0.89	1.1154	0.66	0.8926	0.93	0.8645	0.94
(3, 50, 25)	(25, 0 * 24)	0.8592	0.95	0.9171	0.95	0.9391	0.73	0.8358	0.94	0.8060	0.94
	(1 * 25)	0.8508	0.93	0.9734	0.88	0.9141	0.84	0.8401	0.96	0.8194	0.95
	(0 * 24, 25)	0.9446	0.94	1.0914	0.88	1.0755	0.71	0.8254	0.94	0.8645	0.94
(3, 50, 30)	(1 * 25)	0.8075	0.96	0.8805	0.91	0.8650	0.75	0.7845	0.94	0.7678	0.96
	((2, 0, 0) * 10)	0.7929	0.94	0.8422	0.92	0.8515	0.72	0.7782	0.94	0.7495	0.94
	(0 * 29, 20)	0.8585	0.92	0.9496	0.93	0.9669	0.65	0.8421	0.96	0.8151	0.95

Table 7 Data set (gauge lengths of 10mm)

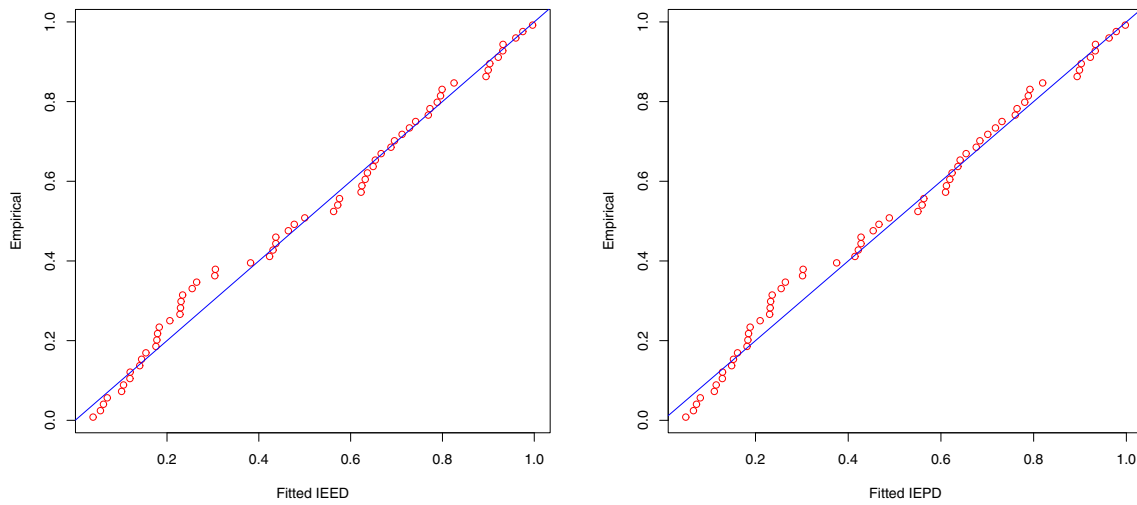
1.901	2.132	2.203	2.228	2.257	2.350	2.361	2.396	2.397
2.445	2.454	2.474	2.518	2.522	2.525	2.532	2.575	2.614
2.616	2.618	2.624	2.659	2.675	2.738	2.740	2.856	2.917
2.928	2.937	2.997	2.977	2.996	3.030	3.125	3.139	3.145
3.220	3.223	3.235	3.243	3.264	3.272	3.294	3.332	3.346
3.377	3.408	3.435	3.493	3.501	3.537	3.554	3.562	3.628
3.852	3.871	3.886	3.971	4.024	4.027	4.225	4.395	5.020

censoring schemes with the last sample. Table 9 presents the censoring schemes and corresponding samples.

We estimate the two parameters and reliability characteristics with MLEs and Bayes estimators in Table 10. And, we compute $r(t)$ and $h(t)$ corresponding to real data by using $t = 4$. This section considers Bayes estimation with non-informative prior, due to that we don't have prior information about the parameters. Table 11 lists the 95% asymptotic, Boot-p, Boot-t CIs and HPD credible intervals for parameters.

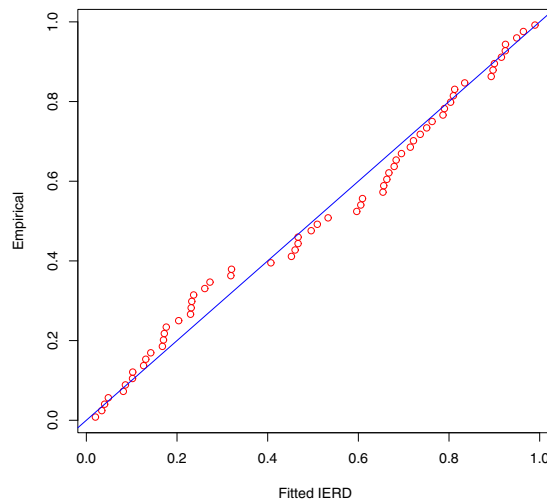
7 Conclusion

Censoring is an ordinary technique to acquire sample from various experiments. Progressively first-failure censored is an extension of censoring, which helps to reduce time and cost. And, the hazard rate of IERD is nonmonotone. It is common that nonmonotone of hazard rate in many practical situations. We study the estimation of parameters of IERD under progressively first-failure censored in our paper. We propose MLE and corresponding asymptotic CI estimate of the parameters with IERD. We then compute confidence intervals with bootstrap method. The Bayesian estimates and the associated HPD interval estimates under square error loss function are developed. It is hard to obtain explicit forms of Bayes estimators, however, we make use of M–H algorithm for Bayes estimation. A simulation is used to evaluate the various estimators' work. And the theoretical results have been applied with real data. Moreover, the methodology is discussed in our article could be helpful to data analyst and reliability practitioners.



(a) Fitted inverted exponentiate exponential distribution

(b) Fitted inverted exponentiated Pareto distribution



(c) Fitted inverted exponentiated Rayleighdistribution

Fig. 2 quantile-quantile plots of three distributions with actual example

Table 8 Fitting of the real data

Parameters	IERD $\hat{\alpha} = 5.6503, \hat{\beta} = 10.7621$	IEED $\hat{\alpha} = 55.7988, \hat{\beta} = 10.0187$	IEPD $\hat{\alpha} = 119.1140, \hat{\beta} = 14.2503$
$-\ln L$	- 47.6455	- 46.9101	- 46.0216
<i>AIC</i>	- 91.2912	- 89.8203	- 88.0430
<i>BIC</i>	- 87.0369	- 85.5660	- 83.7889
<i>K-S</i>	0.0897	0.0903	0.0904
<i>p</i> values	0.7002	0.6929	0.6915

Table 9 Different progressively first-failure censoring samples

(k,n,m)	CS	Progressively first-failure censoring data
(2, 31, 25)	$R_1 = (6, 0 * 24)$	1.382, 1.782, 1.864, 1.868, 1.909, 1.988, 2.106, 2.178, 2.187 2.246, 2.375, 2.395, 2.473, 2.493, 2.522, 2.582, 2.627, 2.685 2.751, 2.804, 2.878, 3.121, 3.221, 3.277, 3.645
(2, 31, 25)	$R_2 = ((1, 0, 0, 0) * 6, 0)$	1.382, 1.600, 1.646, 1.695, 1.724, 1.782, 1.864, 1.868, 1.909 2.106, 2.178, 2.187, 2.246, 2.395, 2.473, 2.493, 2.522, 2.627 2.685, 2.751, 2.804, 3.121, 3.221, 3.277, 3.645
(2, 31, 25)	$R_3 = (0 * 24, 6)$	1.382, 1.478, 1.600, 1.646, 1.695, 1.724, 1.772, 1.782, 1.864 1.868, 1.909, 1.988, 2.106, 2.178, 2.187, 2.246, 2.375, 2.395 2.473, 2.493, 2.522, 2.582, 2.627, 2.685, 2.751

Table 10 MLE and Bayes estimate corresponding to actual example

Estimate	CS		
	R_1	R_2	R_3
$\hat{\alpha}_{ML}$	4.5680	2.1851	2.6503
$\hat{\beta}_{ML}$	14.9623	10.7438	10.3504
$\hat{\alpha}_{Bayes}$	4.5690	2.1886	2.6611
$\hat{\beta}_{Bayes}$	14.9578	10.7547	10.3532
$r(\hat{t})_{ML}$	0.1026	0.2095	0.1401
$h(\hat{t})_{ML}$	0.5093	0.1833	0.2138
$r(\hat{t})_{Bayes}$	0.1025	0.2093	0.1390
$h(\hat{t})_{Bayes}$	0.5093	0.1838	0.2148

Table 11 The 95% CI of asymptotic, Boot-p and Boot-t CI, and HPD credible interval of actual example

Estimate	CS		
	R_1	R_2	R_3
$\hat{\alpha}_{ML}$	(0.6904, 8.4457)	(0.5567, 3.8135)	(0.4465, 4.8541)
$\hat{\beta}_{ML}$	(9.6000, 20.3245)	(6.7000, 14.7876)	(6.3487, 14.3521)
$\hat{\alpha}_{Boot-p}$	(2.5303, 14.9127)	(1.2413, 6.2668)	(1.5070, 9.1892)
$\hat{\beta}_{Boot-p}$	(11.2226, 22.4234)	(7.8579, 16.6567)	(7.5720, 16.7106)
$\hat{\alpha}_{Boot-t}$	(0.1023, 7.0933)	(0.3078, 3.2868)	(0.3570, 4.0318)
$\hat{\beta}_{Boot-t}$	(9.9535, 20.5308)	(6.9551, 14.8850)	(6.7918, 14.5405)
$\hat{\alpha}_{Bayes}$	(2.9770, 6.5026)	(1.4008, 3.1075)	(1.7343, 3.8154)
$\hat{\beta}_{Bayes}$	(14.9533, 14.9648)	(10.7509, 10.7607)	(10.3472, 10.3581)

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