Int J Syst Assur Eng Manag (August 2019) 10(4):552–562
https://doi.org/10.1007/s13198-019-00769-x

ORIGINAL ARTICLE

A two-echelon integrated inventory model under generalized lead time distribution with variable backordering rate

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Received: 24 April 2018 / Revised: 28 January 2019 / Published online: 18 February 2019

Abstract This paper investigates a generalized lead time distribution with a variable backordering rate in a two-echelon supply chain system. The vendor produces a single product and delivers to the buyer in equal sized batches. The delivery lead time follows a generalized stochastic variable. Shortages are allowed to occur and backordered partially. The backorder rate depends on the demand on stock-out period. Based on this notion, we formulate a mixed integer non-linear cost function which needs to be minimized with respect to reorder point, number of deliveries and lot size from the vendor to the buyer, to operate cooperatively in the integrated model. Analytically we proved the convexity of the generalized lead time distribution cost function with respect to the control parameters. Further, the uniqueness of optimality has been proved. To validate the proposed model, uniform, exponential and normal distributed lead times are presented in numerical example section. Sensitivity analysis also performed to the values of the parameters.

Keywords Cost minimization · Integrated economic lot-sizing · Inventory control · Stochastic delivery time · Variable backorder rate

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1 Introduction

This paper develops a two-echelon integrated inventory control with the general distribution of the procurement lead time of a lot size, and variable backorder rate. In this section, we present some of previous works regarding this work.

Today's highly competitive market conditions, the integrated strategy plays a vital role in the supply chain network. Goyal (1977) firstly introduce the single vendor single supplier integrated system. This paper observes that the total cost of the integrated approach is less than the sum of the total cost of the individuals. After this work, till now many researchers incorporate this integrated strategy into their literature (e.g., Ben-Daya and Raouf 1994; Cárdenas-Barrón et al. 2012, 2011; Chung and Wee 2008; Huang 2004; Lee et al. 2007, 2017; Lin 2009; Ouyang et al. 2007; Pan and Yang 2002; Sajadieh et al. 2010; Wangsa and Wee 2017; Wu et al. 2007; Yang 2010).

In all supply chain systems, the lead time is an important factor and many inventory control model involves the deterministic lead times. Controllable lead time is one of the deterministic lead time strategies which is incorporated by many authors (e.g., Ben-Daya and Raouf 1994; Lee et al. 2007; Lin 2009; Ouyang et al. 2007; Pan and Yang 2002; Wu et al. 2007) and it is first introduced by Tersine (1982). According to Tersine model, the lead time may be consisting of many segments like, setup order time, delivery time, distributor lead time etc. These segments can be shortened by additional cost called crashing cost.

A common assumption of aforesaid literature is either deterministic lead time or known variable lead time. In practice, there might be plenty of environmental causes which may influence the delivery lead time, for example, transportation time, late production and so forth, and this outcome the lead time to be uncertain. Liberatore (1977) is the first author addressed an inventory modeling involving stochastic lead times. An inventory model developed by Sphicas and Nasri (1984), deals with constant demand and stochastic lead times. Yano (1987) determined the optimal lead times for two-echelon system, with the aim of minimizing the sum of inventory holding costs and tardiness costs. Nasri et al. (1990) developed inventory control with stochastic lead time and the authors incorporate the investment strategy to minimize the set-up cost. Specifically, the set-up cost reduces as exponential function of investment. Fujiwara and Sedarage (1997) developed an Economic Order Quantity (EOQ) problem for single-item, where the production of an item required number of parts and the lead time is assumed to be a stochastic variable. The problem determines when to order in each part and finds the EOQ, hence the total cost per unit time of the system is minimized. He et al. (2005) presented a probabilistic, finite lead time inventory model with backordering. A generalized EOQ model with backorder is developed and uniformly distributed lead time was employed as a particular case. Paknejad et al. (2005) modified the stochastic lead time inventory model which allows a defectiveness in the system, for that the extra holding cost incurred for the defective items until those items are returned to the vendor. Also the paper deals with the investment to improve the quality. Jokar and Sajadieh (2008) proposed a multi-sourcing inventory model with the stochastic lead time. The authors incorporate more than one suppliers and splitting orders between them to lessen the lead time risk in the unstable environments. The aim of the study is to finding optimal number of suppliers and comparing the results to sole-sourcing model.

Sajadieh and Akbari Jokar (2009) presented a single-vendor single-buyer integrated inventory system for a stochastic lead time. The authors assumed that the lead time follows a uniform distribution. Also, the authors assumed to allow shortages with completely backorder. Sajadieh et al. (2009) developed exponentially distributed stochastic lead time inventory model with shortages and completely backorder. Normally distributed lead time inventory model is investigated by Hoque (2013), the model allows shortages. If the batch Q arrives late at the time t, then the vendor kept the Qt inventory in his warehouse. This extra inventory induces the extra holding cost at the vendor side. Hoque (2013) had taken this extra holding cost into his model. Shu et al. (2015) extends Sajadieh et al. (2009) model by incorporating the transportation cost of lot size which is increasing function of q. Lin (2016) also extends Sajadieh et al. (2009) by allowing shortages and those shortages are partially backordered. Also, Lin (2016) using investment function to reduce the mean value of lead time. The paper by Hossain et al. (2017) is the first literature which considers the general distributions of stochastic lead times and the vendor may have the penalty cost for the delayed delivery. To illustrate the model, the authors used different distributions such as uniform, normal and exponential distributions in the numerical section.

A comparison of our model with these literatures is given in Table 1. The remainder of the paper is arranged as follows. In the next section we put forward Notations and Assumptions. In Sect. 3 we deal the mathematical model with general distribution of lead time, and algorithm of obtaining the minimum cost. Numerical computations for different distributions and parametric sensitivity analysis are carried out in Sect. 4. Conclusions are made in the last section.

2 Notations and assumptions

We adopt the following notations and assumptions to develop the mathematical model.

2.1 Notations

D	demand rate (units/unit time)
A	buyer's ordering cost per order
B	vendor's setup cost per setup
h_b	buyer's holding cost per unit per unit time
h_v	vendor's holding cost per unit per unit time
l	length of the lead time (a random variable)
$f(l)$	probability density function of l
π	buyer's shortage cost per unit per unit time
π_0	marginal profit (i.e., cost of lost demand) per unit per unit time
a	upper limit of lead time (unit time)
b	lower limit of lead time (unit time)
Q	buyer's order quantity (units per order) (a decision variable)
r	reorder level (units) (a decision variable)
β	fraction of shortage will be backordered ($0 \le \beta \le 1$) (a decision variable)
$\mathbb{E}(\cdot)$	mathematical expectation

2.2 Assumptions

1. A two-echelon integrated inventory model is developed with single-vendor and single-buyer for a single-item.
2. Demand rate is constant and production rate is infinite.
3. The buyer follows a classical (Q, r) system, i.e., the inventory is continuously reviewed and the order Q is

Table 1 Comparison with the literature

References	Lead time distribution	Backlog	Variable backorder
Hoque (2013)	Normal	Complete	
Hossain et al. (2017)	General	Complete	
Lin (2016)	Exponential	Partial	
Sajadieh and Akbari Jokar (2009)	Uniform	Complete	
Sajadieh et al. (2009)	Normal	Complete	
Shu et al. (2015)	Exponential	Complete	
Present study	General	Partial	✓

placed when the inventory level reaches the reorder point r.

4. The vendor's rate of production is assumed to be infinite.
5. The lead time of the buyer's order is a random variable.
6. Shortages are allowed at the buyer side and a fraction (β) of shortages is backordered. The backordering parameter β is assumed to inversely proportional to the amount of shortages.

3 Mathematical model

In this section the mathematical model of integrated inventory system is developed. Based on an agreement between the two entities, the buyer orders Q units to the vendor and in order to reduce the setup cost, the vendor produces nQ items and transfers Q items for n times. The vendor produces the items with infinite production rate and production time is assumed to be zero (e.g., Sajadieh et al. 2009; Lin 2016; Hossain et al. 2017). Since the delivery lead time is probabilistic in nature which follows a known randomness with the probability function $f(l)$ within certain range (a, b).

3.1 Buyer's total cost

The buyer's total cost consisting of ordering cost, holding cost and shortage cost (if shortage occur) per unit time. The order Q units are placed when the inventory level drops to the reorder level r. Since the delivery lead time is a probabilistic, the order may arrive early or late with respect to the mean lead time $\mathbb{E}(l) = \frac{r}{D}$. The time is a continuous variable, for that, we develop this model by considering the stochastic lead time follows a continuous random variable. The upper bound and the lower bound of the lead time are b and a respectively, i.e., $-\infty \leq a \leq l \leq b \leq +\infty$ which is the general range of the lead time considered in this paper. Since ordering cost per order is A, the ordering cost per unit time is $\frac{DA}{Q}$. If the order arrives earlier with respect to the mean lead time $\frac{r}{D}$, i.e., $l \leq \frac{r}{D}$, then the time-weighted inventory (area bounded by red color in Fig. 1) is $\frac{1}{2}Dl^2 + l(r - Dl) = l(r - \frac{Dl}{2})$. Therefore the expected holding cost per unit time is $h_b(D/Q)\mathbb{E}\left[\left(r - \frac{Dl}{2}\right)lI_{(a \leq l \leq \min(b,r/D))}\right]$, where I_B is the indicator function, i.e., $I_B := I_B(\omega) = \begin{cases} 1, & \text{if } \omega \in B \\ 0, & \text{if } \omega \notin B. \end{cases}$ Suppose the order arrives late with respect to the mean lead time $\frac{r}{D}$, i.e., $l > \frac{r}{D}$, then the time weighted inventory (area bounded by red color in Fig. 2) is $\frac{r^2}{2D}$. Therefore the expected holding cost per unit time is $\frac{D}{Q}\mathbb{E}\left[\frac{r^2}{2D}I_{\min(b,r/D)<l\leq b}\right]$. Similarly, the shortage area (area bounded by black color in Fig. 2) is $\frac{1}{2D}(Dl - r)^2$. Thus the expected number of backorder per cycle is $\beta\mathbb{E}\left[\frac{1}{2D}(Dl - r)^2 I_{\min(b,r/D)<l\leq b}\right]$ and the expected shortage cost per unit time is $(\pi + \pi_0(1-\beta))\frac{D}{Q}\mathbb{E}\left[\frac{1}{2D}(Dl - r)^2 I_{\min(b,r/D)<l\leq b}\right] = \frac{(\pi+\pi_0(1-\beta))}{2Q}\mathbb{E}\left[(Dl - r)^2 I_{\min(b,r/D)<l\leq b}\right]$. Since the backordering ratio is assumed to inversely proportional to the amount of shortage, we get

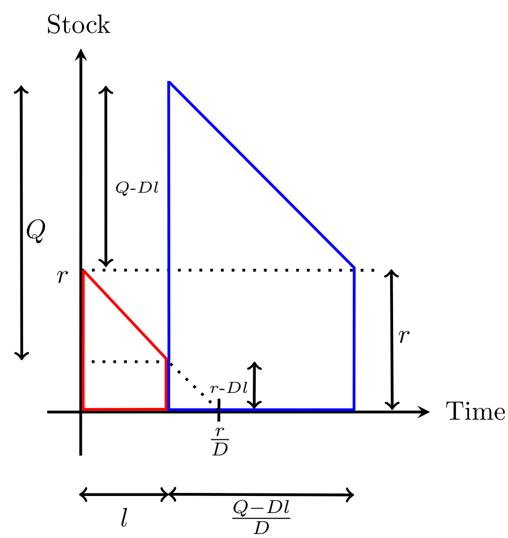

Fig. 1 Buyer's inventory level when $b \leq r/D$ (color figure online)

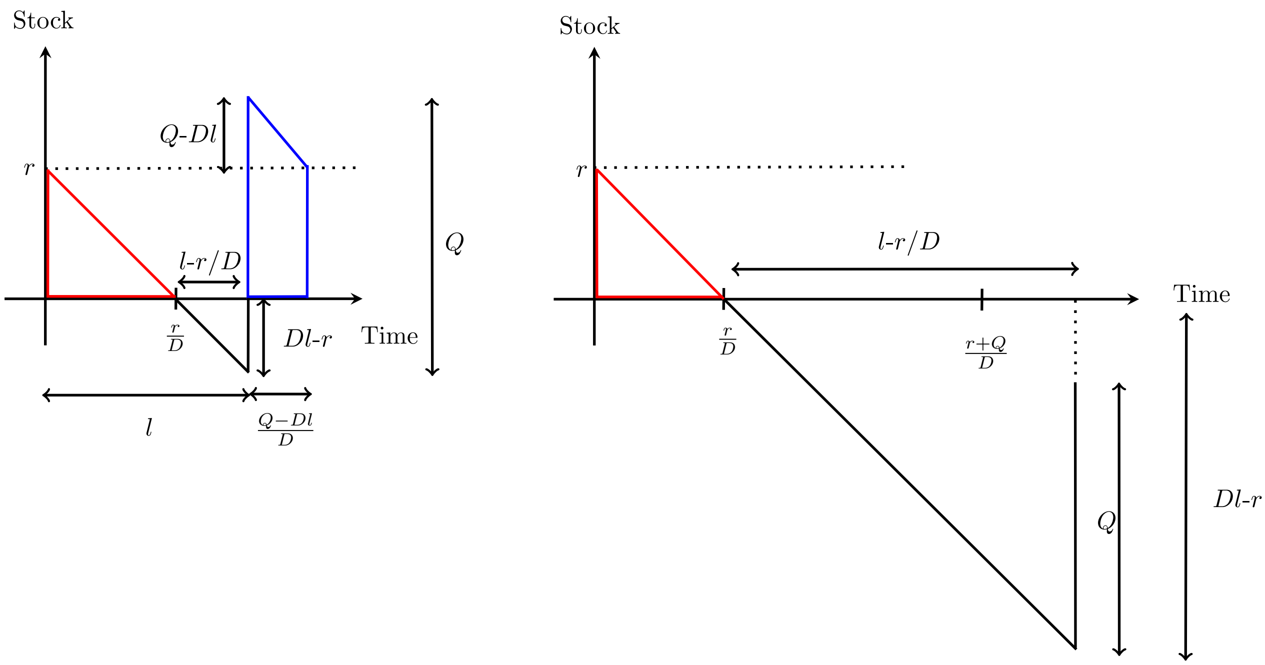

Fig. 2 Buyer's inventory level when $b > r/D$ (color figure online)

$$\beta = \frac{1}{1 + \frac{\rho}{2D}\int_{r/D}^{b}(Dl - r)^2 f(l)\mathrm{d}l} \tag{1}$$

where $\rho > 0$. In both the cases, $l \le \frac{r}{D}$ and $l > \frac{r}{D}$, the maximum inventory level after the arrival is $Q + r - Dl$. Thus the time weighted inventory (area bounded by blue color) after the arrival is $\frac{(Q-Dl)^2}{2D} + \frac{r(Q-Dl)}{D}$. The expected total cost is $\frac{D}{Q}h_b\frac{1}{2D}\mathbb{E}\left[(Q-Dl)^2 I_{(a\le l\le b)}\right] + \frac{D}{Q}h_b\frac{r}{D}\mathbb{E}[(Q-Dl)I_{(a\le l\le b)}] = \frac{h_b}{2Q}\mathbb{E}\left[(Q-Dl)^2 I_{(a\le l\le b)}\right] + \frac{h_b r}{Q}\mathbb{E}[(Q-Dl)I_{(a\le l\le b)}]$. Therefore the expected total cost of the buyer is

$$\begin{aligned} ETC_b(Q,r) = {} & \frac{DA}{Q} + \frac{h_b D}{Q}\mathbb{E}\left[\left(r - \frac{Dl}{2}\right)lI_{(a\le l\le \min(b,r/D))}\right] \\ & + \frac{h_b}{2Q}\mathbb{E}\left[(Q-Dl)^2 I_{(a\le l\le b)}\right] \\ & + \frac{r^2 h_b}{2Q}\mathbb{E}\left[I_{(\min(b,r/D)<l\le b)}\right] \\ & + \frac{rh_b}{Q}\mathbb{E}\left[(Q-Dl)I_{(a\le l\le b)}\right] \\ & + \frac{h_b(1-\beta)}{2Q}\mathbb{E}\left[(Dl-r)^2 I_{(\min(b,r/D)<l\le b)}\right] \\ & + \frac{\pi + \pi_0(1-\beta)}{2Q}\mathbb{E}\left[(Dl-r)^2 I_{(\min(b,r/D)<l\le b)}\right]. \end{aligned} \tag{2}$$

3.2 Vendor's total cost

The vendor produces nQ items and delivers them for n times of Q batches. The vendor's inventory level is depicted in Fig. 3. So the vendor's average holding cost is $\frac{h_v(n-1)Q}{2}$ and the setup cost per unit time is $\frac{B}{nQ/D}$. Due to late delivery, the extra inventory is kept in the vendor's warehouse as in Hoque (2013). The late delivery happens when $l > \frac{r}{D}$, and the order Q arrives by the late time $l - \frac{r}{D}$. Therefore the vendor keeps those extra $Q\left(l - \frac{r}{D}\right)$ inventory per cycle and the extra inventory cost per unit time is $h_v\frac{D}{Q}Q\left(l - \frac{r}{D}\right) = h_v(Dl - r)$. Expected total cost of the vendor is

$$\begin{aligned} ETC_v(Q,n) = {} & \frac{DB}{nQ} + \frac{(n-1)Qh_v}{2} \\ & + h_v\mathbb{E}\left[(Dl-r)_{(r/D<l\le b)}\right]. \end{aligned} \tag{3}$$

3.3 Integrated total cost

Suppose the vendor and the buyer decided to cooperate and agree to follow the integrated approach then the expected total cost of the system will be the sum of vendor total cost and buyer total cost. i.e.,

$$\begin{aligned}ETC &= ETC_b + ETC_v\\ &= \frac{DA}{Q} + \frac{h_b D}{Q}\mathbb{E}\left[\left(r - \frac{Dl}{2}\right) lI_{(a \le l \le \min(b, r/D))}\right]\\ &+ \frac{h_b}{2Q}\mathbb{E}\left[(Q - Dl)^2 I_{(a \le l \le b)}\right]\\ &+ \frac{r^2 h_b}{2Q}\mathbb{E}\left[I_{(\min(b,r/D) < l \le b)}\right]\\ &+ \frac{rh_b}{Q}\mathbb{E}\left[(Q - Dl) I_{(a \le l \le b)}\right]\\ &+ \frac{h_b(1-\beta)}{2Q}\mathbb{E}\left[(Dl - r)^2 I_{(\min(b,r/D) < l \le b)}\right]\\ &+ \frac{\pi + \pi_0(1-\beta)}{2Q}\mathbb{E}\left[(Dl - r)^2 I_{(\min(b,r/D) < l \le b)}\right]\\ &+ \frac{DB}{nQ} + \frac{(n-1)Qh_v}{2} + h_v\mathbb{E}\left[(Dl - r)_{(r/D < l \le b)}\right].\end{aligned} \tag{4}$$

The above equation can be written as

$$\begin{aligned}ETC &= \frac{D}{Q}\left(A + \frac{B}{n}\right) + \frac{h_b D}{Q}\int_a^{\min(b,r/D)}\left(r - \frac{Dl}{2}\right) lf(l)\mathrm{d}l\\ &+ \frac{h_b}{2Q}\int_a^b (Q - Dl)^2 f(l)\mathrm{d}l\\ &+ \frac{r^2 h_b}{2Q}\int_{\min(b,r/D)}^b f(l)\mathrm{d}l + \frac{rh_b}{Q}\int_a^b (Q - Dl) f(l)\mathrm{d}l\\ &+ \frac{h_b(1-\beta)}{2Q}\int_{\min(b,r/D)}^b (Dl - r)^2 f(l)\mathrm{d}l\\ &+ \frac{\pi + \pi_0(1-\beta)}{2Q}\int_{\min(b,r/D)}^b (Dl - r)^2 f(l)\mathrm{d}l\\ &+ \frac{(n-1)Qh_v}{2} + h_v\int_{\min(b,r/D)}^b (Dl - r) f(l)\mathrm{d}l,\end{aligned} \tag{5}$$

subject to, n integer, $q > 0$, $r > 0$, $0 \le \beta \le 1$.

Now the problem is to find the EOQ, reorder level r, number of shipments n and backordering rate β that minimize the expected total cost in Eq. (5). Next we discuss total cost of earlier delivery and late delivery in two cases. Then we use the classical optimization techniques to minimize Eq. (5) with respect to Q, r, n and β. With the help of MATLAB, we iteratively solve simultaneous equations which are obtained from differentiating Eq. (5) with respect to Q and r. To prove the convexity of

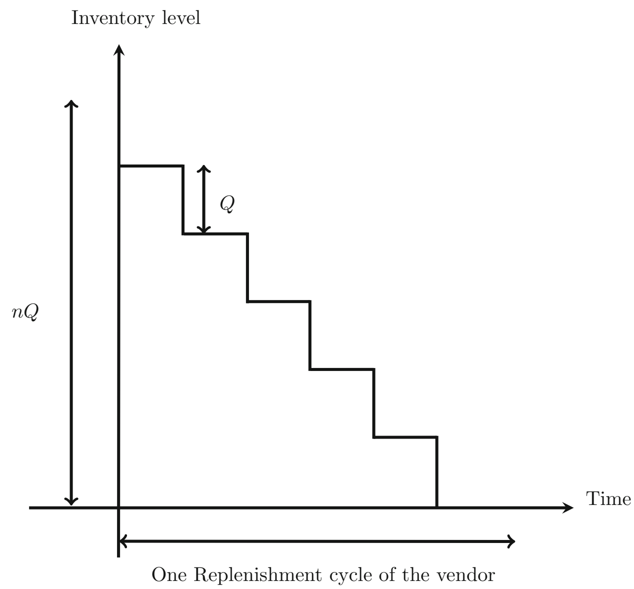

Fig. 3 Vendor's inventory level

$ETC_i,\ i = 1, 2$ over n we assumed that n is a continuous variable.

Case 1 If $b \le r/D$: Then Eq. (5) will becomes

$$\begin{aligned}ETC_1 &= \frac{D}{Q}\left(A + \frac{B}{n}\right) + \frac{h_b D}{Q}\int_a^b \left(r - \frac{Dl}{2}\right) lf(l)\mathrm{d}l\\ &+ \frac{h_b}{2Q}\int_a^b (Q - Dl)^2 f(l)\mathrm{d}l\\ &+ \frac{rh_b}{Q}\int_a^b (Q - Dl) f(l)\mathrm{d}l + \frac{(n-1)Qh_v}{2},\\ &= \frac{D}{Q}\left(A + \frac{B}{n}\right) + \frac{h_b}{2Q}\\ &\int_a^b 2D\left(r - \frac{Dl}{2}l + (Q - Dl)^2 + 2r(Q - Dl)\right) f(l)\mathrm{d}l\\ &+ \frac{(n-1)Qh_v}{2},\\ &= \frac{D}{Q}\left(A + \frac{B}{n}\right) + h_b\int_a^b \left(\frac{Q}{2} + r - Dl\right) f(l)\mathrm{d}l\\ &+ \frac{(n-1)Qh_v}{2}.\end{aligned} \tag{6}$$

Here $\frac{\partial^2}{\partial n^2}ETC_1 = \frac{2DB}{Qn^3} > 0 \ \ \forall \ \ n > 0$. This shows that ETC_1 is convex in n for given Q and r. $\frac{\partial^2}{\partial Q^2}ETC_1 = \frac{2D}{Q^3}\left(A + \frac{B}{n}\right) > 0$, $\frac{\partial^2}{\partial r^2}ETC_1 = 0$. This shows that ETC_1 is convex in Q and r for given n. On setting $\frac{\partial ETC_1}{\partial Q}=0$ and solving this with respect to Q, we get

$$Q^* = \sqrt{\frac{2D\left(A+\frac{B}{n}\right)}{(n-1)h_v + h_b\int_a^b f(l)\mathrm{d}l}}. \tag{7}$$

Again $\frac{\partial ETC_1}{\partial r} = h_b\int_a^b f(l)\mathrm{d}l$ which is a constant function; leads the fact that the optimal solution of r is the expected value of the reorder point. i.e., $r^* = D\mathbb{E}(l)$.

Case 2 If $b > r/D$: Then Eq. (5) will become

$$\begin{aligned} ETC_2 = {} & \frac{D}{Q}\left(A+\frac{B}{n}\right) + \frac{h_b D}{Q}\int_a^{r/D}\left(r-\frac{Dl}{2}\right)lf(l)\mathrm{d}l \\ & + \frac{h_b}{2Q}\int_a^b (Q-Dl)^2 f(l)\mathrm{d}l + \frac{r^2 h_b}{2Q}\int_{r/D}^b f(l)\mathrm{d}l \\ & + \frac{rh_b}{Q}\int_a^b (Q-Dl)f(l)\mathrm{d}l + \frac{h_b(1-\beta)}{2Q}\int_{r/D}^b (Dl-r)^2 f(l)\mathrm{d}l \\ & + \frac{\pi+\pi_0(1-\beta)}{2Q}\int_{r/D}^b (Dl-r)^2 f(l)\mathrm{d}l \\ & + \frac{(n-1)Qh_v}{2} + h_v\int_{r/D}^b (Dl-r)f(l)\mathrm{d}l. \end{aligned} \tag{8}$$

As in the previous case $\frac{\partial^2}{\partial n^2}ETC_2 = \frac{2DB}{Qn^3} > 0 \;\forall\; n>0$. This shows that ETC_2 is convex in n for given Q and r. Putting $\frac{\partial}{\partial Q}ETC_2 = 0$.

$$\begin{aligned} \frac{\partial}{\partial Q}ETC_2 = {} & -\frac{D}{Q^2}\left(A+\frac{B}{n}\right) - \frac{h_b D}{Q^2}\int_a^{r/D}\left(r-\frac{Dl}{2}\right)lf(l)\mathrm{d}l \\ & + h_b\int_a^b\left(\frac{1}{2}-\frac{D^2l^2}{2Q^2}+\frac{rDl}{Q^2}\right)f(l)\mathrm{d}l - \frac{h_b r^2}{2Q^2} \\ & \int_{r/D}^b f(l)\mathrm{d}l - \frac{h_b(1-\beta)}{2Q^2}\int_{r/D}^b (Dl-r)^2 f(l)\mathrm{d}l \\ & - \frac{\pi+\pi_0(1-\beta)}{2Q^2}\int_{r/D}^b (Dl-r)^2 f(l)\mathrm{d}l \\ & + \frac{(n-1)h_v}{2} = 0, \end{aligned} \tag{9}$$

$$\begin{aligned} & \frac{1}{Q^2}\left[D\left(A+\frac{B}{n}\right) + \frac{h_b+\pi+\pi_0(1-\beta)}{2}\right. \\ & \left.\int_{r/D}^b (Dl-r)^2 f(l)\mathrm{d}l + \frac{h_b(1-\beta)}{2}\int_{r/D}^b (Dl-r)^2 f(l)\mathrm{d}l\right] \\ & \quad = \frac{h_b}{2}\int_a^b f(l)\mathrm{d}l + \frac{(n-1)h_v}{2}, \end{aligned} \tag{10}$$

and solving with respect to Q we get

$$Q^* = \sqrt{\frac{2D\left(A+\frac{B}{n}\right) + [(\pi_0+h_b)(1-\beta)+(\pi+h_b)]\int_{r/D}^b (Dl-r)^2 f(l)\mathrm{d}l}{(n-1)h_v + h_b\int_a^b f(l)\mathrm{d}l}}. \tag{11}$$

Now differentiating Eq. (8) over r using the Leibniz rule, we get

$$\begin{aligned} \frac{\partial}{\partial r}ETC_2 = {} & h_b\int_a^b f(l)\mathrm{d}l - h_v\int_{r/D}^b f(l)\mathrm{d}l \\ & - \frac{1}{Q}[(h_b+\pi)+(h_b+\pi_0)(1-\beta)]\int_{r/D}^b (Dl-r)f(l)\mathrm{d}l \\ & - \frac{(\pi_0+h_b)\rho\beta^2}{2QD}\int_{r/D}^b (Dl-r)f(l)\mathrm{d}l\int_{r/D}^b (Dl-r)^2 f(l)\mathrm{d}l. \end{aligned} \tag{12}$$

Putting $\frac{\partial}{\partial r}ETC_2 = 0$, we get

$$\begin{aligned} & h_v\int_{r/D}^b f(l)\mathrm{d}l + \frac{1}{Q}[(h_b+\pi)+(h_b+\pi_0)(1-\beta)] \\ & \int_{r/D}^b (Dl-r)f(l)\mathrm{d}l + \frac{(\pi_0+h_b)\rho\beta^2}{2QD} \\ & \int_{r/D}^b (Dl-r)f(l)\mathrm{d}l\int_{r/D}^b (Dl-r)^2 f(l)\mathrm{d}l \\ & \quad = h_b\int_a^b f(l)\mathrm{d}l. \end{aligned} \tag{13}$$

Now taking second derivative of Eq. (8) over Q we get

$$\frac{\partial^2}{\partial Q^2}ETC_2 = \frac{2D}{Q^3}\left(A+\frac{B}{n}\right)+\frac{1}{Q^3}[(\pi+h_b)+(\pi_0+h_b)(1-\beta)]\int_{r/D}^{b}(Dl-r)^2 f(l)\mathrm{d}l. \tag{14}$$

Since $1-\beta \geq 0$ and $Dl - r > 0$, the above expression is positive for all $Q > 0$. This shows that ETC_2 is convex in Q. Now taking second derivative of Eq. (8) over r we get

$$\frac{\partial^2}{\partial r^2}ETC_2 = \frac{h_v}{D}f(r/D)+\frac{1}{Q}[(h_b+\pi)+(h_b+\pi_0)(1-\beta)]\int_{r/D}^{b}f(l)\mathrm{d}l+\frac{(h_b+\pi_0)\rho\beta^2}{2DQ}\times\left[\int_{r/D}^{b}(Dl-r)^2 f(l)\mathrm{d}l\int_{r/D}^{b}f(l)\mathrm{d}l+2\left(\int_{r/D}^{b}(Dl-r)f(l)\mathrm{d}l\right)^2\left(1-\rho\beta\int_{r/D}^{b}(Dl-r)^2 f(l)\mathrm{d}l+\frac{D}{\beta^4}\right)\right]>0, \tag{15}$$

provided $-\rho\beta\int_{r/D}^{b}(Dl-r)^2 f(l)\mathrm{d}l+\frac{D}{\beta^4}>0$. This shows that ETC_2 is convex provided $\frac{D}{\beta^4}>\rho\beta\int_{r/D}^{b}(Dl-r)^2 f(l)\mathrm{d}l$. The cost function in Eq. (8) is non-linear and it is difficult to find the closed form to get the optimality. So we follow an iterative algorithm. Since the number of deliveries is a positive integer and we already proved that the convex nature of the cost function in n for both the cases. At first, the solution process can be started by setting $n = 1$. Then we need to find which case will be suitable for the problem. So first we start with Case 2. To find the extreme points Q and r from Eqs. (11) and (13) respectively, but Q and r are interlinked with each other. So we start the algorithm with $n = 1$ and $Q = \sqrt{\frac{2DA}{h_v+h_b}}$ (simple EOQ formula) and finding r from Eq. (13). Then substitute r value obtained from the previous step into Eq. (11). Repeat this process until no change occur in the values of Q and r. Now we can say which case is suitable for the problem using the value of r. i.e., if $r/D \geq b$ then this problem is suitable for Case 1 otherwise the problem is suitable for Case 2.

In our proposed model, if l follows a uniform distribution and the late delivery holding cost is not incorporated then, our model becomes the model proposed by Sajadieh and Akbari Jokar (2009). If l follows an exponential distribution then this model will be same as Sajadieh et al. (2009) model. We cannot compare our proposed model numerically to Hossain et al. (2017) model, because in Hossain model the late delivery cost is assumed in terms of time (c_p per unit time), whereas here we consider the extra holding cost at the vendor side due to the late delivery. Anyhow, our proposed model is general one when compare to Hossain in the sense of variable backorder, since Hossain assumed that, the shortages are completely backorder. If $\beta = 1$ then our proposed model will become Hossain model. In our proposed model, if β is any constant lies between 0 and 1, then it will become model of Lin (2016) for without investment case.

The algorithmic solution procedure can therefore be stated as follows.

3.4 Algorithm

1. Set $n = 1$.
2. Find r by solving Eq. (13) with respect to r using $Q = \sqrt{\frac{2DA}{h_v+h_b}}$.
3. Put Q in Eq. (13) and solve for r.
4. Using this r, find Q from Eq. (11). Repeat this process until no change occur in the values of Q and r. This (Q^*, r^*) is the optimal solution for given n. Find ETC_2 using (Q^*, r^*, n).
5. Put $n = n + 1$ and go to step 1.
6. Find n^* such that it satisfy $ETC_2(Q^*, r^*, n^* - 1) > ETC_2(Q^*, r^*, n^*) \leq ETC_2(Q^*, r^*, n^* + 1)$.
7. This (Q^*, r^*, n^*) is the optimal solution of the system. Substitute r^* value in Eq. (1) and find β^*. $ETC_2(Q^*, r^*, n^*, \beta^*)$ is the minimum total cost of the proposed model.

4 Numerical example

To illustrate the aforementioned searching procedure, we study a numerical example. All the numerical parameters are taken from Hossain et al. (2017), and we add π_0, ρ as additional parameters.

Example 4.1 First assume that the lead time is uniformly distributed with the limits 0 to 35 days, i.e., $l \sim U(0, 35)$ days with other parameters $D = 1000$ units per unit time, $B = 400$ per setup, $h_v = 4$ per unit per unit time, $A = 25$

per order, $h_b = 5$ per unit per unit time, $\pi = 30$ per unit, $\pi_0 = 50$ per unit and $\rho = 0.4$. According to algorithmic procedure, fix $n = 1$, and find $Q = \sqrt{\frac{2DA}{h_v + h_b}} = 74.54$. Now substitute Q in Eq. (13) and getting $r = 59.83$. Evaluate Q from Eq. (11) using $r = 59.83$. By using this Q, determine the value of r from Eq. (13). Repeat this process until no change occur in the values of Q and r, we get $Q = 421.01$ and $r = 33.51$. From this value of r we can find the backorder ratio $\beta = 0.8556$. This $(Q^*, r^*, \beta^*) = (421.01, 33.51, 0.8556)$ is the optimal solution for given $n = 1$ and the total cost $ETC_2(Q^*, r^*, \beta^*) = ETC_2(421.01, 33.51, 0.8556) = 2114.03$. Now put $n = 2$ repeat the same aforesaid procedure we get the optimal solutions $(Q^*, r^*, \beta^*) = (228.57, 43.36, 0.9084)$, and the corresponding total cost $ETC_2(Q^*, r^*, \beta^*) = ETC_2(228.57, 43.36, 0.9084, 2091.72) = 2091.72$. Note that ETC_2 for $n = 1$ is greater than ETC_2 for $n = 2$. This assures $n = 1$ cannot be optimal, and now to check whether $n = 2$ is optimum or not, thus we need ETC_2 for $n = 3$. Now fix $n = 3$ and we get the optimal solutions $(Q^*, r^*, \beta^*) = (159.47, 48.92, 0.9328)$ and the corresponding total cost $ETC_2 = 2194.04$. Therefore the optimal solutions are $(Q^*, r^*, \beta^*, n^*) = (228.57, 43.36, 0.9084, 2)$ and the minimum total cost is 2091.72. Also note that $r^*/D = 43.36/1000 = 0.0434$ years $= 15.83$ days, which is less than $b = 35$ days. Hence, this problem is suit for Case 2. For different values of ρ, the optimal values are given below in Table 2. From Table 2, we observe that for any amount of shortages, for $\rho = 0$ and $\rho = \infty$ we get $\beta = 1$ (completely backordered) and $\beta = 0$ (completely lost) respectively as per Eq. (1). The graphical representation of the total cost with respect to r is depicted in Fig. 4 for $n = 1$. This shows that the total cost is convex in r.

Example 4.2 Now assume that, if the lead time follows an exponential distribution with mean 18.25 days, which remains the limits 0 to ∞ days, i.e., $l \sim \mathrm{Exp}(1/18.25)$, and other parameters are $D = 1000$, $B = 50$, $h_v = 1$, $A = 40$, $h_b = 4$, $\pi = 6$, $\pi_0 = 10$ and $\rho = 0.4$. Here $b = \infty$, and for all value of r^*, $r^*/D < b$. So, this example fit for Case 2. Using the algorithmic procedure, for $n = 1$ we get $ETC_2(Q^*, r^*, \beta^*) = ETC_2(247.61, 10.93, 0.5544) = 874.35$, for $n = 2$, we get $ETC_2(Q^*, r^*, \beta^*) = ETC_2(190.96, 20.02, 0.5988) = 868.36$ and for $n = 3$ we get $ETC_2(Q^*, r^*, \beta^*) = ETC_2(162.74, 25.71, 0.6258) = 909.15$. Therefore the optimal solution of the system is $(Q^*, r^*, \beta^*, n^*) = (190.96, 20.02, 0.5988, 2)$ and the minimum total cost is

Table 2 Optimal solutions of uniformly distributed lead time

ρ	n	Q	r	β	ETC
0	2	229.22	38.63	1.0000	2084.73
0.4	2	228.57	43.36	0.9084	2091.72
1	2	228.22	46.41	0.8260	2097.40
10	2	227.85	53.73	0.4342	2116.62
100	2	228.39	55.62	0.0809	2127.64
∞	2	228.61	55.67	0.0000	2129.85

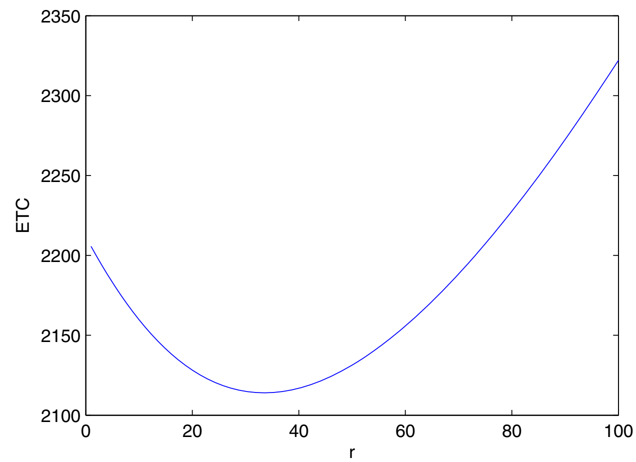

Fig. 4 Total cost with respect to r of Example 4.1 when $n = 1$

Table 3 Optimal solutions of exponentially distributed lead time

ρ	n	Q	r	β	ETC
0	2	189.74	0	1.0000	798.68
0.4	2	190.96	20.2	0.5988	868.36
1	2	192.53	25.44	0.3995	894.45
10	1	255.84	17.56	0.0538	928.78
100	1	255.84	17.56	0.0538	928.78
∞	1	257.11	17.42	0.0056	933.40

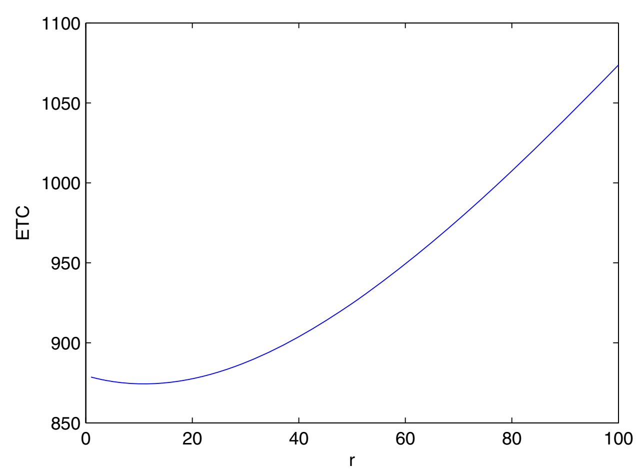

Fig. 5 Total cost with respect to r of Example 4.2 when $n = 1$

868.36. Sensitivity analysis on ρ is employed in Table 3. The graphical representation of the total cost with respect to r is depicted in Fig. 5 when $n = 1$.

Example 4.3 Finally, assume that if the lead time is normally distributed. Hoque (2013) states that, "probability of arrival of a batch earlier or late appears to be smaller than the probability of arrival of a batch in the mean lead time. Thus normal distribution of lead time seems to be a better fit to the problem". Also he states that, in the exponential distribution, the probability of earlier arrival is higher than the probability of late arrival. Now, assume that the mean of the lead time is 35 days and the standard deviation is 4 days, which remains the limits 0 to ∞ days, i.e., $l \sim N(35, 4^2)$. Other parameters are $D = 120{,}000$, $B = 1000$, $h_v = 1$, $A = 400$, $h_b = 1.25$, $\pi = 1.5$, $\pi_0 = 3$ and $\rho = 0.4$. Like previous example, $b = \infty$, here also $r/D < b$ for all $r > 0$. Therefore the problem fits for Case 2. By algorithm we get the optimal solutions as $(Q^*, r^*, \beta^*, n^*) = (16{,}767.17, 10{,}513.42, 0.1937, 1)$ and the minimum total cost is 20,881.49. Sensitivity analysis on ρ is employed in Table 4.

Table 4 Optimal solutions of normally distributed lead time

ρ	n	Q	r	β	ETC
0	1	16,751.69	9586.74	1.0000	20,501.78
0.4	1	16,767.17	10,513.42	0.1937	20,881.49
1	1	16,788.31	10,528.23	0.0888	20,914.97
10	1	16,806.82	10,531.39	0.0096	20,939.62
100	1	16,809.00	10,531.33	0.0009	20,942.32
∞	1	16,809.25	10,531.32	0.0000	20,942.62

In Table 5, we tabulate optimal solutions for different distribution and for different parameters. From Tables 2, 3, 4 and 5, it can be observed that, increasing the value of ρ will result in a decrease in the order quantity and reorder level. This will happen because if ρ increases will make the backorder ratio decreases to zero, thus the system will become completely lost case. So that to avoid the shortages, the reorder will increase. Also, increasing shortage and ρ will lead to the increasing the total cost. Most of all cases the increasing the shortage and ρ will not affect the number of shipments.

Table 5 Some arbitrary examples for three different distributions of lead time

Vendor's parameter			Buyer's parameter					ρ	Optimal solutions				ETC^*
B	h_v	l (days)	A	h_b	π	D	π_0		n^*	Q^*	r^*	β^*	
								0	2	1996.36	4.7	1	6581.48
855	0.5	Exp(1/60)	175	4.5	4.9	5200	9	100	2	2067.53	679.74	0.0001	9742.90
								∞	2	2068.20	679.04	0	9743.24
								0	1	4181.31	491.81	1	4758.70
650	0.75	Exp(1/70)	150	1.25	1.4	8000	3	100	1	4563.03	1378.58	0.00003	5977.62
								∞	1	4562.9	1378.61	0	5977.69
								0	2	2184.93	0	1	7478.42
780	1.15	Exp(1/12)	120	2.6	2.75	15,000	5	100	2	2277.21	205.94	0.0009	8166.30
								∞	2	2276.9	206.83	0	8166.8
								0	3	41,984.32	30,844.54	1	135,688.86
570	0.65	U(0, 3.5)	35	2.15	2.85	8,250,000	5	100	2	51,970.66	43,643.51	0.0008	159,476.1
								∞	2	51,976.32	43,641.74	0	159,487.44
								0.4	4	13,776.36	12,186.51	0.1416	70,183.74
2000	0.75	U(0, 15)	150	2.55	2.9	550,000	4.8	10	4	13,894.23	12,201.06	0.0065	70,781.55
								∞	4	13,901.37	12,198.61	0	70,810.44
								0.4	3	4459.36	3746.91	0.3225	23,319.78
800	1.15	$N(14, 3^2)$	150	3.2	2.45	115,000	5	10	3	4499.26	3795.2	0.0199	23,652.06
								∞	3	4503.25	3794.59	0	23,672.60
								0.4	1	26,025.57	6532.53	0.1970	57,911.85
4500	1.75	$N(20, 2.5^2)$	650	2.25	2.45	145,000	4.75	10	1	26,052.47	6553.93	0.0099	57,986.07
								∞	1	26,054.2	6553.92	0	57,989.93
								0.4	2	35,186.12	10,829.65	0.1488	49,406.08
1290	0.35	$N(7, 1^2)$	450	1.15	1.25	785,000	3.25	10	2	35,293.25	10,876.45	0.0070	49,604.64
								∞	2	35,300.08	10,875.8	0	49,614.36

5 Conclusion

This article presents an integrated single buyer single vendor inventory control problem for generalized lead time distribution. The demand rate was assumed to be deterministic (constant), and the lead time was probabilistic variable with a known probability distribution. Unlike the existing vendor-buyer integrated models, the presented model assumed that the lead time to be a generalized stochastic variable. Furthermore the shortages are allowed and it will be backordered, and it depends on the amount of shortages. Previous works on the stochastic lead time, are focused only on completely backorder case, whereas this proposed model dealt with the partial backorder case. Also in this model, the extra holding cost due to late delivery was incorporated in the vendor side which is one of the practical aspect, where the vendor takes the full responsibility of goods in a good manner until they are delivered to the customer. We formulated the cost function as a non-linear mixed integer problem to determine the EOQ, reorder level, backorder rate and the number of shipments. The convexity of total cost with respect to the Q, n and r are also provided and this revealed that the solution is a global minimum. This convexity holds for any probability distribution of lead time. We presented a step by step algorithmic procedure to find the optimal solutions. Numerical examples were also given, for different distributions such as uniform, normal, exponential and different parameters. Sensitivity analysis also provided for different value of ρ.

The model can be further extended to multi-item inventory problems. Another extension that the authors are working on is to consider the stochastic demand together the stochastic lead time. It will be more relevant to the real market.

Acknowledgements The work of the authors are supported by Department of Science and Technology-Science and Engineering Research Board (DST-SERB), Government of India, New Delhi, under the Grant Number DST-SERB/SR/S4/MS: 814/13-Dated 24.04.2014.

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