Sankhyā : The Indian Journal of Statistics
2023, Volume 85-A, Part 2, pp. 1260-1306

On some Bivariate Semi Parametric Families of Distributions with a Singular Component

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Abstract

The main idea of this paper is to present families of bivariate distributions that depend in their formation on adding a shape parameter to the powers of the hazard and reversed hazard functions in different manners, which would provide additional flexibility in applications. Different baseline distributions were used namely, exponential, inverse exponential, uniform, inverse uniform, inverse Rayleigh, Gompertz and Pareto. Many of the mathematical properties of these families are discussed in detail. Moreover, it is observed that the new bivariate distributions also can make appropriate modeling of three real data sets.

Keywords. Exponential distribution, Inverse exponential distribution, Uniform distribution, inverse uniform distribution, Inverse Rayleigh distribution, Gompertz Distribution and Pareto distribution

1 Introduction

The hazard and reversed hazard rates play important roles in the statistical literatures because of their applicability in many fields. The concept of hazard rate is very well known in the literature and lifetime distributions are usually characterized using the concept of failure rate $h(t)$, defined as

$$h(t) = \lim_{\Delta t \to 0} P(t < T < t + \Delta t / T > t)/\Delta t.$$

which can be equivalently written as

$$h(t) = \frac{f(t)}{1 - F(t)}.$$

where $f(t)$and $F(t)$ are the pdf and cdf of life time T.

The failure rate $h(t)$ measures the instantaneous rate of failure or death at time t, given that an individual survives up to time t. The failure rate is also known as conditional failure rate in reliability, the hazard rate in survival analysis, the force of mortality in demography, the age-specific failure rate in epidemiology. In extreme-value theory, it is known as the intensity rate and its reciprocal is termed as Mill's ratio in economics.

It can be shown that $h(x)$ uniquely determines the distribution. When X is non-negative and has a distribution function absolutely continuous with respect to the Lebesgue measure $h(t)$ can provides

$$\overline{F}(x) = e^{-\int_0^x h(t)\,dt} = e^{-H(x)}$$

Where $H(x)$ is the cumulative hazard rate.

In many practical situations reversed hazard (RH) rate is more appropriate to analyze the survival data. Reversed hazard rate was proposed as a dual to the hazard rate respectively, as

$$r(t) = \lim_{\Delta t \to 0} P(t-\Delta t < T < t/T \leq t)/\Delta t.$$

$$r(t) = \frac{f(t)}{F(t)}.$$

The reversed hazard rate specifies the instantaneous rate of death or failure at time t, given that it failed before time t. Thus in a small interval, $r(t)\Delta t$ is the approximate probability of failure in the interval $(t - \Delta t,\ t]$, given failure before the end of the interval.

It can be shown that $r(t)$ determines the cdf through the following relation

$$F(x) = e^{\int_0^x r(t)\,dt} = e^{R(x)}$$

Where $R(x) = logF(x)$ denotes the cumulative reversed hazard rate.

There are many methods for adding a shape parameter to a family of distributions based on the survival and failure functions that produced the so-called, proportional hazard family and proportional reversed hazard family, along the same line the hazard and reversed hazard functions can also be used to adding a power parameter (shape parameter) that are producing two important families of distributions namely, hazard power parameter and reversed hazard power parameter. The aim of this paper is to introduce the bivariate extensions of these families based on an idea similar to that of Theorem 3.2 proposed by Marshall and Olkin (1967). These authors introduced a multivariate exponential distribution whose marginals have exponential distributions and proposed a bivariate Weibull distribution. The proposed bivariate distributions are constructed from three independent distributions using both minimization and maximization process. These new distributions are singular distributions, and they can be used quit conveniently if there are ties in the data.

Our new models can be extended to different data sets for example, survival times of failure of paired organs like kidneys, lungs, eyes, ears, dental implants etc. In industrial applications, the breakdown times of dual generators in a power plant or failure times of two engines in a two-engine airplanes. In the infectious diseases, time to infection of two or more family members who might visit an infected person and all of them become infected. Some examples are the human lifetimes for which natural disasters or accidents lead to the death of several persons at the same time. They are also widely used in life insurance and the design of multiple life insurance products. Furthermore, they are used in statistics and reliability in shock model, competing risks model, stress model, maintenance model and longevity model, as well as warranty polices based on failure time and warranty servicing time.

The paper is organized as follows: In Section 2, some baseline distributions are introduced. The bivariate reversed hazard power parameter (BRPP) family of distributions is introduced in Section 3. The bivariate hazard power parameter (BHPP) family of distributions is introduced in Section 4. The bivariate power parameter (BPP) family of distributions is introduced Section 5. The bivariate proportional hazard (BPHP) family of distributions is discussed in Section 6. The bivariate proportional reversed hazard (BPRP) family of distributions is discussed in Section 7. A numerical study is discussed in Section 8. Finally, conclude the paper in Section 9.

2 Baseline Distributions

Some baseline distributions with the interesting properties will presented below.

i) Exponential Distribution

If a continuous random variable X follows the exponential distribution then the pdf, survival function, hazard function, and cumulative hazard function are respectively:

$$f_B(x) = \lambda e^{-\lambda x} \tag{2.1}$$

$$S_B(x) = e^{-\lambda x} \tag{2.2}$$

$$h_B(x) = \lambda \tag{2.3}$$

$$H_B(x) = \lambda x. \tag{2.4}$$

ii) **Inverse Exponential Distribution**

If a continuous random variable X follows the inverse exponential distribution then the cdf, pdf, reversed hazard function, and cumulative reversed hazard function are respectively:

$$F_B(x) = e^{-\frac{\lambda}{x}} \qquad x > 0 \text{ and } \lambda > 0 \tag{2.5}$$

$$f_B(x) = \frac{\lambda}{x^2} \qquad e^{-\frac{\lambda}{x}} \qquad , \tag{2.6}$$

$$r_B(x) = \frac{\lambda}{x^2}, \tag{2.7}$$

$$R_B(x) = -\frac{\lambda}{x} \quad . \tag{2.8}$$

iii) **Inverse Rayleigh Distribution**

If a continuous random variable X follows the inverse Rayleigh distribution then the cdf, pdf, reversed hazard function, and cumulative reversed hazard function are respectively:

$$F_B(x) = e^{-\left(\frac{\sigma}{x}\right)^2} \quad x > 0 \text{ and } \boldsymbol{\sigma} > 0 \tag{2.9}$$

$$f_B(x) = \frac{2\,\boldsymbol{\sigma}^2}{x^3}\, e^{-\left(\frac{\sigma}{x}\right)^2}, \tag{2.10}$$

$$r_B(x) = \frac{2\,\sigma^2}{x^3}, \tag{2.11}$$

$$R_B(x) = -\left(\frac{\sigma}{x}\right)^2. \tag{2.12}$$

iv) **Uniform Distribution**

If a continuous random variable X follows the uniform distribution then the cdf, pdf, hazard function,and cumulative hazard function are respectively:

$$F_B(x) = x \qquad 0 < x < 1. \tag{2.13}$$

$$f_B(x) = 1. \tag{2.14}$$

$$h_B(x) = \frac{1}{1-x}. \tag{2.15}$$

$$H_B(x) = -\log(1-x). \tag{2.16}$$

v) **Inverse Uniform Distribution**

The inverse uniform (IU) distribution is defined by using the transformation $X = \frac{1}{T} - 1$ where $T \sim U(0,\ 1)$. Then the cdf, pdf, reversed hazard function, and cumulative reversed hazard function for invers uniform distribution are respectively:

$$\mathrm{F}_B(x) = \frac{x}{x+1} \qquad ,0 < x < \infty. \tag{2.17}$$

$$f_B(x) = \frac{1}{(x+1)^2} \tag{2.18}$$

$$r_B(x) = \frac{1}{x(x+1)} \tag{2.19}$$

$$R_B(x) = \log\left(\frac{x}{x+1}\right). \tag{2.20}$$

i) **Gompertz Distribution**:

The pdf, survival function, hazard function, and cumulative hazard function of a continuous random variable X follows a Gompertz distribution are given respectively as:

$$f_B(x) = \lambda\xi\exp\left\{\lambda x - \xi\left(e^{\lambda x} - 1\right)\right\}, \quad \xi, \lambda > 0, x > 0 \tag{2.21}$$

$$S_B(x) = e^{-\xi\left(e^{\lambda x}-1\right)} \tag{2.22}$$

$$h_B(x) = \lambda\xi e^{\lambda x} \tag{2.23}$$

$$H_B(x) = \xi\left(e^{\lambda x} - 1\right). \tag{2.24}$$

ii) **Pareto Type I Distribution**

The pdf, survival function, hazard function, and cumulative hazard function of a continuous random variable X follows a Pareto distribution are given respectively as:

$$f_B(x) = \frac{\lambda}{1+\lambda x^2}, \quad \lambda > 0, x > 0 \tag{2.25}$$

$$S_B(x) = [1+\lambda x]^{-1} \tag{2.26}$$

$$h_B(x) = \frac{\lambda}{1+\lambda x}, \tag{2.27}$$

$$H_B(x) = \log(1 + \lambda x). \tag{2.28}$$

3 Bivariate Reversed Hazard Power Parameter (BRPP) Family of Distributions

A reversed hazard power parameter model can be defined by adding a power parameter ($\boldsymbol{\alpha}$) to the formula $F(x) = e^{-R(x)}$, as following

$$F_{RPP}(x; \boldsymbol{\alpha}) = e^{-[R_B(x)]^{\alpha}}, \ \ \forall x \tag{3.1}$$

where $\boldsymbol{\alpha} > 0$ and $R(x) = -\log F(x)$ is a cumulative reversed hazard function.

$$f_{RPP}(x; \boldsymbol{\alpha}) = \alpha \ r_B(\mathrm{x})[R_B(x)]^{\alpha-1}\exp\{-[R_B(x)]^{\alpha}\}, \quad \forall \boldsymbol{\alpha} > 0 \tag{3.2}$$

Where $r_B(.)$ and $R_B(.)$ are the baseline reversed hazard and cumulative reversed hazard functions respectively. Accordingly, the reversed hazard function for RPP family is given as

$$r_{RPP}(x; \boldsymbol{\alpha}) = \alpha \ r_B(\mathrm{x})[R_B(x)]^{\alpha-1} \tag{3.3}$$

For more details in this manner [see Marshall and Olkin (2007), p257].

Now, the bivariate extension for this family is given as: Assume $U_1 \sim RPP(\boldsymbol{\alpha}_1)$, $U_2 \sim RPP(\boldsymbol{\alpha}_2)$ and $U_3 \sim RPP(\boldsymbol{\alpha}_3)$ and $U's$ are independent random variables. Let $X_1 = max\ (U_1, \ U_3)$ and $X_2 = max\ (U_2, \ U_3)$.

Then, $(X_1, \ X_2)$ constitute a BRPP class of distributions denoted by $BRPP(\boldsymbol{\alpha}_1, \ \boldsymbol{\alpha}_2, \ \boldsymbol{\alpha}_3)$ with the following cdf and pdf

$$F_{BRPP}(x_1, x_2) = \exp\{-[R_B(x_1)]^{\alpha_1} - [R_B(x_2)]^{\alpha_2} - [R_B(x_3)]^{\alpha_3}\}$$

where $x_3 = \min(x_1, \ x_2)$.

The joint cdf of BRPP models can be stretching in the following form

$$F_{BRPP}(x_1, x_2) = \begin{cases} F_1(x_1, x_2), & x_1 < x_2 \\ F_2(x_1, x_2), & x_1 > x_2 \\ F_3(x), & x_1 = x_2 = x \end{cases} \tag{3.4}$$

Where

$$\begin{aligned} F_1(x_1, x_2) &= \exp\{-[R_B(x_1)]^{\alpha_1} - [R_B(x_2)]^{\alpha_2} - [R_B(x_1)]^{\alpha_3}\}, \\ F_2(x_1, x_2) &= \exp\{-[R_B(x_1)]^{\alpha_1} - [R_B(x_2)]^{\alpha_2} - [R_B(x_2)]^{\alpha_3}\}, \\ F_3(x) &= \exp\{-[R_B(x)]^{\alpha_1} - [R_B(x)]^{\alpha_2} - [R_B(x)]^{\alpha_3}\}. \end{aligned}$$

Accordingly, the joint pdf of BRPP model can be obtained by the following proposition.

Proposition 1 Assume $(X_1,\ X_2) \sim BRPP(\alpha_1,\ \alpha_2,\ \alpha_3)$ with the cdf $F_{BRPP}(x_1,\ x_2)$ defined in (3.4). Then the joint pdf for this class denoted by $f_{BRPP}(x_1,\ x_2)$ is given as

$$f_{BRPP}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & x_1 < x_2 \\ f_2(x_1, x_2), & x_1 > x_2 \\ f_3(x), & x_1 = x_2 = x \end{cases} \tag{3.5}$$

Where

$$f_1(x_1, x_2) = \left\{\alpha_1 r_B(x_1)[R_B(x_1)]^{\alpha_1 - 1} + \alpha_3 r_B(x_1)[R_B(x_1)]^{\alpha_3 - 1}\right\}$$
$$.\alpha_2 r_B(x_2)[R_B(x_2)]^{\alpha_2 - 1}.\exp\{-[R_B(x_1)]^{\alpha_1} - [R_B(x_2)]^{\alpha_2} - [R_B(x_1)]^{\alpha_3}\},$$

$$f_2(x_1, x_2) = \left\{\alpha_2 r_B(x_2)[R_B(x_2)]^{\alpha_2 - 1} + \alpha_3 r_B(x_2)[R_B(x_2)]^{\alpha_3 - 1}\right\}$$
$$.\alpha_1 r_B(x_1)[R_B(x_1)]^{\alpha_1 - 1}.\exp\{-[R_B(x_1)]^{\alpha_1} - [R_B(x_2)]^{\alpha_2} - [R_B(x_2)]^{\alpha_3}\},$$

and $f_3(x) = \alpha_3\ r_B(x)[R_B(x)]^{\alpha_3 - 1}\exp\{-[R_B(x)]^{\alpha_1} - [R_B(x)]^{\alpha_2} - [R_B(x)]^{\alpha_3}\}$.

Proof Let $x_1 < x_2$. In this case $F_{BRPP}(x_1,\ x_2)$ in (3.4) becomes

$$F_1(x_1, x_2) = \exp\{-[R_B(x_1)]^{\alpha_1} - [R_B(x_2)]^{\alpha_2} - [R_B(x_1)]^{\alpha_3}\}.$$

Hence, by differentiation we get $f_1(x_1, x_2) = \frac{\partial^2 F_1(x_1, x_2)}{\partial x_1 \partial x_2}$.

Similarly for $x_1 > x_2$ we can get the expression of $f_2(x_1,\ x_2)$ by the mixed derivatives $\frac{\partial^2 F_2(x_1, x_2)}{\partial x_1 \partial x_2}$ and hence $f_2(x_1, x_2) = \frac{\partial^2 F_2(x_1, x_2)}{\partial x_1 \partial x_2}$.

But the expression of $f_3(x)$ can not be obtained by the similar manner. For this reason the following identity will be used

$$\int_0^\infty \int_0^{x_2} f_1(x_1, x_2)\, dx_1\, dx_2 + \int_0^\infty \int_0^{x_1} f_2(x_1, x_2)\, dx_2\, dx_1 + \int_0^\infty f_3(x) = 1 \tag{3.6}$$

One can verify that

$$\begin{aligned} I_1 &= \int_0^\infty \int_0^{x_2} f_1(x_1, x_2)\, dx_1\, dx_2 \\ &= \int_0^\infty \alpha_2\ r_B(x_2)[R_B(x_2)]^{\alpha_2 - 1}\exp\{-[R_B(x_2)]^{\alpha_1} - [R_B(x_2)]^{\alpha_2} - [R_B(x_2)]^{\alpha_3}\}\, dx_2 \end{aligned}$$

And

$$I_2 = \int_0^\infty \int_0^{x_1} f_2(x_1, x_2)\, dx_2\, dx_1$$
$$= \int_0^\infty \alpha_1\; r_B(x_1)[R_B(x_1)]^{\alpha_1 - 1} \exp\{-[R_B(x_1)]^{\alpha_1} - [R_B(x_1)]^{\alpha_2} - [R_B(x_1)]^{\alpha_3}\}\, dx_1$$

Since

$$I_1 + I_2 = \int_0^\infty \left\{ \alpha_2 r_B(x)[R_B(x)]^{\alpha_2 - 1} + \alpha_1 r_B(x)[R_B(x)]^{\alpha_1 - 1} \right\} .\exp\{-[R_B(x)]^{\alpha_1} - [R_B(x)]^{\alpha_2} - [R_B(x)]^{\alpha_3}\}\, dx. \tag{3.7}$$

Then, from (3.6) and (3.7) we can readily obtain

$$f_3(x) = \alpha_3 r_B(x)[R_B(x)]^{\alpha_3 - 1} \exp\{-[R_B(x)]^{\alpha_1} - [R_B(x)]^{\alpha_2} - [R_B(x)]^{\alpha_3}\}\, dx.$$

Which completes the proof.

The joint reversed hazard function of $(X_1,\ X_2) \sim BRPP(\alpha_1,\ \alpha_2,\ \alpha_3)$ is obtained as follows

$$r_{BRPP}(x_1, x_2) = \begin{cases} r_1(x_1, x_2), & x_1 < x_2 \\ r_2(x_1, x_2), & x_1 > x_2 \\ r_3(x), & x_1 = x_2 = x \end{cases} \tag{3.8}$$

Where

$$r_1(x_1, x_2) = r_{RPP}(x_2; \alpha_2)\{r_{RPP}(x_1; \alpha_1) + r_{RPP}(x_1; \alpha_3)\}$$
$$= \alpha_2 r_B(x_2)[R_B(x_2)]^{\alpha_2 - 1} . \left\{ \alpha_1 r_B(x_1)[R_B(x_1)]^{\alpha_1 - 1} + \alpha_3 r_B(x_1)[R_B(x_1)]^{\alpha_3 - 1} \right\},$$
$$r_2(x_1, x_2) = r_{RPP}(x_1; \alpha_1)\{r_{RPP}(x_2; \alpha_2) + r_{RPP}(x_2; \alpha_3)\}$$
$$= \alpha_1 r_B(x_1)[R_B(x_1)]^{\alpha_1 - 1} . \left\{ \alpha_2 r_B(x_2)[R_B(x_2)]^{\alpha_2 - 1} + \alpha_3 r_B(x_2)[R_B(x_2)]^{\alpha_3 - 1} \right\},$$

and $r_3(x) = r_{RPP}(x; \alpha_3) = \alpha_3\; r_B(x)[R_B(x)]^{\alpha_3 - 1}$.

3.1 Marginal and Conditional Densities

Assume $(X_1,\ X_2) \sim BRPP(\alpha_1,\ \alpha_2,\ \alpha_3)$, then the marginal cdf and pdf of X_1 and X_2 are given respectively, as follows

$$F_{X_i}(x_i) = \exp\{-[R_B(x_i)]^{\alpha_i} - [R_B(x_i)]^{\alpha_3}\}, \quad i = 1, 2$$
$$f_{X_i}(x_i) = \left\{ \alpha_i\; r_B(x_i)[R_B(x_i)]^{\alpha_i - 1} + \alpha_3\; r_B(x_i)[R_B(x_i)]^{\alpha_3 - 1} \right\} .\exp\{-[R_B(x_i)]^{\alpha_i} - [R_B(x_i)]^{\alpha_3}\}, i = 1, 2.$$

Further, for $(X_1,\ X_2)\sim BRPP(\alpha_1,\ \alpha_2,\ \alpha_3)$, the conditional density of X_{1i} given $X_{2j} = x_{2j}$ is given by

$$f_{X_i/X_j}(x_1, x_2) = \begin{cases} f^{(1)}_{i/j}(x_i/x_j) \text{if} x_i < x_j \\ f^{(2)}_{i/j}(x_i/x_j) \text{if} x_j > x_i \\ f^{(3)}_{i/j}(x_i/x_j) \text{if} x_i = x_j, \end{cases} \tag{3.9}$$

where

$$f^{(1)}_{i/j}(x_i/x_j) = \frac{\alpha_2 r_B(x_i)[\mathrm{R}_B(x_i)]^{\alpha_2-1}\left[\alpha_1 r_B(x_j)\left[\mathrm{R}_B(x_j)\right]^{\alpha_1-1} + \alpha_3 r_B(x_j)\left[\mathrm{R}_B(x_j)\right]^{\alpha_3-1}\right]}{\left[\alpha_2 r_B(x_j)\left[\mathrm{R}_B(x_j)\right]^{\alpha_2-1} + \alpha_3 r_B(x_j)\left[\mathrm{R}_B(x_j)\right]^{\alpha_3-1}\right]}$$
$$.\exp\left\{-[\mathrm{R}_B(x_i)]^{\alpha_1} - [\mathrm{R}_B(x_i)]^{\alpha_3} + \left[\mathrm{R}_B(x_j)\right]^{\alpha_3}\right\},$$
$$f^{(2)}_{i/j}(x_i/x_j) = \alpha_1 r_B(x_j)\left[\mathrm{R}_B(x_j)\right]^{\alpha_1-1}\exp\left\{-\left[\mathrm{R}_B(x_j)\right]^{\alpha_1}\right\},$$
$$f^{(3)}_{i/j}(x_i/x_j) = \left\{\frac{\alpha_3 r_B(x_i)[\mathrm{R}_B(x_i)]^{\alpha_3-1}}{\left[\alpha_2 r_B(x_j)\left[\mathrm{R}_B(x_j)\right]^{\alpha_2-1} + \alpha_3 r_B(x_j)\left[\mathrm{R}_B(x_j)\right]^{\alpha_3-1}\right]}\right\}$$
$$.\exp\left\{-[\mathrm{R}_B(x_i)]^{\alpha_1} - [\mathrm{R}_B(x_i)]^{\alpha_2} - [\mathrm{R}_B(x_i)]^{\alpha_3} + \left[\mathrm{R}_B(x_j)\right]^{\alpha_2} + \left[\mathrm{R}_B(x_j)\right]^{\alpha_3}\right\}.$$

3.2 Absolutely Continuous BRPP Family of Distributions

An absolutely continuous BRPP ($BRPP_{ac}$) family of distributions will be introduced by removing the singular part and remaining only the absolutely continuous part.

A random vector $(Y_1,\ Y_2)$ follows a $BRPP_{ac}$ family if its pdf is given by

$$f_{Y_1,Y_2}(y_1, y_2) = \begin{cases} Cf_1(y_1, y_2) \text{if} y_1 < y_2 \\ Cf_2(y_1, y_2) \text{if} y_1 > y_2 \end{cases},$$

Where

$$f_1(y_1, y_2) = \left\{\alpha_1 r_B(y_1)[R_B(y_1)]^{\alpha_1-1} + \alpha_3 r_B(y_1)[R_B(y_1)]^{\alpha_3-1}\right\}$$
$$.\alpha_2 r_B(y_2)[R_B(y_2)]^{\alpha_2-1}.\exp\{-[R_B(y_1)]^{\alpha_1} - [R_B(y_2)]^{\alpha_2} - [R_B(y_1)]^{\alpha_3}\},$$
$$f_2(y_1, y_2) = \left\{\alpha_2 r_B(y_2)[R_B(y_2)]^{\alpha_2-1} + \alpha_3 r_B(y_2)[R_B(y_2)]^{\alpha_3-1}\right\}$$
$$.\alpha_1 r_B(y_1)[R_B(y_1)]^{\alpha_1-1}.\exp\{-[R_B(y_1)]^{\alpha_1} - [R_B(y_2)]^{\alpha_2} - [R_B(y_2)]^{\alpha_3}\}.$$

and C is a normalizing constant. It will be denoted as $(Y_1,\ Y_2)\sim BRPP_{ac}(\alpha_1,\ \alpha_2,\ \alpha_3)$.

It is easy to check that the marginal distributions in this case are not univariate RPP models.

3.3 Parameters Estimation for BRPP Family of Distributions

In this section the shape parameters of BRPP models are estimated based on MLE method. Assume that $\{(x_{11}, x_{21}), (x_{12}, x_{22}), ..., (x_{1n}, x_{2n})\}$ be a complete random sample from $BRPP(\alpha_1, \alpha_2, \alpha_3)$ family of distributions whose pdf and cdf are given in (3.5) and (3.4). Consider the following notation

$$I_1 = \{i; x_{1i} < x_{2i}\}, I_2 = \{i; x_{1i} > x_{2i}\}, I_3 = \{x_{1i} = x_{2i} = x_i\}, I = I_1 \cup I_2 \cup I_3,$$
$$|I_1| = n_1, |I_2| = n_2, |I_3| = n_3, \text{and } n_1 + n_2 + n_3 = n.$$

The log-likelihood function of the sample of size n from $BRPP(\alpha_1, \alpha_2, \alpha_3)$ is given by

$$\begin{aligned} l(\underline{\alpha}) = {} & n_1 \log\alpha_2 + n_2 \log\alpha_1 + n_3 \log\alpha_3 + (\alpha_2 - 1) \sum_{I_1} \log[R_B(x_{2i})] \\ & + (\alpha_1 - 1) \sum_{I_2} \log[R_B(x_{1i})] + (\alpha_3 - 1) \sum_{I_3} \log[R_B(x_i)] \\ & - \sum_I [R_B(x_{1i})]^{\alpha_1} + [R_B(x_{1i})]^{\alpha_1} + [R_B(x_i)]^{\alpha_1} \\ & - \sum_I [R_B(x_{2i})]^{\alpha_2} + [R_B(x_{2i})]^{\alpha_2} + [R_B(x_{2i})]^{\alpha_2} \\ & - \sum_I [R_B(x_{2i})]^{\alpha_3} + [R_B(x_{1i})]^{\alpha_3} + [R_B(x_i)]^{\alpha_3} \\ & + \sum_I \log[r_B(x_{2i})] + \log[r_B(x_{1i})] + \log[r_B(x_i)] \\ & + \sum_{I_1} \Phi(x_{1i}; \alpha_2, \alpha_3) + \sum_{I_2} \Phi(x_{2i}; \alpha_1, \alpha_3). \end{aligned}$$

Where $\underline{\alpha} = (\alpha_1 , \alpha_2 , \alpha_3)$,

$$\Phi(x_{ki}; \alpha_k, \alpha_3) = \log\left[\alpha_j r_B(x_{ki})[R_B(x_{ki})]^{\alpha_j - 1} + \alpha_3 r_B(x_{ki})[R_B(x_{ki})]^{\alpha_3 - 1}\right], \text{for } k$$
$$= 1, 2 \text{ and } j = 1, 2, k \neq j.$$

Accordingly, the likelihood equations can be written as

$$\frac{n_2}{\widehat{\boldsymbol{\alpha}}_1}+\sum_{\mathrm{I}_2}\log[R_B(x_{1i})]+\sum_{\mathrm{I}_2}\Psi(x_{2i};\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_3)$$
$$=\sum_{I}\log[R_B(x_{1i})]\left\{R_B(x_{1i})^{\widehat{\boldsymbol{\alpha}}_1}+R_B(x_{1i})^{\widehat{\boldsymbol{\alpha}}_1}\right\}+\log[R_B(x_i)][R_B(x_i)]^{\widehat{\boldsymbol{\alpha}}_1},$$
$$\frac{n_1}{\widehat{\boldsymbol{\alpha}}_2}+\sum_{\mathrm{I}_1}\log[R_B(x_{2i})]+\sum_{\mathrm{I}_1}\Psi(x_{1i};\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)$$
$$=\sum_{I}\log[R_B(x_{2i})]\left\{R_B(x_{2i})^{\widehat{\boldsymbol{\alpha}}_2}+R_B(x_{2i})^{\widehat{\boldsymbol{\alpha}}_2}\right\}+\log[R_B(x_i)][R_B(x_i)]^{\widehat{\boldsymbol{\alpha}}_2}$$
$$\frac{n_3}{\widehat{\boldsymbol{\alpha}}_3}+\sum_{\mathrm{I}_3}\log[R_B(x_i)]+\sum_{\mathrm{I}_1\cup\mathrm{I}_2}\eta(x_{2i};\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_3)+\eta(x_{1i};\boldsymbol{\alpha}_1,\boldsymbol{\alpha}_3)$$
$$=\sum_{\mathrm{I}}\log[R_B(x_{2i})][R_B(x_{2i})]^{\widehat{\boldsymbol{\alpha}}_3}+\log[R_B(x_{1i})][R_B(x_{1i})]^{\widehat{\boldsymbol{\alpha}}_3}$$
$$+\log[R_B(x_i)][R_B(x_i)]^{\widehat{\boldsymbol{\alpha}}_3}.$$

Where

$$\eta(x_{ki};\boldsymbol{\alpha}_k,\boldsymbol{\alpha}_3)=\frac{[R_B(x_{ki})]^{\alpha_3-1}[1+\boldsymbol{\alpha}_3\log[R_B(x_{ki})]]}{\boldsymbol{\alpha}_j[R_B(x_{ki})]^{\boldsymbol{\alpha}_j-1}+\boldsymbol{\alpha}_3[R_B(x_{ki})]^{\alpha_3-1}},k=1,2,\ j=1,2;\ i\neq j.$$

and $\Psi(x_{ki};\boldsymbol{\alpha}_k,\boldsymbol{\alpha}_3)=\frac{[R_B(x_{ki})]^{\boldsymbol{\alpha}_j-1}\left[1+\boldsymbol{\alpha}_j\log[R_B(x_{ki})]\right]}{\boldsymbol{\alpha}_j[R_B(x_{ki})]^{\boldsymbol{\alpha}_j-1}+\boldsymbol{\alpha}_3[R_B(x_{ki})]^{\alpha_3-1}},k=1,2,\ j=1,2;\ i\neq j.$

Consequently, the second derivatives are given as follows

$$\frac{\partial^2 l(\underline{\alpha})}{\partial\alpha_1^2} = -\frac{n_2}{\alpha_1^2} + \sum_{\mathrm{I}_2} \xi(x_{2i};\alpha_1,\alpha_3) +$$

$$\sum_I (\log[R_B(x_{1i})])^2 \left\{ R_B(x_{1i})^{\widehat{\alpha}_1} + R_B(x_{1i})^{\widehat{\alpha}_1} \right\} + (\log[R_B(x_i)])^2 [R_B(x_i)]^{\widehat{\alpha}_1},$$

$$\frac{\partial^2 l(\underline{\alpha})}{\partial\alpha_2^2} = -\frac{n_1}{\alpha_2^2} + \sum_{\mathrm{I}_1} \xi(x_{1i};\alpha_2,\alpha_3) -$$

$$\sum_I (\log[R_B(x_{2i})])^2 \left\{ R_B(x_{2i})^{\widehat{\alpha}_2} + R_B(x_{2i})^{\widehat{\alpha}_2} \right\} + (\log[R_B(x_i)])^2 [R_B(x_i)]^{\widehat{\alpha}_1}$$

$$\frac{\partial^2 l(\underline{\alpha})}{\partial\alpha_3^2} = -\frac{n_3}{\alpha_3^2} + \sum_{\mathrm{I}_1\cup\mathrm{I}_2} \epsilon(x_{1i};\alpha_2,\alpha_3) + \epsilon(x_{2i};\alpha_1,\alpha_3) -$$

$$\sum_I (\log[R_B(x_{2i})])^2 [R_B(x_{2i})]^{\widehat{\alpha}_3} + (\log[R_B(x_{1i})])^2 [R_B(x_{1i})]^{\widehat{\alpha}_3}$$

$$+(\log[R_B(x_i)])^2 [R_B(x_i)]^{\widehat{\alpha}_1},$$

$$\frac{\partial^2 l(\underline{\alpha})}{\partial\alpha_1\partial\alpha_3} = \sum_{\mathrm{I}_2} \delta(x_{2i};\alpha_1,\alpha_3), \frac{\partial^2 l(\underline{\alpha})}{\partial\alpha_2\partial\alpha_3} = \sum_{\mathrm{I}_1} \delta(x_{1i};\alpha_2,\alpha_3) \text{ and } \frac{\partial^2 l(\underline{\alpha})}{\partial\alpha_1\partial\alpha_2} = 0.$$

Where

$$\xi(x_{ki};\alpha_k,\alpha_3) = \frac{[R_B(x_{ki})]^{\alpha_j-1}\log[R_B(x_{ki})]\left[2+\alpha_j\log[R_B(x_{ki})]\right]}{\alpha_j[R_B(x_{ki})]^{\alpha_j-1} + \alpha_3[R_B(x_{ki})]^{\alpha_3-1}} - \left(\frac{[R_B(x_{ki})]^{\alpha_j-1}\left[1+\alpha_j\log[R_B(x_{ki})]\right]}{\alpha_j[R_B(x_{ki})]^{\alpha_j-1} + \alpha_3[R_B(x_{ki})]^{\alpha_3-1}}\right)^2,$$

$$\delta(x_{ki};\alpha_k,\alpha_3) = \frac{-[R_B(x_{ki})]^{\alpha_3-1}[R_B(x_{ki})]^{\alpha_j-1}[1+\alpha_3\log[R_B(x_{\mathrm{k}i})]]\left[1+\alpha_j\log[R_B(x_{ki})]\right]}{\left(\alpha_j[R_B(x_{ki})]^{\alpha_j-1} + \alpha_3[R_B(x_{ki})]^{\alpha_3-1}\right)^2},$$

$$\epsilon(x_{ki};\alpha_k,\alpha_3) = \frac{[R_B(x_{ki})]^{\alpha_3-1}\log[R_B(x_{ki})][1+\alpha_3\log[R_B(x_{ki})]]}{\alpha_j[R_B(x_{ki})]^{\alpha_j-1} + \alpha_3[R_B(x_{ki})]^{\alpha_3-1}} - \left(\frac{[R_B(x_{ki})]^{\alpha_3-1}[1+\alpha_3\log[R_B(x_{ki})]]}{\alpha_j[R_B(x_{ki})]^{\alpha_j-1} + \alpha_3[R_B(x_{ki})]^{\alpha_3-1}}\right)^2.$$

3.3.1 Asymptotic Confidence Intervals

The asymptotic variance-covariance matrix of $\widehat{\alpha}_1, \widehat{\alpha}_2$and $\widehat{\alpha}_3$ is obtained by inverting the Fisher information matrix with elements that are negatives of expected values of the second order derivatives of logarithms of the likelihood function. In the present situation, it seems appropriate to approximate the expected values by their maximum likelihood estimates. Accordingly; the asymptotic variance –covariance matrix can be written as follows

$$F^{-1} = \begin{bmatrix} I_{11} I_{12} I_{13} \\ I_{21} I_{22} I_{23} \\ I_{31} I_{32} I_{33} \end{bmatrix}^{-1} \Bigg|_{\alpha=\widehat{\alpha}}$$

Where $I_{ij} = -\frac{\partial^2 l(\underline{\alpha})}{\partial\alpha_i \partial\alpha_j}\Big|_{\alpha=\widehat{\alpha}}$

Now, the asymptotic normality results will be stated to obtain the asymptotic confidence intervals of α_1, α_2 and α_3. Under particular regularity conditions it can be stated as follows.

$$\sqrt{n}\left[\left(\widehat{\alpha}_1 - \alpha_1\right), \left(\widehat{\alpha}_2 - \alpha_2\right), \left(\widehat{\alpha}_3 - \alpha_3\right)\right] \rightarrow N_3\left(0, F^{-1}\right) \text{ as } n \rightarrow \infty$$

Where F^{-1} is the variance-covariance matrix, $\widehat{\alpha} = (\widehat{\alpha}_1, \widehat{\alpha}_2, \widehat{\alpha}_3)$ and $\alpha = (\alpha_1, \alpha_2, \alpha_3)$.

3.4 A New Bivariate Distributions Belongs to BRPP Class

i) **Bivariate Inverse Weibull Distribution**

A new bivariate inverse Weibull distribution is obtained by substituting Eqs. (2.7)–(2.8) in Eqs. (3.4)–(3.5) as follows.

The joint cdf for the bivariate inverse Weibull (BIW) distribution is given as

$$F_{BIW}(x_1, x_2) = exp\left\{-\left[\frac{\lambda}{x_1}\right]^{\alpha_1} - \left[\frac{\lambda}{x_2}\right]^{\alpha_2} - \left[\frac{\lambda}{x_3}\right]^{\alpha_3}\right\}$$

It can be rewritten in the following form

$$F_{BIW}(x_1, x_2) = \begin{cases} \exp\left\{-\left[\frac{\lambda}{x_1}\right]^{\alpha_1} - \left[\frac{\lambda}{x_2}\right]^{\alpha_2} - \left[\frac{\lambda}{x_1}\right]^{\alpha_3}\right\}, x_1 < x_2 \\ \exp\left\{-\left[\frac{\lambda}{x_1}\right]^{\alpha_1} - \left[\frac{\lambda}{x_2}\right]^{\alpha_2} - \left[\frac{\lambda}{x_2}\right]^{\alpha_3}\right\}, x_1 > x_2 \\ \exp\left\{-\left[\frac{\lambda}{x}\right]^{\alpha_1} - \left[\frac{\lambda}{x}\right]^{\alpha_2} - \left[\frac{\lambda}{x}\right]^{\alpha_3}\right\}, x_1 = x_2 = x \end{cases}$$

where $x_3 = \min(x_1, x_2)$

The corresponding joint pdf is given as

$$f_{BIW}(\mathrm{x}_1,\mathrm{x}_2)=\begin{cases} f_1(\mathrm{x}_1,\mathrm{x}_2),\mathrm{x}_1<\mathrm{x}_2\\ f_2(\mathrm{x}_1,\mathrm{x}_2),\mathrm{x}_1>\mathrm{x}_2\\ f_3(x),\mathrm{x}_1=\mathrm{x}_2=x\end{cases} \tag{3.10}$$

where

$$f_1(\mathrm{x}_1,\mathrm{x}_2)=\frac{\alpha_2^2\lambda^2}{(\mathrm{x}_1\mathrm{x}_2)^2}\left(\frac{\lambda}{\mathrm{x}_2}\right)^{\alpha_2-1}\left\{\alpha_1\left(\frac{\lambda}{\mathrm{x}_1}\right)^{\alpha_1-1}+\alpha_3\left(\frac{\lambda}{\mathrm{x}_1}\right)^{\alpha_3-1}\right\}$$
$$\exp\left\{-\left[\frac{\lambda}{\mathrm{x}_1}\right]^{\alpha_1}-\left[\frac{\lambda}{\mathrm{x}_2}\right]^{\alpha_2}-\left[\frac{\lambda}{\mathrm{x}_1}\right]^{\alpha_3}\right\}$$
$$f_2(\mathrm{x}_1,\mathrm{x}_2)=\frac{\alpha_1^2\lambda^2}{(\mathrm{x}_1\mathrm{x}_2)^2}\left(\frac{\lambda}{\mathrm{x}_1}\right)^{\alpha_1-1}\left\{\alpha_2\left(\frac{\lambda}{\mathrm{x}_2}\right)^{\alpha_2-1}+\alpha_3\left(\frac{\lambda}{\mathrm{x}_2}\right)^{\alpha_3-1}\right\}$$
$$\exp\left\{-\left[\frac{\lambda}{\mathrm{x}_1}\right]^{\alpha_1}-\left[\frac{\lambda}{\mathrm{x}_2}\right]^{\alpha_2}-\left[\frac{\lambda}{\mathrm{x}_2}\right]^{\alpha_3}\right\},$$

and

$$f_3(x)=\frac{\alpha_3\lambda}{x^2}\left(\frac{\lambda}{\mathrm{x}}\right)^{\alpha_3-1}\exp\left\{-\left[\frac{\lambda}{\mathrm{x}}\right]^{\alpha_1}-\left[\frac{\lambda}{\mathrm{x}}\right]^{\alpha_2}-\left[\frac{\lambda}{\mathrm{x}}\right]^{\alpha_3}\right\}$$

ii) Bivariate Generalized Invers Rayleigh Distribution

A new bivariate Generalized inverse Rayleigh distribution is defined by using Eqs. (2.11)–(2.12) in Eqs. (3.4)–(3.5) with the following joint cdf and pdf respectively

$$F_{BGIR}(x_1,x_2)=\exp\left\{-\left[\left(\frac{\sigma}{x_1}\right)^2\right]^{\alpha_1}-\left[\left(\frac{\sigma}{x_2}\right)^2\right]^{\alpha_2}-\left[\left(\frac{\sigma}{x_3}\right)^2\right]^{\alpha_3}\right\}$$

where $x_3=\min(x_1,\ x_2)$. and denoted by $BGIR(\alpha_1,\ \alpha_2,\ \ \alpha_3,\ \sigma)$.

The joint cdf of BGIR model can be stretching in the following form

$$F_{BGIR}(x_1,x_2)=\begin{cases} F_1(x_1,x_2), & x_1<x_2\\ F_2(x_1,x_2), & x_1>x_2\\ F_3(x), & x_1=x_2=x\end{cases}$$

Where

$$F_1(x_1,x_2)=\exp\left\{-\left[\left(\frac{\sigma}{x_1}\right)^2\right]^{\alpha_1}-\left[\left(\frac{\sigma}{x_2}\right)^2\right]^{\alpha_2}-\left[\left(\frac{\sigma}{x_1}\right)^2\right]^{\alpha_3}\right\},$$
$$F_2(x_1,x_2)=\exp\left\{-\left[\left(\frac{\sigma}{x_1}\right)^2\right]^{\alpha_1}-\left[\left(\frac{\sigma}{x_2}\right)^2\right]^{\alpha_2}-\left[\left(\frac{\sigma}{x_2}\right)^2\right]^{\alpha_3}\right\},$$
$$F_3(x)=\exp\left\{-\left[\left(\frac{\sigma}{x}\right)^2\right]^{\alpha_1}-\left[\left(\frac{\sigma}{x}\right)^2\right]^{\alpha_2}-\left[\left(\frac{\sigma}{x}\right)^2\right]^{\alpha_3}\right\}.$$

Accordingly, the joint pdf of BGIR model can be obtained as

$$f_{BGIR}(x_1,x_2)=\begin{cases} f_1(x_1,x_2), & x_1<x_2 \\ f_2(x_1,x_2), & x_1<x_2 \\ f_3(x), & x_1=x_2=x \end{cases} \tag{3.11}$$

Where

$$f_1(x_1,x_2)=\alpha_2\frac{4\sigma^4}{x_1^3x_2^3}\left[\left(\frac{\sigma}{x_2}\right)^2\right]^{\alpha_2-1}\left\{\alpha_1\left[\left(\frac{\sigma}{x_1}\right)^2\right]^{\alpha_1-1}+\alpha_3\left[\left(\frac{\sigma}{x_1}\right)^2\right]^{\alpha_{3-1}}\right\}.$$
$$.\exp\left\{-\left[\left(\frac{\sigma}{x_1}\right)^2\right]^{\alpha_1}-\left[\left(\frac{\sigma}{x_2}\right)^2\right]^{\alpha_2}-\left[\left(\frac{\sigma}{x_1}\right)^2\right]^{\alpha_3}\right\},$$
$$f_2(x_1,x_2)=\alpha_1\frac{4\sigma^4}{x_1^3x_2^3}\left[\left(\frac{\sigma}{x_1}\right)^2\right]^{\alpha_2-1}\left\{\alpha_2\left[\left(\frac{\sigma}{x_2}\right)^2\right]^{\alpha_1-1}+\alpha_3\left[\left(\frac{\sigma}{x_2}\right)^2\right]^{\alpha_{3-1}}\right\}.\cdot$$
$$.\exp\left\{-\left[\left(\frac{\sigma}{x_1}\right)^2\right]^{\alpha_1}-\left[\left(\frac{\sigma}{x_2}\right)^2\right]^{\alpha_2}-\left[\left(\frac{\sigma}{x_2}\right)^2\right]^{\alpha_3}\right\},$$
$$\text{and } f_3(x)=\alpha_3.\left[\left(\frac{\sigma}{x}\right)^2\right]^{\alpha_3-1}\exp\left\{-\left[\left(\frac{\sigma}{x}\right)^2\right]^{\alpha_1}-\left[\left(\frac{\sigma}{x}\right)^2\right]^{\alpha_2}-\left[\left(\frac{\sigma}{x}\right)^2\right]^{\alpha_3}\right\}.$$

iii) **Bivariate Generalized Inverse Uniform Distribution**

A bivariate generalized inverse uniform distribution denoted by $BGIU(\alpha_1,\ \alpha_2,\ \alpha_3)$ can be introduced by substituting Eqs. (2.15)–(2.16) in Eqs. (3.4)–(3.5) to get the joint cdf and pdf as follows

$$F_{BGIU}(x_1,x_2)=\exp\left\{-\left[\log\left(\frac{x_1+1}{x_1}\right)\right]^{\alpha_1}-\left[\log\left(\frac{x_2+1}{x_2}\right)\right]^{\alpha_2}-\left[\log\left(\frac{x_3+1}{x_3}\right)\right]^{\alpha_3}\right\}$$

where $x_3 = \min(x_1,\ x_2)$.

The joint cdf of BGIU model can be stretching in the following form

$$F_{BGIU}(x_1,x_2)=\begin{cases} F_1(x_1,x_2), & x_1<x_2 \\ F_2(x_1,x_2), & x_1>x_2 \\ F_3(x), & x_1=x_2=x \end{cases}$$

Where

$$F_1(x_1,x_2)=\exp\left\{-\left[\log\left(\frac{x_1+1}{x_1}\right)\right]^{\alpha_1}-\left[\log\left(\frac{x_2+1}{x_2}\right)\right]^{\alpha_2}-\left[\log\left(\frac{x_1+1}{x_1}\right)\right]^{\alpha_3}\right\},$$
$$F_2(x_1,x_2)=\exp\left\{-\left[\log\left(\frac{x_1+1}{x_1}\right)\right]^{\alpha_1}-\left[\log\left(\frac{x_2+1}{x_2}\right)\right]^{\alpha_2}-\left[\log\left(\frac{x_2+1}{x_2}\right)\right]^{\alpha_3}\right\},$$
$$F_3(x)=\exp\left\{-\left[\log\left(\frac{x+1}{x}\right)\right]^{\alpha_1}-\left[\log\left(\frac{x+1}{x}\right)\right]^{\alpha_2}-\left[\log\left(\frac{x+1}{x}\right)\right]^{\alpha_3}\right\}.$$

Accordingly, the joint pdf of BGIU model can be obtained as

$$f_{BGIU}(x_1,x_2)=\begin{cases} f_1(x_1,x_2), & x_1<x_2 \\ f_2(x_1,x_2), & x_1<x_2 \\ f_3(x), & x_1=x_2=x \end{cases} \tag{3.12}$$

where

$$\begin{aligned} f_1(x_1,x_2)=&\frac{\boldsymbol{\alpha}_2}{x_2(x_2+1)}\left[\log\left(\frac{x_2+1}{x_2}\right)\right]^{\alpha_2-1}\left\{\boldsymbol{\alpha}_1\left[\log\left(\frac{x_1+1}{x_1}\right)\right]^{\alpha_1-1}+\boldsymbol{\alpha}_3\left[\log\left(\frac{x_1+1}{x_1}\right)\right]^{\alpha_3-1}\right\}. \\ &.\exp\left\{-\left[\log\left(\frac{x_1+1}{x_1}\right)\right]^{\alpha_1}-\left[\log\left(\frac{x_2+1}{x_2}\right)\right]^{\alpha_2}-\left[\log\left(\frac{x_1+1}{x_1}\right)\right]^{\alpha_3}\right\}, \\ f_2(x_1,x_2)=&\frac{\boldsymbol{\alpha}_1}{x_1(x_1+1)}\left[\log\left(\frac{x_1+1}{x_1}\right)\right]^{\alpha_1-1}\left\{\boldsymbol{\alpha}_2\left[\log\left(\frac{x_2+1}{x_2}\right)\right]^{\alpha_2-1}+\boldsymbol{\alpha}_3\left[\log\left(\frac{x_2+1}{x_2}\right)\right]^{\alpha_3-1}\right\}. \\ &.\exp\left\{-\left[\log\left(\frac{x_1+1}{x_1}\right)\right]^{\alpha_1}-\left[\log\left(\frac{x_2+1}{x_2}\right)\right]^{\alpha_2}-\left[\log\left(\frac{x_2+1}{x_2}\right)\right]^{\alpha_3}\right\}, \\ f_3(x)=&\frac{\boldsymbol{\alpha}_3}{x(x+1)}\left[\log\left(\frac{x+1}{x}\right)\right]^{\alpha_3-1} \\ &.\exp\left\{-\left[\log\left(\frac{x+1}{x}\right)\right]^{\alpha_1}-\left[\log\left(\frac{x+1}{x}\right)\right]^{\alpha_2}-\left[\log\left(\frac{x+1}{x}\right)\right]^{\alpha_3}\right\}. \end{aligned}$$

4 Bivariate Hazard Power Parameter (BHPP)Family of Distributions

The survival function $S(.)$ and its corresponding cumulative hazard function $H(.)$ can related by the following formula $S(x) = e^{-H(x)}, \quad \forall\ x.$

So, The hazard power parameter model can be defined as follows

$$S_{HPP}(x;\boldsymbol{\alpha}) = \exp\{-[H_B(x)]^{\alpha}\}, \quad \forall \boldsymbol{\alpha} > 0. \tag{4.1}$$

The corresponding pdf is given by differentiating (4.1) as

$$f_{HPP}(x;\boldsymbol{\alpha}) = \alpha\ h_B(\mathrm{x})[H_B(x)]^{\alpha-1}\exp\{-[H_B(x)]^{\alpha}\}, \quad \forall \boldsymbol{\alpha} > 0 \tag{4.2}$$

Where $h_B(.)$ and $H_B(.)$ are the baseline hazard and cumulative hazard functions respectively. Accordingly, the hazard function for HPP family is given as

$$h_{HPP}(x;\boldsymbol{\alpha}) = \alpha\ h_B(\mathrm{x})[H_B(x)]^{\alpha-1} \tag{4.3}$$

It is follows if h_B increasing and $\boldsymbol{\alpha} \geq 1$, then h_{HPP} is increasing; if h_B decreasing and $0 < \boldsymbol{\alpha} < 1$, then h_{HPP} is decreasing

Now, to get the bivariate HPP class of distributions. Assume the univariate hazard power parameter model is denoted by $HPP(\boldsymbol{\alpha},\ \Theta)$ where $\boldsymbol{\alpha}$ is the hazard power parameter and Θ may be a vector of parameters for an underlying distribution. Now suppose that $U_i \sim HPP(\boldsymbol{\alpha}_i,\ \Theta)$, $i = 1, 2, 3$ such that $U_i's$ are mutually independent random variables and define $X_j = \ min\,(U_j,\ U_3\,)$, $j = 1$, 2. Such that; $X_j's$ are dependent random variables. Hence BHPP model denoted by $BHPP(\boldsymbol{\alpha}_1,\ \boldsymbol{\alpha}_2,\ \boldsymbol{\alpha}_3)$ is defined with the following joint survival function.

$$\begin{aligned} S_{BHPP}(x_1, x_2) &= S_{HPP}(x_1;\boldsymbol{\alpha}_1)S_{HPP}(x_2;\boldsymbol{\alpha}_2)S_{HPP}(x_3;\boldsymbol{\alpha}_3). \\ &= \exp\{-[H_B(x_1)]^{\alpha_1}-[H_B(x_2)]^{\alpha_2}-[H_B(x_3)]^{\alpha_3}\}. \end{aligned}$$

where $x_3 = \max(x_1,\ x_2)$.

The joint survival function of $(X_1,\ X_2) \sim BHPP(\boldsymbol{\alpha}_1,\ \boldsymbol{\alpha}_2,\ \boldsymbol{\alpha}_3)$ can be stretching in the following form

$$S_{BHPP}(x_1, x_2) = \begin{cases} S_1(x_1, x_2), & x_1 < x_2 \\ S_2(x_1, x_2), & x_1 > x_2 \\ S_3(x), & x_1 = x_2 = x \end{cases} \tag{4.4}$$

Where

$$S_1(x_1,x_2)=\exp\{-[H_B(x_1)]^{\alpha_1}-[H_B(x_2)]^{\alpha_2}-[H_B(x_2)]^{\alpha_3}\},$$
$$S_2(x_1,x_2)=\exp\{-[H_B(x_1)]^{\alpha_1}-[H_B(x_1)]^{\alpha_3}-[H_B(x_2)]^{\alpha_2}\},$$
$$\text{and } S_3(x)=\exp\{-[H_B(x)]^{\alpha_1}-[H_B(x)]^{\alpha_2}-[H_B(x)]^{\alpha_3}\}.$$

Accordingly, the joint pdf of BHPP model can be obtained by the following proposition.

Proposition 2 Assume $(X_1,\ X_2)\sim BHPP(\alpha_1,\ \alpha_2,\ \alpha_3)$ with the survival function $S_{BHPP}(x_1,\ x_2)$ defined in (4.4). Then the joint pdf for this family denoted by $f_{BHPP}(x_1,\ x_2)$ is given as

$$f_{BHPP}(x_1,x_2)=\begin{cases} f_1(x_1,x_2), & x_1<x_2 \\ f_2(x_1,x_2), & x_1<x_2 \\ f_3(x), & x_1=x_2=x \end{cases} \tag{4.5}$$

where

$$\begin{aligned} f_1(x_1,x_2)&=\left\{\alpha_2 h_B(x_2)[H_B(x_2)]^{\alpha_2-1}+\alpha_3 h_B(x_2)[H_B(x_2)]^{\alpha_3-1}\right\} \\ &\quad .\alpha_1 h_B(x_1)[H_B(x_1)]^{\alpha_1-1}.\exp\{-[H_B(x_1)]^{\alpha_1}-[H_B(x_2)]^{\alpha_2}-[H_B(x_2)]^{\alpha_3}\}, \\ f_2(x_1,x_2)&=\left\{\alpha_1 h_B(x_1)[H_B(x_1)]^{\alpha_1-1}+\alpha_3 h_B(x_1)[H_B(x_1)]^{\alpha_3-1}\right\} \\ &\quad .\alpha_2 h_B(x_2)[H_B(x_2)]^{\alpha_2-1}.\exp\{-[H_B(x_2)]^{\alpha_2}-[H_B(x_1)]^{\alpha_1}-[H_B(x_1)]^{\alpha_3}\}, \\ f_3(x)&=\alpha_3 h_B(x)[H_B(x)]^{\alpha_3-1}\exp\{-[H_B(x)]^{\alpha_1}-[H_B(x)]^{\alpha_2}-[H_B(x)]^{\alpha_3}\}. \end{aligned}$$

Proof By Following the same idea as in Proposition 1 the pdf is derived

The joint hazard function of the dependent variables $(X_1,\ X_2)\sim BHPP(\alpha_1,\ \alpha_2,\ \alpha_3)$ is obtained as follows

$$h_{BHPP}(x_1,x_2)=\begin{cases} h_1(x_1,x_2), & x_1<x_2 \\ h_2(x_1,x_2), & x_1>x_2 \\ h_3(x), & x_1=x_2=x \end{cases} \tag{4.6}$$

where

$$h_1(x_1,x_2) = h_{HPP}(x_1;\boldsymbol{\alpha}_1)\{h_{HPP}(x_2;\boldsymbol{\alpha}_2)+h_{HPP}(x_2;\boldsymbol{\alpha}_3)\}$$
$$= \boldsymbol{\alpha}_1 h_B(x_1)[H_B(x_1)]^{\boldsymbol{\alpha}_1-1}.\Big\{\boldsymbol{\alpha}_2 h_B(x_2)[H_B(x_2)]^{\boldsymbol{\alpha}_2-1} + \boldsymbol{\alpha}_3 h_B(x_2)[H_B(x_2)]^{\boldsymbol{\alpha}_3-1}\Big\},$$
$$h_2(x_1,x_2) = h_{HPP}(x_2;\boldsymbol{\alpha}_2)\{h_{HPP}(x_1;\boldsymbol{\alpha}_1)+h_{HPP}(x_1;\boldsymbol{\alpha}_3)\}$$
$$= \boldsymbol{\alpha}_2 h_B(x_2)[H_B(x_2)]^{\boldsymbol{\alpha}_2-1}.\Big\{\boldsymbol{\alpha}_1 h_B(x_1)[H_B(x_1)]^{\boldsymbol{\alpha}_1-1} + \boldsymbol{\alpha}_3 h_B(x_1)[H_B(x_1)]^{\boldsymbol{\alpha}_3-1}\Big\},$$

and

$$h_3(x) = h_{HPP}(x;\boldsymbol{\alpha}_3) = \boldsymbol{\alpha}_3 h_B(x)[H_B(x)]^{\boldsymbol{\alpha}_3-1}.$$

4.1 Marginal and Conditional Densities

Assume $(X_1,\ X_2)\sim BHPP(\boldsymbol{\alpha}_1,\ \boldsymbol{\alpha}_2,\ \boldsymbol{\alpha}_3)$.Then, the marginal survival functions and densities of X_1 and X_2 are given respectively, as follows

$$S_{X_i}(x_i) = \exp\{-[H_B(x_i)]^{\boldsymbol{\alpha}_i} - [H_B(x_i)]^{\boldsymbol{\alpha}_3}\},\quad i=1,2\quad .$$
$$f_{X_i}(x_i) = \Big\{\boldsymbol{\alpha}_i\ h_B(x_i)[H_B(x_i)]^{\boldsymbol{\alpha}_i-1} + \boldsymbol{\alpha}_3\ h_B(x_i)[H_B(x_i)]^{\boldsymbol{\alpha}_3-1}\Big\}$$
$$.\exp\{-[H_B(x_i)]^{\boldsymbol{\alpha}_i} - [H_B(x_i)]^{\boldsymbol{\alpha}_3}\}, i=1,2.$$

Further, for $(X_1,\ X_2)\sim BHPP(\boldsymbol{\alpha}_1,\ \boldsymbol{\alpha}_2,\ \boldsymbol{\alpha}_3)$, the conditional density of X_{1i} given $X_{2j} = x_{2j}$ is given by

$$f_{X_i/X_j}(x_1,x_2) = \begin{cases} f^{(1)}_{i/j}(x_i/x_j)\text{if} x_i < x_j \\ f^{(2)}_{i/j}(x_i/x_j)\text{if} x_j > x_i \\ f^{(3)}_{i/j}(x_i/x_j)\text{if} x_i = x_j, \end{cases} \tag{4.7}$$

where

$$f^{(1)}_{i/j}(x_i/x_j) = \boldsymbol{\alpha}_1 h_B(x_i)[H_B(x_i)]^{\boldsymbol{\alpha}_1-1}\exp\{-[H_B(x_i)]^{\boldsymbol{\alpha}_1}\},$$
$$f^{(2)}_{i/j}(x_i/x_j) = \boldsymbol{\alpha}_2 h_B(x_j)\left[H_B(x_j)\right]^{\boldsymbol{\alpha}_2-1}\exp\left\{-\left[H_B(x_j)\right]^{\boldsymbol{\alpha}_2}\right\},$$
$$f^{(3)}_{i/j}(x_i/x_j) = \left\{\frac{\boldsymbol{\alpha}_3 h_B(x_j)\left[H_B(x_j)\right]^{\boldsymbol{\alpha}_3-1}}{\left[\boldsymbol{\alpha}_2 h_B(x_j)\left[H_B(x_j)\right]^{\boldsymbol{\alpha}_2-1} + \boldsymbol{\alpha}_3 h_B(x_j)\left[H_B(x_j)\right]^{\boldsymbol{\alpha}_3-1}\right]}\right\}.$$
$$.\exp\left\{-[H_B(x_i)]^{\boldsymbol{\alpha}_1} - [H_B(x_i)]^{\boldsymbol{\alpha}_2} - [H_B(x_i)]^{\boldsymbol{\alpha}_3} + \left[H_B(x_j)\right]^{\boldsymbol{\alpha}_2} + \left[H_B(x_j)\right]^{\boldsymbol{\alpha}_3}\right\}.$$

4.2 Absolutely Continuous BHPP Family of Distributions

An absolutely continuous BHPP ($BHPP_{ac}$) family of distributions will be introduced by removing the singular part and remaining only the absolutely continuous part.

A random vector $(Y_1, \ Y_2)$ follows a $BHPP_{ac}$ family if its pdf is given by

$$f_{Y_1,Y_2}(y_1,y_2) = \begin{cases} Cf_1(y_1,y_2)\text{if} y_1 < y_2 \\ Cf_2(y_1,y_2)\text{if} y_1 > y_2 \end{cases},$$

where

$$\begin{aligned} f_1(y_1,y_2) = &\left\{\alpha_2 h_B(y_2)[H_B(y_2)]^{\alpha_2-1} + \alpha_3 h_B(y_2)[H_B(y_2)]^{\alpha_3-1}\right\} \\ &.\alpha_1 h_B(y_1)[H_B(y_1)]^{\alpha_1-1}.\exp\{-[H_B(y_1)]^{\alpha_1}-[H_B(y_2)]^{\alpha_2}-[H_B(y_2)]^{\alpha_3}\}, \\ f_2(y_1,y_2) = &\left\{\alpha_1 h_B(y_1)[H_B(y_1)]^{\alpha_1-1} + \alpha_3 h_B(y_1)[H_B(y_1)]^{\alpha_3-1}\right\} \\ &.\alpha_2 h_B(y_2)[H_B(y_2)]^{\alpha_2-1}.\exp\{-[H_B(y_2)]^{\alpha_2}-[H_B(y_1)]^{\alpha_1}-[H_B(y_1)]^{\alpha_3}\}, \end{aligned}$$

and C is a normalizing constant. It will be denoted as $(Y_1, \ Y_2)\sim BHPP_{ac}(\alpha_1, \ \alpha_2, \ \alpha_3)$.

Again the marginal distributions in this case are not univariate HPP models.

4.3 Parameter Estimation for BHPP Family of Distributions

Assume that $\{(x_{11}, \ x_{21}), \ (x_{12}, \ x_{22}), \ \ldots, \ (x_{1n}, \ x_{2n})\}$ be a complete random sample from $BHPP(\alpha_1, \ \alpha_2, \ \alpha_3)$ family of distributions whose pdf and survival function are given in Eqs. (4.5) and (4.4). Again, consider the following notation

$$I_1 = \{i; x_{1i} < x_{2i}\}, I_2 = \{i; x_{1i} > x_{2i}\}, I_3 = \{x_{1i} = x_{2i} = x_i\}, I = I_1 \cup I_2 \cup I_3, \\ |I_1| = n_1, |I_2| = n_2, |I_3| = n_3, \text{and } n_1 + n_2 + n_3 = n.$$

The log-likelihood function of the sample of size n from $BHPP(\alpha_1, \ \alpha_2, \ \alpha_3)$ is given by

$$\begin{aligned} l(\underline{\alpha}) = &\ n_1\log\alpha_1 + n_2\log\alpha_2 + n_3\log\alpha_3 + (\alpha_1-1)\textstyle\sum_{I_1}\log[H_B(x_{1i})] \\ &+ (\alpha_2-1)\textstyle\sum_{I_2}\log[H_B(x_{2i})] + (\alpha_3-1)\sum_{I_3}\log[H_B(x_i)] \\ &-\textstyle\sum_I [H_B(x_{1i})]^{\alpha_1} + [H_B(x_{1i})]^{\alpha_1} + [H_B(x_i)]^{\alpha_1} \\ &-\textstyle\sum_I [H_B(x_{2i})]^{\alpha_2} + [H_B(x_{2i})]^{\alpha_2} + [H_B(x_{2i})]^{\alpha_2} \\ &-\textstyle\sum_I [H_B(x_{2i})]^{\alpha_3} + [H_B(x_{1i})]^{\alpha_3} + [H_B(x_i)]^{\alpha_3} \\ &+ \textstyle\sum_I \log[h_B(x_{1i})] + \log[h_B(x_{2i})] + \log[h_B(x_i)] \\ &+ \textstyle\sum_{I_1}\Phi(x_{2i};\alpha_2,\alpha_3) + \sum_{I_2}\Phi(x_{1i};\alpha_1,\alpha_3). \end{aligned}$$

Where $\underline{\alpha} = (\alpha_1\ , \alpha_2, \alpha_3)$,

$$\Phi(x_{ki}; \alpha_k, \alpha_3) = \log\Big[\alpha_k h_B(x_{ki})[H_B(x_{ki})]^{\alpha_k - 1} + \alpha_3 h_B(x_{ki})[H_B(x_{ki})]^{\alpha_3 - 1}\Big],$$

for $k = 1, 2$.

Accordingly, the likelihood equations can be written as

$$\frac{n_1}{\widehat{\alpha}_1} + \sum_{\mathrm{I}_1} \log[H_B(x_{1i})] + \sum_{\mathrm{I}_2} \Psi(x_{1i}; \alpha_1, \alpha_3)$$

$$= \sum_{I} \log[H_B(x_{1i})]\Big\{H_B(x_{1i})^{\widehat{\alpha}_1} + H_B(x_{1i})^{\widehat{\alpha}_1}\Big\}$$

$$+ \log[H_B(x_i)][H_B(x_i)]^{\widehat{\alpha}_1},$$

$$\frac{n_2}{\widehat{\alpha}_2} + \sum_{\mathrm{I}_2} \log[H_B(x_{2i})] + \sum_{\mathrm{I}_1} \Psi(x_{2i}; \alpha_2, \alpha_3)$$

$$= \sum_{I} \log[H_B(x_{2i})]\Big\{H_B(x_{2i})^{\widehat{\alpha}_2} + H_B(x_{2i})^{\widehat{\alpha}_2}\Big\}$$

$$+ \log[H_B(x_i)]\ [H_B(x_i)]^{\widehat{\alpha}_2},$$

$$\frac{n_3}{\widehat{\alpha}_3} + \sum_{\mathrm{I}_3} \log[H_B(x_i)] + \sum_{\mathrm{I}_1 \cup \mathrm{I}_2} \xi(x_{2i}; \alpha_2, \alpha_3)$$
$$+ \xi(x_{1i}; \alpha_1, \alpha_3) = \sum_{\mathrm{I}} \log[H_B(x_{2i})][H_B(x_{2i})]^{\widehat{\alpha}_3}$$

$$+ \log[H_B(x_{1i})][H_B(x_{1i})]^{\widehat{\alpha}_3} + \log[H_B(x_i)][H_B(x_i)]^{\widehat{\alpha}_3}.$$

Where

$$\xi(x_{ki}; \alpha_k, \alpha_3) = \frac{[H_B(x_{ki})]^{\alpha_3 - 1}[1 + \alpha_3 \log[H_B(x_{ki})]]}{\alpha_k [H_B(x_{ki})]^{\alpha_k - 1} + \alpha_3 [H_B(x_{ki})]^{\alpha_3 - 1}}, k = 1, 2.$$

and $\Psi(x_{ki}\ ; \alpha_k\ , \alpha_3) = \frac{[H_B(x_{ki})]^{\alpha_k - 1}[1 + \alpha_k \log[H_B(x_{ki})]]}{\alpha_k [H_B(x_{ki})]^{\alpha_k - 1} + \alpha_3 [H_B(x_{ki})]^{\alpha_3 - 1}}, k = 1, 2.$

The second derivatives are given as follows

$$\frac{\partial^2 l(\underline{\alpha})}{\partial \alpha_1^2} = -\frac{n_1}{\alpha_1^2} + \sum_{\mathrm{I}_2} \eta(x_{1i}; \alpha_1, \alpha_3) -$$

$$\sum_{I} (\log[H_B(x_{1i})])^2 \left\{ H_B(x_{1i})^{\widehat{\alpha}_1} + H_B(x_{1i})^{\widehat{\alpha}_1} \right\} + (\log[H_B(x_i)])^2 [H_B(x_i)]^{\widehat{\alpha}_1},$$

$$\frac{\partial^2 l(\underline{\alpha})}{\partial \alpha_2^2} = -\frac{n_2}{\alpha_2^2} + \sum_{\mathrm{I}_1} \eta(x_{2i}; \alpha_2, \alpha_3) -$$

$$\sum_{I} (\log[H_B(x_{2i})])^2 \left\{ H_B(x_{2i})^{\widehat{\alpha}_2} + H_B(x_{2i})^{\widehat{\alpha}_2} \right\} + (\log[H_B(x_i)])^2 [H_B(x_i)]^{\widehat{\alpha}_1},$$

$$\frac{\partial^2 l(\underline{\alpha})}{\partial \alpha_3^2} = -\frac{n_3}{\alpha_3^2} + \sum_{\mathrm{I}_1 \cup \mathrm{I}_2} \delta(x_{2i}; \alpha_2, \alpha_3) + \delta(x_{1i}; \alpha_1, \alpha_3) -$$

$$\sum_{I} (\log[H_B(x_{2i})])^2 [H_B(x_{2i})]^{\widehat{\alpha}_3} + (\log[H_B(x_{1i})])^2 [H_B(x_{1i})]^{\widehat{\alpha}_3}$$

$$+ (\log[H_B(x_i)])^2 [H_B(x_i)]^{\widehat{\alpha}_1},$$

$$\frac{\partial^2 l(\underline{\alpha})}{\partial \alpha_1 \partial \alpha_3} = \sum_{\mathrm{I}_2} \epsilon(x_{1i}; \alpha_1, \alpha_3), \text{and} \frac{\partial^2 l(\underline{\alpha})}{\partial \alpha_2 \partial \alpha_3} = \sum_{\mathrm{I}_1} \epsilon(x_{2i}; \alpha_2, \alpha_3).$$

Where

$$\begin{aligned}
\mathrm{A}(x_{ki}; \alpha_k, \alpha_3) &= \alpha_k [H_B(x_{ki})]^{\alpha_k - 1} + \alpha_3 [H_B(x_{ki})]^{\alpha_3 - 1} \; k = 1, 2. \\
\mathrm{B}(x_{ki}; \alpha_k) &= [H_B(x_{ki})]^{\alpha_k - 1} [1 + \alpha_k \log[H_B(x_{ki})]] \; k = 1, 2. \\
\mathrm{C}(x_{ki}; \alpha_3) &= [H_B(x_{ki})]^{\alpha_3 - 1} [1 + \alpha_3 \log[H_B(x_{ki})]] \; k = 1, 2. \\
\mathrm{E}(x_{ki}; \alpha_k) &= [H_B(x_{ki})]^{\alpha_k - 1} \log[H_B(x_{ki})] [2 + \log[H_B(x_{ki})]] \; k = 1, 2. \\
\mathrm{G}(x_{ki}; \alpha_k) &= [H_B(x_{ki})]^{\alpha_3 - 1} \log[H_B(x_{ki})] [2 + \log[H_B(x_{ki})]] \; k = 1, 2. \\
\eta(x_{ki}; \alpha_k, \alpha_3) &= \frac{\mathrm{A}(x_{ki}; \alpha_k, \alpha_3) . \mathrm{E}(x_{ki}; \alpha_k) - [\mathrm{B}(x_{ki}; \alpha_k)]^2}{[\mathrm{A}(x_{ki}; \alpha_k, \alpha_3)]^2}, k = 1, 2. \\
\delta(x_{ki}; \alpha_k, \alpha_3) &= \frac{[\mathrm{A}(x_{ki}; \alpha_k, \alpha_3)]^2 . \mathrm{G}(x_{ki}; \alpha_k) - [\mathrm{C}(x_{ki}; \alpha_k)]^2}{[\mathrm{A}(x_{ki}; \alpha_k, \alpha_3)]^2}, k = 1, 2. \\
\text{and } \epsilon(x_{ki}; \alpha_k, \alpha_3) &= -\frac{\mathrm{B}(x_{ki}; \alpha_k) \; \mathrm{C}(x_{ki}; \alpha_k)}{[\mathrm{A}(x_{ki}; \alpha_k, \alpha_3)]^2}, k = 1, 2.
\end{aligned}$$

The asymptotic variance-covariance matrix for the parameters of BHPP family of distributions can be obtained by using the above second derivatives and doing the same steps explained in Section (3.3).

4.4 A New Bivariate Distributions Belongs to BHPP Class

i) Bivariate Weibull Distribution

Using Eqs. (2.3)-(2–4) in Eqs. (4.4)–(4.5). A new bivariate Weibull distribution denoted by $BW(\alpha_1,\ \alpha_2,\ \ \alpha_3,\ \lambda)$ can be defined by the joint survival function

$$S_{BW}(x_1,x_2)=\exp\{-(\lambda x_1)^{\alpha_1}\}\exp\{-(\lambda x_2)^{\alpha_2}\}\ \exp\{-(\lambda x_3)^{\alpha_3}\}$$

where $x_3 = \max(x_1,\ x_2)$.

The joint survival function of BW model can be stretching in the following form

$$S_{BW}(x_1,x_2)=\begin{cases} S_1(x_1,x_2), & x_1<x_2\\ S_2(x_1,x_2), & x_1>x_2\\ S_3(x), & x_1=x_2=x\end{cases}$$

Where

$$\begin{aligned} &S_1(x_1,x_2)=\exp\{-(\lambda x_1)^{\alpha_1}-(\lambda x_2)^{\alpha_2}-(\lambda x_2)^{\alpha_3}\},\\ &S_2(x_1,x_2)=\exp\{-(\lambda x_1)^{\alpha_1}-(\lambda x_1)^{\alpha_3}-(\lambda x_2)^{\alpha_2}\},\\ &S_3(x)=\exp\{-(\lambda x)^{\alpha_1}-(\lambda x)^{\alpha_2}-(\lambda x)^{\alpha_3}\}.\end{aligned}$$

Accordingly, the joint pdf of BW model can be obtained as

$$f_{BW}(x_1,x_2)=\begin{cases} f_1(x_1,x_2), & x_1<x_2\\ f_2(x_1,x_2), & x_1<x_2\\ f_3(x), & x_1=x_2=x\end{cases} \tag{4.8}$$

where

$$\begin{aligned} &f_1(x_1,x_2)=\alpha_1\lambda^{\alpha_1}x_1^{\alpha_1-1}\left\{\alpha_2\lambda^{\alpha_2}x_2^{\alpha_2-1}+\alpha_3\lambda^{\alpha_3}x_2^{\alpha_3-1}\right\}.\exp\{-(\lambda x_1)^{\alpha_1}-(\lambda x_2)^{\alpha_2}-(\lambda x_2)^{\alpha_3}\},\\ &f_2(x_1,x_2)=\alpha_2\lambda^{\alpha_2}x_2^{\alpha_2-1}\left\{\alpha_1\lambda^{\alpha_1}x_1^{\alpha_1-1}+\alpha_3\lambda^{\alpha_3}x_1^{\alpha_3-1}\right\}.\exp\{-(\lambda x_1)^{\alpha_1}-(\lambda x_2)^{\alpha_2}-(\lambda x_1)^{\alpha_3}\},\\ &\text{and } f_3(x)=\alpha_3\lambda^{\alpha_3}x^{\alpha_3-1}\exp\{-(\lambda x)^{\alpha_1}-(\lambda x)^{\alpha_2}-(\lambda x)^{\alpha_3}\}.\end{aligned}$$

Surface plots of the joint pdf of the BW model are given in Fig. 1. Where the values of $(\alpha_1,\ \alpha_2,\ \alpha_3,\ \lambda)$ are taken to be as follows a = *(5,5,5,0.5)*, b = *(2,3,10,0.6)*, c = *(8,4,3,0.5)* d = *(2,2,2,1)*, e = *(2,2.5,2,2)* and f = *(1,1,1,0.05)*

ii) Bivariate Generalized Gompertz (BGG) Distribution

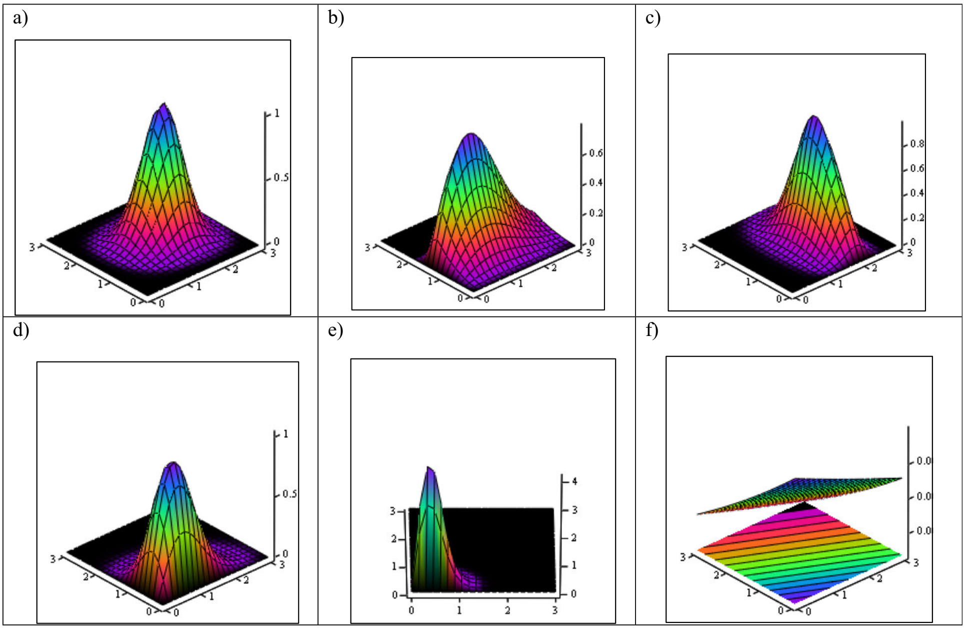

Fig. 1: 3D plots for the pdf of the absolutely continuous part of the BW model

Using Eqs. (2.23)-(2–24) in Eqs. (4.4)–(4.5). A new bivariate generalized Gompertz distribution denoted by $BGG(\alpha_1,\ \alpha_2,\ \ \alpha_3,\ \lambda,\ \xi)$ can be defined by the following joint survival function

$$S_{BGG}(x_1, x_2) = \exp\left\{-\left[\xi\left(e^{\lambda x_1}-1\right)\right]^{\alpha_1}-\left[\xi\left(e^{\lambda x_2}-1\right)\right]^{\alpha_2}-\left[\xi\left(e^{\lambda x_3}-1\right)\right]^{\alpha_3}\right\}$$

where $x_3 = \max(x_1,\ x_2)$.

Or, the joint survival function of BGG model can be written as

$$S_{BGG}(x_1, x_2) = \begin{cases} S_1(x_1, x_2), & x_1 < x_2 \\ S_2(x_1, x_2), & x_1 > x_2 \\ S_3(x), & x_1 = x_2 = x \end{cases}$$

Where

$$\begin{aligned} &S_1(x_1, x_2) = \exp\left\{-\left[\xi\left(e^{\lambda x_1}-1\right)\right]^{\alpha_1}-\left[\xi\left(e^{\lambda x_2}-1\right)\right]^{\alpha_2}-\left[\xi\left(e^{\lambda x_2}-1\right)\right]^{\alpha_3}\right\}, \\ &S_2(x_1, x_2) = \exp\left\{-\left[\xi\left(e^{\lambda x_1}-1\right)\right]^{\alpha_1}-\left[\xi\left(e^{\lambda x_2}-1\right)\right]^{\alpha_2}-\left[\xi\left(e^{\lambda x_1}-1\right)\right]^{\alpha_3}\right\}, \\ &\text{and } S_3(x) = \exp\left\{-\left[\xi\left(e^{\lambda x}-1\right)\right]^{\alpha_1}-\left[\xi\left(e^{\lambda x}-1\right)\right]^{\alpha_2}-\left[\xi\left(e^{\lambda x}-1\right)\right]^{\alpha_3}\right\}. \end{aligned}$$

The joint pdf of BGG model can be written as

$$f_{BGG}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & x_1 < x_2 \\ f_2(x_1, x_2), & x_1 < x_2 \\ f_3(x), & x_1 = x_2 = x \end{cases} \tag{4.9}$$

where

$$\begin{aligned}
&f_1(x_1, x_2) = (\lambda\xi)^2 \alpha_1\ e^{\lambda x_1} e^{\lambda x_2} \left[\xi\left(e^{\lambda x_1} - 1\right)\right]^{\alpha_1 - 1} . \Big\{\alpha_2\ \left[\xi\left(e^{\lambda x_2} - 1\right)\right]^{\alpha_2 - 1} \\
&+ \alpha_3\ \left[\xi\left(e^{\lambda x_2} - 1\right)\right]^{\alpha_3 - 1}\Big\} . \exp\left\{-\left[\xi\left(e^{\lambda x_1} - 1\right)\right]^{\alpha_1} - \left[\xi\left(e^{\lambda x_2} - 1\right)\right]^{\alpha_2} - \left[\xi\left(e^{\lambda x_2} - 1\right)\right]^{\alpha_3}\right\}, \\
&f_2(x_1, x_2) = (\lambda\xi)^2 \alpha_2\ e^{\lambda x_1} e^{\lambda x_2} \left[\xi\left(e^{\lambda x_2} - 1\right)\right]^{\alpha_2 - 1} . \Big\{\alpha_1\ \left[\xi\left(e^{\lambda x_1} - 1\right)\right]^{\alpha_1 - 1} \\
&+ \alpha_3\ \left[\xi\left(e^{\lambda x_1} - 1\right)\right]^{\alpha_3 - 1}\Big\} . \exp\left\{-\left[\xi\left(e^{\lambda x_1} - 1\right)\right]^{\alpha_1} - \left[\xi\left(e^{\lambda x_1} - 1\right)\right]^{\alpha_3} - \left[\xi\left(e^{\lambda x_2} - 1\right)\right]^{\alpha_2}\right\}, \\
&\text{and} \\
&f_3(x) = \lambda\xi\alpha_3\ e^{\lambda x} \left[\xi\left(e^{\lambda x} - 1\right)\right]^{\alpha_3 - 1} \exp\left\{-\left[\xi\left(e^{\lambda x} - 1\right)\right]^{\alpha_1} - \left[\xi\left(e^{\lambda x} - 1\right)\right]^{\alpha_2} - \left[\xi\left(e^{\lambda x} - 1\right)\right]^{\alpha_3}\right\}.
\end{aligned}$$

Surface plots of the joint pdf of the BGG model are given in Fig. 2. Where the values of (α_1, α_2, α_3, λ, ξ) are taken to be as follows a = *(2, 2.5, 2, 1, 1)*, b = *(2, 2, 2, 1, 0.1)*, c = *(1.5, 1.5,1,1,0.2)*, d = *(1.2,1.5,1.1, 1, 0.2)*, e = *(1,1,1,1,0.2)* and f = *(1,2.4,1,1,1)*.

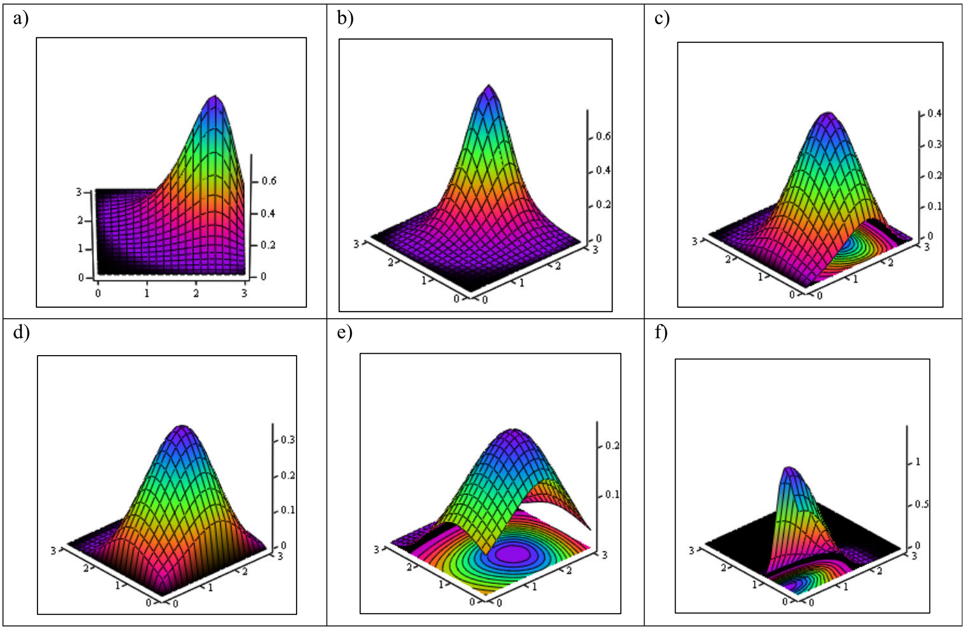

Fig. 2: 3D plots for the pdf of the absolutely continuous part of the BGG model

iii) Bivariate Generalized Pareto (BGP) Distribution

Using Eqs. (2.27)-(2-28) in Eqs. (4.4)–(4.5). A BGP distribution denoted by $BGP(\boldsymbol{\alpha}_1, \ \boldsymbol{\alpha}_2, \ \ \boldsymbol{\alpha}_3, \ \lambda)$ can be introduced by the joint survival function

$$S_{BGP}(x_1, x_2) = \exp\{-[\log(1+\lambda x_1)]^{\alpha_1}\}\exp\{-[\log(1+\lambda x_2)]^{\alpha_2}\} \\ .\exp\{-[\log(1+\lambda x_3)]^{\alpha_3}\}$$

where $x_3 = \max(x_1, \ x_2)$.

That is,

$$S_{BGP}(x_1, x_2) = \begin{cases} S_1(x_1, x_2), & x_1 < x_2 \\ S_2(x_1, x_2), & x_1 > x_2 \\ S_3(x), & x_1 = x_2 = x \end{cases}$$

Where

$$S_1(x_1, x_2) = \exp\{-[\log(1+\lambda x_1)]^{\alpha_1} - [\log(1+\lambda x_2)]^{\alpha_2} - [\log(1+\lambda x_2)]^{\alpha_3}\},$$
$$S_2(x_1, x_2) = \exp\{-[\log(1+\lambda x_1)]^{\alpha_1} - [\log(1+\lambda x_1)]^{\alpha_3} - [\log(1+\lambda x_2)]^{\alpha_2}\},$$
$$S_3(x) = \exp\{-[\log(1+\lambda x)]^{\alpha_1} - [\log(1+\lambda x)]^{\alpha_2} - [\log(1+\lambda x)]^{\alpha_3}\}.$$

The joint pdf of BGP model can be written as

$$f_{BGP}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & x_1 < x_2 \\ f_2(x_1, x_2), & x_1 < x_2 \\ f_3(x), & x_1 = x_2 = x \end{cases} \tag{4.10}$$

where

$$f_1(x_1, x_2) = \frac{\lambda^2 \boldsymbol{\alpha}_1}{(1+\lambda x_1)(1+\lambda x_2)} \ [\log(1+\lambda x_1)]^{\alpha_1 - 1}\Big\{\boldsymbol{\alpha}_2[\log(1+\lambda x_2)]^{\alpha_2 - 1} + \\ \boldsymbol{\alpha}_3[\log(1+\lambda x_2)]^{\alpha_3 - 1}. \ \exp\{-[\log(1+\lambda x_1)]^{\alpha_1} - [\log(1+\lambda x_2)]^{\alpha_2} - [\log(1+\lambda x_2)]^{\alpha_3}\},$$
$$f_2(x_1, x_2) = \frac{\lambda^2 \boldsymbol{\alpha}_2}{(1+\lambda x_1)(1+\lambda x_2)} \ [\log(1+\lambda x_2)]^{\alpha_2 - 1}\Big\{\boldsymbol{\alpha}_1[\log(1+\lambda x_1)]^{\alpha_1 - 1} + \\ \boldsymbol{\alpha}_3[\log(1+\lambda x_1)]^{\alpha_3 - 1}\Big\}.\exp\{-[\log(1+\lambda x_1)]^{\alpha_1} - [\log(1+\lambda x_1)]^{\alpha_3} - [\log(1+\lambda x_2)]^{\alpha_2}\},$$
$$f_3(x) = \frac{\lambda \boldsymbol{\alpha}_3}{(1+\lambda x)} \ [\log(1+\lambda x)]^{\alpha_3 - 1}\exp\{-[\log(1+\lambda x)]^{\alpha_1} - [\log(1+\lambda x)]^{\alpha_2} - [\log(1+\lambda x)]^{\alpha_3}\}.$$

Surface plots of the joint pdf of the BGP model are given in Fig. 3. Where the values of $(\boldsymbol{\alpha}_1, \ \boldsymbol{\alpha}_2, \ \boldsymbol{\alpha}_3, \ \lambda)$ are taken to be as follows a = *(3,2,1, 0.5)*, b = *(3,2,2,5)*, c = *(3,2,1,0,05)*, d = *(2,3,2,0.5)* e = *(3,2,1,5)* and f = *(1,1,1,0.5)*

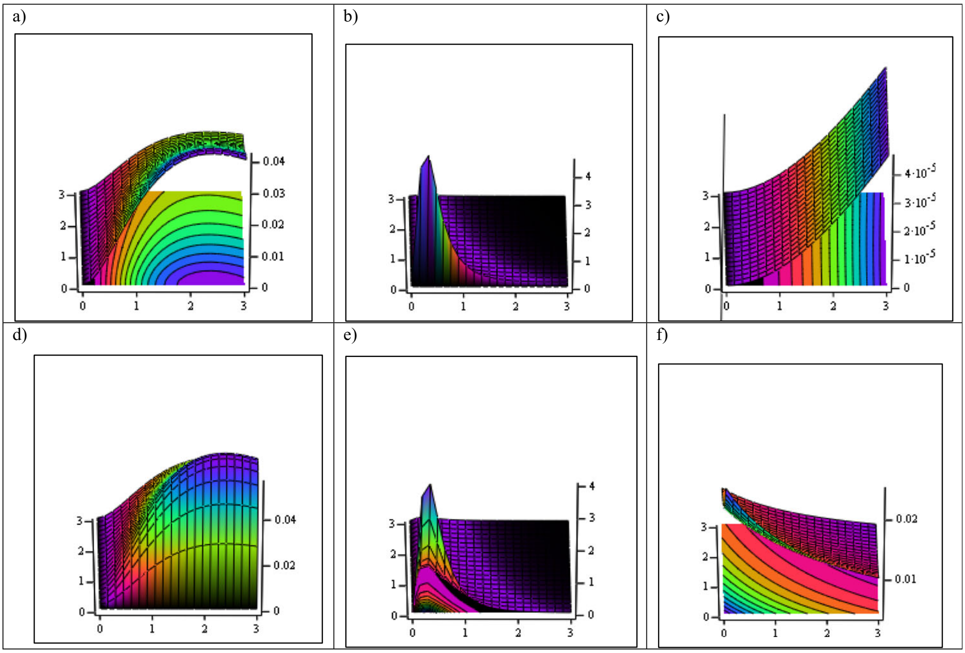

Fig. 3: 3D plots for the pdf of the absolutely continuous part of the BGP model

iv) Bivariate Generalized Uniform (BGU) Distribution

Using Eqs. (2.15)-(2-16) in Eqs. (4.4)–(4.5). A new BGU distribution denoted by $BGU(\alpha_1,\ \alpha_2,\ \ \alpha_3)$ can be introduced by the joint survival function

$$S_{BGU}(x_1,x_2) = \exp\{-[-\log(1-x_1)]^{\alpha_1}\}\exp\{-[-\log(1-x_2)]^{\alpha_2}\} \\ .\exp\{-[-\log(1-x_3)]^{\alpha_3}\}.$$

where $x_3 = \max(x_1,\ x_2)$.

$$S_{BGU}(x_1,x_2) = \begin{cases} S_1(x_1,x_2), & x_1 < x_2 \\ S_2(x_1,x_2), & x_1 > x_2 \\ S_3(x), & x_1 = x_2 = x \end{cases}$$

where

$$S_1(x_1,x_2) = \exp\{-[-\log(1-x_1)]^{\alpha_1}-[-\log(1-x_2)]^{\alpha_2}-[-\log(1-x_2)]^{\alpha_3}\},$$
$$S_2(x_1,x_2) = \exp\{-[-\log(1-x_1)]^{\alpha_1}-[-\log(1-x_2)]^{\alpha_2}-[-\log(1-x_1)]^{\alpha_3}\},$$
$$S_3(x) = \exp\{-[-\log(1-x)]^{\alpha_1}-[-\log(1-x)]^{\alpha_2}-[-\log(1-x)]^{\alpha_3}\}.$$

The joint pdf of BGU model can be written as

$$f_{BGU}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & x_1 < x_2 \\ f_2(x_1, x_2), & x_1 < x_2 \\ f_3(x), & x_1 = x_2 = x \end{cases} \tag{4.11}$$

where

$$f_1(x_1, x_2) = \alpha_1(1-x_1)^{-1}\ (1-x_2)^{-1}[-\log(1-x_1)]^{\alpha_1-1}.\Big\{\alpha_2[-\log(1-x_2)]^{\alpha_2-1}$$
$$+\alpha_3[-\log(1-x_2)]^{\alpha_3-1}.\exp\{-[-\log(1-x_1)]^{\alpha_1}-[-\log(1-x_2)]^{\alpha_2}-[-\log(1-x_2)]^{\alpha_3}\},$$
$$f_2(x_1, x_2) = \alpha_2(1-x_1)^{-1}\ (1-x_2)^{-1}[-\log(1-x_2)]^{\alpha_2-1}.\Big\{\alpha_1[-\log(1-x_1)]^{\alpha_1-1}$$
$$+\alpha_3[-\log(1-x_1)]^{\alpha_3-1}.\exp\{-[-\log(1-x_1)]^{\alpha_1}-[-\log(1-x_2)]^{\alpha_2}-[-\log(1-x_1)]^{\alpha_3}\},$$
and
$$f_3(x) = \alpha_3(1-x)^{-1}\ [-\log(1-x)]^{\alpha_3-1}.\exp\{-[-\log(1-x)]^{\alpha_1}-[-\log(1-x)]^{\alpha_2}-[-\log(1-x)]^{\alpha_3}\}$$

5 Bivariate Power Parameter Family of Distributions (BPP)

Let F_B be a baseline cdf. Suppose that that $F_{PPF}(.;\ \alpha)$ is defined in terms of F_B by the formula

$$F_{PPF}(x; \alpha) = F_B(x^{\alpha}),\ \ \alpha > 0 \tag{5.1}$$

Then α is called a power parameter and $\{F_{PPF}(.;\ \alpha),\ \alpha > 0\}$ is a power parameter family with underling distribution F_B.

The corresponding pdf and hazard function is given respectively, as

$$f_{PPF}(x; \alpha) = \alpha\ x^{\alpha-1}\ f_B(x^{\alpha}) \tag{5.2}$$

$$h_{PPF}(x; \alpha) = \alpha\ x^{\alpha-1}\ h_B(x^{\alpha}). \tag{5.3}$$

Where $f_B$and h_B are a baseline pdf and hazard functions respectively.

The bivariate version correspondence to this family can introduced as follows Assuming that $U_1,\ U_2$ and U_3 are mutually independent random variables such that

$U_1 \sim PP(\alpha_1),\ U_2 \sim PP(\alpha_2)$ and $U_3 \sim PP(\alpha_3)$. Define $X_1 = Max\ (U_1,\ U_3)$ and $X_2 = Max\ (U_2,\ U_3)$ then by using Eqs. (5.1) and (5.2), the bivariate Power parameter family of distributions denoted by $BPP(\alpha_1,\ \alpha_2,\ \alpha_3)$ is defined by the following joint cdf

$$F_{BPP}(x_1,x_2) = F_B(x_1^{\alpha_1})\ F_B(x_2^{\alpha_2})\, F_B\big(x_3^{\alpha_3}\big) \text{ where } x_3 = \min(x_1,x_2)$$

$$F_{BPP}(x_1,x_2) = \begin{cases} F_1(x_1,x_2), & x_1 < x_2 \\ F_2(x_1,x_2), & x_1 > x_2 \\ F_3(x), & x_1 = x_2 = x \end{cases}$$

Where

$$F_1(x_1,x_2) = F_B(x_1^{\alpha_1})\ F_B(x_1^{\alpha_3})\, F_B(x_2^{\alpha_2}),$$

$$F_2(x_1,x_2) = F_B(x_1^{\alpha_1})\ F_B(x_2^{\alpha_2})\, F_B(x_2^{\alpha_3}),$$

And $F_3(x) = F_B(x^{\alpha_1})\ F_B(x^{\alpha_2})\, F_B(x^{\alpha_3})$.

6 Bivariate Proportional Hazard Family of Distributions (BPHP)

Let S_B be a baseline survival function with cumulative hazard function $H_B(x) = -\ log\ S_B(x)$ Suppose that $S_{FPF}(.;\ \boldsymbol{\alpha})$ is defined in terms of S_B by the formula

$$S_{FP}(x;\boldsymbol{\alpha}) = [S_B(x)]^{\alpha} = \exp\{-\boldsymbol{\alpha} H_B(x)\},\ \ \boldsymbol{\alpha} > 0 \tag{6.1}$$

In this case $\boldsymbol{\alpha}$ is called a frailty parameter and $\{S_{FP}(.;\ \boldsymbol{\alpha}),\ \boldsymbol{\alpha} > 0\}$ is a frailty parameter family, or alternatively, a proportional hazard family with underlying survival functionS_B.

The corresponding pdf and hazard function is given respectively, as

$$f_{FP}(x;\boldsymbol{\alpha}) = \boldsymbol{\alpha}\ [S_B(x)]^{\alpha-1}\ f_B(x) \tag{6.2}$$

$$h_{FP}(x;\boldsymbol{\alpha}) = \boldsymbol{\alpha}\ \ h_B(x). \tag{6.3}$$

Where $f_B$and h_B are a baseline pdf and hazard functions respectively.

The bivariate version correspondence to this family has introduced by Shoaee (2020) as follows:

Assuming that U_1, U_2 and U_3 are mutually independent random variables such that

$U_1 \sim UFP(\alpha_1)$, $U_2 \sim UFP(\alpha_2)$ and $U_3 \sim UFP(\alpha_3)$. Define $X_1 = \ min\ (U_1,\ U_3)$ and $X_2 = \ min\ (U_2,\ U_3)$ then by using Eqs. (6.1) and (6.2), the bivariate frailty parameter family of distributions(or bivariate proportional hazard models) denoted by $BFP(\alpha_1,\ \alpha_2,\ \alpha_3)$ is defined by the joint survival and density functions respectively, as follows

$$S_{BFP}(x_1, x_2) = [S_B(x_1)]^{\alpha_1}[S_B(x_2)]^{\alpha_2}[S_B(x_3)]^{\alpha_3}, \text{such that } x_3 = \max(x_1, x_2)$$

$$\text{or } S_{BFP}(x_1, x_2) = \exp\{-\alpha_1 H_B(x_1) - \alpha_2 H_B(x_2) - \alpha_3 H_B(x_3)\},$$

and

$$f_{BFP}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & x_1 < x_2 \\ f_2(x_1, x_2), & x_1 > x_2 \\ f_3(x), & x_1 = x_2 = x \end{cases}$$

Where

$$f_1(x_1, x_2) = \alpha_1(\alpha_2 + \alpha_3) f_B(x_1)\ f_B(x_2)[S_B(x_1)]^{\alpha_1 - 1}[S_B(x_1)]^{\alpha_2 + \alpha_3 - 1},$$

$$f_2(x_1, x_2) = (\alpha_1 + \alpha_3)\alpha_2 f_B(x_1)\ f_B(x_2)[S_B(x_1)]^{\alpha_1 + \alpha_3 - 1}[S_B(x_1)]^{\alpha_2 - 1},$$

and $f_3(x) = \alpha_3\ f_B(x)\ [S_B(x_1)]^{\alpha_1 + \alpha_2 + \alpha_3 - 1}$.

7 Bivariate Proportional Reversed Hazard Family of Distributions (BPRP)

Suppose that $F_{RP}(.;\ \alpha)$ is defined in terms of F_B by the formula

$$F_{RP}(x; \alpha) = [F_B(x)]^{\alpha} = \exp\{\alpha R_B(x)\}, \quad \alpha > 0 \tag{7.1}$$

In this case α is called a resilience parameter and $\{F_{RP}(.;\ \alpha),\ \alpha > 0\}$ is a resilience parameter family, or alternatively, a proportional reversed hazard family with baseline cdf F_B.

The corresponding pdf and reversed hazard function is given respectively, as

$$f_{RP}(x; \alpha) = \alpha\ [F_B(x)]^{\alpha - 1}\ f_B(x) \tag{7.2}$$

$$r_{RP}(x; \alpha) = \alpha\ \ r_B(x). \tag{7.3}$$

Where $f_B$and r_B are a baseline pdf and reversed hazard functions respectively.

Kundu and Gupta (2010) introduced a bivariate proportional reversed hazard family of distributions with the joint cdf and pdf respectively, as follows

$F_{BRP}(x_1\ , x_2) = [F_B(x_1)]^{\alpha_1}[F_B(x_2)]^{\alpha_2}[F_B(x_3)]^{\alpha_3}$, such that $x_3 = \min(x_1, x_2)$

$$f_{BRP}(x_1, x_2) = \begin{cases} f_1(x_1, x_2), & x_1 < x_2 \\ f_2(x_1, x_2), & x_1 > x_2 \ , \\ f_3(x), & x_1 = x_2 = x \end{cases}$$

where

$$f_1(x_1, x_2) = (\boldsymbol{\alpha}_1 + \boldsymbol{\alpha}_3)\boldsymbol{\alpha}_2 f_B(x_1)\ f_B(x_2)[F_B(x_1)]^{\alpha_1+\alpha_3-1}[F_B(x_2)]^{\alpha_2-1},$$

$$f_2(x_1, x_2) = \boldsymbol{\alpha}_1(\boldsymbol{\alpha}_2 + \boldsymbol{\alpha}_3) f_B(x_1)\ f_B(x_2)[F_B(x_1)]^{\alpha_1-1}[F_B(x_2)]^{\alpha_2+\alpha_3-1},$$

and $f_3(x) = \boldsymbol{\alpha}_3\ f_B(x)\ [F_B(x)]^{\alpha_1+\alpha_2+\alpha_3-1}$.

8 Numerical Study

8.1 Simulation Study

As be mentioned above the BRPP and BHPP contain different distributions. So it is better to use the different distributions in both BRPP and BHPP families. Such as bivariate Weibull (BW), bivariate generalized Gompertz (BGG), bivariate generalized inverse uniform (BGIU) distributions. Also, the following algorithm can be used to simulate these families in general.

Algorithm to generate from BRPP models

Step 1. Generate U_1, U_2 and U_3 from U(0, 1).

Step 2. Compute $Z_1 = R_B^{-1}\left([-\log U_1]^{1/\alpha_1}\right)$, $Z_2 = R_B^{-1}\left([-\log U_2]^{1/\alpha_2}\right)$,

and $Z_3 = R_B^{-1}\left([-\log U_3]^{1/\alpha_3}\right)$.

Step3. Obtain $X_1 = \ min\ (Z_1,\ Z_3)$ and $X_2 = \ min\ (Z_2,\ Z_3)$.

Step4. Define the indicator functions as

$$\delta_{1i} = \begin{cases} 1; & x_{1i} < x_{1i} \\ 0; & otherwise \end{cases}, \quad \delta_{2i} = \begin{cases} 1; & x_{1i} > x_{1i} \\ 0; & otherwise \end{cases} \quad and\ \ \delta_{3i} = \begin{cases} 1; & x_{1i} = x_{1i} \\ 0; & otherwise \end{cases}.$$

Step5. The corresponding sample size n must satisfy $n = n_1 + n_2 + n_3$

Such that $n_1 = \sum_{i=1}^{n} \delta_{1i}, \quad n_2 = \sum_{i=1}^{n} \delta_{2i}$ *and* $\quad n_3 = \sum_{i=1}^{n} \delta_{1i}.$

To generate from BHPP model apply the same steps from 1 to 5 except in step 2 the quantile function is exchanged to be

$$Z_1 = H_B^{-1}\left([-\log U_1]^{1/\alpha_1}\right), \quad Z_2 = H_B^{-1}\left([-\log U_2]^{1/\alpha_2}\right), \text{and } Z_3 = H_B^{-1}\left([-\log U_3]^{1/\alpha_3}\right).$$

A Monte Carlo simulation study testing the performance of MLE for the BRPP and BHPP models parameters will be introduced in general and especially for BGIU model which defined by Eq. (3.12) and denoted by $BGIU(\alpha_1, \alpha_2, \alpha_3)$ and belongs to the BRPP family, BGG model which defined by Eq. (4.9) and denoted by BGG $(\alpha_1, \alpha_2, \alpha_3, \lambda, \xi)$ and belongs to the BHPP family, and BW model which defined by Eq. (4.8) and denoted by $BW(\alpha_1, \alpha_2, \alpha_3, \lambda)$ and belongs to the BHPP family.

The evaluation of the MLE was performed based on the following quantities for each sample size: the mean of the MLEs (MLE) and the corresponding Mean Squared Error, (*MSE*). For different choices for the sample sizes and different sets of parameters real values which are as follows

Group1:

For BGG model $(\alpha_1, \alpha_2, \alpha_3, \lambda, \xi) = (0.8, 0.7, 0.7, 0.2, 0.002)$

For BW model $(\alpha_1, \alpha_2, \alpha_3, \lambda) = (0.2, 0.3, 0.5, 0.2)$

For BGIU model $(\alpha_1, \alpha_2, \alpha_3) = (1.3, 1.5, 1.2)$

Group 2:

For BGG model $(\alpha_1, \alpha_2, \alpha_3, \lambda, \xi) = (0.7, 0.7, 0.6, 0.3, 0.002)$

For BW model $(\alpha_1, \alpha_2, \alpha_3, \lambda) = (0.8, 0.7, 0.7, 0.2)$

For BGIU model $(\alpha_1, \alpha_2, \alpha_3) = (2, 2.5, 2)$

The results of these simulations are presented in Tables 1 and 2. These results are useful, and it is observed that in most of the cases as the sample size increases, the MSEs decrease. This represents that the MLEs are consistent.

8.2 Application to Real Data Sets

In this section, three real data sets will be examined. Here, these data will be fitted to seven sub models. Four of them belong to BHPP family namely: (i) bivariate Weibull (BW) distribution, which defined by Eq. (4.8) (ii) bivariate generalized Gompertz (BGG) distribution which defined by Eq. (4.9), (iii) bivariate generalized Pareto (BGP) distribution which defined by Eq. (4.10) and (iv) bivariate generalized uniform distribution (BGU) which defined by Eq. (4.11). And three of them belong to BRPP family namely (i) bivariate inverse Weibull (BIW) distribution which defined by Eq. (3.10), (ii) bivariate generalized invers Rayleigh (BGIR) distribution

Table 1: The MLE and MSE for BGG, BW and BGIU models for Group1

BGG										
n	α_1		α_2		α_3		λ		ξ	
	0.8		0.7		0.7		0.2		0.002	
	MLE	**MSE**	**MLE**	**MSE**	**MLE**	**MSE**	**MLE**	**MSE**	**MLE**	**MSE**
20	0.099	0.491	0.357	0.118	0.1	0.36	0.0041	0.038	0.007	0.00002
40	0.074	0.528	0.48	0.048	0.073	0.393	0.0039	0.038	0.006	0.00001
70	0.107	0.48	0.544	0.024	0.104	0.335	0.0019	0.039	0.001	0.0000007
120	0.135	0.443	0.866	0.027	0.129	0.327	0.0014	0.039	0.0009	0.0000012
150	0.071	0.531	0.495	0.042	0.071	0.396	0.0018	0.039	0.0013	0.0000004
BW										
n	α_1		α_2		α_3		λ			
	0.2		0.3		0.5		0.2			
	MLE	**MSE**	**MLE**	**MSE**	**MLE**	**MSE**	**MLE**			
30	0.362	0.026	0.383	0.0068	0.313	0.035	0.002	0.039		
50	0.319	0.014	0.355	0.0029	0.29	0.044	0.0019	0.039		
70	0.339	0.019	0.349	0.0024	0.308	0.037	0.0014	0.039		
100	0.323	0.015	0.347	0.0022	0.291	0.044	0.0014	0.039		
150	0.336	0.018	0.362	0.0039	0.307	0.037	0.0015	0.039		
200	0.328	0.016	0.338	0.0015	0.294	0.042	0.0014	0.039		
BGIU										
n	α_1		α_2		α_3					

Table 1 (continued)

	1.3		1.5		1.2	
	MLE	**MSE**	**MLE**	**MSE**	**MLE**	**MSE**
30	1.277	0.00054	1.41	0.0081	1.193	0.000047
50	1.364	0.0041	1.547	0.0022	1.094	0.011
70	1.275	0.00064	1.407	0.0086	0.934	0.071
100	1.271	0.00011	1.436	0.0042	0.903	0.088

Table 2: The MLE and MSE for BGG, BW and BGIU models for Group 2

BGG										
n	α_1		α_2		α_3		λ		ξ	
	0.7		0.7		0.6		0.3		0.002	
	MLE	**MSE**	**MLE**	**MSE**	**MLE**	**MSE**	**MLE**	**MSE**	**MLE**	**MSE**
30	0.094	0.368	0.262	0.192	0.044	0.309	0.0024	0.089	0.02	0.0003
50	0.152	0.301	0.989	0.084	0.146	0.206	0.0017	0.089	0.0022	0.00000005
70	0.114	0.344	0.624	0.0057	0.091	0.259	0.0018	0.089	0.016	0.000204
90	0.166	0.285	0.818	0.014	0.129	0.221	0.0016	0.089	0.0036	0.0000026
120	0.126	0.33	0.677	0.00052	0.11	0.24	0.0018	0.089	0.0059	0.000015
170	0.114	0.344	0.83	0.017	0.094	0.256	0.0021	0.089	0.0011	0.0000009
BW										
n	α_1		α_2		α_3		λ			
	0.8		0.7		0.7		0.2			
	MLE	**MSE**	**MLE**	**MSE**	**MLE**	**MSE**	**MLE**	**MSE**		
30	0.564	0.056	0.502	0.039	0.492	0.043	0.0087	0.037		
50	0.58	0.049	0.542	0.025	0.508	0.037	0.0098	0.036		
80	0.593	0.043	0.605	0.009	0.548	0.023	0.011	0.036		
100	0.543	0.071	0.538	0.026	0.493	0.043	0.0097	0.036		
150	0.581	0.048	0.578	0.015	0.523	0.031	0.01	0.036		
200	0.567	0.054	0.574	0.016	0.503	0.039	0.0092	0.036		

Table 2 (continued)

BGIU

n	α_1		α_2		α_3	
	2		2.5		2	
	MLE	**MSE**	**MLE**	**MSE**	**MLE**	**MSE**
20	2.092	0.0084	2.658	0.025	1.64	0.13
30	2.351	0.123	2.474	0.0011	1.608	0.154
50	2.174	0.03	2.394	0.011	1.707	0.086
70	2.091	0.0083	2.432	0.0046	1.439	0.314
100	1.983	0.00029	2.524	0.00056	1.604	0.157

which defined by Eq. (3.11), and (iii) bivariate generalized inverse uniform (BGIU) distribution which defined by Eq. (3.12).

8.2.1 **Data Set 1***: UEFA Champion's League Data*
The data set has been obtained from Meintanis (2007) and represented in Table 3. He explained that: the data represent the football (soccer) data where at least one goal scored by the home team and at least one goal scored directly from a penalty kick, foul kick or any other direct kick (all of them together will be called as kick goal) by any team have been considered. Here X_1 represents the time in minutes of the first kick goal scored by any team and X_2 represents the first goal of any type scored by the home team. In this case all possibilities are open, for example $X_1 < X_2$ or $X_1 > X_2$ or $X_1 = X_2 = X$.

8.2.2 **Data Set 2***: Cholesterol Levels*
This data set contains cholesterol levels at 5 and 25 weeks after treatment in 30 patients and represented in Table 4. Before analyzing this data, the transformation (X − 150)/100 is applied to all data, this transformation will not effect on the analysis and are for computational reasons only. This data set was used by Shoaee (2020). Again, in this case all possibilities are exist, i.e., $X_1 < X_2$ or $X_1 > X_2$ or $X_1 = X_2 = X$.

8.2.3 **Data Set 3***: Burr Data*
The data set has been obtained from Shoaee (2020) and represented in Table 5.

Table 3: UEFA Champion's League data

S.N.	X_1	X_2	S.N.	X_1	X_2	S.N.	X_1	X_2	S.N.	X_1	X_2
1	26	20	11	72	72	21	53	39	31	49	49
2	63	18	12	66	62	22	54	7	32	24	24
3	19	19	13	25	9	23	51	28	33	44	30
4	66	85	14	41	3	24	76	64	34	42	3
5	4	4	15	16	75	25	64	15	35	27	47
6	49	49	16	18	18	26	26	48	36	28	28
7	8	8	17	22	14	27	16	16	37	2	2
8	69	71	18	42	42	28	44	6			
9	39	39	19	36	52	29	25	14			
10	82	48	20	34	34	30	55	11			

Table 4: Cholesterol levels at 5 and 25 weeks after treatment in 30 patients

S.N.	X_1	X_2	S.N.	X_1	X_2	S.N.	X_1	X_2
1	325	246	11	217	252	21	316	283
2	278	245	12	248	305	22	243	245
3	257	212	13	225	225	23	305	272
4	192	192	14	287	208	24	197	197
5	276	325	15	233	217	25	243	247
6	262	294	16	198	198	26	315	283
7	309	232	17	229	179	27	205	205
8	287	287	18	310	352	28	315	255
9	304	245	19	214	274	29	263	215
10	215	261	20	253	209	30	210	271

This dataset contains 50 observations on the burr. In the first component, the hole diameter is 12 mm and the sheet thickness is 3.15 mm. In the second component, the hole diameter is 9 mm and the sheet thickness is 2 mm. These two datasets are derived from two different machines. Also, in this case all possibilities are exist, for example $X_1 < X_2$ or $X_1 > X_2$ or $X_1 = X_2 = X$.

The marginal distributions of both families are fitted to each data set separately which are: Weibull (W), generalized Gompertz (GE), generalized Pareto (GP), generalized uniform (GU), inverse Weibull (IW), generalized inverse Rayleigh (GIR) and generalized inverse uniform (GIU). The MLEs, The Kolmogorov-Smirnov (K-S) distances between the fitted distribution and the empirical distribution function for X_1 and X_2 and their maximum are shown in Tables 6 and 7 separately.

Now, the three data sets will fit to the seven bivariate sub-models (BW, BGG, BGP, BGU, BIW, BGIR, BGIU) defined above, the MLEs, the standard error (SE) and the confidence intervals(CI) with confidence interval lengths (CIL) will be calculated for each data set and listed in Tables 8, 9, and 10. To compare these models with each other or with any other bivariate models that represent this data the Akaike information criterion (AIC), Bayesian information criterion (BIC), the consistent Akaike information criterion (CAIC) and Hannan-Quinn information criterion (HQIC) are calculated for each model and each data set and listed in Table 11.

Table 5: Burr Data

S.N.	X_1	X_2	S.N.	X_1	X_2	S.N.	X_1	X_2	S.N.	X_1	X_2	S.N.	X_1	X_2
1	0.04	0.06	11	0.24	0.16	21	0.24	0.12	31	0.24	0.14	41	0.02	0.16
2	0.02	0.12	12	0.04	0.12	22	0.22	0.24	32	0.16	0.06	42	0.18	0.32
3	0.06	0.14	13	0.14	0.24	23	0.12	0.06	33	0.32	0.04	43	0.22	0.18
4	0.12	0.04	14	0.16	0.06	24	0.18	0.02	34	0.18	0.14	44	0.14	0.24
5	0.14	0.14	15	0.08	0.02	25	0.24	0.18	35	0.24	0.22	45	0.06	0.22
6	0.08	0.16	16	0.26	0.18	26	0.32	0.22	36	0.22	0.14	46	0.04	0.04
7	0.22	0.08	17	0.32	0.22	27	0.16	0.14	37	0.16	0.06	47	0.14	0.14
8	0.12	0.26	18	0.28	0.14	28	0.14	0.02	38	0.12	0.04	48	0.26	0.26
9	0.08	0.32	19	0.14	0.22	29	0.08	0.18	39	0.24	0.16	49	0.18	0.18
10	0.26	0.22	20	0.16	0.16	30	0.16	0.22	40	0.06	0.24	50	0.16	0.16

Table 6: The Marginals Fittings of BHPP Models Parameters

Data	Model	Variables	α	λ	ξ	K-S
Data 1	W	X_1	2.358	0.182		0.542
		X_2	1.01	0.182		0.440
		$Min\ (X_1, X_2)$	3.139	0.182		0.433
	GG	X_1	0.836	0.705	0.358	0.336
		X_2	0.915	0.705	0.358	0.975
		$Min\ (X_1, X_2)$	1.346	0.705	0.358	0.285
	GP	X_1	3.743	0.11		1.007
		X_2	3.018	0.11		0.964
		$Min\ (X_1, X_2)$	4.96	0.11		0.955
	GU	X_1	3.07			0.084
		X_2	2.528			0.176
		$Min\ (X_1, X_2)$	4.146			0.142
Data 2	W	X_1	2.526	0.26		0.508
		X_2	1.946	0.26		0.370
		$Min\ (X_1, X_2)$	3.022	0.26		0.338
	GG	X_1	2.678	0.6	0.805	0.215
		X_2	2.858	0.6	0.805	1.517
		$Min\ (X_1, X_2)$	4.23	0.6	0.805	0.284
	GP	X_1	2.216	0.13		0.742
		X_2	2.796	0.13		0.787
		$Min\ (X_1, X_2)$	3.901	0.13		0.661
	GU	X_1	0.869			0.757
		X_2	0.914			0.718
		$Min\ (X_1, X_2)$	1.33			0.872
Data 3	W	X_1	3.432	4.694		0.201
		X_2	3.633	4.694		0.182
		$Min\ (X_1, X_2)$	5.481	4.694		0.488
	GG	X_1	1.958	1.8	1.772	0.255
		X_2	2.099	1.8	1.772	1.147
		$Min\ (X_1, X_2)$	3.14	1.8	1.772	0.220
	GP	X_1	3.631	0.129		1.013
		X_2	2.872	0.129		0.954
		$Min\ (X_1, X_2)$	4.471	0.129		0.929
	GU	X_1	1.204			0.344
		X_2	1.349			0.389
		$Min\ (X_1, X_2)$	1.943			0.295

Table 7: The Marginals Fittings of BRPP Models Parameters

Data	Model	Variables	α	λ	σ	K-S
Data 1	IW	X_1	2.055	0.128		0.906
		X_2	2.281	0.128		0.917
		$Max\,(X_1, X_2)$	3.317	0.128		0.938
	GIR	X_1	2.677		0.134	0.958
		X_2	1.265		0.134	0.917
		$Max\,(X_1, X_2)$	3.216		0.134	0.944
	GIU	X_1	2.536			0.943
		X_2	2.033			0.916
		$Max\,(X_1, X_2)$	3.345			0.917
Data 2	IW	X_1	3.269	0.18		0.985
		X_2	2.705	0.18		0.921
		$Max\,(X_1, X_2)$	4.194	0.18		0.897
	GIR	X_1	1.793	0.26		0.753
		X_2	1.755	0.26		0.776
		$Max\,(X_1, X_2)$	3.255	0.26		0.771
	GIU	X_1	6.549			0.944
		X_2	7.022			0.965
		$Max\,(X_1, X_2)$	10.336			0.963
Data 3	IW	X_1	2.979	0.05		0.959
		X_2	2.615	0.05		0.938
		$Max\,(X_1, X_2)$	1.46	0.05		0.98
	GIR	X_1	0.227	0.013		0.796
		X_2	0.534	0.013		0.738
		$Max\,(X_1, X_2)$	0.672	0.013		0.897
	GIU	X_1	1.357			0.953
		X_2	1.359			0.933
		$Max\,(X_1, X_2)$	2.074			0.898

Table 8: The MLE, the CIL and the SE for both BHPP and BRPP Models parameters for Data 1

Models	α_1	α_2	α_3	λ σ	ξ
	MLE(SE) [CI] CIL	MLE(SE) [CI] CIL	MLE(SE) [CI] CIL	MLE(SE) [CI] CIL	MLE(SE) [CI] CIL
BW	2.129 (0.468) [1.909,2.35] 0.441	0.781 (0.02) [0.734,0.828] 0.094	0.229 (0.0013) [0.217,0.24] 0.023	0.182 (0.00093) [0.172,0.192] 0.02	
BGG	0.431 (0.0083) [0.402, 0.46] 0.059	0.51 (0.011) [0.476, 0.544] 0.069	0.405 (0.0062) [0.38,0.43] 0.051	0.705 (0.044) [0.638, 0.773] 0.135	0.358 (0.014) [0.32, 0.397] 0.077
BGP	1.942 (0.018) [1.899, 1.986] 0.087	1.217 (0.0049) [1.194, 1.239] 0.045	1.801 (0.02) [1.754,1.848] 0.094	0.11 (0.00022) [0.106, 0.115] 0.0096	
BGU	1.618 (0.136) [1.499,1.737] 0.237	1.076 (0.037) [1.014, 1.138] 0.124	1.452 (0.069) [1.367,1.537] 0.17		
BIW	1.036 (0.035) [0.976, 1.097] 0.12	1.262 (0.039) [1.198, 1.325] 0.127	1.019 (0.024) [0.969, 1.069] 0.1	0.128 (0.0003) [0.123, 0.134] 0.011	
BGIR	1.951 (0.078) [1.861, 2.041] 0.18	0.539 (0.008) [0.51, 0.568] 0.058	0.726 (0.0069) [0.699, 0.753] 0.054	0.134 (0.00009) [0.131, 0.137] 0.0061	
BGIU	1.312 (0.041) [1.247, 1.377] 0.131	0.809 (0.042) [0.743, 0.875] 0.133	1.224 (0.031) [1.167, 1.281] 0.114		

Table 9: The MLE, the CIL and the SE for both BHPP and BRPP Models parameters for Data 2

Models	α_1	α_2	α_3	λ σ	ξ
	MLE(SE) [CI] CIL	MLE(SE) [CI] CIL	MLE(SE) [CI] CIL	MLE(SE) [CI] CIL	MLE(SE) [CI] CIL
BW	1.076 (0.096) [0.965,1.187] 0.222	0.496 (0.011) [0.458,0.534] 0.076	1.45 (0.363) [1.235, 1.666] 0.431	0.26 (0.0081) [0.228, 0.292] 0.064	
BGG	1.371 (0.064) [1.28, 1.462] 0.181	1.551 (0.075) [1.454, 1.649] 0.196	1.307 (0.065) [1.215, 1.399] 0.183	0.6 (0.045) [0.524, 0.676] 0.152	0.805 (0.177) [0.655, 0.956] 0.301
BGP	1.106 (0.0079) [1.074, 1.138] 0.063	1.686 (0.037) [1.617,1.755] 0.138	1.11 (0.00802) [1.078, 1.142] 0.064	0.13 (0.00032) [0.124, 0.136] 0.013	
BGU	0.415 (0.0016) [0.381, 0.449] 0.068	0.416 (0.0023) [0.427, 0.494] 0.067	0.415 (0.00112) [0.415, 0.492] 0.077		
BIW	1.49 (0.096) [1.379, 1.6] 0.222	0.925 (0.025) [0.869, 0.982] 0.113	1.78 (0.514) [1.523,2.036] 0.513	0.18 (0.0012) [0.167,0.193] 0.513	
BGIR	1.5 (0.021) [1.449, 11.551] 0.103	1.462 (0.051) [1.381, 1543] 0.162	0.293 (0.0027) [0.274, 0.311] 0.037	0.26 (0.00043) [0.253, 0.267] 0.015	
BGIU	3.314 (0.413) [3.084, 3.544] 0.46	3.786 (0.514) [3.53, 4.043] 0.474	3.236 (0.438) [2.999, 3.473] 0.474		

Table 10: The MLE, the CIL and the SE for both BHPP and BRHPP Models parameters for Data 3

Models	α_1	α_2	α_3	λ σ	ξ
	MLE(SE) [CI] CIL	MLE(SE) [CI] CIL	MLE(SE) [CI] CIL	MLE(SE) [CI] CIL	MLE(SE) [CI] CIL
BW	1.848 (0.099) [1.76, 1.935] 0.175	2.049 (0.105) [1.959, 2.139] 0.18	1.584 (0.095) [1.499, 1.67] 0.171	4.694 (0.072) [4.62, 4.769] 0.149	
BGG	1.04 (0.029) [0.993, 1.088] 0.095	1.182 (0.036) [1.129, 1.234] 0.105	0.918 (0.026) [0.873, 0.963] 0.089	1.8 (0.112) [1.707, 1.893] 0.186	1.772 (0.159) [1.661, 1.882] 0.221
BGP	1.599 (0.014) [1.566, 1.631] 0.065	0.84 (0.00055) [0.834, 0.846] 0.013	2.032 (0.0034) [2.006, 2.058] 0.051	0.129 (0.00017) [0.125, 0.132] 0.00725	
BGU	0.594 (0.00726) [0.75, 0.618] 0.047	0.739 (0.01) [0.711, 0.767] 0.055	0.61 (0.0081) [0.585, 0.635] 0.05		
BIW	1.519 (0.032) [1.469, 1.569] 0.1	1.155 (0.024) [1.112, 1.198] 0.086	1.46 (0.037) [1.407, 1.514] 0.106	0.05 (0.000015) [0.049, 0.051] 0.00214	
BGIR	1.138 (0.0005) [0.132, 0.143] 0.012	0.445 (0.0099) [0.417, 0.473] 0.055	0.089 (0.00028) [0.084, 0.093] 0.0092	0.013 (0.000015) [0.012, 0.014] 0.00213	
BGIU	0.715 (0.009) [0.689, 0.742] 0.052	0.717 (0.011) [0.687, 0.747] 0.059	0.642 (0.016) [0.606, 0.678] 0.071		

Table 11: Comparison between Bivariate Models

Data Sets	models	-2lnl	AIC	BIC	AICC	HQIC
Data 1	BW	212.849	220.849	227.292	222.099	223.12
	BGG	129.096	139.096	147.151	141.032	141.936
	BGP	472.503	480.503	486.946	481.753	482.774
	BGU	29.996	35.996	40.828	36.723	37.699
	BIW	71.068	79.068	85.511	80.318	81.339
	BGIR	137.224	145.224	151.668	146.474	147.496
	BGIU	224.383	230.383	235.216	231.111	232.087
Data 2	BW	174.28	182.28	187.885	183.88	184.074
	BGG	82.642	92.642	99.648	95.142	94.883
	BGP	352.79	360.79	366.395	362.39	362.583
	PGU	188.208	194.208	198.412	195.131	195.553
	BIW	187.607	195.607	201.212	197.207	197.4
	BGIR	207.529	215.529	221.134	217.129	217.322
	BGIU	52.922	58.922	63.125	59.845	60.266
Data 3	BW	−181.969	−173.969	−166.321	−173.08	−171.057
	BGG	−160.678	−150.678	−141.117	−149.314	−147.037
	BGP	770.136	778.136	785.784	779.024	781.048
	PGU	−53.68	−47.68	−41.944	−47.158	−45.496
	BIW	−106.59	−98.59	−90.942	−97.701	−95.677
	BGIR	64.957	72.957	80.605	73.845	75.869
	BGIU	384.869	390.869	396.605	391.391	393.054

9 Conclusion

In this study new bivariate families of distributions are proposed by adding an extra shape parameter to the base distributions by different manners using the hazard and reversed hazard functions. In most of the cases the joint probability distribution, joint distribution and joint hazard and joint reversed hazard functions can be expressed in compact forms. The maximum likelihood estimation is considered for the vector of the unknown parameters. A simulation study is performed to see the performances of the estimators. For illustrative purposes three data sets has been re-analyzed and the performances are quite satisfactory.

Acknowledgments The author is profoundly grateful to the anonymous referees and editor for their valuable suggestions and comments which improved the earlier draft of the article.

Funding Not applicable

Declarations

Conflicts of Interest/Competing Interests The author declare that she have no conflict of interest

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Paper received: 27 January 2020.