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Maximum Likelihood Estimation in Single Server Queues

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Abstract

In this paper, maximum likelihood estimation for the parameters in a single server queues are investigated. The queues are observed over a continuous time interval $(0, T]$, where T is determined by a suitable stopping rule. The existence of the maximum likelihood estimator is proved by applying Rolle's theorem. Also, we have obtained the limiting distribution of the error of estimation.

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1 Introduction

The problem of statical estimation for queueing parameters such as the arrival rate, service rate, traffic intensity *etc.*, plays an extremely important role in the decision analysis of queues due to its practical application. The pioneering work in this direction was due due to Clarke (1957), in which the parameters of an $M/M/1$ queue have been estimated by the maximum likelihood principle. Since then, some notable works in this field have been done. The papers by Cox (1965), Wolff (1965), Goyal and Harris (1972), & Basawa and Prabhu (1981) and Bhat and Rao (1987) are also worth mentioning. Basawa and Prabhu (1988) have investigated the large sample properties of the parameters for single server queues in the context of exponential families of interarrival and service times density, and have established the consistency and asymptotic normality properties of the maximum likelihood estimators. Acharya (1999) has extended the research work of Basawa and Prabhu (1988) to find the rate of convergence of the distribution of the maximum likelihood estimators. Acharya and Mishra (2007) have proved the Bernstein-von Mises theorem for the arrival process in a single server $M/M/1$ queue.

Recently, Acharya and Singh (2019) have established the asymptotic properties of the maximum likelihood estimator from single server queues by applying the martingale technique. Singh and Acharya (2019a) have obtained the bound for the equivalence of the Bayes and maximum likelihood estimator and also pointed out the bound on the difference between the Bayes estimator from their true values of arrival and service rate parameter in an $M/M/1$ queue. Singh and Acharya (2019b) have studied the normal approximation of the joint posterior distribution of the arrival and service rate parameters in $GI/G/1$ queueing system. Singh (2020) has found a moderate deviation result of the maximum likelihood estimator for the single server queueing model under certain regular conditions. Singh and Acharya (2021a) have proved the Bernstein-von Mises theorem and investigated the large sample properties of Bayes estimator from single server queues. Singh and Acharya (2021b) have derived the rate of convergence in the Bernstein-von Mises theorem for $M/M/1$ queueing system by extending the work of Acharya and Mishra (2007).

In this study, we consider the method of maximum likelihood estimation for the arrival and service rate parameters in a single server $GI/G/1$ queueing system by assuming that the interarrival density and service time density belong to an exponential family. The idea that using maximum likelihood estimation to estimate for the arrival rate and service rate parameters is different from previous literatures. The results in this paper are due to Basawa and Prabhu (1988) in which they have used random sum central limit theorem and Cramer-Wold argument. But we provide simple elementary analysis and probability techniques to show the consistency and asymptotic normality properties of the maximum likelihood estimators. At first, we find an interval in which the likelihood function gets the same value at two endpoints. Then, we apply Rolle's theorem to prove the existence of the maximum likelihood estimators on which the likelihood function gets the local maximum. Finally, the consistency in probability of the maximum likelihood estimator is proved and the limiting distribution of the error of estimation is obtained.

The rest of the paper is organised as follows. Section 2 introduces the queueing system of our interest and constructs the likelihood function based on the sample data which is observed over a continuous time interval $(0, T]$. The main results are provided in Section 3 where the existence and consistency in probability of the maximum likelihood estimator are proved and the limiting distribution of the error of estimation is investigated. In Section 4, we provide two examples to illustrate the main results. Finally, Section 5 gives concluding remarks of the paper.

2 Preliminaries

Let us consider a single server queueing system where the interarrival times $\{u_k, k \geq 1\}$ and the service times $\{v_k, k \geq 1\}$ are two independent sequences of independent and identically distributed (i.i.d.) non-negative random variables with probability density functions $f(u;\theta)$ and $g(v;\phi)$, respectively, where the arrival rate parameter θ and the service rate parameter ϕ are unknown parameters. Let us assume that the probability density functions of interarrival times, $f(u;\theta)$, and service times $g(v;\phi)$ belong to the continuous exponential families given by

$$f(u;\theta) = a_1(u)\exp\{\theta h_1(u) - k_1(\theta)\} \tag{2.1}$$

$$g(v;\phi) = a_2(v)\exp\{\phi h_2(v) - k_2(\phi)\} \tag{2.2}$$

and $f(u;\theta) = g(v;\phi) = 0$ on $(-\infty, 0)$. The moment generating function of the random variables $h_1(u)$ is

$$M_u(s) = \exp\left[k_1(s+\theta) - k_1(\theta)\right]$$

and consequently we get

$$\eta_1(\theta) = \mathbb{E}[h_1(u)] = k_1'(\theta), \quad \sigma_1^2(\theta) = \mathbb{V}(h_1(u)) = k_1''(\theta). \tag{2.3}$$

Similarly, for the random variable $h_2(v)$ we have

$$\eta_2(\phi) = \mathbb{E}[h_2(v)] = k_2'(\phi), \quad \sigma_2^2(\phi) = \mathbb{V}(h_2(v)) = k_2''(\phi). \tag{2.4}$$

Let the first customer arrive at time $t = 0$ and the service starts at the arrival of the first customer. We observe the queueing system over a continuous time interval $(0, T]$, where T is a suitable stopping time. The sample observations for the arrival and departure processes are

$$\{A(T), D(T), u_1, u_2, ..., u_{A(T)}, v_1, v_2, ..., v_{D(T)}\}, \tag{2.5}$$

where $A(T)$ and $D(T)$ are the number of arrivals and the number of departures respectively during the time interval $(0, T]$. Notice that there are no arrivals during the time interval $[\sum_{i=1}^{A(T)} u_i, T]$ and no departures during the time interval $[\gamma(T) + \sum_{i=1}^{D(T)} v_i, T]$, in which $\gamma(T)$ is the total idle period in $(0, T]$.

The followings are the some possible stopping rules to determine T according to Basawa and Prabhu (1988).

Rule 1. Observe the system until a fixed time t. Here, $T = t$ with probability one and $A(T)$ and $D(T)$ are both random variables.

Rule 2. Observe the system until d departures have occurred so that $D(T) = d$. Here, $T = \gamma(T) + v_1 + v_2 + \cdots + v_d$ and $A(T)$ are random variables.

Rule 3. Observe the system until m arrivals take place so that $A(T) = m$. Here, $T = u_1 + u_2 + u_3 + \cdots + u_m$ and $D(T)$ are random variables.

Rule 4. Stop at the nth transition epoch. Here, $T, A(T)$ and $D(T)$ are all random variables and $A(T) + D(T) = n$.

Under Rule 4, we stop either with an arrival or in a departure. If we stop with an arrival, then $\sum_{i=1}^{A(T)} u_i = T$, and no departures during $[\gamma(T) + \sum_{i=1}^{D(T)} v_i, T]$. Similarly, if we stop in a departure, then $\gamma(T) + \sum_{i=1}^{D(T)} v_i = T$, and there are no arrivals during $[\sum_{i=1}^{A(T)} u_i, T]$.

The likelihood function based on sample data Eq. 2.5 is given by

$$\begin{aligned} L_{A(T),D(T)}(\theta, \phi) &= \prod_{i=1}^{A(T)} f(u_i, \theta) \prod_{i=1}^{D(T)} g(v_i, \phi) \\ &\quad \times \left[1 - F_\theta\left(T - \sum_{i=1}^{A(T)} u_i\right)\right] \left[1 - G_\phi\left(T - \gamma(T) - \sum_{i=1}^{D(T)} v_i\right)\right] \end{aligned} \tag{2.6}$$

where F and G are distribution functions corresponding to the probability density functions of the interarrival times, f, and the service times, g, respectively. The likelihood function $L_T(\theta, \phi)$ remains valid under all the stopping rules.

According to Basawa and Prabhu (1988), the approximate likelihood function $L_T^a(\theta, \phi)$ is given by

$$L^a_{A(T),D(T)}(\theta, \phi) = \prod_{i=1}^{A(T)} f(u_i, \theta) \prod_{i=1}^{D(T)} g(v_i, \phi). \tag{2.7}$$

The maximum likelihood estimators of θ and ϕ obtained from Eq. 2.7 are asymptotically equivalent to those obtained from Eq. 2.6 provided that the following two conditions are satisfied as $T \to \infty$:

$$\frac{1}{\sqrt{A(T)}} \frac{\partial}{\partial \theta} \log \left[1 - F_\theta\left(T - \sum_{i=1}^{A(T)} u_i\right)\right] \xrightarrow{P} 0 \tag{2.8}$$

and

$$\frac{1}{\sqrt{D(T)}}\frac{\partial}{\partial\phi}\log\left[1-G_\phi\left(T-\gamma(T)-\sum_{i=1}^{D(T)}v_i\right)\right]\xrightarrow{P}0. \tag{2.9}$$

To understand the implications of the above two conditions observe that

$$T-\sum_{i=1}^{A(T)}u_i=u'_{A(T)+1},\qquad T-\gamma(T)-\sum_{i=1}^{D(T)}v_i=v'_{A(T)+1}$$

where $u'_{A(T)+1}$ is the last interarrival time, and $v'_{A(T)+1}$ is the residual service time of the customer (if any) still being served at time T. If $T=t$ with probability one (stopping rule 1) then from renewal theory we know that $u'_{A(T)+1}$ has a limit distribution as $t\to\infty$. Since

$$\frac{\partial}{\partial\theta}\log\left[1-F_\theta\left(T-\sum_{i=1}^{A(T)}u_i\right)\right]$$

is a continuous function of $u'_{A(T)+1}$ so this has also a limit distribution as $t\to\infty$. Thus, the condition (2.8) is satisfied under rule 1. Using similar argument one can also verify condition (2.9). For further implications of these conditions we refer (Basawa and Prabhu, 1988). From Eq. 2.7, we have the likelihood function

$$\begin{aligned}L^a_{A(T),D(T)}(\theta,\phi) &= \prod_{i=1}^{A(T)}a_1(u_i)\prod_{i=1}^{D(T)}a_1(v_i)\\ &\quad\times\exp\left\{\sum_{i=1}^{A(T)}[\theta h_1(u_i)-k_1(\theta)]+\sum_{i=1}^{D(T)}[\phi h_2(v_i)-k_2(\theta)]\right\}.\end{aligned} \tag{2.10}$$

The log-likelihood function can be written as

$$\begin{aligned}\log L^a_{A(T),D(T)}(\theta,\phi) &= \sum_{i=1}^{A(T)}\log a_1(u_i)+\sum_{i=1}^{A(T)}[\theta h_1(u_i)-k_1(\theta)]\\ &\quad+\sum_{i=1}^{D(T)}\log a_2(v_i)+\sum_{i=1}^{D(T)}[\phi h_2(v_i)-k_2(\theta)]\\ &= \ell^a_{A(T)}(\theta)+\ell^a_{D(T)}(\phi)\end{aligned} \tag{2.11}$$

where

$$\ell^a_{A(T)}(\theta) = \sum_{i=1}^{A(T)} \log a_1(u_i) + \sum_{i=1}^{A(T)} \left[\theta h_1(u_i) - k_1(\theta)\right] \tag{2.12}$$

and

$$\ell^a_{D(T)}(\phi) = \sum_{i=1}^{D(T)} \log a_2(v_i) + \sum_{i=1}^{D(T)} \left[\phi h_2(v_i) - k_2(\phi)\right]. \tag{2.13}$$

We assume that the following stability conditions on our stopping time holds:

$$\frac{A(T)}{\mathbb{E}(A(T))} \xrightarrow{P} 1 \quad \text{as } T \to \infty \text{ a.s.} \tag{2.14}$$

and

$$\frac{D(T)}{\mathbb{E}(D(T))} \xrightarrow{P} 1 \quad \text{as } T \to \infty \text{ a.s.} \tag{2.15}$$

From Eqs. 2.12 and 2.13 we note that $\ell^a_{A(T)}(\theta)$ and $\ell^a_{D(T)}(\phi)$ are of the same form. So, from here onwards we will deal only with the arrival rate parameter, that is, θ. The results, we will prove for the parameter θ using the condition (2.14), can be obtained for service parameter ϕ in similar fashion under the condition (2.15). From here onwards we denote $\ell^a_{A(T)}(\theta)$ by $\ell_{A(T)}(\theta)$ only.

From Eq. 2.12, applying first and second derivatives with respect to θ we have

$$\ell'_{A(T)}(\theta) = \frac{\partial \ell^a_{A(T)}(\theta)}{\partial \theta} = \sum_{i=1}^{A(T)} h_1(u_i) - A(T)k'_1(\theta) \tag{2.16}$$

and

$$\ell''_{A(T)}(\theta) = \frac{\partial^2 \ell^a_{A(T)}(\theta)}{\partial \theta^2} = -A(T)k''_1(\theta). \tag{2.17}$$

Let us define

$$\Psi_{A(T)}(\theta) = \ell_{A(T)}\left(\theta + \frac{1}{\sqrt{\mathbb{E}(A(T))}}\right) - \ell_{A(T)}\left(\theta - \frac{1}{\sqrt{\mathbb{E}(A(T))}}\right). \tag{2.18}$$

The expression in Eq. 2.18 will be used in the next section to find out an interval in which the likelihood function gets the same value at two points.

3 Main Results

In this section, we prove our main two results in the form of theorems. The first theorem says about the existence and consistency of the maximum likelihood estimator which satisfies the equation $\ell'_{A(T)}(\theta) = 0$. And, the second theorem gives the limiting distribution of the error of estimation.

Theorem 3.1. *Under the stability condition,* (2.14), *for any $\epsilon > 0$ there exists a solution $\widehat{\theta}_{A(T)}$ to the equation $\ell'_{A(T)}(\theta) = 0$ with probability greater than $1 - \epsilon$ and $\widehat{\theta}_{A(T)} \xrightarrow{P} \theta$ as $T \to \infty$.*

Proof. In the first step, we will find an element $\overline{\theta}_{A(T)}$ that satisfies the equation

$$\ell_{A(T)}\left(\theta + \frac{1}{\sqrt{\mathbb{E}(A(T))}}\right) - \ell_{A(T)}\left(\theta - \frac{1}{\sqrt{\mathbb{E}(A(T))}}\right) = 0. \tag{3.1}$$

For any τ in the neighbourhood of θ,

$$\begin{aligned}\ell_{A(T)}\left(\tau \pm \frac{1}{\sqrt{\mathbb{E}(A(T))}}\right) &= \ell_{A(T)}(\theta) + \left(\tau \pm \frac{1}{\sqrt{\mathbb{E}(A(T))}}\right)\ell'_{A(T)}(\theta) \\ &\quad + \frac{1}{2}\left(\tau \pm \frac{1}{\sqrt{\mathbb{E}(A(T))}}\right)^2 \ell''_{A(T)}\left(\theta + \delta\left(\tau \pm \frac{1}{\sqrt{\mathbb{E}(A(T))}} - \theta\right)\right)\end{aligned} \tag{3.2}$$

where $0 < \delta < 1$.

Then, we have

$$\begin{aligned}\Psi_{A(T)}(\tau) &= \ell_{A(T)}\left(\tau + \frac{1}{\sqrt{\mathbb{E}(A(T))}}\right) - \ell_{A(T)}\left(\tau - \frac{1}{\sqrt{\mathbb{E}(A(T))}}\right) \\ &= \frac{2}{\sqrt{\mathbb{E}(A(T))}}\ell'_{A(T)}(\theta) + \frac{1}{2}(\tau-\theta)^2\Bigg[\ell''_{A(T)}\left(\theta + \delta\left(\tau + \frac{1}{\sqrt{\mathbb{E}(A(T))}} - \theta\right)\right) \\ &\qquad - \ell''_{A(T)}\left(\theta + \delta\left(\tau - \frac{1}{\sqrt{\mathbb{E}(A(T))}} - \theta\right)\right)\Bigg] \\ &\quad + \frac{1}{2\mathbb{E}(A(T))}\Bigg[\ell''_{A(T)}\left(\theta + \delta\left(\tau + \frac{1}{\sqrt{\mathbb{E}(A(T))}} - \theta\right)\right) \\ &\qquad - \ell''_{A(T)}\left(\theta + \delta\left(\tau - \frac{1}{\sqrt{\mathbb{E}(A(T))}} - \theta\right)\right)\Bigg] \\ &\quad + \frac{\tau-\theta}{\sqrt{\mathbb{E}(A(T))}}\Bigg[\ell''_{A(T)}\left(\theta + \delta\left(\tau + \frac{1}{\sqrt{\mathbb{E}(A(T))}} - \theta\right)\right) \\ &\qquad - \ell''_{A(T)}\left(\theta + \delta\left(\tau - \frac{1}{\sqrt{\mathbb{E}(A(T))}} - \theta\right)\right)\Bigg].\end{aligned} \tag{3.3}$$

Thus, for any constant $M > 0$ form Eq. 3.3 we get

$$\begin{aligned}
\Psi_{A(T)}\left(\theta+\frac{M}{\sqrt{\mathbb{E}(A(T))}}\right) &= \ell_{A(T)}\left(\theta+\frac{M+1}{\sqrt{\mathbb{E}(A(T))}}\right)-\ell_{A(T)}\left(\theta+\frac{M-1}{\sqrt{\mathbb{E}(A(T))}}\right)\\
&= \frac{2}{\sqrt{\mathbb{E}(A(T))}}\ell'_{A(T)}(\theta)+\frac{M^2}{\mathbb{E}(A(T))}\left[\ell''_{A(T)}\left(\theta+\frac{\delta(M+1)}{\sqrt{\mathbb{E}(A(T))}}\right)\right.\\
&\quad\left.-\ell''_{A(T)}\left(\theta+\frac{\delta(M-1)}{\sqrt{\mathbb{E}(A(T))}}\right)\right]\\
&\quad+\frac{1}{2\mathbb{E}(A(T))}\left[\ell''_{A(T)}\left(\theta+\frac{\delta(M+1)}{\sqrt{\mathbb{E}(A(T))}}\right)-\ell''_{A(T)}\left(\theta+\frac{\delta(M-1)}{\sqrt{\mathbb{E}(A(T))}}\right)\right]\\
&\quad+\frac{M}{\mathbb{E}(A(T))}\left[\ell''_{A(T)}\left(\theta+\frac{\delta(M+1)}{\sqrt{\mathbb{E}(A(T))}}\right)+\ell''_{A(T)}\left(\theta+\frac{\delta(M-1)}{\sqrt{\mathbb{E}(A(T))}}\right)\right] \quad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
\Psi_{A(T)}\left(\theta-\frac{M}{\sqrt{\mathbb{E}(A(T))}}\right) &= \ell_{A(T)}\left(\theta-\frac{M-1}{\sqrt{\mathbb{E}(A(T))}}\right)-\ell_{A(T)}\left(\theta-\frac{M+1}{\sqrt{\mathbb{E}(A(T))}}\right)\\
&= \frac{2}{\sqrt{\mathbb{E}(A(T))}}\ell'_{A(T)}(\theta)+\frac{M^2}{\mathbb{E}(A(T))}\left[\ell''_{A(T)}\left(\theta-\frac{\delta(M-1)}{\sqrt{\mathbb{E}(A(T))}}\right)\right.\\
&\quad\left.-\ell''_{A(T)}\left(\theta-\frac{\delta(M+1)}{\sqrt{\mathbb{E}(A(T))}}\right)\right]\\
&\quad+\frac{1}{2\mathbb{E}(A(T))}\left[\ell''_{A(T)}\left(\theta-\frac{\delta(M-1)}{\sqrt{\mathbb{E}(A(T))}}\right)-\ell''_{A(T)}\left(\theta-\frac{\delta(M+1)}{\sqrt{\mathbb{E}(A(T))}}\right)\right]\\
&\quad-\frac{M}{\mathbb{E}(A(T))}\left[\ell''_{A(T)}\left(\theta-\frac{\delta(M-1)}{\sqrt{\mathbb{E}(A(T))}}\right)+\ell''_{A(T)}\left(\theta-\frac{\delta(M+1)}{\sqrt{\mathbb{E}(A(T))}}\right)\right] \quad (3.5)
\end{aligned}$$

where we assume that $\frac{M}{\sqrt{\mathbb{E}(A(T))}}$ is small enough when $T \to \infty$.

Let θ be the true value of arrival rate parameter.

Now, using the fact $\frac{M}{\sqrt{\mathbb{E}(A(T))}} \to 0$ as $T \to \infty$ in the Taylor's expansion of $\ell''_{A(T)}\left(\theta \pm \frac{\delta(M+1)}{\sqrt{\mathbb{E}(A(T))}}\right)$ about θ, and then applying Eq. 2.17 we get that

$$\begin{aligned}
\ell''_{A(T)}\left(\theta \pm \frac{\delta(M+1)}{\sqrt{\mathbb{E}(A(T))}}\right) &= \ell''_{A(T)}(\theta) + \left(\theta \pm \frac{\delta(M+1)}{\sqrt{\mathbb{E}(A(T))}} - \theta\right)\ell'''_{A(T)}(\theta^*_{A(T)})\\
&\to \ell''_{A(T)}(\theta) = -A(T)k''_1(\theta) \quad \text{as } T \to \infty, \quad (3.6)
\end{aligned}$$

where $|\theta^*_{A(T)} - \theta| \leq \frac{\delta(M+1)}{\sqrt{\mathbb{E}(A(T))}}$.

Similarly,

$$\ell''_{A(T)}\left(\theta \pm \frac{\delta(M-1)}{\sqrt{\mathbb{E}(A(T))}}\right) \to -A(T)k_1''(\theta) \quad \text{as } T \to \infty. \tag{3.7}$$

Thus, using Eqs. 3.6 and 3.7 in 3.4, and then applying Eq. 2.14 we have

$$\begin{aligned} \Psi_{A(T)}\left(\theta + \frac{M}{\sqrt{\mathbb{E}(A(T))}}\right) &\to \frac{2}{\sqrt{\mathbb{E}(A(T))}}\ell'_{A(T)}(\theta) - 2M\frac{A(T)}{\mathbb{E}(A(T))}k_1''(\theta) \\ &\xrightarrow{a.s.} \frac{2}{\sqrt{\mathbb{E}(A(T))}}\ell'_{A(T)}(\theta) - 2Mk_1''(\theta) \quad \text{as } T \to \infty. \end{aligned} \tag{3.8}$$

Using similar arguments from Eq. 3.5, we get

$$\Psi_{A(T)}\left(\theta - \frac{M}{\sqrt{\mathbb{E}(A(T))}}\right) \xrightarrow{a.s.} \frac{2}{\sqrt{\mathbb{E}(A(T))}}\ell'_{A(T)}(\theta) + 2Mk_1''(\theta) \quad \text{as } T \to \infty. \tag{3.9}$$

Let θ be the true value of the arrival rate parameter.
Now, from Eq. 2.16 we have

$$\begin{aligned} \mathbb{E}\left[\ell'_{A(T)}(\theta)\right] &= \mathbb{E}\left[\sum_{i=1}^{A(T)} h_1(u_i) - A(T)k_1(\theta)\right] \\ &= \mathbb{E}(A(T))\mathbb{E}(h_1(u_i)) - \mathbb{E}(A(T))k_1'(\theta) \\ &= 0, \quad (\text{using Eq. 3, } \mathbb{E}(h_1(u_i)) = k_1'(\theta)) \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}\left[\ell'_{A(T)}(\theta)\right]^2 &= \mathbb{E}\left[\sum_{i=1}^{A(T)} \{h_1(u_i) - k_1(\theta)\}\right]^2 \\ &= \mathbb{E}\left[\sum_{i=1}^{A(T)} [h_1(u_i) - k_1(\theta)]^2\right] \quad (\text{Since } u_i\text{'s are i.i.d.}) \\ &= \mathbb{E}(A(T))\mathbb{V}(h_1(u_i)) \\ &= \mathbb{E}(A(T))k_1''(\theta), \quad (\text{using Eq. 2.3, } \mathbb{V}(h_1(u_i)) = k_1''(\theta)). \end{aligned}$$

Thus, using the above expressions we have

$$\frac{1}{\sqrt{\mathbb{E}(A(T))}}\ell'_{A(T)}(\theta) \sim N\left(0, k_1''(\theta)\right). \tag{3.10}$$

Therefore, applying Eq. 3.10 in Eqs 3.8 and 3.9 respectively, we get

$$\Psi_{A(T)}\left(\theta+\frac{M}{\sqrt{\mathbb{E}(A(T))}}\right)\xrightarrow{d}N\left(-2Mk_1''(\theta),4k_1''(\theta)\right)\quad\text{as }T\to\infty. \tag{3.11}$$

and

$$\Psi_{A(T)}\left(\theta-\frac{M}{\sqrt{\mathbb{E}(A(T))}}\right)\xrightarrow{d}N\left(2Mk_1''(\theta),4k_1''(\theta)\right)\quad\text{as }T\to\infty. \tag{3.12}$$

Since $k_1''(\theta)>0$, for any $\epsilon>0$ and for large T and M, it follows that

$$\mathbb{P}\left[\Psi_{A(T)}\left(\theta+\frac{M}{\sqrt{\mathbb{E}(A(T))}}\right)<0\right]>1-\frac{\epsilon}{4}, \tag{3.13}$$

and

$$\mathbb{P}\left[\Psi_{A(T)}\left(\theta-\frac{M}{\sqrt{\mathbb{E}(A(T))}}\right)>0\right]>1-\frac{\epsilon}{4}. \tag{3.14}$$

Let the sets

$$\mathcal{A}=\left\{\Psi_{A(T)}\left(\theta+\frac{M}{\sqrt{\mathbb{E}(A(T))}}\right)<0\right\}$$

and

$$\mathcal{B}=\left\{\Psi_{A(T)}\left(\theta-\frac{M}{\sqrt{\mathbb{E}(A(T))}}\right)<0\right\}.$$

Then, we have

$$\mathbb{P}(\mathcal{A})>1-\frac{\epsilon}{4}\ \text{ and }\ \mathbb{P}(\mathcal{B})>1-\frac{\epsilon}{4}.$$

Thus, it follows that

$$\begin{aligned}\mathbb{P}(\mathcal{A}\cap\mathcal{B}) &= \mathbb{P}(\mathcal{A})+\mathbb{P}(\mathcal{B})-\mathbb{P}(\mathcal{A}\cup\mathcal{B})\\ &> \left(1-\frac{\epsilon}{4}\right)+\left(1-\frac{\epsilon}{4}\right)-1\\ &= 1-\frac{\epsilon}{2}.\end{aligned}$$

Because $\Psi_{A(T)}(\tau)$ is continuous in τ, it follows that there exists an element $\overline{\theta}_{A(T)}$ satisfying

$$\Psi_{A(T)}(\overline{\theta}_{A(T)}) = \ell_{A(T)}\left(\overline{\theta}_{A(T)} + \frac{1}{\sqrt{\mathbb{E}(A(T))}}\right) - \ell_{A(T)}\left(\overline{\theta}_{A(T)} - \frac{1}{\sqrt{\mathbb{E}(A(T))}}\right) = 0$$

and

$$\overline{\theta}_{A(T)} \in \left[\theta - \frac{M}{\sqrt{\mathbb{E}(A(T))}}, \theta + \frac{M}{\sqrt{\mathbb{E}(A(T))}}\right]$$

with probability greater than $1 - \frac{\epsilon}{2}$.

Since $\ell_{A(T)}(\theta)$ is continuous and differentiable on $\left[\theta - \frac{1}{\sqrt{\mathbb{E}(A(T))}}, \theta + \frac{1}{\sqrt{\mathbb{E}(A(T))}}\right]$, applying Rolle's theorem, there exists an element $\widehat{\theta}_{A(T)}$ such that $\ell'_{A(T)}(\widehat{\theta}_{A(T)}) = 0$ and $\widehat{\theta}_{A(T)} \in \left(\overline{\theta}_{A(T)} - \frac{1}{\sqrt{\mathbb{E}(A(T))}}, \overline{\theta}_{A(T)} + \frac{1}{\sqrt{\mathbb{E}(A(T))}}\right)$.

From the above arguments, we have

$$\widehat{\theta}_{A(T)} \in \left(\theta - \frac{M+1}{\sqrt{\mathbb{E}(A(T))}}, \theta + \frac{M+1}{\sqrt{\mathbb{E}(A(T))}}\right), \tag{3.15}$$

with probability greater than $1 - \epsilon$.

From Eq. 3.15, we get that

$$\mathbb{P}\left(\left|\widehat{\theta}_{A(T)} - \theta\right| < \frac{M+1}{\sqrt{\mathbb{E}(A(T))}}\right) > 1 - \epsilon.$$

For $\forall \varrho > 0$, let $A(T_0) = \left(\frac{M+1}{\varrho}\right)^2$. When $A(T) > A(T_0)$,

$$\left|\widehat{\theta}_{A(T)} - \theta\right| < \frac{M+1}{\sqrt{\mathbb{E}(A(T))}} < \varrho. \tag{3.16}$$

Then, it follows that

$$\mathbb{P}\left(\left|\widehat{\theta}_{A(T)} - \theta\right| < \varrho\right) > \mathbb{P}\left(\left|\widehat{\theta}_{A(T)} - \theta\right| < \frac{M+1}{\sqrt{\mathbb{E}(A(T))}}\right)$$

which implies that

$$\mathbb{P}\left(\left|\widehat{\theta}_{A(T)} - \theta\right| < \varrho\right) > 1 - \epsilon. \tag{3.17}$$

Therefore, from Eq. 3.17 we have

$$\widehat{\theta}_{A(T)} \xrightarrow{P} \theta \quad \text{as } T \to \infty. \tag{3.18}$$

Next, we will show that $\ell''_{A(T)}(\widehat{\theta}_{A(T)}) < 0$ to verify $\widehat{\theta}_{A(T)}$ is a maximum likelihood estimator.

From the Taylor's expansion of $\ell''_{A(T)}(\widehat{\theta}_{A(T)})$ about θ, we have

$$\ell''_{A(T)}(\widehat{\theta}_{A(T)}) = \ell''_{A(T)}(\theta) + \left(\widehat{\theta}_{A(T)} - \theta\right) \ell''_{A(T)}(\theta^{**}_{A(T)}).$$

where $|\theta^{**}_{A(T)} - \theta| \leq |\widehat{\theta}_{A(T)} - \theta|$.

Using the facts $\ell''_{A(T)}(\theta_{A(T)}) = -A(T)k''_1(\theta) < 0$ and $\widehat{\theta}_{A(T)} \xrightarrow{P} \theta$, we get that

$$\ell''_{A(T)}(\widehat{\theta}_{A(T)}) < 0 \quad \text{as } T \to \infty. \tag{3.19}$$

Hence, maximum likelihood estimator $\widehat{\theta}_{A(T)}$ exists and $\widehat{\theta}_{A(T)} \xrightarrow{P} \theta$ as $T \to \infty$.

Thus, we complete the proof.

Theorem 3.2. *Under the stability condition,* Eq. 2.14*,*

$$\sqrt{\mathbb{E}(A(T))}\left(\widehat{\theta}_{A(T)} - \theta\right) \xrightarrow{d} N\left(0, \frac{1}{k''_1(\theta)}\right)$$

as $T \to \infty$.

Proof. From the Taylor's expansion of $\ell'_{A(T)}(\theta)$ about $\widehat{\theta}_{A(T)}$ we get

$$\ell'_{A(T)}(\theta) = \ell'_{A(T)}(\widehat{\theta}_{A(T)}) + \left(\theta - \widehat{\theta}_{A(T)}\right) \ell''_{A(T)}(\widetilde{\theta}_{A(T)}). \tag{3.20}$$

where $|\widetilde{\theta}_{A(T)} - \theta| \leq |\widehat{\theta}_{A(T)} - \theta|$.

Using the result $\ell'_{A(T)}(\widehat{\theta}_{A(T)}) = 0$ from the Theorem 3.1, Eq. 3.20 becomes

$$\ell'_{A(T)}(\theta) = \left(\theta - \widehat{\theta}_{A(T)}\right) \ell''_{A(T)}(\widetilde{\theta}_{A(T)}). \tag{3.21}$$

From the proof of the Theorem 3.1, Eq. 3.10, we have

$$\frac{1}{\sqrt{\mathbb{E}(A(T))}}\ell'_{A(T)}(\theta) \sim N\left(0, k''_1(\theta)\right) \quad \text{as } T \to \infty. \tag{3.22}$$

And, Since $\ell''_{A(T)}(\theta) = -A(T)k''_1(\theta)$, Eq. 2.17, using condition, Eq. 2.14, we get

$$\frac{1}{\mathbb{E}(A(T))}\ell''_{A(T)}(\theta) \xrightarrow{a.s.} -k''_1(\theta) \quad \text{as } T \to \infty. \tag{3.23}$$

Using Taylor's expansion of $\frac{1}{\mathbb{E}(A(T))}\ell''_{A(T)}\left(\widetilde{\theta}_{A(T)}\right)$ about θ, we can write

$$\frac{1}{\mathbb{E}(A(T))}\ell''_{A(T)}\left(\widetilde{\theta}_{A(T)}\right) = \frac{1}{\mathbb{E}(A(T))}\ell''_{A(T)}(\theta) + \frac{1}{\mathbb{E}(A(T))}\left(\widetilde{\theta}_{A(T)} - \theta\right)\ell'''_{A(T)}\left(\widetilde{\theta}^*_{A(T)}\right) \quad (3.24)$$

where $\left|\widetilde{\theta}^*_{A(T)} - \theta\right| \leq \left|\widetilde{\theta}_{A(T)} - \theta\right|$.

Since $\left|\widetilde{\theta}_{A(T)} - \theta\right| \leq \left|\widehat{\theta}_{A(T)} - \theta\right|$, then using $\widehat{\theta}_{A(T)} \xrightarrow{P} \theta$ we have

$$\widetilde{\theta}_{A(T)} \xrightarrow{P} \theta \quad \text{as } T \to \infty. \quad (3.25)$$

Thus, using Eqs. 3.23 and 3.7 in 3.24 we get

$$\begin{aligned} \frac{1}{\mathbb{E}(A(T))}\ell''_{A(T)}\left(\widetilde{\theta}_{A(T)}\right) \quad &\xrightarrow{P} \frac{1}{\mathbb{E}(A(T))}\ell''_{A(T)}(\theta) \\ &\xrightarrow{P} -k''_1(\theta) \quad \text{as } T \to \infty. \end{aligned} \quad (3.26)$$

Now, multiplying $\frac{1}{\sqrt{\mathbb{E}(A(T))}}$ in the both sides of Eq. 3.21 we have

$$\frac{1}{\sqrt{\mathbb{E}(A(T))}}\ell'_{A(T)}(\theta) = \frac{1}{\sqrt{\mathbb{E}(A(T))}}\left(\theta - \widehat{\theta}_{A(T)}\right)\ell'_{A(T)}\left(\widetilde{\theta}_{A(T)}\right)$$

which implies that

$$\sqrt{\mathbb{E}(A(T))}\left(\widehat{\theta}_{A(T)} - \theta\right) = -\frac{\frac{1}{\sqrt{\mathbb{E}(A(T))}}\ell'_{A(T)}(\theta)}{\frac{1}{\mathbb{E}(A(T))}\ell''_{A(T)}(\widetilde{\theta}_{A(T)})}. \quad (3.27)$$

Finally, by combining Eqs. 3.22, 3.26 and 3.27 we obtain

$$\sqrt{\mathbb{E}(A(T))}\left(\widehat{\theta}_{A(T)} - \theta\right) \xrightarrow{d} \frac{1}{k''_1(\theta)}N\left(0, k''_1(\theta)\right) = N\left(0, \frac{1}{k''_1(\theta)}\right) \text{ as } T \to \infty.$$

Now, the proof is complete.

4 Example

4.1. $M/M/1$ Queue Let us consider a single server Markovian queueing system used in practice, that is, an $M/M/1$ queue. The arrivals are assumed to occur in a Poisson process with rate θ and the service time distribution follows exponential distribution with mean $1/\phi$. Therefore, we have

$$f(u, \theta) = \theta e^{-\theta u} \quad \text{and} \quad g(v, \phi) = \phi e^{-\phi v}.$$

The log-likelihood function, Eq. 2.11, becomes

$$\log L^a_{A(T),D(T)}(\theta,\phi) = A(T)\log\theta - \theta\sum_{i=1}^{A(T)} u_i + D(T)\log\phi - \phi\sum_{i=1}^{D(T)} v_i.$$

Here,

$$\ell^a_{A(T)}(\theta) = A(T)\log\theta - \theta\sum_{i=1}^{A(T)} u_i \quad \text{and} \quad \ell^a_{D(T)}(\phi) = D(T)\log\phi - \phi\sum_{i=1}^{D(T)} v_i.$$

Let $\varepsilon(T)\downarrow 0$ as $T\to\infty$. If we choose $\varepsilon(T) = T^{-\frac{2}{5}}$, then

$$\begin{aligned}
\mathbb{P}\left[\left|\frac{A(T)}{\mathbb{E}(A(T))} - 1\right| \geq \varepsilon(T)\right] &\leq \frac{\mathbb{E}\left[A(T) - \mathbb{E}(A(T))\right]^2}{\varepsilon^2(T)\left[\mathbb{E}(A(T))\right]^2} \\
&= \frac{\mathbb{V}(A(T))}{\varepsilon^2(T)\left[\mathbb{E}(A(T))\right]^2} \\
&= \frac{T\theta}{\varepsilon^2(T)(T\theta)^2} \text{ (since the arrival process is Poission)} \\
&= \frac{T^{-1/5}}{\theta} \to 0 \text{ as } T\to\infty.
\end{aligned}$$

Thus, the stability condition, Eq. 2.14, holds for $M/M/1$ queueing system. Hence, the results of the Section 3 can be used for this queueing system.

4.2. $E_r/M/1$ Queue Consider a single server queue with Erlang distributed interarrival times with probability density function

$$f(u,\theta,r) = \frac{\theta r}{\Gamma(r)}(\theta r u)^{r-1}e^{-\theta r u}, \; u > 0$$

with mean $1/\theta$ and r is positive integer (known), service times being exponentially distributed with $1/\phi$. The Erlang arrival model may also be thought of as a model with arrival in r-exponential stages where arrival at each stage is exponential at rate $r\theta$. In Kendall notation this queueing system denoted as $E_r/M/1$ queue (Wiper, 1998).

The log-likelihood function is

$$\begin{aligned}
\log L^a_{A(T),D(T)}(\theta,\phi) &= A(T)[\log(\theta r) - \log(\Gamma(r))] + (r-1)\log(\theta r) \\
&\quad + (r-1)\sum_{i=1}^{A(T)}\log u_i - \theta\sum_{i=1}^{A(T)} u_i + D(T)\log\ \phi - \phi\sum_{i=1}^{D(T)} v_i.
\end{aligned}$$

Thus,

$$\ell^a_{A(T)}(\theta) = A(T)[\log(\theta r) - \log(\Gamma(r))] + (r-1)\log(\theta r) + (r-1)\sum_{i=1}^{A(T)} \log\ u_i - \theta \sum_{i=1}^{A(T)} u_i$$

and

$$\ell^a_{D(T)}(\phi) = D(T)\log\phi - \phi\sum_{i=1}^{D(T)} v_i.$$

As in the previous example, choosing $\varepsilon(T) = T^{-\frac{2}{5}}$, the condition in Eq. 2.14

$$\mathbb{P}\left[\left|\frac{A(T)}{\mathbb{E}(A(T))} - 1\right| \geq \varepsilon(T)\right] \leq \frac{\mathbb{V}(A(T))}{\varepsilon^2(T)\,[\mathbb{E}(A(T))]^2} = \frac{T^{-1/5}}{r\theta} \to 0 \ \text{ as } T \to \infty$$

holds. Therefore, the results we proved in this paper can be used for the $E_r/M/1$ queueing system.

5 Concluding Remarks

This paper discussed the maximum likelihood estimator on which the likelihood function gets a local maximum, the consistency in probability of the maximum likelihood estimator and the limiting distribution of the error of estimation for a single server $GI/G/1$ queueing system by observing the system during a continuous time interval $(0, T]$. This queueing system has been studied in Basawa and Prabhu (1988), in which they have proved the consistency and asymptotic normality of maximum likelihood estimators under stability condition, Eq. 2.14, on stopping time T. To prove their results, they have followed the random sum central limit theorem from Billingsley (1961) and the Cramer-Wold argument. But, the methodology, we have applied here is different from their. We have used simple elementary analysis and probability to prove our main theorems under the stability condition on stopping time T.

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