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Cliques and Chromatic Number in Multiregime Random Graphs

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Abstract

In this paper, we study cliques and chromatic number in the random subgraph G_n of the complete graph K_n on n vertices, where each edge is independently open with a probability p_n. Associating G_n with the probability measure $\mathbb{P}_n$, we say that the sequence $\{\mathbb{P}_n\}$ is *multiregime* if the edge probability sequence $\{p_n\}$ is not convergent. Using a recursive method we obtain uniform bounds on the maximum clique size and chromatic number for such multiregime random graphs.

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1 Introduction

The study of cliques and chromatic numbers in random subgraphs of the complete graph K_n on n vertices is of importance from both theoretical and application perspectives. For the case where every edge of K_n is independently open with probability p_n with $\lim_n p_n \in \{0, p, 1\}$ for some $0 < p < 1$, bounds on cliques and chromatic numbers have been obtained using a combination of second moment method and martingale inequalities (see Alon and Spencer (2003) & Bollobás (2001)) and references therein). For the intermediate regime where $p_n = p$ is a constant, Shamir and Spencer (1987) studied the concentration of the chromatic number around its mean using martingale techniques. Sharp bounds for the chromatic number was then derived in Bollobás (1988) by exploring a maximal disjoint set of cliques. Alex Scott (2008) uses a combinatorial argument to sharpen the interval of concentration of the chromatic number to within an interval of length $\frac{\sqrt{n}}{\log n}$ and Panagiotou and Steger (2009) use a counting argument to obtain sharp lower bounds on the chromatic number.

For the sparse regime where $p_n \to 0$ as $n \to \infty$, Frieze (1990) uses martingales to investigate the independence number of homogenous random graphs and Luczak (1991) studies the chromatic number using a more delicate second moment method. Alon and Krivelevich (1997) use $k-$choosable graphs to study two point concentration of the chromatic number when the edge probability $p_n = n^{-1/2-\delta}$ for $\delta > 0$. More recently, Achlioptas and Naor (2005) use an analytic approach combining sharp threshold results with second moment methods to obtain the two possible values of the chromatic number when $p_n = \frac{d}{n}$ and $d > 0$ is a constant.

In this paper, we consider random graphs where the edge probability $\{p_n\}$ is not necessarily convergent and use a recursive method to obtain bounds for the clique numbers and consequently the chromatic numbers. In the rest of this subsection, we briefly describe the random graph model under consideration and state our main results (Theorems 1 and 2) regarding the cliques and chromatic number.

1.1. Clique Number For $n \geq 2$, let K_n be the labelled complete graph with vertex set $\{1, 2, \ldots, n\}$ and for $1 \leq i < j \leq n$, let $e(i, j)$ be the edge between the vertices i and j. Let $\{X(i, j)\}_{1 \leq i < j \leq n}$ be independent Bernoulli random variables with

$$\mathbb{P}_n(X(i,j) = 1) = 1 - \mathbb{P}_n(X(i,j) = 0) = p_n.$$

We say that edge $e(i, j)$ is *open* if $X(i, j) = 1$ and *closed* otherwise. The resulting random graph $G = G(n, p_n)$ is an Erdős-Rényi (ER) random graph, defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}_n)$. We say that the sequence of probability measures $\{\mathbb{P}_n\}$ is *multiregime* if $\{p_n\}$ is not convergent. For notational simplicity we henceforth drop the subscript from the probability measure $\mathbb{P}_n$ and denote it simply as $\mathbb{P}$.

A subset $U \subseteq \{1, \ldots, n\}$ is said to be an open *clique* of G if for all $u, v \in U$, the edge $e(u, v)$ is open. The *clique number*

$$\omega(G) := \max\{\#U : U \subseteq \{1, 2, \ldots, n\} \text{ is an open clique}\}$$

is the size of the largest open clique in G, where $\#U$ denotes the cardinality of a set U. To estimate $\omega(G)$, we let

$$W_n := \frac{\log n}{\log\left(\frac{1}{p_n}\right)} \tag{1.1}$$

and define

$$\alpha_1 := \limsup_n \frac{\log\left(\frac{1}{p_n}\right)}{\log n} \text{ and } \alpha_2 := \limsup_n \frac{\log\left(\frac{1}{1-p_n}\right)}{\log n}. \tag{1.2}$$

Theorem 1. *Suppose $\alpha_1 < 2$ and $\alpha_2 < 1$. There are positive constants η and γ such that*

$$\eta < 1 - \alpha_2 \text{ and } 2(\alpha_2 + \eta - \gamma) > \max(\alpha_1, 2\alpha_2). \tag{1.3}$$

For all $\epsilon > 0$ and η, γ satisfying (1.3), there is a positive integer $N \geq 1$ so that

$$\begin{aligned} &\mathbb{P}\left((1-\eta-\alpha_2)W_n \leq \omega(G(n,p_n)) \leq (2+2\epsilon)W_n + 1\right) \\ &\quad \geq 1 - 3\exp\left(-n^{\theta_{clq}}\right) - \exp\left(-\epsilon(2+2\epsilon)W_n \log n\right) \qquad (1.4) \\ &\quad \geq 1 - 4n^{-\frac{2\epsilon}{\alpha_1+\epsilon}} \qquad (1.5) \end{aligned}$$

for all $n \geq N$, where

$$\theta_{clq} := 2(\alpha_2 + \eta - \gamma) - \max(\alpha_1, \alpha_2) > 0. \tag{1.6}$$

In words, the clique number $\omega(G)$ is of the order of W_n with high probability, i.e., with probability converging to one as $n \to \infty$.

We briefly outline the proof of Theorem 1. To obtain the lower bound in Eq. 1.4, we let $1 - t_L(n)$ denote the probability of finding a clique of size L in the random graph G. We first obtain a recursive estimate roughly of the form (see Eq. 2.12 of Lemma 6 for a more precise formulation)

$$t_L(n) \leq t_{L-1}(\beta n) + e^{-\theta n^2} \tag{1.7}$$

where $\beta = \beta_n \in (0,1)$ and $\theta = \theta_n > 0$. Using the recursion (1.7) we then find an explicit upper bound of the form $t_L(n) \leq e^{-A_1} + 2e^{-A_2}$ where $A_1 = A_1(L, n, p_n)$ and $A_2 = A_2(L, n, p_n)$ (see Lemma 7 for explicit expressions for A_1 and A_2). Choosing $L = (1-\eta-\alpha_2)W_n$, we then show that both A_1 and A_2 are at least $n^{\theta_{clq}}$ (see proof of Theorem 1 in Section 3).

For the upper bound in Eq. 1.4, we use a union bound and estimate the probability that there exists an open clique of size L for $L = (2+2\epsilon)W_n + 1$, to be at most $\exp\left(-\epsilon(2+2\epsilon)W_n \log n\right)$. Combining with the lower bound estimates described in the previous paragraph, we then get Eq. 1.4. To obtain Eq. 1.5 from Eq. 1.4, we use the fact that $p_n \geq \frac{1}{n^{\alpha_1+\epsilon}}$ for all n large (see Eq. 1.2) and so $W_n = \frac{\log n}{\log\left(\frac{1}{p_n}\right)} \geq \frac{1}{\alpha_1+\epsilon}$ for all n large. This implies that

$$\epsilon(2+2\epsilon)W_n \log n \geq \frac{2\epsilon}{\alpha_1+\epsilon}\log n$$

and so

$$\max\left(\exp\left(-n^{\theta_{clq}}\right), \exp\left(-\epsilon(2+2\epsilon)W_n \log n\right)\right) \leq n^{-\frac{2\epsilon}{\alpha_1+\epsilon}}$$

for all n large. This obtains Eq. 1.5.

1.2. Chromatic Number Let $G = G(n, p_n)$ be the random graph obtained in the previous subsection. The *chromatic number* $\chi(G)$ is defined as follows. A $k-$colouring of G is a map $h : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, k\}$; i.e., each vertex in G is assigned one of k colours. The $k-$colouring h is said to be *proper* if $h(v_1) \neq h(v_2)$ for v_1, v_2 that are endvertices of an open edge; i.e., no two endvertices of an open edge have the same colour. Using a distinct colour for each of the n vertices of G, we obtain a proper $n-$colouring. Define

$$\chi(G) = \min\{k : 1 \leq k \leq n \text{ and } G \text{ has a proper } k - \text{colouring}\}$$

to be the chromatic number of G.

Letting α_1, α_2 be as in Eq. 1.2 and W_n be as in Eq. 1.1, we have the following result regarding the chromatic number of G.

Theorem 2. *Suppose $\{p_n\}$ is such that $\alpha_1 < 1, \alpha_2 < \frac{1}{2}$ and $1 - \alpha_2 > \max(\alpha_1, \alpha_2)$. There are positive constants η, γ and c such that*

$$\max(\eta, c) < 1 - \alpha_2, \quad \alpha_2 < c(\alpha_2 + \eta) < 1, 2c(\alpha_2 + \eta - \gamma) > \max(\alpha_1, 2\alpha_2) \tag{1.8}$$

and

$$\theta_{chr} := 2(\alpha_2 + \eta - \gamma) - \frac{1}{c}\max(\alpha_1, \alpha_2) > 1. \tag{1.9}$$

For all $\epsilon > 0$ and η, γ, c satisfying (1.8) and (1.9), there is a positive integer $N \geq 1$ so that

$$\mathbb{P}\left(\frac{n}{(2+2\epsilon)W_n + 1} \leq \chi(G(n, 1 - p_n)) \leq \frac{(1+\epsilon)n}{c(1 - \eta - \alpha_2)W_n}\right)$$
$$\geq 1 - 3\exp\left(-\frac{1}{2}n^{c\theta_{chr}}\right) - 2\exp\left(-\epsilon(2+2\epsilon)W_n \log n\right) \tag{1.10}$$
$$\geq 1 - 4n^{-\frac{2\epsilon}{\alpha_1 + \epsilon}} \tag{1.11}$$

for all $n \geq N$.

Thus the chromatic number $\chi(G)$ is of the order of $\frac{n}{W_n}$ with high probability. As in Alon and Spencer (2003), weuse the estimates in Theorem 1 regarding the clique number and consider the complement graph $G(n, 1-p_n)$ to find upper and lower bounds on the chromatic number (see Lemma 9 in Section 4).

<u>*Remark*</u>: Because we use recursion (see Lemma 6) to estimate the probability of occurrence of cliques, the information regarding the graph G becomes coarser at each successive step and this reflects in the difference between the upper and lower bounds for the clique number in Theorem 1.

Since we use the clique number estimates to bound the chromatic number (see Lemma 9 in Section 4), the corresponding difference appears in the chromatic number bounds (Theorem 2) as well.

1.3. Special Cases For completeness and illustration, we use our results above to state and provide brief proofs for the special cases where the probability sequence $\{p_n\}$ belongs to one of the three regimes, sparse, intermediate and dense. For stronger versions of below stated results, we refer to Alon and Spencer (2003) and references therein.

Theorem 3. *(i) Suppose $p_n = \frac{1}{n^{\theta_1}}$ for some $0 < \theta_1 < 2$. For every $\xi > 0$, there is a positive integer $N \geq 1$ so that*

$$\mathbb{P}\left(\frac{2-\theta_1}{2\theta_1} - \xi \leq \omega(G(n,p_n)) \leq \frac{2+\theta_1}{\theta_1} + \xi\right) \geq 1 - 4n^{-\xi} \tag{1.12}$$

for all $n \geq N$.

(ii) Suppose $p_n = p \in (0,1)$ for all n. For every $\xi > 0$, there is a positive integer $N \geq 1$ so that

$$\mathbb{P}\left(\frac{(1-\xi)\log n}{\log\left(\frac{1}{p}\right)} \leq \omega(G(n,p_n)) \leq \frac{(2+\xi)\log n}{\log\left(\frac{1}{p}\right)}\right) \geq 1 - 4\exp\left(\frac{-\xi}{\log\left(\frac{1}{p}\right)}(\log n)^2\right) \tag{1.13}$$

for all $N \geq N$.

(iii) Suppose $p_n = 1 - \frac{1}{n^{\theta_2}}$ for some $0 < \theta_2 < 1$. For every $\xi > 0$, there is a positive integer $N \geq 1$ so that

$$\mathbb{P}\left((1-\theta_2)(1-\xi)n^{\theta_2}\log n \leq \omega(G(n,p_n)) \leq (2+\xi)n^{\theta_2}\log n\right) \geq 1 - 4\exp\left(-\frac{\xi}{4}n^{\theta_2}(\log n)^2\right) \tag{1.14}$$

for all $n \geq N$.

Using the bounds for the clique number in Theorem 3, we derive bounds for the chromatic number of ER graphs where each edge is independently open with probability r_n, belonging to one of the three regimes sparse, intermediate or dense. As before, we discuss separate cases depending on the asymptotic behaviour of r_n.

Theorem 4. *(i) Suppose* $r_n = \frac{1}{n^{\theta_2}}$ *for some* $0 < \theta_2 < \frac{1}{2}$*. For all* $\xi > 0$*, there is a constant* $N \geq 1$ *so that*

$$\mathbb{P}\left((1-\xi)\frac{n^{1-\theta_2}}{2\log n} \leq \chi(G(n,r_n)) \leq \frac{2(1+\xi)}{1-2\theta_2}\frac{n^{1-\theta_2}}{\log n}\right) \geq 1 - 4\exp\left(-\frac{\xi}{4}n^{\theta_2}(\log n)^2\right) \quad (1.15)$$

for all $n \geq N$*.*

(ii) Suppose $r_n = p$ *for some constant* $0 < p < 1$ *and for all* n*. For all* $\xi > 0$*, there is a constant* $N \geq 1$ *so that*

$$\mathbb{P}\left((1-\xi)\frac{n\log\left(\frac{1}{1-p}\right)}{2\log n} \leq \chi(G(n,r_n)) \leq 2(1+\xi)\frac{n\log\left(\frac{1}{1-p}\right)}{\log n}\right) \geq 1 - 4\exp\left(-\frac{\xi}{3\log\left(\frac{1}{1-p}\right)}(\log n)^2\right) \quad (1.16)$$

for all $n \geq N$*.*

(iii) Suppose $r_n = 1 - \frac{1}{n^{\theta_1}}$ *for some* $0 < \theta_1 < 1$*. For all* $\xi > 0$*, there is a constant* $N \geq 1$ *so that*

$$\mathbb{P}\left((1-\xi)\frac{\theta_1 n}{2+\theta_1} \leq \chi(G(n,r_n)) \leq (1+\xi)\frac{2\theta_1 n}{1-\theta_1}\right) \geq 1 - 4n^{-\frac{\xi^2}{3\theta_1}} \quad (1.17)$$

for all $n \geq N$*.*

We remark here that Alon and Spencer (2003) consider the random graph G as a whole and use the extended Janson's inequality to obtain sharp concentration type estimates for the clique number $\omega(G)$ for example, when the edge probability is $p_n = \frac{1}{2}$. Using the bounds for the chromatic number in terms of the clique number (see Lemma 9 in Section 4) then results in the property that $\chi(G)$ is concentrated around $\frac{n}{\log_2 n}$ with high probability. Because we use recursion to estimate the probability of occurrence of cliques, this inherently results in a “loss of information” at each successive step and reflects in the differences between our upper and lower bounds for the clique number and correspondingly the chromatic number. Indeed, we compute the chromatic number using the clique number estimates (Lemma 9) analogous to Alon and Spencer (2003) (see Section 4 for more details). For additional results regarding the chromatic number in the sparse regime, we also refer to Bollobás (2001).

The paper is organized as follows. In Section 2, we obtain preliminary estimates used for proving the main Theorems 1 and 2. In Section 3, we prove Theorem 1 and in Section 4, we prove Theorem 1. Finally, in Appendix, we prove Theorems 3 and 4.

2 Preliminary Estimates

We use the following estimates throughout.

***Logarithm estimate*:** The following logarithmic estimate is used throughout. For $0 < x < 1$,

$$x < -\log(1-x) = \sum_{k\geq 1} \frac{x^k}{k} < \sum_{k\geq 1} x^k < \frac{x}{1-x}. \tag{2.1}$$

***Binomial Estimate*:** Let $\{X_j\}_{1\leq j\leq m}$ be independent Bernoulli random variables with $\mathbb{P}(X_j = 1) = p_j = 1 - \mathbb{P}(X_j = 0)$ and fix $0 < \epsilon < \frac{1}{6}$. If $T_m = \sum_{j=1}^{m} X_j$ and $\mu_m = \mathbb{E}T_m$, then

$$\mathbb{P}\left(|T_m - \mu_m| \geq \mu_m \epsilon\right) \leq 2\exp\left(-\frac{\epsilon^2}{4}\mu_m\right) \tag{2.2}$$

for all $m \geq 1$. For a proof of Eq. 2.2, we refer to Corollary A.1.14, pp. 312 of Alon and Spencer (2003).

The rest of the section is divided into three parts: The first part concerns a technical Lemma (see Lemma 5) that estimates a number δ_n and a decreasing sequence $\{q_i\}_{i\geq 0}$ both of which occur in the recursive estimate of the second part. In the second part, we estimate the probability $t_L(q_i)$ of finding a $L-$open clique (i.e., an open clique formed by L vertices) from among a total of q_i vertices, in terms of $t_{L-1}(q_{i+1})$ (see estimate (2.12) in Lemma 6). The final part uses the recursion estimate (2.12) to explicitly compute $t_L(q_0)$ for arbitrary q_0 and L (see Lemma 7).

In Sections 3 and 4, we use the estimate for $t_L(q_0)$ computed in Lemma 7 with appropriately chosen values of $L = L_n$ and $q_0 = q_0(n)$ to prove Theorems 1 and 2, respectively.

2.1. A Technical Lemma In this subsection, we define and estimate certain quantities that are used throughout in the proofs of the main theorems. Let $0 < \epsilon < \frac{1}{6}$ and let $M = M(\epsilon) \geq 2$ be large such that

$$\frac{1+\epsilon}{1 - \frac{1+\epsilon}{M}} < 1 + 2\epsilon. \tag{2.3}$$

Fixing such an M, we let $\epsilon_1 = \epsilon_1(\epsilon, M) \leq \epsilon$ be small so that

$$\frac{\log\left(\frac{1}{1-\epsilon_1}\right)}{\log\left(\frac{M}{M-1}\right)} \leq \epsilon. \tag{2.4}$$

For $n \geq 1$, we now set

$$\delta_n := \begin{cases} \epsilon_1 p_n & \text{if } p_n < 1 - \frac{1}{M} \\ \epsilon(1-p_n) & \text{if } p_n \geq 1 - \frac{1}{M}. \end{cases} \tag{2.5}$$

For a positive integer q let $q_0 = q$ and for $i \geq 1$, let

$$q_i = q_i(n) := \lfloor (p_n - \delta_n)(q_{i-1} - 1) \rfloor \tag{2.6}$$

be the largest integer less than or equal to $(p_n - \delta_n)(q_{i-1} - 1)$.

In the next subsection, we see that the number δ_n and the numbers $\{q_i\}$ both occur in the main recursive estimate used in the proof of Theorem 1. Indeed, the quantity δ_n occurs in the exponent of a term in Eq. 2.12 of Lemma 6 below and is a part of an estimate that relates the probability $t_L(q_i)$ of finding an $L-$open clique among q_i vertices, in terms of $t_{L-1}(q_{i+1})$. For convenience, we therefore record properties of δ_n and the numbers of $\{q_i\}$ for future use. We recall from Eq. 1.1 that $W_n = \frac{\log n}{\log\left(\frac{1}{p_n}\right)}$.

Lemma 5. *For any $n \geq 2$, the difference $p_n - \delta_n > 0$ and there is a constant $K \geq 1$ such that for all $n \geq K$,*

$$W_n \leq \begin{cases} \frac{\log n}{\log\left(\frac{M}{M-1}\right)} & \textit{if } p_n < 1 - \frac{1}{M} \\ n^{\alpha_2+\epsilon}\log n & \textit{if } p_n \geq 1 - \frac{1}{M}, \end{cases} \tag{2.7}$$

and

$$R_n := \frac{\log\left(\frac{1}{p_n - \delta_n}\right)}{\log\left(\frac{1}{p_n}\right)} < 1 + 2\epsilon. \tag{2.8}$$

The numbers $\{q_i\}$ *satisfy the following properties. For* $i \geq 1$,

$$(p_n - \delta_n)(q_{i-1} - 1) - 1 \leq q_i \leq (p_n - \delta_n)q_{i-1} \leq q_{i-1} \leq q \tag{2.9}$$

and

$$v_i := (p_n - \delta_n)^i q - \frac{2}{1 - p_n + \delta_n} \leq q_i \leq (p_n - \delta_n)^i q. \tag{2.10}$$

PROOF OF EQS. 2.7 AND 2.8 IN LEMMA 5. If $p_n < 1 - \frac{1}{M}$ then $\delta_n = \epsilon_1 p_n$ and so $p_n - \delta_n = (1-\epsilon_1)p_n > 0$. Also $\log\left(\frac{1}{p_n}\right) \geq \log\left(\frac{M}{M-1}\right)$. Consequently

$$W_n = \frac{\log n}{\log\left(\frac{1}{p_n}\right)} \leq \frac{\log n}{\log\left(\frac{M}{M-1}\right)} \text{ and } R_n \leq 1 + \frac{\log\left(\frac{1}{1-\epsilon_1}\right)}{\log\left(\frac{M}{M-1}\right)} \leq 1 + \epsilon,$$

by the choice of ϵ_1 in Eq. 2.4.

If $p_n \geq 1 - \frac{1}{M}$, then $\delta_n = \epsilon(1-p_n)$ and so

$$p_n - \delta_n = p_n(1+\epsilon) - \epsilon \geq \left(1 - \frac{1}{M}\right)(1+\epsilon) - \epsilon \geq \frac{1+\epsilon}{1+2\epsilon},$$

using Eq. 2.4. Moreover using the bound $-\log(1-x) > x$ (see Eq. 2.1) with $x = 1 - p_n$ we get

$$\log\left(\frac{1}{p_n}\right) = -\log(1-(1-p_n)) > 1 - p_n \geq \frac{1}{n^{\alpha_2+\epsilon}}$$

for all n large, where $\alpha_2 = \limsup_n \frac{\log\left(\frac{1}{1-p_n}\right)}{\log n}$ is as defined in Eq. 1.2. This means that $W_n = \frac{\log n}{\log\left(\frac{1}{p_n}\right)} \leq n^{\alpha_2+\epsilon}\log n$, giving Eq. 2.7.

To prove (2.8), we use the fact that $(1+\epsilon)(1-p_n) \leq \frac{1+\epsilon}{M} < 1$, since $M \geq 2$ and $0 < \epsilon < 1$. Therefore using the upper bound $-\log(1-x) < \frac{x}{1-x}$ from Eq. 2.1 with $x = (1+\epsilon)(1-p_n) = 1 - (p_n - \delta_n)$, we have that

$$-\log(p_n - \delta_n) \leq \frac{(1+\epsilon)(1-p_n)}{1-(1+\epsilon)(1-p_n)} \leq \frac{(1+\epsilon)(1-p_n)}{1 - \frac{1+\epsilon}{M}}.$$

Similarly using the lower bound estimate $-\log(1-x) > x$ from Eq. 2.1, we have $-\log p_n = -\log(1-(1-p_n)) > 1 - p_n$. Thus

$$R_n = \frac{\log\left(\frac{1}{p_n-\delta_n}\right)}{\log\left(\frac{1}{p_n}\right)} \leq \frac{1+\epsilon}{1 - \frac{1+\epsilon}{M}} \leq 1 + 2\epsilon,$$

by choice of M in Eq. 2.3. This proves (2.8).

Proof of Eqs. 2.9 and 2.10 in Lemma 5. The relation (2.9) is obtained using the property $x-1 \leq \lfloor x \rfloor \leq x$ for any $x > 0$. Set $\Delta = p_n - \delta_n \in (0,1)$ and apply the upper bound in Eq. 2.9 recursively, to get $q_i \leq \Delta^i q_0 = \Delta^i q$. This proves the upper bound in Eq. 2.10. For the lower bound we again proceed iteratively and first obtain for $i \geq 2$ that $q_i \geq \Delta(q_{i-1} - 1) - 1 = \Delta q_{i-1} - (1+\Delta)$. Continuing iteratively we obtain $q_i \geq \Delta^i q_0 - (1+\Delta)\sum_{j=0}^{i} \Delta^{j-1}$. Since $\Delta < 1$, the sum $\sum_{j=0}^{i} \Delta^j \leq \frac{1}{1-\Delta}$ and so $q_i \geq \Delta^i q_0 - \frac{1+\Delta}{1-\Delta} \geq \Delta^i q - \frac{2}{1-\Delta}$, since $\Delta \in (0,1)$. This proves (2.10).

2.2. Recursion Estimate Recall that the random graph $G = G(n, p_n)$ is said to have an open $L-$clique if there exists a subset $S \subset \{1, 2, \ldots, n\}$ such that $\#S = L$ and for any $u, v \in S$, the edge $e(u,v)$ is open (see statement prior to Eq. 1.1). For integer $2 \leq q \leq n$, let $S_q := \{1, 2, \ldots, q\}$ and let $G(S_q)$ be the induced subgraph of $G = G(n, p_n)$, with vertex set S_q. For integer $L \geq 2$, let $B_L(S_q)$ denote the event that the random subgraph $G(S_q)$ contains an open $L-$clique and set

$$t_L(q) := \mathbb{P}(B_L^c(S_q)). \tag{2.11}$$

By definition $t_L(q) = 0$ if $L > q$.

In this subsection, we obtain a recursive estimate for the probability $t_L(q)$ for arbitrary L and q in terms of $t_{L-1}(q_1)$ where q_1 is as defined in Eq. 2.6 with $i = 1$. In the next subsection, we use the recurrence relation obtained below to obtain an explicit estimate for $t_L(q)$.

Lemma 6. *Fix $0 < \epsilon < \frac{1}{6}$ and let ϵ_1 and δ_n be as defined in* Eqs. 2.4 *and* 2.5*, respectively. There exists a constant $N_0 = N_0(\epsilon)$ such that the following holds for all $n \geq N_0$: If $q, L \geq 2$ are such that $L - 1 \geq 2$ and $q_1 = \lfloor (p_n - \delta_n)(q-1) \rfloor \geq 2$ (see* Eq. 2.6*) then*

$$t_L(q) \leq q t_{L-1}(q_1) + \exp\left(-\frac{\epsilon_1 \delta_n}{16} q^2\right). \tag{2.12}$$

In Lemma 7 stated below, we use recursion to estimate the first term in Eq. 2.12 and obtain explicit estimates for $t_L(q)$ (see proof of Lemma 7).

Proof of Lemma 6. Recall from Eq. 2.5 that $\delta_n \in \{\epsilon_1 p_n, (1-\epsilon)p_n\}$. For simplicity, we write $p = p_n, \delta = \delta_n$ and first prove that Eq. 2.12 is satisfied with $\delta = p\epsilon_1$.

If N_e denotes the number of open edges in the random graph $G(S_q)$, then $\mathbb{E}N_e = p\binom{q}{2}$. Fixing $0 < \epsilon < \frac{1}{6}$, setting $\delta = p\epsilon_1$ and applying the binomial estimate (2.2) with $T_m = N_e$ then gives

$$\mathbb{P}\left(N_e \geq p(1-\epsilon_1)\binom{q}{2}\right) \geq 1-\exp\left(-\frac{\epsilon_1\delta}{4}\binom{q}{2}\right) \geq 1-\exp\left(-\frac{\epsilon_1\delta}{16}q^2\right) \tag{2.13}$$

since $\frac{1}{4}\binom{q}{2} \geq \frac{q^2}{16}$ for all $q \geq 2$.

We now write $\mathbb{P}(B_L^c(S_q)) = I_1 + I_2$, where

$$I_1 := \mathbb{P}\left(B_L^c(S_q)\bigcap\left\{N_e \geq (p-\delta)\binom{q}{2}\right\}\right) \tag{2.14}$$

and $I_2 := \mathbb{P}\left(B_L^c(S_q)\bigcap\left\{N_e < (p-\delta)\binom{q}{2}\right\}\right)$ is bounded above as

$$I_2 \leq \mathbb{P}\left(N_e < (p-\delta)\binom{q}{2}\right) \leq \exp\left(-\frac{\epsilon_1\delta}{16}q^2\right), \tag{2.15}$$

by Eq. 2.13.

It remains to estimate I_1. Suppose that the event $N_e \geq (p-\delta)\binom{q}{2}$ occurs. If $d(v)$ denotes the degree of vertex $v \in S_q$ in the random graph $G(S_q)$, then $\sum_{1\leq v\leq q} d(v) = 2N_e \geq (p-\delta)q(q-1)$ and so there exists a vertex w such that $d(w) \geq (p-\delta)(q-1) \geq q_1$, where q_1 is as defined in the statement of the Lemma. Thus

$$\begin{aligned} I_1 &\leq \mathbb{P}\left(B_L^c(S_q)\bigcap\left(\bigcup_{1\leq z\leq q}\{d(z)\geq q_1\}\right)\right) \\ &\leq \sum_{1\leq z\leq q}\mathbb{P}\left(B_L^c(S_q)\bigcap\{d(z)\geq q_1\}\right). \end{aligned} \tag{2.16}$$

If $N(z) = N(z, G(S_q))$ is the set of neighbours of z in the random graph $G(S_q)$, then

$$\mathbb{P}\left(B_L^c(S_q)\bigcap\{d(z)\geq q_1\}\right) = \sum_{S\subseteq S_q}\mathbb{P}\left(B_L^c(S_q)\bigcap\{N(z)=S\}\right), \tag{2.17}$$

where the summation is over all subsets S of S_q such that $\#S \geq q_1$ and $z \notin S$.

Suppose now that the event $B_L^c(S_q)\bigcap\{N(z) = S\}$ occurs for some fixed set $S \subseteq S_q$. We recall that since $B_L^c(S_q)$ occurs, there is no open $L-$clique in the random graph $G(S_q)$ with vertex set S_q. This necessarily means that there is no open $(L-1)-$clique in the random induced subgraph of $G(S_q)$ formed by the vertices of S; i.e., the event $B_{L-1}^c(S)$ occurs. This is because each vertex of S is connected to z by an open edge. Therefore $\mathbb{P}\left(B^c_L(S_q)\bigcap\{N(z)=S\}\right)$ is bounded above by

$$\mathbb{P}\left(B^c_{L-1}(S)\cap\{N(z)=S\}\right)=\mathbb{P}\left(B^c_{L-1}(S)\right)\mathbb{P}\left(N(z)=S\right), \tag{2.18}$$

where the final equality is true since the event $\{N(z)=S\}$ depends only on the state of edges containing z as an endvertex. On the other hand, the event $B^c_{L-1}(S)$ depends only on the state of edges having both their endvertices in S. Since the set S does not contain the vertex z (see Eq. 2.17), the events $\{N(z)=S\}$ and $B^c_{L-1}(S)$ are independent.

To obtain the desired recursion (2.12) using Eq. 2.18, let T denote the set of the q_1 least indices in S, where $q_1 \leq \#S$ is as defined in the statement of the Lemma. Since $T\subseteq S$, the events $B^c_{L-1}(S)\subseteq B^c_{L-1}(T)$; i.e., there is no open $(L-1)-$clique in the random induced subgraph formed by the vertices of T. From Eq. 2.18, we then get that $\mathbb{P}\left(B^c_L(S_q)\bigcap\{N(z)=S\}\right)$ is bounded above by

$$\mathbb{P}\left(N(z)=S\right)\mathbb{P}\left(B^c_{L-1}(T)\right)=\mathbb{P}\left(N(z)=S\right)t_{L-1}(q_1), \tag{2.19}$$

since $\#T=q_1$ (see Eq. 2.11).

Substituting (2.19) into (2.17) we get that $\mathbb{P}\left(B^c_L(S_q)\bigcap\{N(z)\geq q_1\}\right)$ is bounded above by

$$\sum_{S\subseteq S_q:\#S\geq q_1,\, z\notin S}\mathbb{P}\left(N(z)=S\right)t_{L-1}(q_1)=\mathbb{P}\left(d(z)\geq q_1\right)t_{L-1}(q_1)\leq t_{L-1}(q_1), \tag{2.20}$$

where the equality (2.20) is true since the events $\{N(z)=S\}$ are disjoint for distinct S. Substituting (2.20) into (2.16) gives

$$I_1\leq\sum_{1\leq z\leq q}t_{L-1}(q_1)=qt_{L-1}(q_1).$$

Using this estimate for I_1 and Eq. 2.15 we get

$$\mathbb{P}(B^c_L(S_q))\leq qt_{L-1}(q_1)+\exp\left(-\frac{\epsilon_1\delta}{16}q^2\right).$$

This proves (2.12) for $\delta=p\epsilon_1$.

Suppose now that $\delta=\epsilon(1-p)$. Recalling that N_e denotes the number of open edges in the random graph $G(S_q)$ (see the first paragraph of this proof), we let $W_e=\binom{q}{2}-N_e$ denote the number of closed edges so that $\mathbb{E}W_e=(1-p)\binom{q}{2}$. If $W_e\leq(1-p)(1+\epsilon)\binom{q}{2}$, then $W_e-\mathbb{E}W_e\leq$

$\epsilon\mathbb{E}W_e$. Applying the binomial estimate (2.2) with $T_m = W_e$ therefore gives that $\mathbb{P}\left(W_e \le (1-p)(1+\epsilon)\binom{q}{2}\right)$ is bounded below by

$$1-\exp\left(-\frac{\epsilon^2}{4}\mathbb{E}W_e\right) = 1-\exp\left(-\frac{\epsilon^2}{4}(1-p)\binom{q}{2}\right) = 1-\exp\left(-\frac{\epsilon\delta}{4}\binom{q}{2}\right) \tag{2.21}$$

for all $q \ge 2$, since $\delta = \epsilon(1-p)$. Moreover,

$$\left\{W_e \le (1-p)(1+\epsilon)\binom{q}{2}\right\} = \left\{N_e \ge (p-\delta)\binom{q}{2}\right\}$$

and so we again obtain (2.13) with ϵ instead of ϵ_1. Arguing as before, we then get $\mathbb{P}(B_L^c(S_q)) \le qt_{L-1}(q_1) + \exp\left(-\frac{\epsilon\delta}{16}q^2\right)$. Since $\epsilon_1 \le \epsilon$ (see Eq. 2.4), this proves (2.12).

2.3. Small Cliques Estimate We now use the recursion obtained in the Lemma 6 iteratively, to obtain an explicit estimate for the probability $t_L(q)$.

Lemma 7. *Fix $0 < \epsilon < \frac{1}{6}$ and let ϵ_1, δ_n be as defined in* Eqs. 2.4 *and* 2.5, *respectively. There is a constant $N_0 = N_0(\epsilon) \ge 1$ so that the following holds for all $n \ge N_0$: If $q = q_n$ and $L = L_n$ are such that*

$$v_L = (p_n - \delta_n)^L q - \frac{2}{1-p_n+\delta_n} \ge 2, \tag{2.22}$$

then $t_L(q) \le e^{-A_1} + 2e^{-A_2}$ where

$$A_1 := \frac{p_n}{4}v_L^2 - L\log q \text{ and } A_2 := \frac{\epsilon_1\delta_n}{16}v_L^2 - L\log q.$$

We recall that v_L is as defined in Eq. 2.10. In the proof of Theorem 1, we use a auxiliary lemma (Lemma 8) to first check that the condition (2.22) holds and then use Lemma 7 to obtain lower bounds on the clique number.

PROOF OF LEMMA 7. For simplicity, let $p = p_n, \delta = \delta_n$ and $r(q) := \exp\left(-\frac{\epsilon_1\delta}{16}q^2\right)$. Recalling the definition of $\{q_i\}$ in Eq. 2.6 and applying the recursion (2.12) twice we get $t_L(q) \le qt_{L-1}(q_1) + r(q) \le q(q_1 t_{L-2}(q_2) + r(q_1)) + r(q)$ provided $L-2$ and q_2 are both at least 2. Since $q_2 \le q_1 \le q_0 = q$, (see Eq. 2.9), we further get $t_L(q) \le q^2 t_{L-2}(q_2) + qr(q_1) + r(q)$. Proceeding iteratively, we therefore get that

$$t_L(q) \le q^{L-2}t_2(q_{L-2}) + \sum_{j=0}^{L-3} q^j r(q_j), \tag{2.23}$$

provided $q_{L-2} \ge 2$.

Since $q_{L-2} \geq q_L \geq v_L$ (see Esqs. 2.9 and 2.10), it is enough to prove the estimate for $t_L(q)$ in the Lemma from Eq. 2.23, assuming that $v_L \geq 2$. We now show that if $v_L \geq 2$, then the first term in Eq. 2.23 is at most e^{-A_1} and the second summation in Eq. 2.23 is at most $2e^{-A_2}$, where A_1 and A_2 are as in the statement of the Lemma. This proves the Lemma. To estimate the summation term in Eq. 2.23, we use $q_j \geq q_L \geq v_L$ for $1 \leq j \leq L-1$ (see Eqs. 2.9 and 2.10) to get that $r(q_j) \leq r(v_L)$. Thus

$$\sum_{j=0}^{L-3} q^j r(q_j) \leq \left(\sum_{j=0}^{L-3} q^j\right) r(v_L) = \frac{q^{L-2}-1}{q-1} r(v_L) \leq 2q^{L-3} r(v_L) \leq 2q^L r(v_L), \tag{2.24}$$

where the second inequality in Eq. 2.24 follows from the fact that $\frac{q^{L-2}-1}{q-1} \leq 2q^{L-3}$ for all $q \geq 2$. Since the final term of Eq. 2.24 is in fact $2e^{-A_2}$, we have bounded the summation in Eq. 2.23.

To bound the first term in Eqs. 2.23, we use the estimate

$$t_2(q_{L-2}) = (1-p)^{\binom{q_{L-2}}{2}} \leq \exp\left(-p\binom{q_{L-2}}{2}\right) \leq \exp\left(-p\binom{v_L}{2}\right), \tag{2.25}$$

where the first equality in Eq. 2.25 is true since there is no open 2−clique among a set of vertices if and only if all the edges between the vertices are closed. The final inequality in Eq. 2.25 follows from the fact that $q_{L-2} \geq q_L \geq v_L$ (see Eqas. 2.9 and 2.10). Since $\binom{v_L}{2} \geq \frac{v_L^2}{4}$ for all $v_L \geq 2$, we get from Eq. 2.25 that the first term in Eq. 2.23 is at most e^{-A_1}, proving the Lemma.

3 Proof of Theorem 1

We use the small cliques estimate in Lemma 7 with appropriately chosen value of $L = L_n$, to obtain the lower bound on the clique number. For the upper bound on the clique number, we use the union bound to estimate the probability that there an open clique of size L, chosen sufficiently large.

We begin with the proof of the lower bound on the clique number. To apply the small cliques estimate in Lemma 7, we first state and prove an auxiliary result (Lemma 8 below) that describes the feasibility of using Lemma 7. Let $\eta, \gamma > 0$ and $0 < c \leq 1$ be constants such that

$$\alpha_2 < c(\eta + \alpha_2) < 1 \text{ and } 2c(\alpha_2 + \eta - \gamma) > \max(\alpha_1, 2\alpha_2). \tag{3.1}$$

For $c = 1$, this corresponds to the conditions mentioned in Eq. 1.3 of the statement of Theorem 1. To see that are constants η, γ and c satisfying (3.1), we recall that $\alpha_1 < 2$ and $\alpha_2 < 1$. Therefore $2 > \max(\alpha_1, 2\alpha_2)$ and choosing $\alpha_2 + \eta_0$ and c_0 close enough to one and γ_0 close enough to zero, we get that Eq. 3.1 is satisfied with γ_0, η_0 and c_0. Arguing similarly, we also have that there are constants η and γ such that Eq. 3.1 holds with $c = 1$.

For future use, we perform the analysis below for a general $0 < c \leq 1$. We recall the definition of ϵ, ϵ_1 and δ_n prior to Eq. 2.5 and the term $W_n = \frac{\log n}{\log\left(\frac{1}{p_n}\right)}$ from Eq. 1.1. We now set

$$L_n = (1 - \eta - \alpha_2)\frac{\log(n^c)}{\log\left(\frac{1}{p_n}\right)} = c(1 - \eta - \alpha_2)W_n \tag{3.2}$$

and show that the quantities v_{L_n}, A_1 and A_2 defined in Lemma 7 satisfy the following estimates.

Lemma 8. *There are constants $\epsilon > 0$ small and $K = K(\epsilon) \geq 1$ large such that for all $n \geq K$*

$$v_{L_n} \geq n^{c(\alpha_2+\eta-3\epsilon)}, A_1 \geq n^{2c(\alpha_2+\eta-\gamma)-\alpha_1} \text{ and } A_2 \geq n^{2c(\alpha_2+\eta-\gamma)-\max(\alpha_1,\alpha_2)}. \tag{3.3}$$

The first condition in Eq. 3.3 in the above Lemma implies that condition (2.22) holds for all large n and therefore ensures the feasibility of using the small cliques estimate Lemma 7. Below, in the proof of Theorem 1, we use the estimates in Eq. 3.3 with $c = 1$ together with Lemma 7, to get the following lower bound for $\omega(G(n, p_n))$:

$$\mathbb{P}(\omega(G(n, p_n)) \geq (1 - \eta - \alpha_2)W_n) \geq 1 - \exp(-n^{\theta_{clq}}), \tag{3.4}$$

where $\theta_{clq} = 2(\alpha_2 + \eta - \gamma) - \max(\alpha_1, \alpha_2)$ is as defined in Eq. 1.6.

PROOF OF LEMMA 8. Let $0 < \epsilon < \frac{1}{6}$ be arbitrary but fixed. We first estimate v_{L_n} by writing

$$v_{L_n} = \exp\left(L_n \log(p_n - \delta_n) + c\log n\right) - \frac{2}{1 - p_n + \delta_n} \tag{3.5}$$

(see Lemma 7 for the expression for v_L) and use Eq. 2.8 (which states that $\frac{\log(p_n-\delta_n)}{\log p_n} < 1 + 2\epsilon$) and the fact that $L_n \log p_n < 0$ to get

$$L_n \log(p_n - \delta_n) \geq (1 + 2\epsilon)L_n \log p_n = -(1 + 2\epsilon)(1 - \eta - \alpha_2)c\log n, \tag{3.6}$$

by the definition of L_n in Eq. 3.2. Thus

$$L_n \log(p_n - \delta_n) + c\log n \quad \geq \quad (\alpha_2 + \eta - 2\epsilon(1 - \eta - \alpha_2))\, c\log n$$

$$\geq \quad (\alpha_2 + \eta - 3\epsilon)c \log n \tag{3.7}$$

for all n large and so the exponent term in the expression for v_{L_n} in Eq. 3.5 is at least $n^{c(\alpha_2+\eta-3\epsilon)}$.

To evaluate the term $\frac{1}{1-p_n+\delta_n}$ in Eq. 3.5, we consider two cases depending on whether $\delta_n = \epsilon_1 p_n$ or $\delta_n = \epsilon(1-p_n)$ (see Eq. 2.5). If $\delta_n = \epsilon_1 p_n$, then

$$p_n - \delta_n = p_n(1-\epsilon_1) \leq 1 - \epsilon_1$$

and so $\frac{1}{1-p_n+\delta_n} \leq \frac{1}{\epsilon_1}$. If $\delta_n = \epsilon(1-p_n)$, then

$$1 - p_n + \delta_n = (1+\epsilon)(1-p_n) \geq (1+\epsilon)\frac{1}{n^{\alpha_2+\epsilon}}$$

for all n large, by definition of $\alpha_2 = \limsup_n \frac{\log\left(\frac{1}{1-p_n}\right)}{\log n}$ in Eq. 1.2. In any case, $\frac{1}{1-p_n+\delta_n} \leq \frac{n^{\alpha_2+\epsilon}}{1+\epsilon}$ for all n large and so combining with the estimate (3.7) obtained in the previous paragraph we get that

$$v_{L_n} \geq n^{c(\alpha_2+\eta-2\epsilon)} - \frac{2n^{\alpha_2+\epsilon}}{1+\epsilon}$$

for all n large. Using the fact that $c(\alpha_2+\eta) > \alpha_2$ (see Eq. 3.1), we choose $\epsilon > 0$ small so that

$$c(\alpha_2 + \eta - 2\epsilon) > c(\alpha_2 + \eta - 3\epsilon) > \alpha_2 + \epsilon.$$

Fixing such an ϵ, we get $v_{L_n} \geq n^{c(\alpha_2+\eta-3\epsilon)}$ for all n large, proving the estimate for v_{L_n} in Eq. 3.3.

To estimate A_1 and A_2, we consider two separate cases depending on whether $p_n < 1 - \frac{1}{M}$ and $p_n \geq 1 - \frac{1}{M}$. If $p_n < 1 - \frac{1}{M}$, then $\delta_n = \epsilon_1 p_n$ (see the definition of δ_n in Eq. 2.5). From the estimate for W_n in Eq. 2.7 we have $L_n \leq W_n \leq \frac{\log n}{\log\left(\frac{M}{M-1}\right)}$. Since $\alpha_1 = \limsup_n \frac{\log\left(\frac{1}{p_n}\right)}{\log n} < 1$ (see Eq. 1.2), we have that $p_n \geq \frac{1}{n^{\alpha_1+\epsilon}}$ for all n large. Together with the estimate for v_{L_n} obtained above in Eq. 3.3, we therefore get that

$$A_1 = \frac{p_n}{4} v_{L_n}^2 - L_n \log n \geq \frac{1}{4n^{\alpha_1+\epsilon}} n^{2c(\alpha_2+\eta-3\epsilon)} - \frac{(\log n)^2}{\log\left(\frac{M}{M-1}\right)}, \tag{3.8}$$

for all n large. Using the condition (3.1) and choosing $\epsilon > 0$ smaller if necessary, we have that $2c(\alpha_2+\eta-3\epsilon) - (\alpha_1+\epsilon) > 2c(\alpha_2+\eta-\gamma) - \alpha_1 > 0$.

Thus $A_1 \geq n^{2c(\alpha_2+\eta-\gamma)-\alpha_1}$ and since $\delta_n = \epsilon_1 p_n$, an analogous analysis holds for A_2.

Suppose now that $p_n \geq 1 - \frac{1}{M}$. From Eq. 2.5 we have $\delta_n = \epsilon(1-p_n)$ and from Eq. 2.7 we have that $L_n \leq W_n \leq n^{\alpha_2+\epsilon} \log n$. Substituting into the expression for A_1 (see Eq. 3.8) we get

$$A_1 \geq \frac{1}{4}\left(1 - \frac{1}{M}\right) n^{2c(\alpha_2+\eta-3\epsilon)} - n^{\alpha_2+\epsilon}(\log n)^2 \geq n^{2c(\alpha_2+\eta-\gamma)},$$

for all n large, provided we choose $\epsilon > 0$ smaller if necessary so that

$$2c(\alpha_2 + \eta - 3\epsilon) > 2c(\alpha_2 + \eta - \gamma) > \alpha_2 + \epsilon.$$

This is possible by the conditions on α_2, η and c in Eq. 3.1.

To estimate A_2, we use $\alpha_2 = \limsup_n \frac{\log\left(\frac{1}{1-p_n}\right)}{\log n} < 1$ (see Eq. 1.2) to get that $\delta_n = \epsilon(1-p_n) \geq \frac{\epsilon}{n^{\alpha_2+\epsilon}}$ for all n large. Thus

$$A_2 = \frac{\epsilon_1}{16}\delta_n v_{L_n}^2 - L_n \log n \geq \frac{\epsilon_1}{16}\frac{\epsilon}{n^{\alpha_2+\epsilon}} n^{2c(\alpha_2+\eta-3\epsilon)} - n^{\alpha_2+\epsilon}(\log n)^2.$$

As before, we use Eq. 3.1 and choose $\epsilon > 0$ smaller if necessary to get that

$$2c(\alpha_2 + \eta - 3\epsilon) - (\alpha_2 + \epsilon) > 2c(\alpha_2 + \eta - \gamma) - \alpha_2 > \alpha_2 + \epsilon.$$

This implies that $A_2 \geq n^{2c(\alpha_2+\eta-\gamma)-\alpha_2}$ for all n large, proving the Lemma.

PROOF OF THEOREM 1. To prove the lower bound for $\omega(G)$ in Eq. 3.4, we set $c = 1$ in Eq. 3.1 and get from Lemma 8 that

$$\min(A_1, A_2) \geq n^{2(\alpha_2+\eta-\gamma)-\max(\alpha_1,\alpha_2)} = n^{\theta_{clq}},$$

where θ_{clq} is as defined in Eq. 1.6. Moreover, from Eq. 3.3 we get $v_{L_n} \geq 2$ for all n large and so Lemma 7 is applicable. Thus $t_{L_n}(n) \leq e^{-A_1} + 2e^{-A_2} \leq 3\exp\left(-n^{\theta_{clq}}\right)$. In other words, with probability at least $1 - 3\exp\left(-n^{\theta_{clq}}\right)$, the random graph G contains an open clique of size L_n, proving the lower bound (3.4) for $\omega(G(n,p_n))$.

To obtain the upper bound for $\omega(G(n,p_n))$, we prove below that for any positive sequence $\{H_n\}$, the term

$$\mathbb{P}\left(\omega(G(n,p_n)) \leq H_n\right) \geq 1 - \exp\left(-f_n H_n\right) \tag{3.9}$$

for all $n \geq 2$, where $f_n := \frac{(H_n - 1)}{2}\log\left(\frac{1}{p_n}\right) - \log n$. Setting $H_n = (2 + 2\epsilon)W_n + 1 \geq (2+2\epsilon)W_n$, where $W_n = \frac{\log n}{\log\left(\frac{1}{p_n}\right)}$ as defined in Eq. 1.1, we get that $f_n = \epsilon \log n$. From Eq. 3.9, we then get the desired upper bound (1.4). As argued in the discussion following Theorem 1, the estimate (1.5) follows from Eq. 1.4.

To prove (3.9) we use the fact that if there is an open $L-$clique in G with $L = H_n$, then some set T with $\#T = L$ has the property that every vertex in T is connected to every other vertex in T by an open edge. The number of edges with both endvertices in T is $\binom{L}{2}$ and the number of possible choices for T is $\binom{n}{L}$. Therefore,

$$\begin{aligned} \mathbb{P}(\omega(G(n,p_n)) \geq H_n) &\leq \binom{n}{L} p_n^{\binom{L}{2}} \\ &\leq n^L p_n^{\binom{L}{2}} \\ &= \exp\left(-L\left(\left(\frac{L-1}{2}\right)\log\left(\frac{1}{p_n}\right) - \log n\right)\right) \\ &= e^{-f_n H_n}, \end{aligned}$$

since $L = H_n$. This proves (3.9).

4 Proof of Theorem 2

We begin with the following definition:

Independence Number of a Graph We recall that the graph $G = G(n,p_n)$ is the random subgraph of K_n obtained by allowing each edge to be independently open with probability p_n. Also $\omega(G)$ is the size of the largest clique in G. The independence number $\alpha(G)$ is defined as follows: $\alpha(G) = h$ if and only if there is a set of h vertices, none of which have an open edge between them and every set of $h+1$ vertices have an open edge between them in G.

Let $\overline{G} = (\overline{V}, \overline{E})$ denote the complement of the graph G obtained by flipping the states of all edges in G; i.e., all open edges in G are closed in $\overline{G}$ and all closed edges in G are open in $\overline{G}$. We use the following Lemma (Alon and Spencer (2003)) in the proof of Theorem 2.

Lemma 9. $(a1)$ *The independence number* $\alpha(G) = \omega(\overline{G})$ *and so the chromatic number*

$$\chi(G) \geq \frac{n}{\alpha(G)} = \frac{n}{\omega(\overline{G})}. \tag{4.1}$$

$(a2)$ Suppose for some integer $1 \leq m \leq n$, every set of m vertices in the complement graph $\overline{G}$ contains an open clique of size L. We then have

$$\chi(G) \leq \frac{n}{L} + m. \tag{4.2}$$

PROOF OF THEOREM 2. As in Section 1, let $G = G(n, p_n)$ be the random subgraph of the complete graph K_n where each edge is open with probability p_n. We recall that $\omega(G)$ is the clique number of G as defined prior to Theorem 1. The lower bound in Eq. 1.11 then follows from property $(a1)$ in Lemma 9 and the upper bound for the clique number $\omega(G(n, p_n))$ in Eq. 1.5, since the random graph $\overline{G}(n, 1 - p_n)$ has the same distribution as the random graph $G(n, p_n)$. Indeed, we recall from Eq. 3.9 that

$$\mathbb{P}\left(\omega(G(n, p_n)) \leq (2 + 2\epsilon)W_n + 1\right) \geq 1 - \exp\left(-(2 + 2\epsilon)\epsilon W_n \log n\right) \tag{4.3}$$

where $W_n = \frac{\log n}{\log\left(\frac{1}{p_n}\right)}$ is as defined in Eq. 1.1. Using Eq. 4.3 in Eq. 4.1 we therefore get

$$\mathbb{P}\left(\chi(G(n, 1 - p_n)) \geq \frac{n}{(2 + 2\epsilon)W_n + 1}\right) \geq 1 - \exp\left(-(2 + 2\epsilon)\epsilon W_n \log n\right).$$

For the upper bound in Eq. 1.11, we use property $(a2)$ with $m = n^c$, where c satisfies the conditions in the statement of Theorem 2. For that we first see that there are positive constants η, γ and c such that Eq. 1.8 holds; i.e.,

$$\max(\eta, c) < 1 - \alpha_2,\ \alpha_2 < c(\alpha_2 + \eta) < 1 \text{ and } 2c(\alpha_2 + \eta - \gamma) > \max(\alpha_1, 2\alpha_2). \tag{4.4}$$

Indeed, recalling that $\alpha_1 < 1$ and $\alpha_2 < \frac{1}{2}$ we have that $2(1 - \alpha_2) > \max(\alpha_1, 2\alpha_2)$. Therefore choosing $c_0, \alpha_2 + \eta_0$ and γ_0, close enough to $1 - \alpha_2$, one and zero, respectively, we get that Eq. 4.4 holds. To see that Eq. 1.9 also holds; i.e.,

$$\theta_{chr} = 2(\alpha_2 + \eta - \gamma) - \frac{1}{c}\max(\alpha_1, \alpha_2) > 1, \tag{4.5}$$

we recall that $1 - \alpha_2 > \max(\alpha_1, \alpha_2)$. Therefore $2 - \frac{1}{1-\alpha_2}\max(\alpha_1, \alpha_2) > 1$. Choosing $c_0, \alpha_2 + \eta_0$ and γ_0 closer to $1 - \alpha_2$, one and zero, respectively, if necessary, we get that Eq. 4.5 also holds.

We now let $m = n^c$ where c satisfies (4.4) and let $\mathcal{S}_m$ be the set of subsets of size m in $\{1, 2, \ldots, n\}$. For a set $S \in \mathcal{S}_m$ and integer $L \geq 2$, we recall from Eq. 2.11 that $B_L(S)$ denotes the event that the random induced subgraph of $G(n,p_n)$ with vertex set S contains an open $L-$clique. We set

$$L := (1-\eta-\alpha_2)\frac{\log m}{\log\left(\frac{1}{p_n}\right)} = c(1-\eta-\alpha_2)W_n,$$

by Eq. 1.1 and the fact that $m = n^c$. Also from our choices of c, γ and η in Eqs. 4.4 and 4.5, the conditions in Eq. 3.1 hold. Therefore the estimates for A_1 and A_2 Lemma 8 hold and we get

$$\begin{aligned} \mathbb{P}(B_L^c(S)) &\leq e^{-A_1} + 2e^{-A_2} \leq 3\exp\left(-n^{2c(\alpha_2+\eta-\gamma)-\max(\alpha_1,\alpha_2)}\right) \\ &= 3\exp\left(-m^{\theta_{chr}}\right), \end{aligned} \tag{4.6}$$

where $\theta_{chr} > 1$, by Eq. 4.5.

If $F_n = \bigcap_{S\in\mathcal{S}_m} B_L(S)$ denotes the event that every set of m vertices in the random graph $G(n,p_n)$ contains an open $L-$clique, then

$$\mathbb{P}(F_n^c) \leq \binom{n}{m} 3\exp\left(-m^{\theta_{chr}}\right) \leq n^m 3\exp\left(-m^{\theta_{chr}}\right) = 3e^{-B}, \tag{4.7}$$

where $B = m^{\theta_{chr}} - m\log n \geq \frac{1}{2}m^{\theta_{chr}}$ for all n large, since $\theta_{chr} > 1$ by Eq. 4.5. The first estimate in Eq. 4.7 is obtained from the fact that the number of subsets of size m in the set $\{1,2,\ldots,m\}$ is $\binom{n}{m}$ and the probability that any subset of size m does not contain an open $L-$clique is at most $3\exp\left(-m^{\theta_{chr}}\right)$ by Eq. 4.6. If the event F_n occurs, then using property $(a2)$ in Lemma 9, we have that

$$\chi(G(n,1-p_n)) \leq \frac{n}{L} + m = \frac{n}{c(1-\eta-\alpha_2)W_n} + m. \tag{4.8}$$

We now estimate $\frac{n}{W_n}$ and show that it is much larger than $m = n^c$. Fix $\epsilon > 0$ and recall from Eq. 1.1 that $W_n = \frac{\log n}{\log\left(\frac{1}{p_n}\right)}$. Using the bound $-\log(1-x) > x$ in Eq. 2.1 with $x = 1-p_n$, we get $\log\left(\frac{1}{p_n}\right) = -\log(1-(1-p_n)) > 1 - p_n \geq \frac{1}{n^{\alpha_2+\epsilon}}$ for all n large, by the definition of α_2 in Eq. 1.2. Thus the term $W_n \leq n^{\alpha_2+\epsilon}\log n$ and so $\frac{n}{W_n} \geq \frac{n^{1-\alpha_2-\epsilon}}{\log n}$. Since $c < 1-\alpha_2$ (see Eq. 1.8), we choose $\epsilon > 0$ small so that $c < 1-\alpha_2-\epsilon$. Fixing such an ϵ, we get from Eq. 4.8 that $\chi(G(n,1-p_n)) \leq \frac{n(1+\epsilon)}{c(1-\eta-\alpha_2)W_n}$ for all n large, proving the upper bound in Eq. 1.10. Arguing as in the discussion following Theorem 1, the estimate (1.11) follows from Eq. 1.10.

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Appendix

Appendix: Proof of Theorems 3 and 4

Proof of Theorem 3

We use Theorem 1 with appropriate choices of η and γ.

(i) Here $\alpha_1 = \theta_1 < 2, \alpha_2 = 0$ and $W_n = \frac{\log n}{\log\left(\frac{1}{pn}\right)} = \frac{1}{\theta_1}$. For $0 < \xi < \frac{2-\theta_1}{\theta_1}$, we set $\epsilon = \frac{\xi\theta_1}{2}, \eta = \frac{\theta_1}{2} + \xi\theta_1$ and $\gamma = \frac{\xi\theta_1}{2}$ so that Eq. 1.3 is satisfied and $\theta_{clq} = \xi$. Using $\epsilon(2+2\epsilon)W_n \log n \geq 2\epsilon W_n \log n = \xi \log n$ in Eq. 1.5, we then get Eq. 1.12.

(*ii*) Here $\alpha_1 = \alpha_2 = 0$ and $W_n = \frac{\log n}{\log\left(\frac{1}{p}\right)}$. We let $0 < \xi < 1$ and set $\epsilon = \frac{\xi}{2}, \eta = \xi$ and $\gamma = \frac{\xi}{2}$ and so that Eq. 1.3 is satisfied and $\theta_{clq} = \xi$. As before, we use $\epsilon(2+2\epsilon)W_n \log n \geq 2\epsilon W_n \log n = \xi \frac{(\log n)^2}{\log\left(\frac{1}{p}\right)}$ and get Eq. 1.13 from Eq. 1.5.

(*iii*) Here $\alpha_1 = 0$ and $\alpha_2 = \theta_2 < 1$ and letting $0 < \xi < 1$ we set $\epsilon = \frac{\xi}{4}, \eta = \frac{\xi}{2} - \frac{\theta_2 \xi}{2}$ and $\gamma > 0$ smaller than η. With these choices (1.3) is satisfied and moreover $\theta_{clq} = \theta_2 + 2(\eta - \gamma) > \theta_2$. To evaluate W_n, use the log estimates (2.1) and (2.1) to get that $\frac{\frac{1}{n^{\theta_2}}}{1-n^{-\theta_2}} > \log\left(\frac{1}{p_n}\right) = -\log\left(1 - \frac{1}{n^{\theta_2}}\right) > \frac{1}{n^{\theta_2}}$ and so

$$n^{\theta_2} \log n \left(1 - \frac{1}{n^{\theta_2}}\right) \leq W_n = \frac{\log n}{\log\left(\frac{1}{p_n}\right)} \leq n^{\theta_2} \log n \tag{A.1}$$

for all n large. Moreover

$$(1 - \eta - \alpha_2)W_n = (1 - \theta_2)\left(1 - \frac{\xi}{2}\right) W_n \geq (1 - \theta_2)(1 - \xi)n^{\theta_2} \log n$$

for all n large and $(2+2\epsilon)W_n + 1 \leq (2+\xi)n^{\theta_2} \log n$ and $\epsilon(2+2\epsilon)W_n \log n \geq \frac{\xi}{4} n^{\theta_2} (\log n)^2$ for all n large. Plugging the above into Eq. 1.5 we get Eq. 1.14.

Proof of Theorem 4

We use Theorem 2 with $p_n = 1 - r_n$ and appropriate choices of η and γ.

(*i*) Here $p_n = 1 - r_n$ with $\alpha_1 = 0$ and $\alpha_2 = \theta_2 < \frac{1}{2}$. Thus $\eta_0 := \frac{1}{2(1-\theta_2)} - \theta_2 < 1 - \theta_2$ and for $0 < \xi < 1$ to be determined later, we set

$$\epsilon = \frac{\xi}{6}, c = (1 - \theta_2)\left(1 - \frac{\xi^3}{6}\right), \eta = \eta_0 \left(1 + \frac{\xi^2}{6}\right) \text{ and } \gamma = \frac{\eta_0 \xi^2}{12}. \tag{A.2}$$

We need to ensure that conditions (1.8) and (1.9) hold with $\alpha_1 = 0$ and $\alpha_2 = \theta_2$. By definition $c < 1-\theta_2$ and $\eta_0 < 1-\theta_2$ and so $\max(\eta, c) < 1 - \theta_2$ provided $\xi > 0$ is small. We choose $\xi > 0$ smaller if necessary so that

$$c(\theta_2+\eta) = \frac{1}{2} - \frac{\xi^3}{12} + \frac{\eta_0 \xi^2}{6}(1-\theta_2)\left(1 - \frac{\xi^3}{6}\right) \geq \frac{1}{2} + \frac{\eta_0 \xi^2}{12}(1-\theta_2)\left(1 - \frac{\xi^3}{6}\right)$$

and so $\theta_2 < \frac{1}{2} < c(\theta_2+\eta) < 1$. To ensure the third condition in Eq. 1.8, we have

$$2c(\theta_2+\eta-\gamma)=1-\frac{\xi^3}{6}+\frac{\eta_0\xi^2}{6}(1-\theta_2)\left(1-\frac{\xi^3}{6}\right)\geq 1+\frac{\eta_0\xi^2}{12}(1-\theta_2)\left(1-\frac{\xi^3}{6}\right)$$

provided $\xi > 0$ is small. Fixing such a ξ we get $2c(\theta_2+\eta-\gamma) > 1 > 2\theta_2$, since $\theta_2 < \frac{1}{2}$. Thus Eq. 1.8 holds.

To ensure (1.9), we write $\theta_{chr} = \frac{1}{1-\theta_2} + \frac{\eta_0\xi^2}{6} - \frac{\theta_2}{(1-\theta_2)\left(1-\frac{\xi^3}{6}\right)}$ and choose $\xi > 0$ small so that $\left(1-\frac{\xi^3}{6}\right)^{-1} \leq 1+\frac{\xi^3}{4}$ and so $\theta_{chr} \geq 1+\frac{\eta_0\xi^2}{6} - \frac{\theta_2}{(1-\theta_2)}\frac{\xi^3}{4} \geq 1+\frac{\eta_0\xi^2}{12}$. For future use we choose $\xi > 0$ smaller if necessary so that

$$c\theta_{chr} \geq (1-\theta_2)\left(1-\frac{\xi^3}{6}\right)\left(1+\frac{\eta_0\xi^2}{12}\right) \geq (1-\theta_2)\left(1+\frac{\eta_0\xi^2}{24}\right) > 1-\theta_2. \tag{A.3}$$

Thus the bounds in Eq. 1.11 is true. We now evaluate the upper and lower bounds in Eq. 1.11. From Eq. A.1 and the fact that $0 < \xi < 1$, we get $(2+2\epsilon)W_n + 1 \leq \frac{2n^{\theta_2}\log n}{1-\xi}$. Similarly

$$\frac{1+\epsilon}{c(1-\eta-\theta_2)} = \frac{2\left(1+\frac{\xi}{6}\right)}{\left(1-\frac{\xi^3}{6}\right)\left(1-2\theta_2-\frac{\eta_0\xi^2}{3}(1-\theta_2)\right)} \leq \frac{2(1+\xi)}{1-2\theta_2},$$

provided $\xi > 0$ is small and these estimates obtain the bounds for $\chi(.)$ in Eq. 1.15.

To evaluate the exponents in Eq. 1.11, we use Eq. A.1 to get that $\epsilon(2+\epsilon)W_n \log n \geq \frac{\xi}{4}n^{\theta_2}(\log n)^2$ for all n large. Similarly from Eq. A.3 we get $c\theta_{chr} > 1-\theta_2$ and this obtains (1.15).

(ii) Here $p_n = 1-r_n = 1-p$ and so $\alpha_1 = \alpha_2 = 0$. Letting ξ be small such that

$$\epsilon = \frac{\xi}{6}, \eta = \frac{1}{2} + 2\xi^2 < 1, \gamma = \xi^2 \text{ and } c = 1-\xi^3 \tag{A.4}$$

we get that the conditions in Eq. 1.8 are true. Also $\theta_{chr} = 1+2\xi^2 > 1$ and so Eq. 1.9 is also true. Thus the bounds in Eq. 1.11 hold. Recalling that $W_n = \frac{\log n}{\log\left(\frac{1}{p_n}\right)}$ we have that $(2+2\epsilon)W_n + 1 = \left(2+\frac{\xi}{3}\right)\frac{\log n}{\log\left(\frac{1}{1-p}\right)} + 1 \leq \frac{2}{(1-\xi)}\frac{\log n}{\log\left(\frac{1}{1-p}\right)}$ for all n large. Similarly, the scaling factor in the upper bound in Eq. 1.11 is $\frac{1+\epsilon}{c(1-\eta-\alpha_2)} = \frac{1+\frac{\xi}{6}}{(1-\xi^3)\left(\frac{1}{2}-2\xi^2\right)} \leq (1+\xi)$ if $\xi > 0$ is small. The exponents in Eq. 1.11 evaluate to $c\theta_{chr} = (1-\xi^3)(1+2\xi^2) \geq 1+\xi^2$ for all $\xi > 0$ small and $\epsilon(2+2\epsilon)W_n \geq 2\epsilon W_n = \frac{\xi}{3}\frac{\log n}{\log\left(\frac{1}{1-p}\right)}$. This obtains (1.16).

(*iii*) Here $p_n = 1 - r_n = \frac{1}{n^{\theta_1}}$ and so $\alpha_1 = \theta_1 < 1$ and $\alpha_2 = 0$. Let ξ be small such that

$$\epsilon = \frac{\xi^2}{6}, \eta = \frac{1+\theta_1}{2} + 2\theta_1\xi^2 < 1, \gamma = \theta_1\xi^2 \text{ and } c = 1-\xi^3. \tag{A.5}$$

Recalling condition (1.8), we have $\max(\eta, c) < 1 = 1 - \alpha_2, 0 < c\eta = c(\alpha_2 + \eta) < 1$ and

$$\begin{aligned} 2c(\alpha_2 + \eta - \gamma) &= 2(1-\xi^3)(\eta-\gamma) = 2(\eta-\gamma) - 2\xi^3(\eta-\gamma) \\ &= 1 + \theta_1 + 2\theta_1\xi^2 - 2\xi^3(\eta-\gamma). \end{aligned} \tag{A.6}$$

which is greater than one if $\xi > 0$ small. Thus Eq. 1.8 is true. Also $\theta_{chr} = 1+\theta_1+2\theta_1\xi^2 - \frac{\theta_1}{1-\xi^3}$ and for all $\xi > 0$ small, we have $\frac{1}{1-\xi^3} \leq 1+\xi^2$ and for such ξ, we have $\theta_{chr} \geq 1 + \theta_1 + 2\theta_1\xi^2 - \theta_1(1+\xi^2) = 1 + \theta_1\xi^2 > 1$. For future use we set $\xi > 0$ smaller if necessary so that

$$c\theta_{chr} \geq (1-\xi^3)(1+\theta_1\xi^2) \geq 1 + \frac{\theta_1\xi^2}{2} > 1. \tag{A.7}$$

Thus Eq. 1.9 is also true and consequently, the bounds in Eq. 1.11 hold.

Recalling that $W_n = \frac{\log n}{\log\left(\frac{1}{p_n}\right)} = \frac{1}{\theta_1}$ we have from Eq. A.5 that $(2 + 2\epsilon)W_n + 1 = \left(2 + \frac{\xi^2}{3}\right)\frac{1}{\theta_1} + 1 \leq \frac{2+\theta_1}{\theta_1(1-\xi)}$ for all n large, provided $\xi > 0$ small. Fixing such a ξ, the scaling factor in the upper bound in Eq. 1.11 is

$$\frac{1+\epsilon}{c(1-\eta-\alpha_2)} = \frac{1+\frac{\xi^2}{6}}{(1-\xi^3)\left(1 - \frac{1+\theta_1}{2} - 2\xi^2\right)} \leq (1+\xi)\frac{2}{1-\theta_1} \tag{A.8}$$

provided we set $\xi > 0$ smaller if necessary.

Finally, regarding the exponents in Eq. 1.17, we have from Eq. A.7 that $c\theta_{chr} > 1$ and moreover $\epsilon(2+2\epsilon)W_n \geq 2\epsilon W_n = \frac{\xi^2}{3}\frac{1}{\theta_1}$. This obtains (1.17).

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