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The Min-characteristic Function: Characterizing Distributions by Their Min-linear Projections

Michael Falk
University of Würzburg, Würzburg, Germany
Gilles Stupfler
University of Nottingham, Nottingham, UK

Abstract

Motivated by a (seemingly previously unnoticed) result stating that $d-$dimensional distributions on $(0,\infty)^d$ are characterized by the collection of their min-linear projections, we introduce and study a notion of min-characteristic function (min-CF) of a random vector with strictly positive components. Unlike the related notion of max-characteristic function which has been studied recently, the existence of the min-CF does not hinge on any integrability conditions. It is itself a multivariate distribution function, which is continuous and concave, no matter which properties the initial distribution function has. We show the equivalence between convergence in distribution and pointwise convergence of min-CFs, and we also study the functional convergence of the min-CF of the empirical distribution function of a sample of independent and identically distributed random vectors. We provide some further insight into the structure of the set of min-CFs, and we conclude by showing how transforming the components of an arbitrary random vector by a suitable one-to-one transformation such as the exponential function allows the construction of a notion of min-CF for arbitrary random vectors.

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1 Introduction and motivation

The well-known Cramér-Wold theorem (see Cramér and Wold 1936) states that any distribution on $\mathbb{R}^d$ is determined by the collection of its one-dimensional projections. In other words, if $\boldsymbol{X} = (X_1,\dots,X_d)$ and $\boldsymbol{Y} = (Y_1,\dots,Y_d)$ are two random vectors (rvs), we have

$$\boldsymbol{X} \stackrel{d}{=} \boldsymbol{Y} \Leftrightarrow \forall t_1,\dots,t_d \in \mathbb{R},\ \sum_{k=1}^{d} t_k X_k \stackrel{d}{=} \sum_{k=1}^{d} t_k Y_k.$$

The Cramér-Wold theorem is well adapted to the case when the individual components of $\boldsymbol{X}$ and $\boldsymbol{Y}$ behave nicely with respect to summation. There are, however, important examples of situations in which this is not the case. For instance, in multivariate extreme value theory, the individual components of $\boldsymbol{X}$ may represent marginal financial or actuarial risk variables, whose distributions would typically be modeled by heavy-tailed distributions such as the Pareto distribution (see e.g. Embrechts et al. 1997; Resnick 2007). Another example is the class of multivariate max-stable distributions, where usually the marginals are assumed to be unit Fréchet. Calculations involving sums of Pareto or Fréchet distributed rvs are typically very complicated, even if these rvs are independent (see e.g. Blum 1970; Nadarajah and Pogány 2013; Nadarajah et al. 2018). The relevant operator in this kind of situation is the maximum rather than the sum, leading one to consider instead the collection of *max-linear projections* of $\boldsymbol{X}$, that is:

$$\bigvee_{k=1}^{d} t_k X_k := \max_{1 \le k \le d} t_k X_k, \qquad t_1, \ldots, t_d > 0.$$

It turns out, somewhat surprisingly, that the distributions of such projections also characterize multivariate distributions, if they are assumed to be nonnegative. This is the focus of our first result, which does not seem to have been shown in the literature so far.

Proposition 1.1. *We have, for arbitrary random vectors* $\boldsymbol{X} = (X_1, \ldots, X_d)$ *and* $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$ *with nonnegative components:*

$$\boldsymbol{X} \stackrel{d}{=} \boldsymbol{Y} \Leftrightarrow \forall t_1, \ldots, t_d > 0, \ \bigvee_{k=1}^{d} t_k X_k \stackrel{d}{=} \bigvee_{k=1}^{d} t_k Y_k.$$

Proof. Since, for any $t_1, \ldots, t_d > 0$,

$$\mathbb{P}\left(\bigvee_{k=1}^{d} t_k X_k \le 1\right) = \mathbb{P}\left(X_1 \le 1/t_1, \ldots, X_d \le 1/t_d\right),$$

the knowledge of the distribution of the max-linear projections of $\boldsymbol{X}$ is equivalent to the knowledge of the distribution function (df) F of $\boldsymbol{X}$ at any point $\boldsymbol{x} \in (0, \infty)^d$. By right-continuity of F, this is equivalent to the knowledge of F on $[0, \infty)^d$. Since $\boldsymbol{X}$ is concentrated on $[0, \infty)^d$, the result follows.

That the distribution of a componentwise nonnegative rv is characterized by the collection of distributions of its max-linear projections is nicely linked to the notion of *max-characteristic function* (max-CF), introduced by Falk and Stupfler (2017): if $\boldsymbol{X} = (X_1, \dots, X_d)$ has nonnegative and integrable components, then the knowledge of the mapping

$$\begin{aligned}\varphi_{\boldsymbol{X}}(\boldsymbol{t}) &= \mathbb{E}(\max(1, t_1 X_1, \dots, t_d X_d)) \\ &= \mathbb{E}\left(\max\left(1, \bigvee_{k=1}^{d} t_k X_k\right)\right), \quad t_1, \dots, t_d > 0,\end{aligned}$$

characterizes the distribution of the rv $\boldsymbol{X}$. This notion of max-characteristic function is particularly interesting when considering standard extreme value distributions such as the Generalized Pareto distribution, for which it has a simple closed form, although the standard characteristic function based on taking a Fourier transform does not (see Falk and Stupfler 2017). However, this notion requires the integrability of the components of $\boldsymbol{X}$; its generalization to random vectors without sign constraints, suggested by Falk and Stupfler (2019), even requires an exponential moment. This is of course a serious restriction.

The motivation for this work resides in combining this last remark with the following observation. For $d = 2$ and any $t_1, t_2 > 0$, one clearly has

$$\max(t_1 X_1, t_2 X_2) = t_1 X_1 + t_2 X_2 - \min(t_1 X_1, t_2 X_2).$$

By Proposition 1.1, we know that the distribution of (X_1, X_2) is characterized by the collection of distributions of the max-linear projections $\max(t_1 X_1,\ t_2 X_2)$ when t_1 and t_2 vary. We also know, by the Cramér-Wold theorem, that it is characterized by the collection of distributions of the one-dimensional projections $t_1 X_1 + t_2 X_2$. We may therefore ask whether the distribution of (X_1, X_2) is determined by the collection of distributions of $\min(t_1 X_1, t_2 X_2)$, when t_1, t_2 range over $(0, \infty)$. More generally, we may ask if the distribution of a d-dimensional rv $\boldsymbol{X}$ is characterized by the *min-linear projections*

$$\bigwedge_{k=1}^{d} t_k X_k := \min_{1 \le k \le d} t_k X_k,\ t_1, \dots, t_d > 0.$$

By analogy with the notion of max-CF, this would then suggest to define the following notion of min-characteristic function:

$$\begin{aligned}\psi_{\boldsymbol{X}}(\boldsymbol{t}) &= \mathbb{E}(\min(1, t_1 X_1, \dots, t_d X_d)) \\ &= \mathbb{E}\left(\min\left(1, \bigwedge_{k=1}^{d} t_k X_k\right)\right),\ t_1, \dots, t_d > 0,\end{aligned}$$

which, unlike the max-CF, does not require any integrability on the components of $\boldsymbol{X}$, since

$$\forall t_1, \ldots, t_d > 0,\ 0 \leq \min(1, t_1 X_1, \ldots, t_d X_d) \leq 1 \text{ almost surely.}$$

If we require the distribution of $\boldsymbol{X}$ to be concentrated on $(0, \infty)^d$, then it is indeed characterized by its min-linear projections, as our next result shows. We denote throughout by $\mathbb{X}_d$ the set of all rvs $\boldsymbol{X} = (X_1, \ldots, X_d)$ on $\mathbb{R}^d$ with almost surely positive components (i.e. $X_i > 0$ for any i).

Proposition 1.2. *Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$ be two rvs in $\mathbb{X}_d$. Then*

$$\boldsymbol{X} \stackrel{d}{=} \boldsymbol{Y} \Leftrightarrow \forall t_1, \ldots, t_d > 0,\ \bigwedge_{k=1}^{d} t_k X_k \stackrel{d}{=} \bigwedge_{k=1}^{d} t_k Y_k.$$

Proof. Note that, for any $t_1, \ldots, t_d > 0$,

$$\mathbb{P}\left(\bigwedge_{k=1}^{d} t_k X_k > 1\right) = \mathbb{P}\left(X_1 > 1/t_1, \ldots, X_d > 1/t_d\right).$$

By right-continuity of a multivariate df, the knowledge of the distribution of the min-linear projections of $\boldsymbol{X}$ is therefore equivalent to the knowledge of the probabilities $\mathbb{P}(X_1 > x_1, \ldots, X_d > x_d)$, for any $x_1, \ldots, x_d \geq 0$. Since all components of $\boldsymbol{X}$ are assumed to be strictly positive, we obtain that, for any $k \in \{1, \ldots, d\}$, all indices $i_1 < \ldots < i_k$ in $\{1, \ldots, d\}$ and $x_{i_1}, \ldots, x_{i_k} \geq 0$, the probabilities $\mathbb{P}(X_{i_1} > x_{i_1}, \ldots, X_{i_k} > x_{i_k})$ are also determined. The result now follows by writing

$$\mathbb{P}(X_1 \leq x_1, \ldots, X_d \leq x_d) = 1 - \mathbb{P}\left(\bigcup_{k=1}^{d} \{X_k > x_k\}\right)$$

and using the inclusion-exclusion principle.

In measure-theoretic terms, unlike max-linear projections, min-linear projections cannot in general characterize a distribution which puts mass on $[0, \infty)^d \setminus (0, \infty)^d$, because the class of quadrants

$$\left\{\prod_{k=1}^{d} (a_k, \infty),\ a_1, \ldots, a_d \geq 0\right\}$$

is an intersection-stable system of open sets but only generates the Borel σ-algebra on $(0, \infty)^d$. A simple illustrative example is the following: if X has a

Bernoulli distribution with parameter $p \in (0,1)$, then clearly $\min(t_1 X, t_2(1-X)) = 0$ for all $t_1, t_2 > 0$, but $(X, 1-X)$ does not have the same distribution as the degenerate vector $(0,0)$.

The present work builds on Proposition 1.2. The paper is organized as follows: in Section 2 we show that the function

$$\psi_{\boldsymbol{X}} : \boldsymbol{t} = (t_1, \ldots, t_d) \in [0,\infty)^d \mapsto \mathbb{E}(\min(1, t_1 X_1, \ldots, t_d X_d)) \tag{1.1}$$

indeed characterizes the distribution of any rv $\boldsymbol{X} \in \mathbb{X}_d$. Referring to this function as the *min-characteristic function* (min-CF) of $\boldsymbol{X}$, we derive basic properties of the min-CF, including an inversion formula. One of the most intriguing results we find is that the min-CF induces a continuous and concave df. In Section 3.1 we examine the sequential behavior of min-CFs with respect to convergence in distribution, and we consider the asymptotic properties of the empirical min-CF (that is, the random min-CF generated by the empirical df of a sample of independent and identically distributed rvs) in Section 3.2. Some insight into the structure of the set of min-CFs, such as its convexity, is provided in Section 4. Finally, in Section 5, we initiate the study of an extension of the min-CF to arbitrary, not necessarily componentwise positive rvs $\boldsymbol{X}$, by using transformations of the components of $\boldsymbol{X}$.

2 The Min-characteristic Function for Positive Random Vectors

The fundamental result of this paper, stated below, is that the mapping in Eq. 1.1 characterizes the distribution of any rv $\boldsymbol{X} \in \mathbb{X}_d$. Here and throughout, any operation on vectors such as $+, \geq, >, \ldots$ is meant componentwise.

THEOREM 2.1. *Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ and $\boldsymbol{Y} = (Y_1, \ldots, Y_d)$ be two rvs in $\mathbb{X}_d$. Then*

$$\boldsymbol{X} \overset{d}{=} \boldsymbol{Y} \Leftrightarrow \forall \boldsymbol{t} = (t_1, \ldots, t_d) > \boldsymbol{0},\ \psi_{\boldsymbol{X}}(\boldsymbol{t}) = \psi_{\boldsymbol{Y}}(\boldsymbol{t}).$$

PROOF. By 1-homogeneity of the min operator, we have

$$\begin{aligned} \forall t_1, \ldots, t_d > 0,\ \mathbb{E}(\min(1, t_1 X_1, \ldots, t_d X_d)) &= \mathbb{E}(\min(1, t_1 Y_1, \ldots, t_d Y_d)) \\ \Leftrightarrow \forall x, t_1, \ldots, t_d > 0,\ \mathbb{E}(\min(x, t_1 X_1, \ldots, t_d X_d)) &= \mathbb{E}(\min(x, t_1 Y_1, \ldots, t_d Y_d)). \end{aligned}$$

Now, for any positive rv Z and any $x > 0$,

$$\mathbb{E}(\min(x, Z)) = \int_0^\infty \mathbb{P}(\min(x, Z) > u)\, du = \int_0^x \mathbb{P}(Z > u)\, du.$$

Differentiating from the right with respect to x entails that the knowledge of $\mathbb{E}(\min(x, Z))$, for any $x > 0$, entails that of $\mathbb{P}(Z > x)$ for any $x > 0$ and thus of the distribution of the positive rv Z. Applying this to the rv $Z = \min(t_1 X_1, \ldots, t_d X_d)$ for arbitrary $t_1, \ldots, t_d > 0$ and using Proposition 1.2 concludes the proof.

DEFINITION 2.2. *The min-characteristic function (min-CF) of* $\boldsymbol{X} = (X_1, \ldots, X_d) \in \mathbb{X}_d$ *is the function* $\psi_{\boldsymbol{X}}$ *on* $[0, \infty)^d$ *defined by*

$$\forall \boldsymbol{t} = (t_1, \ldots, t_d) \in [0, \infty)^d, \ \psi_{\boldsymbol{X}}(\boldsymbol{t}) = \mathbb{E}(\min(1, t_1 X_1, \ldots, t_d X_d)).$$

One immediate benefit of using the min-CF rather than the max-CF is that it does not require any integrability assumption on the components of $\boldsymbol{X}$. At the same time, it can be calculated in much the same way and thus generally applies to the same kind of distributions the max-CF is well-suited to, thanks to the following basic formula.

LEMMA 2.3. *We have, for* $\boldsymbol{X} = (X_1, \ldots, X_d) \in \mathbb{X}_d$*, the identity*

$$\psi_{\boldsymbol{X}}(\boldsymbol{t}) = \int_0^1 \mathbb{P}(X_1 > u/t_1, \ldots, X_d > u/t_d)\, du, \qquad \boldsymbol{t} = (t_1, \ldots, t_d) \in (0, \infty)^d.$$

This formula is indeed similar in spirit to the identity

$$\mathbb{E}(\max(1, t_1 X_1, \ldots, t_d X_d)) = 1 + \int_1^\infty [1 - \mathbb{P}(X_1 \le u/t_1, \ldots, X_d \le u/t_d)]\, du$$

making it possible to calculate the max-CF of a componentwise nonnegative and integrable rv (see Falk and Stupfler 2017).

PROOF. Use the identity

$$\mathbb{E}(\min(1, Z)) = \int_0^1 \mathbb{P}(Z > u)\, du$$

valid for any positive rv Z, with $Z = \min(t_1 X_1, \ldots, t_d X_d)$.

We give a short list of examples next.

EXAMPLE 2.1 (Exponential distribution). *The min-CF of a rv* X *having the exponential distribution with mean* $1/\lambda$*,* $\lambda > 0$*, is given by*

$$\forall t > 0, \ \psi_\lambda(t) = \int_0^1 e^{-\lambda u/t}\, du = \frac{t}{\lambda}\left[1 - e^{-\lambda/t}\right].$$

Example 2.2 (Pareto distribution). *The min-CF of a rv X having the Pareto distribution with tail index $\gamma > 0$, namely, with df $\mathbb{P}(X \le x) = 1 - x^{-1/\gamma}$, $x \ge 1$, is given by*

$$\forall t > 0,\ \psi_\gamma(t) = \int_0^1 \left[\mathbb{1}_{\{u \le t\}} + \left(\frac{u}{t}\right)^{-1/\gamma} \mathbb{1}_{\{u > t\}} \right] du.$$

Consequently,

$$\psi_1(t) = \begin{cases} 1 & \text{if } t \ge 1, \\ t(1 - \log t) & \text{if } t \in (0,1) \end{cases} \quad \text{and}$$

$$\forall \gamma \ne 1,\ \psi_\gamma(t) = \begin{cases} 1 & \text{if } t \ge 1, \\ \dfrac{t - \gamma t^{1/\gamma}}{1 - \gamma} & \text{if } t \in (0,1). \end{cases}$$

Example 2.3 (Generalized Pareto distribution). *The min-CF of a rv X having the Generalized Pareto distribution with location parameter $\mu \ge 0$, scale parameter $\sigma > 0$ and tail index $\xi > 0$, namely, with df*

$$\mathbb{P}(X \le x) = 1 - \left(1 + \xi \frac{x - \mu}{\sigma}\right)^{-1/\xi},\ x \ge \mu,$$

is given by

$$\forall t > 0,\ \psi_{(\mu,\sigma,\xi)}(t) = \int_0^1 \left[\mathbb{1}_{\{u \le t\mu\}} + \left(1 + \frac{\xi}{\sigma}\left[\frac{u}{t} - \mu\right]\right)^{-1/\xi} \mathbb{1}_{\{u > t\mu\}} \right] du.$$

Consequently,

$$\psi_{(\mu,\sigma,1)}(t) = \begin{cases} 1 & \text{if } t \ge 1/\mu, \\ t\left(\mu + \sigma \log\left[1 + \dfrac{1 - \mu t}{\sigma t}\right]\right) & \text{if } t < 1/\mu, \end{cases}$$

and for any $\xi \ne 1$,

$$\psi_{(\mu,\sigma,\xi)}(t) = \begin{cases} 1 & \text{if } t \ge 1/\mu, \\ t\left(\mu + \dfrac{\sigma}{1-\xi} - \dfrac{\sigma}{1-\xi}\left[1 + \xi\dfrac{1 - \mu t}{\sigma t}\right]^{1-1/\xi}\right) & \text{if } t < 1/\mu. \end{cases}$$

This is readily seen to agree with the max-CF calculation in Example 1.3 of Falk and Stupfler (2017)*, for the case $\xi \in (0,1)$, thanks to the identity*

$$\psi_{(\mu,\sigma,\xi)}(t) = \mathbb{E}(\min(1, tX)) = 1 + t\mathbb{E}(X) - \mathbb{E}(\max(1, tX))$$

valid in this case where $\mathbb{E}(X) = \mu + \sigma/(1-\xi) < \infty$.

EXAMPLE 2.4 (Unit Fréchet distribution). *The min-CF of a rv X having a unit Fréchet distribution, namely, with df* $\mathbb{P}(X \le x) = e^{-1/x}$, $x > 0$, *is given by*

$$\forall t > 0,\ \psi(t) = \int_0^1 \left[1 - e^{-t/u}\right] du = 1 - \int_1^\infty \frac{e^{-tv}}{v^2} dv =: 1 - E_2(t)$$

in the notation of Abramovitz and Stegun (1972, Formula 5.1.4 p.228).

EXAMPLE 2.5 (Independent unit Fréchet variables). *The min-CF of a rv $\boldsymbol{X} = (X_1, \ldots, X_d)$, whose components are independent unit Fréchet distributed, is*

$$\begin{aligned}\psi(\boldsymbol{t}) &= \int_0^1 \prod_{k=1}^d \left[1 - e^{-t_k/u}\right] du \\ &= 1 - \int_0^1 \sum_{k=1}^d (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le d} \exp\left(-\frac{1}{u}[t_{i_1} + \cdots + t_{i_k}]\right) du \\ &= 1 - \sum_{k=1}^d (-1)^{k-1} \sum_{1 \le i_1 < \cdots < i_k \le d} E_2(t_{i_1} + \cdots + t_{i_k}),\end{aligned}$$

for any $\boldsymbol{t} = (t_1, \ldots, t_d) \in (0, \infty)^d$, with the notation of Example 2.4.

It is already apparent from the definition of a min-CF that it is a componentwise nondecreasing function on $[0, \infty)^d$. A further list of elementary properties of the min-CF is given in our next result.

PROPOSITION 2.4. *Choose $\boldsymbol{X} = (X_1, \ldots, X_d) \in \mathbb{X}_d$ and let $\psi_{\boldsymbol{X}}$ be its min-CF. We have, for $\boldsymbol{t} = (t_1, \ldots, t_d) \in [0, \infty)^d$:*

(i) $0 \le \psi_{\boldsymbol{X}}(\boldsymbol{t}) \le 1$.

(ii) $\psi_{\boldsymbol{X}}(\boldsymbol{t}) = 0$ *if and only if* $t_i = 0$ *for some* $i \in \{1, \ldots, d\}$.

(iii) $\psi_{\boldsymbol{X}}(\boldsymbol{t}) \to 1$ *as* $\min(t_1, \ldots, t_d) \to \infty$.

(iv) $\boldsymbol{X} \ge \boldsymbol{c} > \boldsymbol{0}$ *almost surely if and only if* $\psi_{\boldsymbol{X}}(\boldsymbol{t}) = 1$ *for* $\boldsymbol{t} \ge 1/\boldsymbol{c}$.

(v) If $\boldsymbol{Y}$ is another rv in $\mathbb{X}_d$, then

$$\sup_{\boldsymbol{t} \ge \boldsymbol{0}} |\psi_{\boldsymbol{X}}(\boldsymbol{t}) - \psi_{\boldsymbol{Y}}(\boldsymbol{t})| \le \sup_{\boldsymbol{t} \ge \boldsymbol{0}} |\mathbb{P}(\boldsymbol{X} > \boldsymbol{t}) - \mathbb{P}(\boldsymbol{Y} > \boldsymbol{t})|.$$

(vi) The function $\psi_{\boldsymbol{X}}$ is a continuous and concave df on $[0, \infty)^d$.

(vii) The function $\boldsymbol{t} \mapsto \psi_{\boldsymbol{X}}(\mathbf{1}/\boldsymbol{t})$, $\boldsymbol{t} > \mathbf{0} \in \mathbb{R}^d$, is a (continuous) survival function, in the sense that there exists a rv $\boldsymbol{Y} \in \mathbb{X}_d$ with $\psi_{\boldsymbol{X}}(\mathbf{1}/\boldsymbol{t}) = \mathbb{P}(\boldsymbol{Y} > \boldsymbol{t})$.

The most interesting result here is probably Proposition 2.4(vi): combined with Theorem 2.1, it shows that any df on $\mathbb{X}_d$, no matter how irregular, is characterized by an associated continuous and concave df, which is its min-CF.

Proof. Assertions (i)–(iv) are elementary. Assertion (v) is a straightforward consequence of Lemma 2.3. We prove (vi). The function $\psi_{\boldsymbol{X}}$ is clearly continuous. Its concavity follows from that of the function $(x_1, \ldots, x_d) \mapsto \min(1, x_1, \ldots, x_d)$ on $[0, \infty)^d$. It only remains to prove that $\psi_{\boldsymbol{X}}$ is a df. Since $\psi_{\boldsymbol{X}}(\mathbf{0}) = 0$ and $\psi_{\boldsymbol{X}}(\boldsymbol{t}) \to 1$ as $\min(t_1, \ldots, t_d) \to \infty$, it is sufficient to prove that $\psi_{\boldsymbol{X}}$ is Δ-monotone (see Reiss 1989, Equation (2.2.19)), i.e., for any $\mathbf{0} \le \boldsymbol{a} \le \boldsymbol{b} \in \mathbb{R}^d$,

$$\Delta_{\boldsymbol{a}}^{\boldsymbol{b}} \psi_{\boldsymbol{X}} := \sum_{T \subset \{1,\ldots,d\}} (-1)^{d-|T|} \psi_{\boldsymbol{X}} \left(b_1^{\mathbb{1}\{1 \in T\}} a_1^{\mathbb{1}\{1 \notin T\}}, \ldots, b_d^{\mathbb{1}\{d \in T\}} a_d^{\mathbb{1}\{d \notin T\}} \right)$$
$$= \mathbb{E} \left(\sum_{T \subset \{1,\ldots,d\}} (-1)^{d-|T|} \min\{1;\, b_i X_i, i \in T;\, a_j X_j, j \notin T\} \right) \ge 0.$$

To show this it suffices to establish that the integrand in the above expectation is always nonnegative. Let U be a random variable which follows the uniform distribution on $[0,1]$. Using repeatedly the identity

$$\mathbb{P}(U \in (s,t], U \le u) = \mathbb{P}(U \le t, U \le u) - \mathbb{P}(U \le s, U \le u)$$

valid for any $s \le t$ and u, we find that for any $\mathbf{0} \le \boldsymbol{c} \le \boldsymbol{d} \in \mathbb{R}^d$, we have,

$$\begin{aligned} 0 &\le \mathbb{P}(U \in (\min(1, c_i), \min(1, d_i)],\ 1 \le i \le d) \\ &= \sum_{T \subset \{1,\ldots,d\}} (-1)^{d-|T|} \mathbb{P}\left(U \le \min(1, d_i), i \in T;\, U \le \min(1, c_j), j \notin T\right) \\ &= \sum_{T \subset \{1,\ldots,d\}} (-1)^{d-|T|} \min\{1; d_i, i \in T;\, c_j, j \notin T\}. \end{aligned}$$

With $\boldsymbol{c} = (a_1 X_1, \ldots, a_d X_d)$ and $\boldsymbol{d} = (b_1 X_1, \ldots, b_d X_d)$ this yields the desired inequality. Finally, part (vii) is an immediate consequence of (vi): $\psi_{\boldsymbol{X}}$ is a (continuous) df of some rv $\boldsymbol{Z} \in \mathbb{X}_d$, i.e. $\psi_{\boldsymbol{X}}(\boldsymbol{t}) = \mathbb{P}(\boldsymbol{Z} \le \boldsymbol{t})$, $\boldsymbol{t} > \mathbf{0} \in \mathbb{R}^d$. Then $\psi_{\boldsymbol{X}}(\mathbf{1}/\boldsymbol{t}) = \mathbb{P}(\boldsymbol{Z} \le \mathbf{1}/\boldsymbol{t}) = \mathbb{P}(\boldsymbol{Y} \ge \boldsymbol{t}) = \mathbb{P}(\boldsymbol{Y} > \boldsymbol{t})$, with $\boldsymbol{Y} := \mathbf{1}/\boldsymbol{Z}$.

Of course, since the min-CF identifies distributions concentrated in the positive orthant of $\mathbb{R}^d$, it is important to find the inversion formula making it possible to go from a min-CF to its pertaining distribution. Since, by Lemma 2.3, computing the min-CF essentially consists in integrating the survival function, it makes sense to expect that a survival function can be recovered by differentiating the pertaining min-CF in a suitable way. Making this intuition rigorous is the focus of the next result.

THEOREM 2.5. *For* $\boldsymbol{X} = (X_1, \dots, X_d) \in \mathbb{X}_d$ *with min-CF* $\psi_{\boldsymbol{X}}$*, we have*

$$\begin{aligned} \forall \boldsymbol{x} &= (x_1, \dots, x_d) \in (0,\infty)^d, \\ \mathbb{P}(X_1 > x_1, \dots, X_d > x_d) &= \frac{\partial_+}{\partial t}\left\{ t\psi_{\boldsymbol{X}}\left(\frac{1}{t\boldsymbol{x}}\right)\right\}\Big|_{t=1} \end{aligned}$$

where $\partial_+/\partial t$ *denotes differentiation from the right with respect to* t*.*

PROOF. By Lemma 2.3, we find, for any $t > 0$ and $(x_1, \dots, x_d) \in (0,\infty)^d$,

$$\begin{aligned} t\psi_{\boldsymbol{X}}\left(\frac{1}{t\boldsymbol{x}}\right) &= t\int_0^1 \mathbb{P}(X_1 > utx_1, \dots, X_d > utx_d)\, du \\ &= \int_0^t \mathbb{P}(X_1 > vx_1, \dots, X_d > vx_d)\, dv. \end{aligned}$$

Conclude by differentiating from the right with respect to t and taking $t = 1$.

It should be apparent from this result that, while a max-CF is adapted to working with the joint df (see Falk and Stupfler 2017, Proposition 2.15), the min-CF is rather adapted to working with the joint *survival function*. The next example illustrates this point nicely.

EXAMPLE 2.6 (Exponential distribution in several dimensions). *Let* $\boldsymbol{X}$ *be a rv in* $\mathbb{R}^d$ *which follows a min-stable distribution with standard exponential margins* $\mathbb{P}(X_i > x) = \exp(-x)$*,* $x \geq 0$*. This is equivalent to assuming that there exists a D-norm* $||\cdot||_D$ *on* $\mathbb{R}^d$ *such that*

$$\mathbb{P}(\boldsymbol{X} > \boldsymbol{x}) = \exp\left(-\|\boldsymbol{x}\|_D\right), \ \boldsymbol{x} \geq \boldsymbol{0} \in \mathbb{R}^d;$$

see Falk (2019, Equation (2.27))*. Then we have, for* $\boldsymbol{t} > \boldsymbol{0} \in \mathbb{R}^d$*,*

$$\begin{aligned} \psi_{\boldsymbol{X}}(\boldsymbol{t}) = \int_0^1 \mathbb{P}\left(\boldsymbol{X} > \frac{u}{\boldsymbol{t}}\right)\, du &= \int_0^1 \exp\left(-u\left\|\frac{1}{\boldsymbol{x}}\right\|_D\right)\, du \\ &= \frac{1}{\|1/\boldsymbol{x}\|_D}\left[1 - \exp\left(-\left\|\frac{1}{\boldsymbol{x}}\right\|_D\right)\right]. \end{aligned}$$

This generalizes Example 2.1 to an arbitrary dimension $d \geq 1$*.*

We may now provide an application of our results to the theory of $D-$norms. Recall that a D-norm on $\mathbb{R}^d$, $d \geq 2$, is a norm of the form

$$\|\boldsymbol{x}\|_D := \mathbb{E}(\max(|x_1|\, Z_1, \ldots, |x_d|\, Z_d))$$

where $\boldsymbol{Z} = (Z_1, \ldots, Z_d)$ is a componentwise nonnegative rv such that $E(Z_i) = 1$, $1 \leq i \leq d$, called the generator of $\|\cdot\|_D$. The concept of D-norms has come to prominence recently for its importance in multivariate extreme value theory, not least because it allows for a simple characterization of max-stable dfs (see Theorem 2.3.3 in Falk 2019). Attached to a D-norm $||\cdot||_D$ is the concept of dual D-norm function

$$\wr\, \boldsymbol{x} \,\wr_D := \mathbb{E}(\min(|x_1|\, Z_1, \ldots, |x_d|\, Z_d))$$

which has recently found applications in the analysis of multivariate records (Dombry et al., 2019; Dombry and Zott, 2018). It is known that the mapping

$$\|\cdot\|_D \mapsto \wr \cdot \wr_D$$

is indeed well-defined, in the sense that two generators $\boldsymbol{Z}$ of the same D-norm also generate the same dual D-norm function, but this mapping is not one-to-one (see Section 1.6 of Falk 2019). The next result shows that if we actually restrict this mapping to componentwise positive generators $\boldsymbol{Z}$, it becomes one-to-one.

Proposition 2.6. *Let $\boldsymbol{Z}^{(1)}, \boldsymbol{Z}^{(2)}$ be componentwise positive generators of two D-norms $\|\cdot\|_{D_1}$ and $\|\cdot\|_{D_2}$. Then*

$$\|\cdot\|_{D_1} = \|\cdot\|_{D_2} \Leftrightarrow \wr \cdot \wr_{D_1} = \wr \cdot \wr_{D_2}.$$

The proof rests on the following lemma.

Lemma 2.7. *Any D-norm with a generator $\boldsymbol{Z} \in \mathbb{X}_d$ also has a generator $\boldsymbol{Z}^* \in \mathbb{X}_d$ with $Z_1^* = 1$.*

Proof. That there is a generator $\boldsymbol{Z}^*$ with $Z_1^* = 1$ follows from Lemma 2.10 in Falk and Stupfler (2019). We need only show that $\boldsymbol{Z}^* \in \mathbb{X}_d$, translating into $\mathbb{P}(Z_i^* > 0) = 1$ for any $i \in \{2, \ldots, d\}$. For any $x > 0$, $\mathbb{E}(\max(Z_1, xZ_i)) = \mathbb{E}(\max(1, xZ_i^*))$, and thus $\mathbb{E}(\min(Z_1, xZ_i)) = \mathbb{E}(\min(1, xZ_i^*))$ by the identity $\max(a, b) + \min(a, b) = a + b$ and the fact that all the Z_j and Z_j^* have expectation 1. Letting $x \uparrow \infty$ and using the dominated convergence theorem entails $1 = \mathbb{P}(Z_i^* > 0)$, as required.

Proof of Proposition 2.6. We only need to show that the equality of the dual D-norm functions implies that of the original D-norms. By

Lemma 2.7, we may assume that the first element of each generator is equal to 1: in particular,

$$\mathbb{E}(\min(|x_1|, |x_2| Z_2^{(1)}, \dots, |x_d| Z_d^{(1)})) = \mathbb{E}(\min(|x_1|, |x_2| Z_2^{(2)}, \dots, |x_d| Z_d^{(2)}))$$

for any $\boldsymbol{x} \in \mathbb{R}^d$. The random vectors $(Z_2^{(1)}, \dots, Z_d^{(1)})$ and $(Z_2^{(2)}, \dots, Z_d^{(2)})$ then have the same distribution, by Theorem 2.1. The result follows.

We conclude this section by discussing an interesting example of interplay between max-stability, min-stability and the notion of min-CF. Recall that a copula C is said to be in the domain of attraction of a standard max-stable df G if

$$\lim_{n\to\infty} C^n\left(1 + \frac{\boldsymbol{x}}{n}\right) = G(\boldsymbol{x}),\ \boldsymbol{x} \in \mathbb{R}^d.$$

In this context, it is a consequence of Falk (2019, Theorem 2.3.3) that $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$, $\boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$, for some D-norm $||\cdot||_D$ which, in this case, describes the extremal dependence within the copula C. We then have the following result.

PROPOSITION 2.8. *Let $\boldsymbol{X} = (X_1, \dots, X_d)$ be a rv that follows a copula C in the domain of attraction of the standard max-stable df $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$, $\boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$. Let (by Proposition 2.4(vii)) $\boldsymbol{Y} \in \mathbb{X}_d$ be a rv with survival function $\boldsymbol{t} \mapsto \psi_{-\log \boldsymbol{X}}(\boldsymbol{1}/\boldsymbol{t})$, $\boldsymbol{t} > \boldsymbol{0} \in \mathbb{R}^d$. Then $\boldsymbol{Y}$ is asymptotically min-stable, in the sense that*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{n}{2} \min_{1\leq i\leq n} \boldsymbol{Y}^{(i)} > \boldsymbol{x}\right) = \exp(-\|\boldsymbol{x}\|_D),\ \boldsymbol{x} > \boldsymbol{0} \in \mathbb{R}^d,$$

where $\boldsymbol{Y}^{(1)}, \boldsymbol{Y}^{(2)}, \dots$ are independent copies of $\boldsymbol{Y}$.

PROOF. The domain of attraction assumption on C is equivalent with the expansion

$$C(\boldsymbol{u}) = 1 - \|\boldsymbol{1} - \boldsymbol{u}\|_D + \mathrm{o}(\|\boldsymbol{1} - \boldsymbol{u}\|) \tag{2.1}$$

as $\boldsymbol{u} \to \boldsymbol{1}$, uniformly for $\boldsymbol{u} \in [0,1]$ (see Proposition 3.1.5 in Falk 2019), in the sense that

$$\forall \varepsilon > 0,\ \exists \delta > 0,\ \boldsymbol{u} \in [1-\delta, 1]^d \Rightarrow \frac{C(\boldsymbol{u}) - (1 - \|\boldsymbol{1} - \boldsymbol{u}\|_D)}{\|\boldsymbol{1} - \boldsymbol{u}\|} \leq \varepsilon.$$

Note then that, from Lemma 2.3,

$$\psi_{-\log \boldsymbol{X}}\left(\frac{\boldsymbol{1}}{s\boldsymbol{x}}\right) = \int_0^1 C(\exp(-st\boldsymbol{x}))\, dt$$

and thus, combining a Taylor expansion of the exponential function around 0 and Eq. 2.1, the min-CF of $-\log(\boldsymbol{X})$ satisfies, for $\boldsymbol{x} > \boldsymbol{0} \in \mathbb{R}^d$,

$$\lim_{s\downarrow 0} \frac{2}{s}\left(1 - \psi_{-\log \boldsymbol{X}}\left(\frac{\boldsymbol{1}}{s\boldsymbol{x}}\right)\right) = \|\boldsymbol{x}\|_D . \tag{2.2}$$

In other words, since $\boldsymbol{Y} \in \mathbb{X}_d$ has survival function $\boldsymbol{t} \mapsto \psi_{-\log \boldsymbol{X}}(\boldsymbol{1}/\boldsymbol{t})$, $\boldsymbol{t} > \boldsymbol{0} \in \mathbb{R}^d$, we have

$$\mathbb{P}\left(\frac{1}{2}\boldsymbol{Y} > s\boldsymbol{x}\right) = 1 - s\|\boldsymbol{x}\|_D + \mathrm{o}(s)$$

as $s \downarrow 0$ for $\boldsymbol{x} > \boldsymbol{0} \in \mathbb{R}^d$. For independent copies $\boldsymbol{Y}^{(1)}, \boldsymbol{Y}^{(2)}, \ldots$ of $\boldsymbol{Y}$, this yields

$$\mathbb{P}\left(\frac{n}{2}\min_{1\le i\le n} \boldsymbol{Y}^{(i)} > \boldsymbol{x}\right) = \left[\mathbb{P}\left(\frac{1}{2}\boldsymbol{Y} > \frac{1}{n}\boldsymbol{x}\right)\right]^n \to \exp(-\|\boldsymbol{x}\|_D),\ \boldsymbol{x} > \boldsymbol{0} \in \mathbb{R}^d,$$

completing the proof.

We highlight the following consequence of Proposition 2.8, which follows from Eqs. 2.1 and 2.2 in its proof. It can be used to suggest estimators of a D-norm as done in Example 3.1 below.

PROPOSITION 2.9. *Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ follow a copula C. If C is in the domain of attraction of a standard max-stable df $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D)$, $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$, then, for all $\boldsymbol{x} \ge \boldsymbol{0} \in \mathbb{R}^d$, the limit*

$$\ell(\boldsymbol{x}) := \lim_{s\downarrow 0} \frac{2}{s}\left(1 - \mathbb{E}\left(\min\left(1, \frac{-\log(X_1)}{sx_1}, \ldots, \frac{-\log(X_d)}{sx_d}\right)\right)\right)$$

exists, and $\ell(\boldsymbol{x}) = \|\boldsymbol{x}\|_D$.

When considering notions of characteristic functions, such as the Fourier transform, the Laplace transform, the moment-generating function, or the max-CF, it is important to examine the connection between convergence of a sequence of characteristic functions and convergence in distribution of the associated rvs. This is the focus of the next section.

3 Sequential Behavior of the Min-characteristic Function

3.1. With Respect to Convergence in Distribution An important result regarding the max-CF is that the pointwise convergence of a sequence of max-CFs to a max-CF is equivalent to the convergence of the pertaining distributions in the Wasserstein metric

$$d_W(P,Q) := \inf\{\mathbb{E}(\|\boldsymbol{X}-\boldsymbol{Y}\|_1)\colon \boldsymbol{X} \text{ has distribution } P,\ \boldsymbol{Y} \text{ has distribution } Q\}.$$

Convergence in this metric is nothing but convergence in distribution plus convergence of first moments, according to Villani (2009, Definition 6.8 and Theorem 6.9). Of course, the use of min-CFs does not require any integrability assumption, so one cannot hope that a similar theorem would link pointwise convergence of min-CFs to convergence in the metric d_W, but we could still anticipate a convergence in distribution of the pertaining dfs. This is precisely the content of our next result.

THEOREM 3.1. *Let $\boldsymbol{X}^{(n)}$, $\boldsymbol{X}$ be rvs in $\mathbb{X}_d$. Then*

$$\boldsymbol{X}^{(n)} \xrightarrow{d} \boldsymbol{X} \Leftrightarrow \psi_{\boldsymbol{X}^{(n)}} \to \psi_{\boldsymbol{X}} \quad \textit{pointwise.}$$

PROOF OF THEOREM 3.1. Suppose that $\boldsymbol{X}^{(n)} \overset{d}{\to} \boldsymbol{X}$. For any $\boldsymbol{t} = (t_1,\dots,t_d) \in (0,\infty)^d$, the function h on $\mathbb{R}^d$ defined by

$$h(x_1,\dots,x_d) := \min(1, t_1x_1,\dots,t_dx_d) \text{ if } x_1,\dots,x_d > 0 \text{ and } 0 \text{ otherwise}$$

is continuous and bounded. Consequently

$$\psi_{\boldsymbol{X}^{(n)}}(\boldsymbol{t}) = \mathbb{E}(h(\boldsymbol{X}^{(n)})) \to \mathbb{E}(h(\boldsymbol{X})) = \psi_{\boldsymbol{X}}(\boldsymbol{t})$$

as required. Suppose conversely that $\psi_{\boldsymbol{X}^{(n)}} \to \psi_{\boldsymbol{X}}$ pointwise. We show that $-\boldsymbol{X}^{(n)} \xrightarrow{d} -\boldsymbol{X}$, or equivalently that

$$G^{(n)}(\boldsymbol{x}) := \mathbb{P}(-\boldsymbol{X}^{(n)} \le \boldsymbol{x}) \to \mathbb{P}(-\boldsymbol{X} \le \boldsymbol{x}) =: G(\boldsymbol{x})$$

at every point of continuity $\boldsymbol{x} \le \boldsymbol{0}$ of G. Let

$$\overline{G}^{(n)}(\boldsymbol{x}) := \mathbb{P}(\boldsymbol{X}^{(n)} \ge \boldsymbol{x}) \text{ and } \overline{G}(\boldsymbol{x}) := \mathbb{P}(\boldsymbol{X} \ge \boldsymbol{x})$$

so that $\overline{G}^{(n)}(\boldsymbol{x}) = G^{(n)}(-\boldsymbol{x})$ and $\overline{G}(\boldsymbol{x}) = G(-\boldsymbol{x})$. From the proof of Theorem 2.5 we know that, for any $\boldsymbol{x} > \boldsymbol{0}$ and $s,t > 0$,

$$t\psi_{\boldsymbol{X}^{(n)}}\left(\frac{1}{t\boldsymbol{x}}\right) - s\psi_{\boldsymbol{X}^{(n)}}\left(\frac{1}{s\boldsymbol{x}}\right) = \int_s^t \mathbb{P}(X_1^{(n)} > vx_1,\dots,X_d^{(n)} > vx_d)\,dv.$$

Using the fact that the distributions of $X_1^{(n)}, \ldots, X_d^{(n)}$ have at most countably many atoms, we get

$$\begin{aligned}\int_s^t \overline{G}^{(n)}(v\boldsymbol{x})\,dv &= t\psi_{\boldsymbol{X}^{(n)}}\left(\frac{1}{t\boldsymbol{x}}\right) - s\psi_{\boldsymbol{X}^{(n)}}\left(\frac{1}{s\boldsymbol{x}}\right) \\ &\to t\psi_{\boldsymbol{X}}\left(\frac{1}{t\boldsymbol{x}}\right) - s\psi_{\boldsymbol{X}}\left(\frac{1}{s\boldsymbol{x}}\right) \\ &= \int_s^t \overline{G}(v\boldsymbol{x})\,dv. \end{aligned} \tag{3.1}$$

Let $\boldsymbol{x} > \boldsymbol{0}$ be a point of continuity of $\overline{G}$. If

$$\limsup_{n\to\infty} \overline{G}^{(n)}(\boldsymbol{x}) > \overline{G}(\boldsymbol{x}) \quad \text{or} \quad \liminf_{n\to\infty} \overline{G}^{(n)}(\boldsymbol{x}) < \overline{G}(\boldsymbol{x})$$

then, by exploiting the monotonicity properties of $\overline{G}^{(n)}$ and the continuity of $\overline{G}$ at $\boldsymbol{x}$, Eq. 3.1 readily produces a contradiction by putting $s = 1$ and $t = 1+\varepsilon$ or $t = 1$ and $s = 1-\varepsilon$ with a small $\varepsilon > 0$. This gives $\overline{G}^{(n)}(\boldsymbol{x}) \to \overline{G}(\boldsymbol{x})$ at any point of continuity $\boldsymbol{x} > \boldsymbol{0}$ of $\overline{G}$, or equivalently

$$G^{(n)}(\boldsymbol{x}) \to G(\boldsymbol{x}) \tag{3.2}$$

at every point of continuity $\boldsymbol{x} < \boldsymbol{0}$ of G. To show that this convergence also holds at the points of continuity $\boldsymbol{x}$ of G with one or several components equal to zero, we fix one such point, and we note that it is enough to prove that every subsequence $G^{(m(n))}(\boldsymbol{x})$ of $G^{(n)}(\boldsymbol{x})$ has itself got a subsequence that converges to $G(\boldsymbol{x})$ (a result known as Cantor's lemma). From Helly's selection theorem, we can take a subsequence $G^{(k(m(n)))}$ of the sequence $G^{(m(n))}$ which converges to some finite measure-generating function G^* on $\mathbb{R}^d$, at all points of continuity of G^*: in other words, we can find a measure μ^* on $\mathbb{R}^d$ with $G^{(k(m(n)))}(\boldsymbol{t}) \to G^*(\boldsymbol{t}) := \mu^*((-\boldsymbol{\infty}, \boldsymbol{t}])$ at every point of continuity $\boldsymbol{t}$ of the limit.

We claim that actually $G^* = G$ on $(-\infty, 0]^d$ irrespective of the choice of the subsequence, which will obviously imply $G^{(k(m(n)))}(\boldsymbol{x}) \to G^*(\boldsymbol{x}) = G(\boldsymbol{x})$ as required. We prove this claim as follows. Clearly $G^* = G$ on $(-\infty, 0)^d$ wherever G and G^* are both continuous, by Eq. 3.2. The set of such points is dense in $(-\infty, 0)^d$, since G and G^* are finite measure-generating functions. Right-continuity of G and G^* then implies that $G^* = G$

everywhere on $(-\infty,0)^d$. Besides, the monotonicity of G^* together with the fact that $\mathbb{P}(-\boldsymbol{X} < \mathbf{0}) = 1$ implies that

$$\begin{aligned} 1 = G(\mathbf{0}) = \lim_{\boldsymbol{\varepsilon}\downarrow\mathbf{0}} G(\mathbf{0}-\boldsymbol{\varepsilon}) = \lim_{\boldsymbol{\varepsilon}\downarrow\mathbf{0}} G^*(\mathbf{0}-\boldsymbol{\varepsilon}) &= \mu^*((-\infty,0)^d) \\ &\leq \mu^*((-\infty,0]^d) = G^*(\mathbf{0}) \leq 1. \end{aligned}$$

Thus each of the $E_j := \{\boldsymbol{y} \in (-\infty,0]^d \,|\, y_j = 0\}$ satisfies $\mu^*(E_j) = 0$. Conclude by letting T be the set of indices for which $x_i < 0$ and by writing

$$\begin{aligned} G^*(\boldsymbol{x}) = \mu^*((-\boldsymbol{\infty},\boldsymbol{x}]) &= \mu^*(\{\boldsymbol{y} \in \mathbb{R}^d \,|\, y_i \leq x_i,\, i \in T,\, y_j \leq 0,\, j \notin T\}) \\ &= \mu^*(\{\boldsymbol{y} \in \mathbb{R}^d \,|\, y_i \leq x_i,\, i \in T,\, y_j < 0,\, j \notin T\}) \\ &= \lim_{\varepsilon\downarrow 0} \mu^*(\{\boldsymbol{y} \in \mathbb{R}^d \,|\, y_i \leq x_i,\, i \in T,\, y_j \leq -\varepsilon,\, j \notin T\}) \\ &= \lim_{\varepsilon\downarrow 0} \mathbb{P}(X_i \leq x_i,\, i \in T,\, X_j \leq -\varepsilon,\, j \notin T) \\ &= \mathbb{P}(X_i \leq x_i,\, i \in T,\, X_j < 0,\, j \notin T) \\ &= \mathbb{P}(X_i \leq x_i,\, i \in T,\, X_j \leq 0,\, j \notin T) = G(\boldsymbol{x}). \end{aligned}$$

Complete the proof by noting that $G(\mathbf{0}) = 1 = G^*(\mathbf{0})$ and thus $G^* = G$.

REMARK 3.1. The pointwise convergence in Theorem 3.1 can be strengthened to uniform convergence on $[0,\infty)^d$. Note that $\psi_{\boldsymbol{X}^{(n)}}$, $n \in \mathbb{N}$, is a sequence of df, which converges pointwise to the continuous df $\psi_{\boldsymbol{X}}$. But this means weak convergence of a sequence of rvs $\boldsymbol{Y}^{(n)}$, having df $\psi_{\boldsymbol{X}^{(n)}}$, to a rv $\boldsymbol{Y}$ having df $\psi_{\boldsymbol{X}}$. Since the limiting df $\psi_{\boldsymbol{X}}$ is continuous, this implies uniform convergence of $\psi_{\boldsymbol{X}^{(n)}}$ to $\psi_{\boldsymbol{X}}$, see, e.g., Billingsley (1968, Problem 3, Section 3). We thus have

$$\boldsymbol{X}^{(n)} \xrightarrow{d} \boldsymbol{X} \Leftrightarrow \sup_{\boldsymbol{t}\geq\mathbf{0}} \left|\psi_{\boldsymbol{X}^{(n)}}(\boldsymbol{t}) - \psi_{\boldsymbol{X}}(\boldsymbol{t})\right| \to 0.$$

3.2. The Empirical min-CF The fact that the min-CF identifies convergence in distribution suggests that it may also be used in estimation settings. We briefly explore this context here from the asymptotic point of view. Let $\boldsymbol{X}^{(1)}, \ldots, \boldsymbol{X}^{(n)}$ be independent copies of a rv $\boldsymbol{X} \in \mathbb{X}_d$. The (random) min-CF induced by the empirical measure $\widehat{P}_n := n^{-1}\sum_{i=1}^n \delta_{\boldsymbol{X}^{(i)}}$ is

$$\widehat{\psi}^{(n)}_{\boldsymbol{X}}(\boldsymbol{t}) = \frac{1}{n}\sum_{i=1}^n \min\left(1, t_1 X_1^{(i)}, \ldots, t_d X_d^{(i)}\right).$$

By the law of large numbers, we have, for any $\boldsymbol{t} = (t_1,\ldots,t_d) \in [0,\infty)^d$, that almost surely:

$$\widehat{\psi}^{(n)}_{\boldsymbol{X}}(\boldsymbol{t}) \to \mathbb{E}(\min(1, t_1X_1, \ldots, t_dX_d)) = \psi_{\boldsymbol{X}}(\boldsymbol{t}) \ \text{ as } \ n \to \infty.$$

Since $\widehat{\psi}_{\boldsymbol{X}}^{(n)}$ is a df with probability 1, we also have uniform almost sure convergence of this estimator, by the same argument as in Remark 3.1:

$$\sup_{\boldsymbol{t}\geq\boldsymbol{0}}\left|\widehat{\psi}_{\boldsymbol{X}}^{(n)}(\boldsymbol{t})-\psi_{\boldsymbol{X}}(\boldsymbol{t})\right|\to 0 \text{ almost surely.}$$

Example 3.1. Our results so far open a way to estimate a D-norm by using the empirical min-CF. Let $\boldsymbol{X}^{(1)},\dots,\boldsymbol{X}^{(n)}$ be independent copies of a rv $\boldsymbol{X}\in\mathbb{X}_d$ following a copula C in the domain of attraction of a standard max-stable df G. From Proposition 2.9 we obtain that the min-CF of $-\log(\boldsymbol{X})$ satisfies

$$\lim_{s\downarrow 0}\frac{2}{s}\left(1-\psi_{-\log\boldsymbol{X}}\left(\frac{\boldsymbol{1}}{s\boldsymbol{x}}\right)\right)=\|\boldsymbol{x}\|_D\,.$$

This suggests to estimate $\|\boldsymbol{x}\|_D$ by

$$\begin{aligned}\widehat{\|\boldsymbol{x}\|_D} &= \frac{2}{s_n}\left(1-\widehat{\psi}_{-\log\boldsymbol{X}}^{(n)}\left(\frac{1}{s_n\boldsymbol{x}}\right)\right)\\ &= \frac{2}{s_n}\left(1-\frac{1}{n}\sum_{i=1}^{n}\min\left(1,\frac{-\log(X_1^{(i)})}{s_nx_1},\dots,\frac{-\log(X_d^{(i)})}{s_nx_d}\right)\right)\end{aligned}$$

where (s_n) is a positive sequence converging to 0. Although the study of this estimator is outside the scope of this paper, it offers a potentially interesting alternative to existing techniques for the estimation of an extremal dependence structure, such as the classical tail dependence estimators developed by Drees and Huang (1998), Schmidt and Stadtmüller (2006) and Einmahl et al. (2008), among others.

Turning to rates of convergence, the central limit theorem implies that $\widehat{\psi}_{\boldsymbol{X}}^{(n)}(\boldsymbol{t})$ is a $\sqrt{n}$–consistent estimator of $\psi_{\boldsymbol{X}}(\boldsymbol{t})$. The above local uniform convergence then naturally raises the question of the weak convergence of the process

$$S_n=(S_n(\boldsymbol{t}))_{\boldsymbol{t}\geq\boldsymbol{0}}:=\sqrt{n}\left(\widehat{\psi}_{\boldsymbol{X}}^{(n)}(\boldsymbol{t})-\psi_{\boldsymbol{X}}(\boldsymbol{t})\right)_{\boldsymbol{t}\geq\boldsymbol{0}}$$

on $[0,\infty)^d$. This stochastic process has continuous sample paths and satisfies $S_n(\boldsymbol{0})=0$. For ease of exposition, we state a result on the weak convergence of this process in the case $d=1$.

Theorem 3.2. *Let $X^{(1)},\dots,X^{(n)}$ be independent copies of a univariate rv $X>0$ with df F. For any $t_0>0$, we have*

$$S_n(t):=\sqrt{n}\left(\widehat{\psi}_X^{(n)}(t)-\psi_X(t)\right)\to S(t):=t\int_0^{1/t}W\circ F(u)\,du$$

weakly in the space $C[0,t_0]$ *of continuous functions over* $[0,t_0]$*, where* W *is a standard Brownian bridge on* $[0,1]$*. The limiting process* S*, which should be read as 0 when* $t=0$*, is a Gaussian process with covariance structure*

$$\mathrm{Cov}(S(t_1),S(t_2)) = \iint_{[0,1]^2} \left[F\left(\min\left\{\frac{x}{t_1},\frac{y}{t_2}\right\}\right) - F\left(\frac{x}{t_1}\right)F\left(\frac{y}{t_2}\right)\right] dx\,dy.$$

Proof. We adapt the proof of Theorem 3.4 in Falk and Stupfler (2019). By Theorem 1, p.93 of Shorack and Wellner (1986), we can construct, on a common probability space, a triangular array of rowwise independent, standard uniform rvs $(U^{(n,1)},\ldots,U^{(n,n)})_{n\geq 1}$, and a Brownian bridge $\widetilde{W}$ such that

$$\sup_{0\leq t\leq 1}\left|\mathbb{W}_n(t)-\widetilde{W}(t)\right| \to 0 \text{ almost surely, with } \mathbb{W}_n(t) := \frac{1}{\sqrt{n}}\sum_{i=1}^n \left[\mathbb{1}_{\{U^{(n,i)}\leq t\}} - t\right].$$

Furthermore, if we denote by q the quantile function of X (i.e. the left-continuous inverse of F) and by $\widetilde{X}^{(n,i)} := q(U^{(n,i)})$, we have, for any $n\geq 1$,

$$S_n(t) \stackrel{d}{=} \widetilde{S}_n(t) := \frac{1}{\sqrt{n}}\sum_{i=1}^n \left[\min\left(1, t\,\widetilde{X}^{(n,i)}\right) - \mathbb{E}(\min(1,tX))\right],$$

as processes in $C[0,t_0]$. Besides, we have for any $t>0$:

$$\min\left(1, t\widetilde{X}^{(n,i)}\right) = t\int_0^{1/t}\left[1-\mathbb{1}_{\{\widetilde{X}^{(n,i)}\leq u\}}\right] du.$$

Since $\widetilde{X}^{(n,i)}\leq u \Leftrightarrow U^{(n,i)}\leq F(u)$, this yields

$$\widetilde{S}_n(0)=0 \text{ and } \forall t>0,\ \widetilde{S}_n(t) = -t\int_0^{1/t}\mathbb{W}_n\circ F(u)\,du.$$

Defining a process $\widetilde{S}$ by $\widetilde{S}(0)=0$ and $\widetilde{S}(t) = -t\int_0^{1/t}\widetilde{W}\circ F(u)\,du$ for $t>0$, we get

$$\sup_{0\leq t\leq t_0}\left|\widetilde{S}_n(t)-\widetilde{S}(t)\right| \leq \sup_{0\leq t\leq 1}\left|\mathbb{W}_n(t)-\widetilde{W}(t)\right| \to 0$$

almost surely. The process $\widetilde{S}$ is then almost surely continuous, as it is the almost sure uniform limit of the sequence of continuous processes $(\widetilde{S}_n)$. By symmetry of the standard Brownian bridge, we conclude that, as processes in $C[0,t_0]$,

$$S_n(t) \stackrel{d}{=} \widetilde{S}_n(t) \xrightarrow{\text{a.s.}} \widetilde{S}(t) \stackrel{d}{=} S(t).$$

This shows the desired weak convergence; the assertion on the covariance structure of the limiting process follows from a simple calculation using the well-known covariance properties of the Brownian bridge.

In the case $d > 1$, and under regularity conditions (e.g. those of Massart 1989), a similar proof can be written to show an analogue of Theorem 3.2, giving the convergence of the process S_n, in a space of continuous functions over compact subsets of $[0, \infty)^d$, to a $d-$dimensional Gaussian process S with covariance structure

$$\operatorname{Cov}(S(\boldsymbol{t}_1), S(\boldsymbol{t}_2)) = \iint_{[0,1]^{2d}} \left[F\left(\min\left\{ \frac{\boldsymbol{x}}{\boldsymbol{t}_1}, \frac{\boldsymbol{y}}{\boldsymbol{t}_2} \right\} \right) - F\left(\frac{\boldsymbol{x}}{\boldsymbol{t}_1} \right) F\left(\frac{\boldsymbol{y}}{\boldsymbol{t}_2} \right) \right] d\boldsymbol{x}\, d\boldsymbol{y}.$$

Note that the asymptotic distribution in Theorem 3.2 bears some similarity to the asymptotic distribution of the empirical max-CF process

$$\sqrt{n}\left(\widehat{\varphi}_X^{(n)}(t) - \varphi_X(t) \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\max\left(1, tX^{(i)}\right) - \mathbb{E}\left(\max\left(1, tX\right)\right) \right]$$

when $\mathbb{E}(X^2) < \infty$, which is obtained as a particular case of Theorem 3.4 of Falk and Stupfler (2019).

4 On the Structure of the Set of Min-characteristic Functions

Theorem 3.1 shows that the convergence of a sequence of min-CFs to a min-CF is equivalent to the convergence of the pertaining distribution functions. The requirement that the limit be a min-CF is necessary: if $X_n = n$ almost surely ($n \in \mathbb{N}$), then the corresponding sequence of min-CFs satisfies

$$\forall x > 0,\ \ \psi_{X_n}(x) = \mathbb{E}(\min(1, nx)) \to 1 \ \text{ as } \ n \to \infty,$$

but the function $\psi(x) = 1$, if $x > 0$, and $\psi(0) = 0$, is not a min-CF because it is not continuous at zero. In other words, the set of min-CFs is not closed in the topology of pointwise convergence. This is certainly not specific to the notion of min-CF; the set of Fourier transforms is not closed either (think for example of the sequence of normal distributions with mean 0 and variance n^2). It is nonetheless interesting to get some further understanding of the structure of the set of min-CFs and the elements it contains: this is the focus of the present section. We start by noting that the set of min-CFs is convex.

LEMMA 4.1. *The convex combination of two min-CFs is again a min-CF.*

PROOF. Let $\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}$ be two rvs in $\mathbb{X}_d$ and $\lambda \in (0,1)$. Let $Z \in \{1,2\}$ be a rv that is independent of $\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}$, with $\mathbb{P}(Z=1) = \lambda = 1 - \mathbb{P}(Z=2)$. Then $\boldsymbol{X} = \boldsymbol{X}^{(Z)}$ is a rv in $\mathbb{X}_d$ with

$$\begin{aligned} \psi_{\boldsymbol{X}}(\boldsymbol{t}) &= \mathbb{E}\left(\min\left(1, t_1 X_1^{(Z)}, \ldots, t_d X_d^{(Z)}\right)\right) \\ &= \lambda\, \mathbb{E}\left(\min\left(1, t_1 X_1^{(1)}, \ldots, t_d X_d^{(1)}\right)\right) \\ &\quad + (1-\lambda)\mathbb{E}\left(\min\left(1, t_1 X_1^{(2)}, \ldots, t_d X_d^{(2)}\right)\right) \\ &= \lambda \psi_{\boldsymbol{X}^{(1)}}(\boldsymbol{t}) + (1-\lambda)\psi_{\boldsymbol{X}^{(2)}}(\boldsymbol{t}). \end{aligned}$$

This shows that the convex combination $\lambda\psi_{\boldsymbol{X}^{(1)}} + (1-\lambda)\psi_{\boldsymbol{X}^{(2)}}$ of the min-CFs $\psi_{\boldsymbol{X}^{(1)}}$ and $\psi_{\boldsymbol{X}^{(2)}}$ is a min-CF again.

Our next result informally states that the set of min-CFs is relatively compact in the space of concave pseudo-distribution functions on $[0,\infty)^d$.

PROPOSITION 4.2. *Any sequence of min-CFs* (ψ_n) *on* $\mathbb{R}^d$ *has a subsequence that converges pointwise to a concave function* ψ *(at each of its points of continuity) such that* $\psi = 0$ *outside of* $[0,\infty)^d$ *and* $\psi(\boldsymbol{t}) \to \psi_\infty \in [0,1]$ *as* $\min(t_1, \ldots, t_d) \to \infty$.

PROOF. Use jointly Proposition 2.4(vi) with Helly's theorem.

We now give some examples of functions which are (or not) min-CFs. Our first example focuses on copula functions. Recall that a copula on $\mathbb{R}^d$ is a $d-$dimensional df with standard uniform marginal distributions.

PROPOSITION 4.3. *The only copula which is also a min-CF is the completely dependent copula*

$$C(\boldsymbol{u}) = \min(u_1, \ldots, u_d),\ \boldsymbol{u} = (u_1, \ldots, u_d) \in [0,1]^d,$$

corresponding to the constant rv $\boldsymbol{X} = (1, \ldots, 1) \in \mathbb{R}^d$.

PROOF. Let C be a copula function which is also a min-CF. In other words, there is a vector $\boldsymbol{U}$ with df C (and in particular, standard uniform marginals) and $\boldsymbol{X} \in \mathbb{X}_d$ such that

$$\forall (t_1, \ldots, t_d) \in [0,\infty)^d,\ \mathbb{P}(U_1 \le t_1, \ldots, U_d \le t_d) = \mathbb{E}(\min(1, t_1 X_1, \ldots, t_d X_d)).$$

Letting, for any i, all t_j except t_i tend to infinity, we obtain, by the dominated convergence theorem,

$$\forall t \ge 0,\ \mathbb{E}(\min(1,t)) = \min(1,t) = \mathbb{P}(U_i \le t) = \mathbb{E}(\min(1, tX_i)).$$

This implies that X_i and the constant 1 have the same min-CF, and thus, by Theorem 2.1, $X_i = 1$ almost surely. Then

$$C(\boldsymbol{u}) = \mathbb{P}(U_1 \leq u_1, \ldots, U_d \leq u_d) = \mathbb{E}(\min(1, u_1, \ldots, u_d)) = \min(u_1, \ldots, u_d)$$

for any $\boldsymbol{u} = (u_1, \ldots, u_d) \in [0,1]^d$, completing the proof.

This result implies that the df of the uniform distribution on $[0,1]$ is also a min-CF. It is straightforward to show (using Proposition 2.4) that actually, a necessary and sufficient condition for the uniform df on $[a,b]$ to be a min-CF is that $a = 0$, corresponding to the min-CF of the constant rv $X = 1/b$.

Proposition 4.3 shows that, although a min-CF is always a df by Proposition 2.4(vi), it can have a rather different structure from the df of its generating rv. We elaborate on this observation in our next result, which shows that the min-CF transformation has no fixed point.

PROPOSITION 4.4. *There is no rv $\boldsymbol{X} \in \mathbb{X}_d$ such that its df F satisfies $\psi_{\boldsymbol{X}} = F$.*

PROOF. Suppose indeed that there were such a rv $\boldsymbol{X} = (X_1, \ldots, X_d) \in \mathbb{X}_d$. Writing $\psi_{\boldsymbol{X}}(\boldsymbol{t}) = F(\boldsymbol{t})$ for any $\boldsymbol{t} = (t_1, \ldots, t_d) \geq \boldsymbol{0}$ and letting $t_2, \ldots, t_d \to \infty$, we find

$$\forall t_1 \geq 0,\ \psi_{X_1}(t_1) = \mathbb{E}(\min(1, t_1 X_1)) = \mathbb{P}(X_1 \leq t_1).$$

It is thus enough to show that no univariate positive rv $X = X_1$ can satisfy this identity. If this were the case then, by Proposition 2.4(vi), F would be continuous on $[0, \infty)$. Using the identity

$$\forall t > 0,\ tF\left(\frac{1}{t}\right) = t\psi_X\left(\frac{1}{t}\right) = \int_0^t \mathbb{P}(X_1 > v)\, dv = \int_0^t [1 - F(v)]\, dv$$

shows that F is actually continuously (and even infinitely) differentiable on $(0, \infty)$. By Theorem 2.5, we get

$$\forall x > 0,\ 1 - F\left(\frac{1}{x}\right) = \frac{\partial}{\partial t}\left\{tF\left(\frac{x}{t}\right)\right\}\Big|_{t=1} = F(x) - xF'(x)\ . \tag{4.1}$$

Replacing x with $1/x$ in this identity immediately entails

$$\forall x > 0,\ \frac{1}{x}F'\left(\frac{1}{x}\right) = xF'(x)$$

and therefore

$$\frac{d}{dx}\left[F(x) + F\left(\frac{1}{x}\right)\right] = F'(x) - \frac{1}{x^2}F'\left(\frac{1}{x}\right) = 0 \ \text{ on } (0, \infty).$$

There is then a constant c such that $F(x)+F(1/x)=c$ on $(0,\infty)$. Letting $x\to\infty$ gives $F(x)+F(1/x)=1$ on $(0,\infty)$. Plugging this back in Eq. 4.1 entails

$$\forall x>0,\ xF'(x)=F(x)+F\left(\frac{1}{x}\right)-1=0$$

and therefore $F'\equiv 0$ on $(0,\infty)$, which finally yields that F is constant on $(0,\infty)$ and thus necessarily equal to 1 on this interval. But F is also right-continuous at 0 with $F(0)=0$, which is an obvious contradiction.

The above result raises the following question: can we compare the df F of a rv $\boldsymbol{X}\in\mathbb{X}_d$ and its min-CF $\psi_{\boldsymbol{X}}$? In other words, although we know that $\psi_{\boldsymbol{X}}\neq F$, can we write that F is in general greater or less than $\psi_{\boldsymbol{X}}$? Our next result examines this question if F is a copula.

Lemma 4.5. *Let $\boldsymbol{X}$ follow a copula. Then the copula C_ψ corresponding to the df $\psi_{\boldsymbol{X}}$ satisfies*

$$C_\psi(\boldsymbol{u})\geq\psi_{\boldsymbol{X}}(\boldsymbol{u}),\ \boldsymbol{u}\in[0,1]^d.$$

Proof. First of all, the univariate margins ψ_i of $\psi_{\boldsymbol{X}}$ are identical and given by

$$\psi(t)=\psi_i(t)=\mathbb{E}\left(\min(1,tX_i)\right)=\int_0^1\mathbb{P}\left(X_i>\frac{s}{t}\right)ds=\begin{cases} t/2, & t\in[0,1],\\ 1-1/(2t), & t\geq 1.\end{cases}$$

The corresponding quantile function is

$$\psi^{-1}(u)=\psi_i^{-1}(u)=\begin{cases} 2u, & u\in[0,1/2],\\ 1/[2(1-u)], & u\in[1/2,1).\end{cases}$$

Note that $\psi^{-1}(u)\geq u$, $u\in[0,1)$. The copula C_ψ is then given by

$$\begin{aligned} C_\psi(\boldsymbol{u}) &= \psi_{\boldsymbol{X}}\left(\psi^{-1}(u_1),\dots,\psi^{-1}(u_d)\right)\\ &= \mathbb{E}\left(\min\left(1,\psi^{-1}(u_1)X_1,\dots,\psi^{-1}(u_d)X_d\right)\right)\\ &\geq \mathbb{E}\left(\min\left(1,u_1X_1,\dots,u_dX_d\right)\right)\\ &= \psi_{\boldsymbol{X}}(\boldsymbol{u}),\ \boldsymbol{u}\in[0,1)^d, \end{aligned}$$

which is the result.

The above proof shows that Lemma 4.5 is actually true for each rv $\boldsymbol{X}$ whose min-CF satisfies $\psi_i^{-1}(u)\geq u$, which is equivalent to $\psi_i(u)\leq u$, for each $u\in(0,1)$ and $1\leq i\leq d$. This is for instance the case if $\mathbb{E}(X_i)\leq 1$, $1\leq i\leq d$, since then

$$\forall t\geq 0,\ \psi_i(t)=\mathbb{E}\left(\min(1,tX_i)\right)\leq\mathbb{E}(tX_i)=t.$$

5 Min-characteristic Functions for Arbitrary Random Vectors

The concept of min-CF as we defined it can be extended to a rv $\boldsymbol{X} = (X_1, \ldots, X_d)$ with not necessarily strictly positive components by applying a continuous one-to-one transformation T mapping $\mathbb{R}$ onto $(0, \infty)$, and considering

$$(t_1, \ldots, t_d) \mapsto \mathbb{E}(\min(1, t_1 T(X_1), \ldots, t_d T(X_d))).$$

The assumptions on T ensure that, by Theorem 2.1, such a mapping identifies the distribution of $\boldsymbol{X}$. The purpose of this section is to show an example of such a construction and explore some of its properties.

A particularly simple and convenient transformation T is the exponential function $T(x) = \exp(x)$, $x \in \mathbb{R}$. For an arbitrary rv $\boldsymbol{X}$, this leads us to consider the mapping

$$\psi_{\exp(\boldsymbol{X})}(\boldsymbol{t}) := \mathbb{E}(\min(1, t_1 \exp(X_1), \ldots, t_d \exp(X_d))), \boldsymbol{t} = (t_1, \ldots, t_d) \in [0, \infty)^d.$$

Example 5.1. Let (U, V) be a bivariate rv which follows a copula, say C. Then the corresponding Kendall df is

$$K(s) := \mathbb{P}(C(U, V) \leq s), \ s \in [0, 1].$$

This function was introduced in Genest and Rivest (1993) in the context of the class of Archimedean copulas

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)),$$

where $\varphi : [0, 1] \to [0, \infty]$ is a convex, continuous and strictly decreasing function with $\varphi(1) = 0$, and

$$\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 \leq t \leq \varphi(0), \\ 0, & \varphi(0) < t \leq \infty; \end{cases}$$

see Theorem 4.1.4 in Nelsen (2006). Such an Archimedean copula has Kendall df

$$K(s) = s - \frac{\varphi(s)}{\varphi'(s)}, \ s \in [0, 1],$$

and the Kendall df characterizes the generator φ; see Genest and Rivest (1993).

Choose for example $\varphi_p(s) = (1 - s)^p$, $s \in [0, 1]$, with $p \in [1, \infty)$. The pertaining Archimedean copula is given by

$$C_p(u, v) = \max\left(0, 1 - \|(1 - u, 1 - v)\|_p\right), \ u, v \in [0, 1],$$

and Kendall's df is

$$K(s) = \frac{1}{p} + s\left(1 - \frac{1}{p}\right), \; s \in [0,1].$$

Thus $K(0) = 1/p > 0$, which means that K has an atom at 0 and therefore we cannot use the ordinary min-CF for the Kendall df. We then transform the rv $C_p(U,V)$ by the exponential function and obtain, for $t \in (0,1]$ and $p \geq 1$,

$$\begin{aligned}
\psi_{\exp(K)}(t) &= \mathbb{E}\left(\min\left(1, t\exp(C_p(U,V))\right)\right) \\
&= \int_0^1 \mathbb{P}\left(\exp(C_p(U,V)) > \frac{u}{t}\right) du \\
&= \begin{cases} t\left(1 + \left(1 - \dfrac{1}{p}\right)(\exp(1) - 2)\right), & 0 < t \leq \exp(-1), \\ 2\left(1 - \dfrac{1}{p}\right) + \left(\dfrac{2}{p} - 1\right)t + \left(1 - \dfrac{1}{p}\right)\log(t), & \exp(-1) \leq t \leq 1. \end{cases}
\end{aligned}$$

This is itself a df having density

$$\psi'_{\exp(K)}(t) = \begin{cases} 1 + \left(1 - \dfrac{1}{p}\right)(\exp(1) - 2), & 0 < t \leq \exp(-1), \\ \dfrac{2}{p} - 1 + \left(1 - \dfrac{1}{p}\right)\dfrac{1}{t}, & \exp(-1) < t \leq 1. \end{cases}$$

The particular case $p = 1$ yields $\psi_{\exp(K)}(t) = t$, $0 \leq t \leq 1$, i.e., the df of the uniform distribution on $[0,1]$.

In general, we clearly have, by monotonicity of the exponential function, that for $t_1, \ldots, t_d > 0$:

$$\psi_{\exp(\boldsymbol{X})}(\boldsymbol{t}) = \mathbb{E}(\exp(\min(0, X_1 + \log(t_1), \ldots, X_d + \log(t_d)))).$$

Replacing $\log(t_i)$ in the above formula by $x_i \in \mathbb{R}$, $1 \leq i \leq d$, leads to the following definition.

DEFINITION 5.1. *The log-min-CF of* $\boldsymbol{X} = (X_1, \ldots, X_d)$ *is the function* $\psi_{\boldsymbol{X}}^{\exp}$ *on* $\mathbb{R}^d$ *defined by*

$$\forall \boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d, \; \psi_{\boldsymbol{X}}^{\exp}(\boldsymbol{t}) = \mathbb{E}(\exp(\min(0, X_1 + x_1, \ldots, X_d + x_d))).$$

Note that obviously $\psi_{\boldsymbol{X}+\boldsymbol{a}}^{\exp}(\boldsymbol{x}) = \psi_{\boldsymbol{X}}^{\exp}(\boldsymbol{x} + \boldsymbol{a})$, for any $\boldsymbol{a}, \boldsymbol{x} \in \mathbb{R}^d$. The following two results are immediate consequences of Theorems 2.1 and 3.1.

Corollary 5.2. *Let $\boldsymbol{X} = (X_1, \dots, X_d)$ and $\boldsymbol{Y} = (Y_1, \dots, Y_d)$ be two rvs. Then*

$$\boldsymbol{X} \overset{d}{=} \boldsymbol{Y} \Leftrightarrow \forall \boldsymbol{x} \in \mathbb{R}^d,\ \psi_{\boldsymbol{X}}^{\exp}(\boldsymbol{x}) = \psi_{\boldsymbol{Y}}^{\exp}(\boldsymbol{x}).$$

Corollary 5.3. *Let $\boldsymbol{X}^{(n)}$, $\boldsymbol{X}$ be rvs. Then*

$$\boldsymbol{X}^{(n)} \xrightarrow{d} \boldsymbol{X} \Leftrightarrow \psi_{\boldsymbol{X}^{(n)}}^{\exp} \to \psi_{\boldsymbol{X}}^{\exp} \quad \textit{pointwise}.$$

We now show two examples of calculation of a log-min-CF.

Example 5.2. *Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion, choose $t > 0$ and put*

$$X := B_t - \frac{t}{2}.$$

Then $\exp(X)$ *follows a log-normal distribution with mean one. From* Falk (2019, Lemma 1.10.6) *we find that, for any x,*

$$\mathbb{E}(\max(1, \exp(X + x))) = \Phi\left(\frac{\sqrt{t}}{2} - \frac{x}{\sqrt{t}}\right) + \exp(x)\Phi\left(\frac{\sqrt{t}}{2} + \frac{x}{\sqrt{t}}\right)$$

where Φ denotes the df of the univariate standard normal distribution. From the identity $\min(a, b) = a + b - \max(a, b)$*, $a, b \in \mathbb{R}$, we obtain the log-min-CF of $X = B_t - t/2$ as:*

$$\begin{aligned} \psi_X^{\exp}(x) &= \mathbb{E}(\min(1, \exp(X + x))) \\ &= 1 - \Phi\left(\frac{\sqrt{t}}{2} - \frac{x}{\sqrt{t}}\right) + \exp(x)\left(1 - \Phi\left(\frac{\sqrt{t}}{2} + \frac{x}{\sqrt{t}}\right)\right), \ x \in \mathbb{R}, \end{aligned}$$

and, thus, that of B_t:

$$\psi_{B_t}^{\exp}(x) = \psi_X^{\exp}(x + t/2) = 1 - \Phi\left(-\frac{x}{\sqrt{t}}\right) + \exp\left(x + \frac{t}{2}\right)\left(1 - \Phi\left(\sqrt{t} + \frac{x}{\sqrt{t}}\right)\right).$$

Example 5.3. *Let $\boldsymbol{\eta} = (\eta_1, \dots, \eta_d)$ follow a max-stable distribution with standard negative exponential margins, i.e. there exists a D-norm $||\cdot||_D$ on $\mathbb{R}^d$ such that $\mathbb{P}(\boldsymbol{\eta} \leq \boldsymbol{x}) = \exp(-||\boldsymbol{x}||_D)$, $\boldsymbol{x} \leq \boldsymbol{0} \in \mathbb{R}^d$. The df of each η_i is $\mathbb{P}(\eta_i \leq x) = \exp(x)$, $x \leq 0$. The log-min-CF of $\boldsymbol{\eta}$ is*

$$\begin{aligned} \psi_{\boldsymbol{\eta}}^{\exp}(\boldsymbol{x}) &= \mathbb{E}\left(1, \min\left(\exp(\eta_1 + x_1), \dots, \exp(\eta_d + x_d)\right)\right) \\ &=: \mathbb{E}\left(\min\left(1, \exp(x_1)U_1, \dots, \exp(x_d)U_d\right)\right), \end{aligned}$$

where $\boldsymbol{U} = (U_1, \dots, U_d)$ follows an extreme value copula on $[0, 1]^d$*:*

$$\mathbb{P}(\boldsymbol{U} \leq \boldsymbol{u}) = \exp\left(-\left\|(\log(u_1), \dots, \log(u_d))\right\|_D\right), \ \boldsymbol{u} = (u_1, \dots, u_d) \in (0, 1]^d.$$

Every extreme value copula has this representation; see Falk (2019, Equation (3.10))*.*

We conclude this section and the paper by an observation regarding the log-min-CF of a sum of independent rvs. Let then $\boldsymbol{X}$ and $\boldsymbol{Y}$ be two independent rv in $\mathbb{R}^d$. The distribution of the sum $\boldsymbol{X}+\boldsymbol{Y}$ is characterized by the product of the corresponding log-min-CFs, defined for $\boldsymbol{s}\in\mathbb{R}^d$ by the product rule

$$\begin{aligned}\left(\psi_{\boldsymbol{X}}^{\exp} * \psi_{\boldsymbol{Y}}^{\exp}\right)(\boldsymbol{s}) &:= \psi_{\boldsymbol{X}+\boldsymbol{Y}}^{\exp}(\boldsymbol{s})\\ &= \mathbb{E}\left(\min\left(1,\exp(X_1+Y_1+s_1),\dots,\exp(X_d+Y_d+s_d)\right)\right).\end{aligned}$$

This multiplication operation can be extended to finitely many rvs in an obvious way. In particular, we can establish a lower bound on the product of the log-min-CFs of univariate rvs. This is the content of the next lemma.

Proposition 5.4. *Let $X_1, X_2, \dots, X_n$ be independent rvs in $\mathbb{R}$. Then we have for $s\in\mathbb{R}$ and $\lambda_1,\dots,\lambda_n\geq 0$, $\sum_{i=1}^n \lambda_i = 1$,*

$$\left(\psi_{X_1}^{\exp} * \cdots * \psi_{X_n}^{\exp}\right)(s) \geq \prod_{i=1}^n \psi_{X_i}^{\exp}(\lambda_i s).$$

Proof. We show the case $n=2$ first. We have, for arbitrary numbers $a,b\geq 0$,

$$\min(1,ab)\geq \min(1,a)\min(1,b).$$

Suppose that X and Y are independent. This yields

$$\begin{aligned}\left(\psi_X^{\exp} * \psi_Y^{\exp}\right)(s) &= \mathbb{E}\left(\min\left(1,\exp(X+Y+s)\right)\right)\\ &= \mathbb{E}\left(\min\left(1,\exp(X+\lambda s)\exp(Y+(1-\lambda)s)\right)\right)\\ &\geq \mathbb{E}\left(\min\left(1,\exp(X+\lambda s)\right)\min\left(1,\exp(Y+(1-\lambda)s)\right)\right)\\ &= \mathbb{E}\left(\min\left(1,\exp(X+\lambda s)\right)\right)\mathbb{E}\left(\min\left(1,\exp(Y+(1-\lambda s))\right)\right)\\ &= \psi_X^{\exp}(\lambda s)\psi_Y^{\exp}((1-\lambda)s),\ \lambda\in[0,1],\ s\in\mathbb{R}.\end{aligned}$$

The result is then shown for $n=2$. The general case follows from a straightforward proof by induction.

In particular we obtain for identical copies $X_1, X_2, \dots, X_n$ of X the lower bound

$$\left(\psi_X^{\exp}\right)^{*n}(s)\geq\left(\psi_X^{\exp}\left(\frac{s}{n}\right)\right)^n.$$

6 Conclusion and Perspectives

This paper introduces the concept of min-CF, as a way to identify probability distributions concentrated on $(0,\infty)^d$. This min-CF is in fact a continuous and concave df. We have worked here on a variety of aspects of this notion, such as a theorem linking convergence in distribution to pointwise convergence of min-CFs, the functional convergence of the sample min-CF for independent and identically distributed random variables, and a construction of the min-CF for arbitrary rvs.

It is natural to think about the probabilistic and statistical applications of the notion of min-CF. In this paper, we use this concept to provide a development of the theory of D-norms by showing that the canonical mapping from the set of D-norms to the set of dual D-norms is one-to-one when restricted to D-norms generated by componentwise positive generators (Proposition 2.6). We further suggest an estimator of a D-norm by using the empirical min-CF, based on our Proposition 2.9, in Example 3.1. D-norms are the skeleton of multivariate extreme value theory (Falk, 2019) which, being the framework adapted to the simultaneous analysis of extremal events, is part of the toolbox for risk management. This estimator suggested by the min-CF provides an alternative to existing methods in multivariate extreme value theory; the investigation of its theoretical and numerical properties appears to be an interesting avenue of research.

Another potential application of the notion of min-CF is goodness-of-fit testing. Theorem 3.2 provides the functional convergence of the sample min-CF to its population counterpart. Paired with examples of explicit calculations of min-CFs for parametric families, such as in Examples 2.1–2.5, it is not hard to see how one may define goodness-of-fit testing procedures by comparing the gap between the empirical min-CF and the min-CF of the hypothesized distribution. Of course, such procedures already exist using standard CFs such as the Fourier or Laplace transforms; however, a goodness-of-fit procedure based on the min-CF is likely to be most interesting in cases such as that of the Generalized Pareto distribution, for which the Fourier transform does not have a simple closed form, although the min-CF does (see Example 2.3). An even more relevant development would be the assessment of the performance of such a goodness-of-fit procedure in the context of Peaks-Over-Threshold modeling, which is a major part of univariate extreme value analysis. In this framework, the Generalized Pareto distribution naturally arises as an approximation of the distribution of exceedances over a high threshold (see Beirlant et al. 2004), and examining the performance of a goodness-of-fit testing procedure based on the min-CF in this context appears to be a stimulating problem, not least because standard post-inference model checking largely seems to be based on the use of graphical tools such as QQ-plots.

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Michael Falk
Institute of Mathematics,
University of Würzburg,
Würzburg, Germany
E-mail: michael.falk@uni-wuerzburg.de

Gilles Stupfler
School of Mathematical Sciences,
University of Nottingham,
Nottingham, UK
E-mail: gilles.stupfler@nottingham.ac.uk

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