Sankhyā : The Indian Journal of Statistics
2018, Volume 80-A, Part 1, pp. 110-120

Multivariate Order Statistics: the Intermediate Case

Michael Falk and Florian Wisheckel
University of Würzburg, Würzburg, Germany

Abstract

Asymptotic normality of intermediate order statistics taken from univariate iid random variables is well-known. We generalize this result to random vectors in arbitrary dimension, where the order statistics are taken componentwise.

AMS (2000) *subject classification.* Primary 62G30, Secondary 62H10.
Keywords and phrases. Multivariate order statistics, Intermediate order statistics, Copula, Domain of attraction, D-norm, von Mises type conditions, Asymptotic normality

1 Introduction

Let $\boldsymbol{X}^{(1)} = \left(X_1^{(1)}, \dots, X_d^{(1)}\right), \dots, \boldsymbol{X}^{(n)} = \left(X_1^{(n)}, \dots, X_d^{(n)}\right)$ be independent copies of a random vector (rv) $\boldsymbol{X} = (X_1, \dots, X_d)$ that realizes in $\mathbb{R}^d$. By

$$X_{1:n,i} \le X_{2:n,i} \le \dots \le X_{n:n,i},$$

we denote the ordered values of the i-th components of $\boldsymbol{X}^{(1)}, \dots, \boldsymbol{X}^{(n)}$, $1 \le i \le d$. Then, $(X_{j_1:n,1}, \dots, X_{j_d:n,d})$ with $1 \le j_1, \dots, j_d \le n$, is a rv of order statistics (os) in each component. We call it a *multivariate* os.

The univariate case $d = 1$ is, clearly, well investigated; standard references are the books by David (1981), Reiss (1989), Galambos (1987), David and Nagaraja (2004), Arnold et al. (2008), among others. In the multivariate case $d \ge 2$, the focus has been on the investigation of the rv of componentwise maxima $(X_{n:n,1}, \dots, X_{n:n,d})$ (Balkema and Resnick (1977), de Haan and Resnick (1977), Resnick (1987), Vatan (1985), Beirlant et al. (2004), de Haan and Ferreira (2006), Falk et al. (2011), among others).

Much less is known in the extremal case $(X_{n-k_1:n,1}, \dots, X_{n-k_d:n})$ with $k_1, \dots, k_d \in \mathbb{N}$ fixed; one reference is Galambos (1975). More recent investigations of this case are Barakat and Nigm (2012) and Barakat et al. (2015). Asymptotic normality of the random vector $(X_{j_1:n,1}, \dots, X_{j_d:n,d})$ in the case of central os is established in Reiss (1989, Theorem 7.1.2). In this case, the indices $j_i = j_i(n)$ depend on n and have to satisfy $j_i(n)/n \to_{n\to\infty} q_i \in (0,1)$, $1 \le i \le d$.

In the case of intermediate os, we require $j_i = j_i(n) = n - k_i$, where $k_i = k_i(n) \to_{n\to\infty} \infty$ with $k_i/n \to_{n\to\infty} 0$. Asymptotic normality of intermediate os in the univariate case under fairly general von Mises conditions was established in Falk (1989). Balkema and de Haan (1978a) and Balkema and de Haan (1978b, Theorem 7.1) proved that for particular underlying distribution function (df) F, $X_{n-k+1:n}$ may have *any* limiting distribution if it is suitably standardized and if the sequence k is chosen appropriately.

As pointed out by Smirnov (1967), a (nondegenerate) limiting distribution of $X_{n-k+1:n}$ different from the normal one can only occur if k has an *exact* preassigned asymptotic behavior. Assuming only $k \to_{n\to\infty} \infty$, $k/n \to_{n\to\infty} 0$, Smirnov (1967) gave necessary and sufficient conditions for F such that $X_{n-k+1:n}$ is asymptotically normal, and he specified the appropriate norming constants (see condition (3.2) below).

Smirnov's result was extended to multivariate intermediate os by Cheng et al. (1997). They identified the class of limiting distributions of $(X_{n-k_1:n,1}, \dots, X_{n-k_d:n,d})$ after suitable normalizing and centering, and gave necessary and sufficient conditions of weak convergence.

Cooil (1985) established multivariate extensions of the univariate case by considering vectors of intermediate os $(X_{n-k_1+1:n}, \dots, X_{n-k_d+1:n})$ taken from the same sample of univariate os $X_{1:n} \le \dots \le X_{n:n}$ but with pairwise different $k_1, \dots, k_d$. Barakat (2001) investigates the limit distribution of bivariate os in all nine possible combinations of central, intermediate and extreme os.

According to Sklar (1959, 1996), the df of $\boldsymbol{X} = (X_1, \dots, X_d)$ can be decomposed into a copula and the df F_i of each component X_i, $1 \le i \le d$. We will establish in this paper asymptotic normality of the vector of multivariate os $(X_{n-k_1:n,1}, \dots, X_{n-k_d:n,d})$ in the intermediate case. This is achieved under the condition that the copula corresponding to $\boldsymbol{X}$ is in the max-domain of attraction of a multivariate extreme value df together with the assumption that each univariate marginal df F_i satisfies a von Mises condition and that the norming constants satisfy Smirnov's condition (3.2) below.

2 Main Results: Copula Case

We consider first the case that the df of the rv $\boldsymbol{X}$ is a copula, C say, on $\mathbb{R}^d$. We require that C is in the max-domain of attraction of a nondegenerate multivariate extreme-value df (evd) G, i.e.,

$$C^n\left(\mathbf{1} + \frac{\boldsymbol{x}}{n}\right) \to_{n\to\infty} G(\boldsymbol{x}), \qquad \boldsymbol{x} \in \mathbb{R}^d, \tag{2.1}$$

where $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^d$ and all operations on vectors are meant componentwise. In this case, there exists a D-norm $\|\cdot\|_D$ on $\mathbb{R}^d$ such that

$$G(\boldsymbol{x}) = \exp\left(- \|\boldsymbol{x}\|_D\right), \qquad \boldsymbol{x} \leq \mathbf{0} \in \mathbb{R}^d. \tag{2.2}$$

A common norm $\|\cdot\|$ on $\mathbb{R}^d$ is a D-norm $\|\cdot\|_D$, if there exists a rv $\boldsymbol{Z} = (Z_1, \dots, Z_d)$ on $\mathbb{R}^d$ with the two properties $Z_i \geq 0$, $E(Z_i) = 1$ for $i = 1, \dots, d$, such that

$$\|\boldsymbol{x}\|_D = E\left(\max_{1 \leq i \leq d} |x_i|\, Z_i\right), \qquad \boldsymbol{x} \in \mathbb{R}^d.$$

The rv $\boldsymbol{Z}$ is called a generator of the D-norm, and we add the index D to the norm symbol, meaning dependence.

Representation (2.2) is just a reformulation of the Pickands-de Haan-Resnick-Vatan characterization of a multivariate evd, using D-norms (see, e.g., Falk et al. (2011, Chapter 4)). Examples of D-norms are the sup-norm $\|\boldsymbol{x}\|_\infty = \max_{1 \leq i \leq d} |x_i|$ and the complete logistic family $\|\boldsymbol{x}\|_p = \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}$, $p \geq 1$. For a systematic treatment of D-norms, we refer to the booklet by Falk (2016).

A straightforward analysis shows that (2.1) and (2.2) are equivalent to the condition that there exists a D-norm on $\mathbb{R}^d$ such that

$$C(\boldsymbol{u}) = 1 - \|\mathbf{1} - \boldsymbol{u}\|_D + o\left(\|\mathbf{1} - \boldsymbol{u}\|\right) \tag{2.3}$$

as $\boldsymbol{u} \to \mathbf{1}$, uniformly for $\boldsymbol{u} \in [0,1]^d$.

Take, for example, an arbitrary *Archimedean copula* on $\mathbb{R}^d$

$$C_\varphi(\boldsymbol{u}) = \varphi^{-1}(\varphi(u_1) + \dots + \varphi(u_d)),$$

where φ is a continuous and strictly decreasing function from $[0,1]$ to $[0,\infty]$ such that $\varphi(1) = 0$ (see, e.g., McNeil and Nešlehová (2009, Theorem 2.2)). Suppose that

$$p := \lim_{s \to 0} \frac{s\varphi'(1-s)}{\varphi(1-s)} \text{ exists in } [1, \infty].$$

Then, C_φ satisfies condition (3) with pertaining D-norm $\|\cdot\|_D = \|\cdot\|_p$, $p \in [1, \infty]$. This follows from Charpentier and Segers (2009, Theorem 4.1) and elementary computations. If $p = 1$, then the margins of C_φ are tail-independent. This concerns the Clayton and Frank copula with generators $\varphi_\lambda(t) = (t^{-\lambda} - 1)/\lambda$, $\lambda \geq 0$, and $\varphi_\lambda(t) = -\log((\exp(-\lambda t) - 1)/(\exp(-\lambda) - 1))$, $\lambda \in \mathbb{R} \setminus \{0\}$, respectively, but not the Gumbel copula with generator $\varphi_\lambda(t) = (-\log(t))^\lambda$, $\lambda > 1$, in which case $p = \lambda$. For an exhaustive account on copulas, we refer to Nelsen (2006).

We are now ready to state asymptotic normality of the vector of multivariate os in the intermediate case with underlying copula. By $\boldsymbol{e}_j := (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^d$ with denote the j-th unit vector, $j = 1, \dots, d$.

THEOREM 2.1 (The Copula Case). *Suppose that the rv $\boldsymbol{X} = (X_1, \dots, X_d)$ follows a copula C, which satisfies expansion (2.3) with some D-norm $\|\cdot\|_D$ on $\mathbb{R}^d$. Let $\boldsymbol{k} = \boldsymbol{k}(n) = (k_1, \dots, k_d) \in \{1, \dots, n-1\}^d$, $n \in \mathbb{N}$, satisfy $k_i/k_j \to k_{ij}^2 \in (0, \infty)$ for all pairs of components $1 \le i, j \le d$, $\|\boldsymbol{k}\| \to \infty$ and $\|\boldsymbol{k}\|/n \to 0$ as $n \to \infty$. Then, the rv of componentwise intermediate os is asymptotically normal:*

$$\left(\frac{n}{\sqrt{k_i}}\left(X_{n-k_i:n,i} - \frac{n-k_i}{n}\right)\right)_{i=1}^d \to_D N(\boldsymbol{0}, \Sigma),$$

where the $d \times d$-covariance matrix is given by

$$\Sigma = (\sigma_{ij}) = \begin{cases} 1, & \text{if } i = j \\ k_{ij} + k_{ji} - \|k_{ij}\boldsymbol{e}_i + k_{ji}\boldsymbol{e}_j\|_D, & \text{if } i \ne j. \end{cases}$$

If, for example, $\|\boldsymbol{x}\|_D = \|\boldsymbol{x}\|_p = \left(\sum_{i=1}^p |x_i|^p\right)^{1/p}$, $p \ge 1$, then $\sigma_{ij} = k_{ij} + k_{ji} - \left(k_{ij}^p + k_{ji}^p\right)^{1/p}$, $i \ne j$.

REMARK 2.2. Note that $\sigma_{ij} = 0$, $i \ne j$, if $\|\cdot\|_D = \|\cdot\|_1$, which is the case if the margins of $G(\boldsymbol{x}) = \exp(-\|\boldsymbol{x}\|_D) = \prod_{i=1}^d \exp(x_i)$, $\boldsymbol{x} \le \boldsymbol{0} \in \mathbb{R}^d$, are independent. Then, the components of $\boldsymbol{X} = (X_1, \dots, X_d)$ are called tail-independent. The reverse implication is true as well, i.e., the preceding result entails that the componentwise intermediate os $X_{n-k_1:n,1}, \dots, X_{n-k_d:n,d}$ are asymptotically independent if, and only if, they are pairwise asymptotically independent. But this is equivalent to the condition that the $\|\cdot\|_D = \|\cdot\|_1$ (see Section 1.3 in Falk (2016)).

Note that $\sigma_{ij} \ge 0$ for each pair i, j, i.e., the componentwise os are asymptotically positively correlated. This follows from the usual triangular inequality, satisfied by each norm, and the fact that a D-norm is in general standardized, i.e., $\|\boldsymbol{e}_j\|_D = 1$, $1 \le j \le d$.

Corollary 2.3. *If we choose identical k_i in the preceding result, i.e., $k_1 = \dots = k_d = k$, then we obtain under the conditions of Theorem 2.1*

$$\frac{n}{\sqrt{k}}\left(X_{n-k:n,i} - \frac{n-k}{n}\right)_{i=1}^d \to_D N(\boldsymbol{0}, \Sigma)$$

with

$$\Sigma = (\sigma_{ij}) = \begin{cases} 1, & \text{if } i = j \\ 2 - \|\boldsymbol{e}_i + \boldsymbol{e}_j\|_D, & \text{if } i \neq j. \end{cases}$$

Let $U_{1:n} \le U_{2:n} \le \cdots \le U_{n:n}$ denote the os of n independent and uniformly on $(0,1)$ distributed rv $U_1, \dots, U_n$. It is well-known that

$$(U_{i:n})_{i=1}^n =_D \left(\frac{\sum_{j=1}^{i} \eta_j}{\sum_{j=1}^{n+1} \eta_j} \right)_{i=1}^n,$$

where $\eta_1, \dots, \eta_{n+1}$ are iid standard exponential rv (see, e.g., Reiss 1989, Corollary 1.6.9).

Let $\xi_1, \xi_2, \dots, \xi_{2(n+1)}$ be iid standard normal distributed rv. From the fact that $\left(\xi_1^2 + \xi_2^2\right)/2$ follows the standard exponential distribution on $(0, \infty)$, we thus obtain (Reiss (1989, Problem 1.17)) the representation

$$(U_{i:n})_{i=1}^n =_D \left(\frac{\sum_{j=1}^{2i} \xi_j^2}{\sum_{j=1}^{2(n+1)} \xi_j^2} \right)_{i=1}^n. \tag{2.4}$$

Corollary 2.3 now opens a way to tackle at least partially and asymptotically a multivariate extension of the above representation (2.4).

Corollary 2.4. *Suppose that the $d \times d$-matrix Λ with entries*

$$\lambda_{ij} = \sigma_{ij}^{1/2} = \begin{cases} 1, & \text{if } i = j \\ \left(2 - \|\boldsymbol{e}_i + \boldsymbol{e}_j\|_D\right)^{1/2}, & \text{if } i \neq j \end{cases}$$

is positive semidefinite and let $\boldsymbol{\xi}^{(1)}, \boldsymbol{\xi}^{(2)}, \dots$ be independent copies of the random vector $\boldsymbol{\xi} = (\xi_1, \dots, \xi_d)$, which follows the normal distribution $N(\mathbf{0}, \Lambda)$ on $\mathbb{R}^d$. Then, we obtain under the conditions of Corollary 2.3

$$\sup_{\boldsymbol{x} \in \mathbb{R}^d} \left| P\left((X_{n-k:n,i})_{i=1}^d \le \boldsymbol{x} \right) - P\left(\left(\frac{\sum_{j=1}^{2(n-k)} \xi_i^{(j)^2}}{\sum_{j=1}^{2(n+1)} \xi_i^{(j)^2}} \right)_{i=1}^d \le \boldsymbol{x} \right) \right| \to_{n \to \infty} 0.$$

Note that the univariate marginal distributions in the above result coincide due to Eq. 2.4. If a matrix is positive semidefinite with nonnegative entries, the matrix of the square roots of its entries is not necessarily semidefinite again. Take, for example, the 3×3-matrix with rows $1, 0, a | 0, 1, a | a, a, 1$. This matrix is positive definite for $a = 3^{-1/2}$, but not for $a = 3^{-1/4}$. The matrix Λ is positive semidefinite, if the value of $\|\boldsymbol{e}_i + \boldsymbol{e}_j\|_D$ does not depend on the pair $i \neq j$, in which case Λ satisfies the *compound symmetry condition.*

PROOF. From Corollary 2.3, we obtain that

$$\frac{n}{\sqrt{k}}\left(X_{n-k:n,i}-\frac{n-k}{n}\right)_{i=1}^{d} \to_D N(\mathbf{0},\Sigma).$$

The assertion follows, if we establish

$$\frac{n}{\sqrt{k}}\left(\frac{\sum_{j=1}^{2(n-k)}\xi_i^{(j)2}}{\sum_{j=1}^{2(n+1)}\xi_i^{(j)2}}-\frac{n-k}{n}\right)_{i=1}^{d} \to_D N(\mathbf{0},\Sigma)$$

as well. But this follows from the central limit theorem and elementary arguments, using the fact that $\mathrm{Cov}(X^2,Y^2)=2c^2$, if (X,Y) is bivariate normal with $\mathrm{Cov}(X,Y)=c$.

The proof of Theorem 2.1 requires a suitable multivariate central limit theorem for arrays. To ease its reference, we state it explicitly here. It follows from the univariate version based on Lindeberg's condition (see, e.g., Billingsley (2012), together with the Cramér-Wold device). Recall that all operations on vectors are meant componentwise.

Lemma 2.5 (Multivariate Central Limit Theorem for Arrays)**.** *Let* $\boldsymbol{X}_n^{(1)},\dots,\boldsymbol{X}_n^{(n)}$ *be iid rv for each* $n\in\mathbb{N}$*, bounded by some constant* $\boldsymbol{c}=(c_1,\dots,c_d)>\mathbf{0}\in\mathbb{R}^d$ *and with mean zero. Suppose there is a sequence* $\boldsymbol{c}^{(n)}\in\mathbb{R}^d$ *with* $nc_i^{(n)}\to_{n\to\infty}\infty$ *for* $i=1,\dots,d$*, such that* $\mathrm{Cov}\left(\boldsymbol{X}_n^{(1)}\right)=C^{(n)}\Sigma^{(n)}C^{(n)}$*,* $n\in\mathbb{N}$*, where* $C^{(n)}=\mathrm{diag}\left(\sqrt{\boldsymbol{c}^{(n)}}\right)$ *and* $\Sigma^{(n)}\to_{n\to\infty}\Sigma$*. Then,*

$$\frac{1}{\sqrt{n\boldsymbol{c}^{(n)}}}\sum_{i=1}^{n}\boldsymbol{X}_n^{(i)} \to_D N(\mathbf{0},\Sigma).$$

PROOF OF THEOREM 2.1. Choose $\boldsymbol{x}=(x_1,\dots,x_d)\in\mathbb{R}^d$. Elementary arguments yield

$$\begin{aligned}
&P\left(\left(\frac{n}{\sqrt{k_i}}\left(X_{n-k_i:n,i}-\frac{n-k_i}{n}\right)\right)_{i=1}^{d}\le\boldsymbol{x}\right)\\
&=P\left(X_{n-k_i:n,i}\le\frac{\sqrt{k_i}}{n}x_i+\frac{n-k_i}{n},\ 1\le i\le d\right)\\
&=P\left(\sum_{j=1}^{n}1_{\left[0,\frac{\sqrt{k_i}}{n}x_i+\frac{n-k_i}{n}\right]}\left(X_i^{(j)}\right)\ge n-k_i,\ 1\le i\le d\right)\\
&=P\left(\left(\frac{1}{\sqrt{k_i}}\sum_{j=1}^{n}\left(\frac{\sqrt{k_i}}{n}x_i+\frac{n-k_i}{n}-1_{\left[0,\frac{\sqrt{k_i}}{n}x_i+\frac{n-k_i}{n}\right]}\left(X_i^{(j)}\right)\right)\right)_{i=1}^{d}\le\boldsymbol{x}\right). \qquad (2.5)
\end{aligned}$$

Put now

$$\boldsymbol{Y}^{(n)} := \left(Y_1^{(n)}, \dots, Y_d^{(n)}\right) := \left(1_{\left[0, \frac{\sqrt{k_i}}{n} x_i + \frac{n-k_i}{n}\right]}(X_i)\right)_{i=1}^d$$

with values in $\{0,1\}^d$. The entries of its covariance matrix $\Sigma^{(n)} = \left(\sigma_{ij}^{(n)}\right)$ are for $i \neq j$ given by

$$\begin{aligned}
\sigma_{ij}^{(n)} &= E\left(Y_i^{(n)} Y_j^{(n)}\right) - E\left(Y_i^{(n)}\right) E\left(Y_j^{(n)}\right) \\
&= P\left(Y_i^{(n)} = Y_j^{(n)} = 1\right) - P\left(Y_i^{(n)} = 1\right) P\left(Y_j^{(n)} = 1\right) \\
&= P\left(X_i \leq \frac{\sqrt{k_i}}{n} x_i + \frac{n-k_i}{n}, X_j \leq \frac{\sqrt{k_j}}{n} x_j + \frac{n-k_j}{n}\right) \\
&\qquad - P\left(X_i \leq \frac{\sqrt{k_i}}{n} x_i + \frac{n-k_i}{n}\right) P\left(X_j \leq \frac{\sqrt{k_j}}{n} x_j + \frac{n-k_j}{n}\right) \\
&= C_{ij}\left(\frac{\sqrt{k_i}}{n} x_i + \frac{n-k_i}{n}, \frac{\sqrt{k_j}}{n} x_j + \frac{n-k_j}{n}\right) \\
&\qquad - \left(\frac{\sqrt{k_i}}{n} x_i + \frac{n-k_i}{n}\right)\left(\frac{\sqrt{k_j}}{n} x_j + \frac{n-k_j}{n}\right)
\end{aligned}$$

if n is large, where

$$C_{ij}(u,v) := C\left(u\boldsymbol{e}_i + v\boldsymbol{e}_j + \sum_{1 \leq m \leq d,\, m \neq i,j} \boldsymbol{e}_m\right), \qquad u, v \in [0,1].$$

Expansion (2.3) now implies in case $i \neq j$

$$\begin{aligned}
\sigma_{ij}^{(n)} &= 1 - \left\| \left(\frac{k_i}{n} - \frac{\sqrt{k_i}}{n} x_i\right) \boldsymbol{e}_i + \left(\frac{k_j}{n} - \frac{\sqrt{k_j}}{n} x_j\right) \boldsymbol{e}_j \right\|_D + o\left(\frac{\sqrt{k_i k_j}}{n}\right) \\
&\qquad - \left(\frac{\sqrt{k_i}}{n} x_i - \frac{k_i}{n} + 1\right)\left(\frac{\sqrt{k_j}}{n} x_j - \frac{k_j}{n} + 1\right) \\
&= - \left\| \left(\frac{k_i}{n} - \frac{\sqrt{k_i}}{n} x_i\right) \boldsymbol{e}_i + \left(\frac{k_j}{n} - \frac{\sqrt{k_j}}{n} x_j\right) \boldsymbol{e}_j \right\|_D + \frac{k_i + k_j}{n} + o\left(\frac{\sqrt{k_i k_j}}{n}\right) \\
&= \frac{\sqrt{k_i k_j}}{n} \left(k_{ij} + k_{ji} - \|k_{ij}\boldsymbol{e}_i + k_{ji}\boldsymbol{e}_j\|_D + o(1)\right).
\end{aligned}$$

For $i = j$, one deduces

$$\sigma_{ii}^{(n)} = \frac{k_i}{n}(1 + o(1)).$$

The asymptotic normality $N(\mathbf{0}, \Sigma)(-\boldsymbol{\infty}, \boldsymbol{x}]$ of the final term in Eq. 2.5 now follows from Lemma 2.5.

3 Main Results: General Case

Let F be a df on $\mathbb{R}^d$ with univariate margins $F_1, \ldots, F_d$. From Sklar's theorem (Sklar 1959, 1996), we know that there exists a copula C on $\mathbb{R}^d$ such that $F(\boldsymbol{x}) = C(F_1(x_1), \ldots, F_d(x_d))$ for each $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

Let $\boldsymbol{X}^{(1)}, \boldsymbol{X}^{(2)}, \ldots$ be independent copies of the random vector $\boldsymbol{X}$, which follows this df F. We can assume the representation

$$\boldsymbol{X} = \left(F_1^{-1}(U_1), \ldots, F_d^{-1}(U_d)\right),$$

where $\boldsymbol{U} = (U_1, \ldots, U_d)$ follows the copula C and $F_i^{-1}(u) := \inf\{t \in \mathbb{R} : F_i(t) \geq u\}$, $u \in (0, 1)$, is the generalized inverse of F_i, $1 \leq i \leq d$. Equally, we can assume the representation

$$\boldsymbol{X}^{(j)} = \left(F_1^{-1}\left(U_1^{(j)}\right), \ldots, F_d^{(-1)}\left(U_d^{(j)}\right)\right), \qquad j = 1, 2, \ldots$$

where $\boldsymbol{U}^{(1)}, \boldsymbol{U}^{(2)}, \ldots$ are independent copies of $\boldsymbol{U}$.

Put $\omega(F_i) := \sup\{x \in \mathbb{R} : F_i(x) < 1\} \in (-\infty, \infty]$, the upper endpoint of the support of F_i, and suppose that the derivative $F_i' = f_i$ exists and is positive throughout some left neighborhood of $\omega(F_i)$. Let $k_i = k_i(n) \in \{1, \ldots, n\}$ satisfy $k_i \to_{n\to\infty} \infty$, $k_i/n \to_{n\to\infty} 0$. It follows from Falk (1989, Theorem 2.1) that under appropriate von Mises type conditions on F_i stated below

$$\frac{X_{n-k_i+1:n,i} - d_{ni}}{c_{ni}} \to_D N(0, 1)$$

for any sequences $c_{ni} > 0$, $d_{ni} \in \mathbb{R}$, which satisfy

$$\lim_{n\to\infty} \frac{c_{ni}}{a_{ni}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{d_{ni} - b_{ni}}{a_{ni}} = 0, \tag{3.1}$$

where

$$b_{ni} := F_i^{-1}\left(1 - \frac{k_i}{n}\right), \quad a_{ni} := \frac{k_i^{1/2}}{n f_i(b_{ni})}, \qquad 1 \leq i \leq d.$$

Theorem 1 of Smirnov (1967) shows that the distribution of $c_n^{-1}(X_{n-k_i+1:n}-d_n)$ converges weakly to $N(0,1)$ for *some* choice of constants $c_n>0$, $d_n\in\mathbb{R}$, if and only if for any $x\in\mathbb{R}$

$$\lim_{n\to\infty}\frac{k_i+n(F_i(c_nx+d_n)-1)}{k_i^{1/2}}=x. \tag{3.2}$$

Next, we state the three von Mises type conditions, under which we have asymptotic normality for intermediate multivariate os in the general case:

$\omega(F_i)\in(-\infty,\infty]$ and

$$\lim_{x\uparrow\omega(F_i)}\frac{f_i(x)\int_x^{\omega(F_i)}1-F_i(t)\,dt}{(1-F_i(x))^2}=1, \qquad \text{(von Mises (1))}$$

$\omega(F_i)=\infty$, and there exists $\alpha_i>0$ such that

$$\lim_{x\to\infty}\frac{xf_i(x)}{1-F_i(x)}=\alpha_i, \qquad \text{(von Mises (2))}$$

$\omega<\infty$, and there exists $\alpha>0$ such that

$$\lim_{x\uparrow\omega(F_i)}\frac{(\omega(F_i)-x)f_i(x)}{1-F_i(x)}=\alpha_i. \qquad \text{(von Mises (3))}$$

The standard normal df as well as the df of the standard exponential df satisfy condition (1); the standard Pareto df $F_\alpha(x)$, $x\geq 1$, $\alpha>0$, satisfies condition (2) and the triangular df on $(-1,1)$ with density $f(x)=1-|x|$, $x\in(-1,1)$, satisfies condition (3) with $\alpha=2$, for example. For a discussion of these well-studied and general conditions, each of which ensures that F_i is in the domain of attraction of a univariate EVD (see, e.g. Falk (1989)).

The following generalization of Theorem 2.1 can now easily be established.

Proposition 3.1. *Suppose that the copula C of F satisfies condition (2.3), i.e., C is in the max-domain of attraction of a nondegenerate multivariate EVD, and suppose that each univariate margin F_i of F satisfies one of the von Mises type conditions (1), (2), or (3).*

Let $\boldsymbol{k}=\boldsymbol{k}^{(n)}\in\{1,\dots,n\}^d$, $n\in\mathbb{N}$ satisfy $k_i/k_j\to_{n\to\infty}k_{ij}^2$ for all pairs of components $i,j=1,\dots,d$, $\|\boldsymbol{k}\|\to_{n\to\infty}\infty$ and $\|\boldsymbol{k}\|/n\to_{n\to\infty}0$. Then, the vector of intermediate multivariate os satisfies

$$\left(\frac{X_{n-k_i+1:n,i}-d_{ni}}{c_{ni}}\right)_{i=1}^d\to_D N(\mathbf{0},\Sigma)$$

with Σ as in Theorem 2.1 for any sequences $c_{ni}>0$, $d_{ni}\in\mathbb{R}$ which satisfy (3.1).

PROOF. We have for $\boldsymbol{x} = (x_1, \dots, x_d) \in \mathbb{R}^d$

$$
\begin{aligned}
&P\left(\frac{X_{n-k_i+1:n,i} - d_{ni}}{c_{ni}} \le x_i,\, 1 \le i \le d\right) \\
&= P\left(F_i^{-1}\left(U_{n-k_i+1:n,i}\right) \le c_{ni}x_i + d_{ni},\, 1 \le i \le d\right) \\
&= P\left(\frac{n}{k_i^{1/2}}\left(U_{n-k_i+1:n,i} - \frac{n-k_i}{n}\right) \le \frac{k_i + n\left(F_i\left(c_{ni}x_i + d_{ni}\right) - 1\right)}{k_i^{1/2}}\right).
\end{aligned}
$$

The assertion is now immediate from Theorem 2.1 and Smirnov's condition (3.2).

References

ARNOLD, B. C., BALAKRISHNAN, N. and NAGARAJA, H. N. (2008). *A First Course in Order Statistics.* Society for Industrial and Applied Mathematics, Philadelphia.

BALKEMA, A. A. and DE HAAN, L. (1978a). Limit distributions for order statistics i. *Theory Probab. Appl.* **23**, 77–92.

BALKEMA, A. A. and DE HAAN, L. (1978b). Limit distributions for order statistics ii. *Theory Probab. Appl.* **23**, 341–358.

BALKEMA, A. A. and RESNICK, S. I. (1977). Max-infinite divisibility. *J. Appl. Probab.* **14**, 309–319. doi: 10.2307/3213001.

BARAKAT, H. M. (2001). The asymptotic distribution theory of bivariate order statistics. *Ann. Inst. Statist. Math.* **53**, 487–497.

BARAKAT, H. M. and NIGM, E. M. (2012). On multivariate order statistics: Asymptotic theory and computing moments. *Kuwait J. Sci. Eng.* **39**, 113–127.

BARAKAT, H. M., NIGM, E. M. and AL-AWADY, M. A. (2015). Asymptotic properties of multivariate order statistics with random index. *Bulletin of the Malaysian Mathematical Sciences Society* **38**, 289–301. doi: 10.1007/s40840-014-0019-7.

BEIRLANT, J., GOEGEBEUR, Y., SEGERS, J. and TEUGELS, J. (2004). *Statistics of Extremes: Theory and Applications.* Wiley Series in Probability and Statistics. Wiley, Chichester. doi: 10.1002/0470012382.

BILLINGSLEY, P. (2012). *Probability and Measure.* Wiley Series in Probability and Statistics, Anniversary ed. Wiley, New York.

CHARPENTIER, A. and SEGERS, J. (2009). Tails of multivariate Archimedean copulas. *J. Multivariate Anal.* **100**, 1521–1537. doi: 10.1016/j.jmva.2008.12.015.

CHENG, S., DE HAAN, L. and YANG, J. (1997). Asymptotic distributions of multivariate intermediate order statistics. *Theory Probab. Appl.* **41**, 646–656. doi: 10.1137/S0040585X97975733.

COOIL, B. (1985). Limiting multivariate distributions of intermediate order statistics. *Ann. Probab.* **13**, 469–477.

DAVID, H. (1981). *Order Statistics.* Wiley Series in Probability and Mathematical Statics, 2nd ed. Wiley, New York.

DAVID, H. and NAGARAJA, H. (2004). *Order Statistics.* Wiley Series in Probability and Mathematical Statics, 3rd ed. Wiley, New York.

FALK, M. (1989). A note on uniform asymptotic normality of intermediate order statistics. *Ann. Inst. Statist. Math.* **41**, 19–29.

FALK, M. (2016) An offspring of multivariate extreme value theory: D-norms, http://www.statistik-mathematik.uni-wuerzburg.de/fileadmin/10040800/D-norms-tutorial_book.pdf.

FALK, M., HÜSLER, J. and REISS, R.-D. (2011). *Laws of Small Numbers: Extremes and Rare Events*, 3rd ed. Springer, Basel. doi: 10.1007/978-3-0348-0009-9.

GALAMBOS, J. (1975). Order statistics of samples from multivariate distributions. *Journal of the American Statistical Association* **70**, 674–680.

GALAMBOS, J. (1987). *The Asymptotic Theory of Extreme Order Statistics*, 2nd ed. Krieger, Malabar.

DE HAAN, L. and FERREIRA, A. (2006). *Extreme Value Theory: An Introduction.* Springer Series in Operations Research and Financial Engineering. Springer, New York. doi: 10.1007/0-387-34471-3. See http://people.few.eur.nl/ldehaan/EVTbook.correction.pdf and http://home.isa.utl.pt/~anafh/corrections.pdf for corrections and extensions.

DE HAAN, L. and RESNICK, S. (1977). Limit theory for multivariate sample extremes. *Probab. Theory Related Fields* **40**, 317–337. doi: 10.1007/BF00533086.

MCNEIL, A. J. and NEŠLEHOVÁ, J. (2009). Multivariate archimedean copulas, d-monotone functions and ℓ_1-norm symmetric distributions. *Ann. Statist.* **37**, 3059–3097. doi: 10.1214/07-AOS556.

NELSEN, R. B. (2006). *An Introduction to Copulas.* Springer Series in Statistics, 2nd ed. Springer, New York. doi: 10.1007/0-387-28678-0.

REISS, R.-D. (1989). *Approximate Distributions of Order Statistics: With Applications to Nonparametric Statistics.* Springer Series in Statistics. Springer, New York. doi: 10.1007/978-1-4613-9620-8.

RESNICK, S. I. (1987). *Extreme Values, Regular Variation, and Point Processes, Applied Probability,* vol. 4. Springer, New York. doi: 10.1007/978-0-387-75953-1. First Printing.

SKLAR, A. (1959). Fonctions de répartition à n dimensions et leurs marges. *Pub. Inst. Stat. Univ. Paris* **8**, 229–231.

SKLAR, A. (1996). Random variables, distribution functions, and copulas – a personal look backward and forward. In *Distributions with fixed marginals and related topics* (L. Rüschendorf, B. Schweizer, and M. D. Taylor, eds.)*, Lecture Notes – Monograph Series,* vol. 28, 1–14. Institute of Mathematical Statistics, Hayward. doi: 10.1214/lnms/1215452606.

SMIRNOV, N. V. (1967). Some remarks on limit laws for order statistics. *Theory. Probab. Appl.* **12**, 337–339.

VATAN, P. (1985). Max-infinite divisibility and max-stability in infinite dimensions. In *Probability in Banach Spaces V: Proceedings of the International Conference held in Medford, USA, July 16-27, 1984* (A. Beck, R. Dudley, M. Hahn, J. Kuelbs, and M. Marcus, eds.)*, Lecture Notes in Mathematics,* vol. 1153, 400–425. Springer, Berlin. doi: 10.1007/BFb0074963.

Michael Falk
Florian Wisheckel
Institute of mathematics,
University of Würzburg,
Emil-Fischer-Str. 30, 97074
Würzburg, Germany
E-mail: michael.falk@uni-wuerzburg.de
florian.wisheckel@uni-wuerzburg.de

Paper received: 20 July 2016.