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New and Fast Block Bootstrap-Based Prediction Intervals for GARCH(1,1) Process with Application to Exchange Rates

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Abstract

In this paper, we propose a new bootstrap algorithm to obtain prediction intervals for generalized autoregressive conditionally heteroscedastic (GARCH(1,1)) process which can be applied to construct prediction intervals for future returns and volatilities. The advantages of the proposed method are twofold: it (a) often exhibits improved performance and (b) is computationally more efficient compared to other available resampling methods. The superiority of this method over the other resampling method-based prediction intervals is explained with Spearman's rank correlation coefficient. The finite sample properties of the proposed method are also illustrated by an extensive simulation study and a real-world example.

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1 Introduction

Measuring volatility and construction of valid predictions for future returns and volatilities have an important role in assessing risk and uncertainty in the financial market. To this end, the generalized autoregressive conditionally heteroscedastic (GARCH) model proposed by Bollerslev (1986) is one of the most commonly used techniques for modeling volatility and obtaining dynamic prediction intervals for returns as well as volatilities. Andersen and Bollerslev (1998), Andersen et al. (2001), Baillie and Bollerslev (1992), & Engle and Patton (2001) provide an excellent overview of research on prediction intervals for future returns in financial time series analysis. However, those works only consider point forecasts of volatility even though prediction intervals provide better inferences taking into account uncertainty of unobservable sequence of volatilities. Technically, construction of such prediction intervals requires some distributional assumptions which are generally unknown in practice. Moreover, the constructed prediction intervals along with the estimated parameter values can be affected due to any departure from the assumptions and may lead us to unreliable results. One of the remedy to construct prediction intervals without considering distributional assumptions is to use the well-known resampling methods, e.g., the bootstrap.

It is well-known that the original nonparametric bootstrap proposed by Efron (1979) fails to provide satisfactory answers to general statistical inference problems for dependent data, since the assumption of independently and identically distributed (i.i.d.) data is violated (see Lahiri 2003 and Hall 1992 for more details). As a way to deal with dependent data, several types of resampling techniques were proposed. Among those, one of the most general tools to approximate the properties of estimators for serially correlated data is the method of block bootstrap. The main idea behind this method is to construct a resample of the data of size n by dividing the data into several blocks and choosing among them till the bootstrap sample is obtained. The commonly used procedures to implement block bootstrap called "non-overlapping" and "overlapping" blocking are proposed by Hall (1985) in the context of spatial data. In the univariate time series context, the non-overlapping block bootstrap (NBB) approach is proposed by Carlstein (1986), and overlapping blocks known as moving block bootstrap (MBB) is proposed by Kunsch (1989). In addition to these methods, a circular block bootstrap (CBB) method is suggested by Politis and Romano (1992), where the data is wrapped around a circle so that each observation in the original data set has an equal probability to appear in a bootstrap sample. Also, the stationary bootstrap (SB) method which deals with random block lengths which have a geometric distribution is proposed by Politis and Romano (1994).

In all of the above blocking techniques, the idea is to specify a sufficiently large block length ℓ so that the data points which are ℓ units apart are practically independent. Then, the dependence structure of the original data is attempted to be captured by these ℓ consecutive observations in each block drawn independently. However while doing so, the correlation structure is broken while moving from one block to another block. Conceptually, obtaining better estimates could be achieved by creating resamples "similar" to the original data, as these could help us in preserving a dependency structure close to the original leading us in obtaining more precise estimates of the actual parameters. Ordered non-overlapping block bootstrap (ONBB), proposed by Beyaztas et al. (2016), improved the performance of the block bootstrap technique by taking into account the correlations between the blocks. The authors empirically proved that the ONBB method often exhibits improved performance over the conventional block bootstrap methods in terms of parameter and coverage probability estimations for univariate linear time series models. They performed a simulation study based on autoregressive (AR) of order 2 and moving average (MA) of order 2 models with different sample sizes and block lengths. Their results show that the ONBB method produces close estimations to the true values of the statistics especially to the second parameter with the increasing ℓ, and this result yields a confidence interval having better coverage probabilities. On the other hand, they failed to provide any information about the correlation structure between the blocks. In this paper, (i) we show that the Spearman's rank correlation between the ONBB resample and original data is always positive (and ≥ 0.5) and stronger than those for conventional block bootstrap methods, which gives a justification for the superiority of the ONBB method. The similarities of the resamples obtained by the block bootstrap to the original data is shown using the dynamic time warping measure. (*ii*) Also, we extend the ONBB method to GARCH(1,1) process to obtain prediction intervals for future returns and volatilities. In summary, our extension works as follows: first, we use the squares of the GARCH process, which have the autoregressive- moving average (ARMA) representation, to make the the parameter estimation process linear. The ordinary least squares estimators of the ARMA model are calculated by a high-order autoregressive model of order m, and the residuals are computed. Then, the ONBB method is applied to the data to obtain the bootstrap sample of the returns which are used to calculate the ONBB estimators of the ARMA coefficients and the bootstrap sample of the volatilities. Finally, the future values of the returns and volatilities of the GARCH process are obtained by means of bootstrap replicates and quantiles of the Monte Carlo estimates of the ONBB distribution.

The rest of the paper is organized as follows. We describe the data in Section 2. In Section 3, we provide a detailed information on the ONBB method and the correlation structures between the resampled and original blocks. In Section 4, we propose a new, computationally efficient bootstrap algorithm based on the ONBB method to obtain prediction intervals for future returns and volatilities of GARCH(1,1) process. An extensive Monte Carlo simulation is conducted to examine the finite sample performance of the proposed method, and the results are presented in Section 5. Finally, the Australian dollar/U.S. dollar (AUD/USD) daily exchange rate data is analyzed using the new method, and the results are presented in Section 6, followed by some concluding remarks in Section 7.

2 Data

The exchange rate is regarded as the value of a specific country's currency in terms of another currency and obtaining valid prediction intervals of exchange rates is often essential to evaluate foreign denominated cash flows related to international transactions. For instance, exchange brokers, central banks, international traders, and investors require prediction intervals of future returns and volatilities for many reasons, e.g., option pricing, to determine the next target zone of the exchange rate of interest, and for international portfolio diversification. In recent years, Australian dollar (AUD) has become one of the most traded currencies in the world, and it has an important position in Asian import market. Also, AUD is an attractive currency for the investors, and it is often used in carrying out trades with other currencies due to the strength of Australian economy. On the other hand, from global perspective, the bilateral exchange rate with U.S. dollar (USD) has a great effect on the trading volume of the AUD in the foreign exchange market because USD is the dominant currency against almost all currencies in the world. Therefore, multi-step ahead prediction intervals of levels and volatilities of the bilateral AUD/USD exchange rate are crucial for the international firms and investors who import and export with Australia and use AUD as an investment.

The AUD/USD daily exchange rate data were obtained starting from 29th July, 2011 and ending on 3rd November, 2015 (available at https://www.stlouisfed.org/). After excluding observations on weekends and inactive days, our final data consisted a total of 1070 observations. From the original data, the daily logarithmic returns

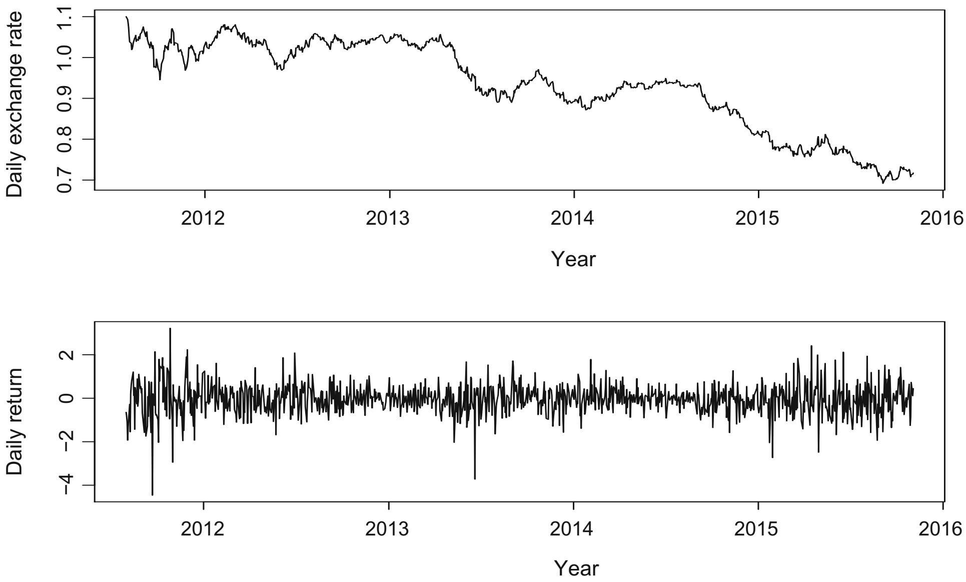

Figure 1: Time series plots of AUD/USD daily exchange rates and returns from 29th July, 2011 to 3rd November, 2015

were obtained as $y_t = 100 * \log(P_t/P_{t-1})$, where P_t was the closing price on t-th day. The time series plots of the exchange rates and returns are presented in Fig. 1. We checked the stationary status of the return series by applying the Ljung-Box (LB) and augmented Dickey-Fuller (ADF) t-statistic tests, and small p values (p value = 0.017 for the LB test and 0.010 for the ADF test) suggest that the return series is a mean-zero stationary process. Table 1 reports the sample statistics of y_t series, and it shows that the estimated kurtosis is higher than 3 which indicates that the distribution of the returns was leptokurtic. Next, we checked for the Gaussianity of the return series, and the p value < 0.001 of Jarque-Bera test indicated that y_t was not Gaussian. Further, we performed the LB test to test for auto-correlations in the absolute and squared returns, and smaller p values indicated that the absolute and squared returns are highly auto-correlated. The auto-correlations of returns, absolute and squared returns are presented in Table 2. All of our preliminary exploratory analyses suggested the presence of conditional heteroscedasticity in the series. To find the optimal lag for the GARCH model to model the return series, we defined many possible subsets of the GARCH(p, q) models with different p and q values. To choose the best model, we used Akaike information criterion (AIC) (since it is proposed to determine the best model for forecasting), and the results show that

Table 1: Sample statistics for y_t

T	Mean	Median	SD	Skewness	Kurtosis	Min.	Max.
1069	−0.04	−0.04	0.7	−0.27	6.15	−4.46	3.21

Table 2: Autocorrelations of y_t at lag k

Autocorrelations	$r(1)$	$r(2)$	$r(3)$	$r(8)$	$r(9)$	$r(10)$	$r(19)$	$r(20)$
y_t	−0.021	0.015	−0.027	0.042	0.030	−0.097	0.072	−0.019
$\|y_t\|$	0.058	0.101	0.114	0.120	0.167	0.064	0.134	0.089
y_t^2	0.016	0.040	0.166	0.102	0.146	0.046	0.096	0.061

GARCH(1,1) model is optimal according to AIC. Also, Hansen and Lunde (2005) compares a large number of volatility models to describe the conditional variance in an extensive empirical study based on exchange rate data. Their results show that the GARCH(1,1) model provides significantly better forecasts for exchange rates than the other models. In light of these results, we consider a GARCH(1,1) model as a suitable choice to model the return series.

3 Methodology

Let $\chi_n = \{X_1, \ldots, X_n\}$ be a sequence of stationary dependent random variables of size n having an unknown common distribution function F, whose parameter θ_0 is of our interest. We further assume that the distribution has a finite mean μ and a finite variance σ^2, both unknown. Let $\hat{\theta}_n$ be the estimator of θ_0 based on χ_n. Suppose $B_1, \ldots, B_b$ be the non-overlapping blocks where $B_i = (X_{(i-1)\ell+1}, \ldots, X_{i\ell})$ for $i = 1, \ldots, b$. In conventional NBB, b blocks are drawn independently from $B_1, \ldots, B_b$ and pasted end-to-end to form a bootstrap sample. ONBB is proposed as ordering the bootstrapped blocks according to given labels to each original block for capturing more dependence structure compared to the conventional NBB method. In more detail, suppose the data is divided into the four independent blocks which are non-overlapping. In this case, the labels are determined as $B_1 = 1$, $B_2 = 2$, $B_3 = 3$, and $B_4 = 4$, and let the bootstrapped blocks are $B_1^* = B_4$, $B_2^* = B_2$, $B_3^* = B_3$, and $B_4^* = B_3$. As a consequence, the new data is obtained using the NBB method as $\chi^*_{\text{NBB}} = \{B_4 \vdots B_2 \vdots B_3 \vdots B_3\}$ whereas it is obtained as $\chi^*_{\text{ONBB}} = \{B_2 \vdots B_3 \vdots B_3 \vdots B_4\}$ when ONBB is used. From this example, it can be seen immediately that more "representative" data sets can be formed by the ONBB method.

To provide a statistical explanation on the superiority of ONBB, we use Spearman's rank correlation between the given labels of the original and bootstrapped blocks. Let j and $k_{(j)}$ be the given labels of the original and NBB blocks in the j-th order, respectively, where $j, k_{(j)} = 1, \ldots, b$. Also, let $R_{k_{(j)}}$ and m_k denote the rank of $k_{(j)}$ and frequency of the block k respectively, where $R_k = \sum_{i=1}^{k-1} m_{k-i} + (m_k+1)/2$, $0 \leq R_k \leq b$, $\sum_{k=1}^{b} m_k = b$, and $m_k = 0, 1, \ldots, b$. Then, the Spearman's rank correlation coefficient between the original and NBB block labels is obtained as $\rho_{\text{Original,NBB}} = 1 - (6\sum_{j=1}^{b} d_j^2)/(b^3 - b)$, where $\sum_{j=1}^{b} d_j^2 = \sum_{j=1}^{b} (j - R_{k_{(j)}})^2 = (1 - R_{k_{(1)}})^2 + \ldots + (b - R_{k_{(b)}})^2$.

Based on the above notations, the following theorem provides an explanation for why ONBB-based sample should be better representative of the original sample than that obtained using the NBB method. The proof of the theorem has been relegated to Appendix.

Theorem 1. *It can be shown that,*

$$\rho_{\text{Original,NBB}} \leq \rho_{\text{Original,ONBB}}.$$

Corollary 1. *It can further be shown that the individual Spearman's correlation coefficients have the following ranges:*

$$-1 \leq \rho_{\text{Original,NBB}} \leq 1, \ \textit{and}, \ 0.5 \leq \rho_{\text{Original,ONBB}} \leq 1.$$

Theorem 1 shows that the bootstrap data obtained by ONBB is more representative of the original data than the one obtained by the NBB method. Thus, ONBB allows us to obtain better estimates and more reliable bootstrap quantities such as confidence interval of $\hat{\theta}_n$.

Remark 1. *For the process $\{X_t\}_{t\in Z}$ which has a strong $\alpha(\cdot)$ mixing condition (see* Bilingsley 1994*),* Athreya and Lahiri (1994) *show that under mild moment conditions, the NBB variance estimator of the statistic $T_n = \sqrt{n}(\bar{X}_n - \mu)$, $T_n^* = \sqrt{n}(\bar{X}_n^* - E^*(\bar{X}_n))$, is $Var(T_n^*) \to \sigma_\infty^2$ as $n \to \infty$ where $Z_i = X_i - \mu$ and $\sigma_\infty^2 = \sum_{i=-\infty}^{\infty} EZ_1 Z_{i+1}$. Also, Theorem 17.4.3 in* Athreya and Lahiri (1994) *show that $\sup_{x\in R} |P^*(T_n^* \leq x) - P(T_n \leq x)| \to 0$ as $n \to \infty$. This means the sampling distribution G_n and its NBB estimator $\hat{G}_n$ are consistent when ℓ goes to infinity at a slower rate compared to sample size n ($\ell^{-1}n = o(1)$ as $n \to \infty$). It is clear to say that the ONBB method provides consistent estimators since sorting the bootstrapped blocks does not change the estimated values such as μ, σ^2, and median. As mentioned in Section 1, the ONBB method improves the performance of the block bootstrap technique by taking into account the correlations between the blocks, which leads us to have closer results to the true values of the statistics of interest. Also, the better estimates obtained by changing the correlation structure of the resampled time series affects the size of the prediction interval and provides more reliable results (please see the numerical results given in Section 5). It should be noted that the superiority of the ONBB is not a consequence of either Edgeworth or Cornish-Fisher expansion. The main reason is based on having more representative bootstrap data sets by this method as shown in Theorem 1.*

Remark 2. *The construction of the ONBB bears the question of whether the new bootstrap method produces bootstrap samples that generate enough "new" information on the time series or if the suggested ordering might limit bootstrap samples look too "similar". As it is known, each block has an equal probability $1/b$ to appear in a resample so that there are b^b number of distinct NBB samples. On the other hand, for the ONBB method, there are $\binom{2b-1}{b}$ number of distinct bootstrap samples since the order of the bootstrapped blocks is not important (it automatically puts in order the bootstrapped blocks). Let #NBB and #ONBB denote the number of distinct resamples generated by the NBB and ONBB methods, respectively. To answer this question, we carry out a simulation with $B = 10000$ bootstrap resamples, and calculate the proportion of (#NBB $\Big/$ #ONBB) along with the number of blocks b. The results are shown in Fig. 2. Note that by Fig. 2 we can say that the ONBB*

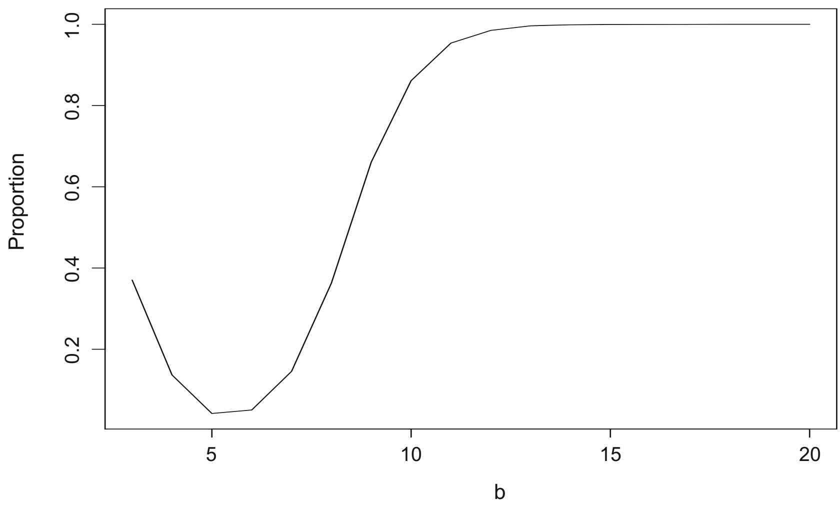

Figure 2: Proportion of the number of distinct resamples generated by the NBB and ONBB methods

produces bootstrap samples that generate enough new information on the time series as $n \to \infty$ *and* $\ell^{-1} + n^{-1}\ell = o(1)$.

To show the superiority of the ONBB, we use the dynamic time warping (DTW) dissimilarity measure. DTW considers time axis offsets of two time series, say $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. The computed value is based on a distance matrix $D_{n \times m}$ whose elements $d_{i,j}$ represents the distance $d(x_i, y_j) = (x_i - y_j)^2$, that is the (i, j) element measures the strength of alignment of points x_i and y_j. This measure is used to find the best synchronization of these two time series by minimizing warping cost $\mathrm{DTW}(X, Y) = \min\left\{\sqrt{\sum_{k=1}^{K} w_k}\right\}$, $\max(m, n) \leq K \leq m + n - 1$, where w_k is the element of warping path W, a function of the distance matrix D. That is, this measure can be used to find out which block bootstrap method has the best matched resample with the original time series. A detailed discussion on the DTW measure can be found in Giorgino (2009) and Ratanamahatana and Keogh (2004). DTW measure has also been implemented in `TSclust` and `dtw` packages in the `R` software.

Generally, DTW is an algorithm for comparing and aligning two sequences of data. The aim of the algorithm is to find an optimal match between two sequences by warping the time axis. To compare the block bootstrap methods in terms of their representativeness to the original dataset, we performed thorough simulation studies for AR(1) and ARMA(1,1) models with various parameters and sample sizes with block length $\ell = n^{1/3}$. Since the results and our conclusions do not vary significantly with different choices of parameter values, therefore to save space, we present only the results obtained for the choices of autoregressive parameter $\alpha = 0.2$ and moving average parameter $\beta = 0.4$. The number of bootstrap replications B and

Monte Carlo simulations MC are set at $B = \text{MC} = 1000$. For each simulation, we record DTW distances for all block bootstrap methods. The results are presented in Fig. 3. Since the calculated values are distances, the method which has the smallest DTW values can be considered as the best method compared to others. As it is shown in Fig. 3, the DTW distance values obtained by the NBB, MBB, CBB, and SB are very close to each other and their lines are overlapping. On the other hand, it is clear that the ONBB has considerable small values compared to other block bootstrap methods. Also, in Fig. 4, we plot the DTW densities of the block bootstrap methods for a simulated AR(1) sequence with the sample size $n = 64$. In this figure, `Reference index` and `Query index` stand for the time index of original time series and resampled series, respectively, obtained by the block bootstrap method. The blue trace represents the warping path of the corresponding block bootstrap method. In this figure, the best alignment between two sequences is equivalent as finding the shortest path to go from the bottom-left to the top-right of the plot. Thus, the block bootstrap method which has a working path close to the diagonal produces more representative resamples of the original time series data. Clearly, considering the DTW analysis results in Figs. 3 and 4, the most representative resamples are produced by the ONBB method compared to other block bootstrap methods, and these results further support Theorem 1.

4 ONBB Prediction Intervals for GARCH(1,1) Model

As noted earlier, construction of prediction intervals for future returns and volatilities is an important problem in financial markets. However, the estimation of parameters and the construction of prediction intervals may be affected by a great amount due to any departure from these assumptions and may lead us to

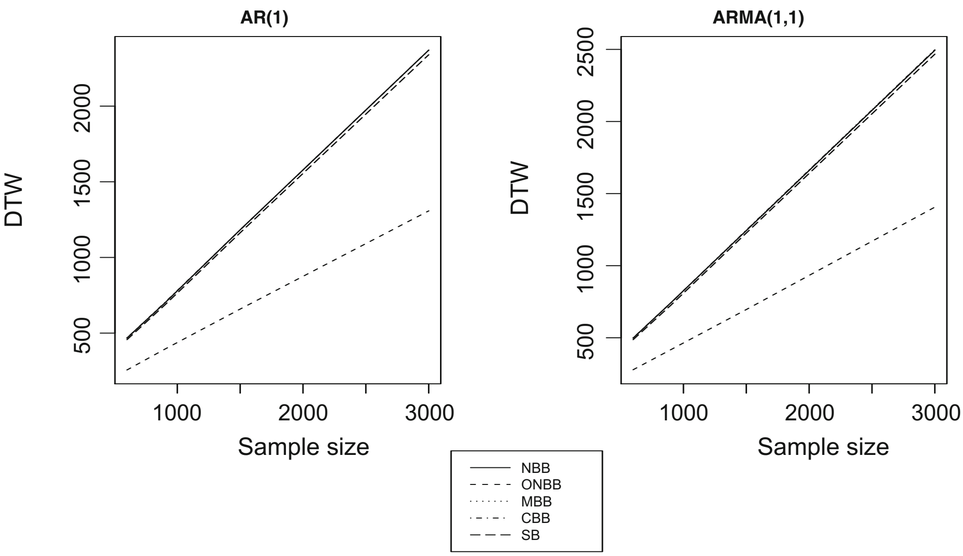

Figure 3: DTW distance values of block bootstrap methods

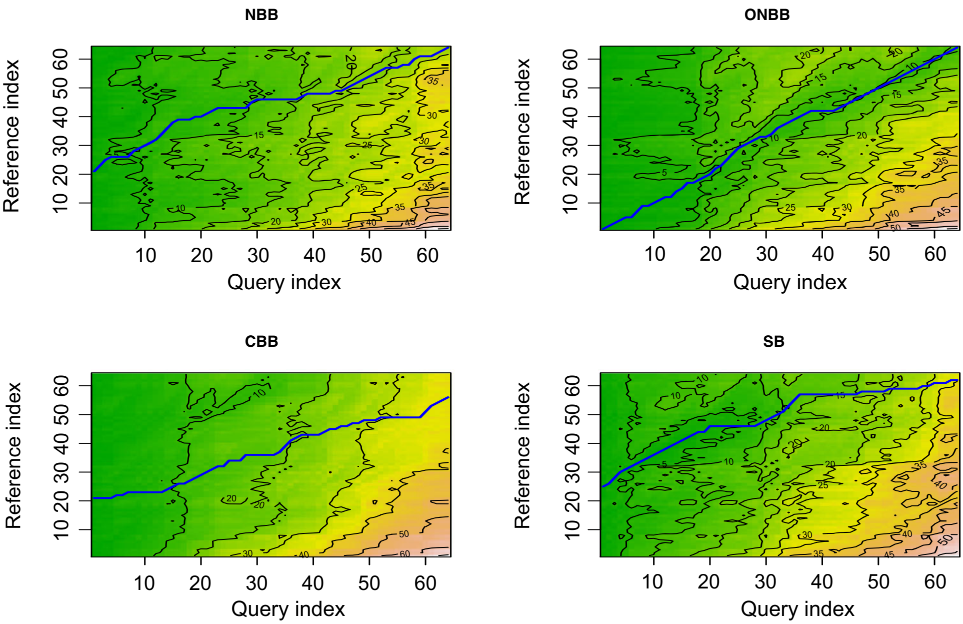

Figure 4: DTW density plots of block bootstrap methods

unreliable results. Resampling-based prediction interval is one of the possible solutions to overcome this vexing issue since it does not require the full knowledge of the underlying data and distributional assumptions. In this context, bootstrap-based prediction intervals of autoregressive conditionally heteroscedastic (ARCH) model for future returns and volatilities by resampling residuals are proposed by Miguel and Olave (1999) and Reeves (2005). Pascual et al. (2006) further extend the previous works to construct bootstrap-based prediction intervals for returns and volatilities for GARCH(1,1) models. Later, Chen et al. (2011) suggest a computationally efficient bootstrap prediction intervals for ARCH and GARCH processes in the context for financial time series. Also, Hwang and Shin (2013) develop a stationary bootstrap prediction interval for GARCH models and provide a mathematical justification for this method. Generally, block bootstrap is not suitable for construction of prediction intervals in conditionally heteroscedastic time series models because of its poor finite sample performance. However, our ONBB method overcomes this shortcoming and can be used to obtain reliable prediction intervals for future returns and volatilities.

To start with, we use ARMA representation of a GARCH(1,1) model and its least squares (LS) estimators in order to employ ONBB method for constructing prediction intervals. The GARCH(1,1) process has the following representation:

$$
\begin{aligned}
y_t &= \sigma_t \epsilon_t, \\
\sigma_t^2 &= \omega + \alpha y_{t-1}^2 + \beta \sigma_{t-1}^2, \quad t = 1, \ldots, T,
\end{aligned}
$$

where $\{\epsilon_t\}$ is a sequence of i.i.d. random variables with zero mean, unit variance, and $E(\epsilon^4) < \infty$, and ω, α, and β are unknown parameters satisfying $\omega \geq 0$, $\alpha \geq 0$, and $\beta \geq 0$. The stochastic process σ_t is assumed to be independent of ϵ_t. Throughout this paper, we assume that the process $\{y_t\}$ is strictly stationary, i.e., $E[\log(\beta + \alpha\epsilon_t^2)] < 1$ and the strict stationary conditions of y_t as in Nelson (1990) hold. A GARCH(1,1) process $\{y_t\}$ is represented in the form of ARMA(1,1) as follows:

$$y_t^2 = \omega + (\alpha + \beta)y_{t-1}^2 + \nu_t - \beta\nu_{t-1},$$

where the innovation $\nu_t = y_t^2 - \sigma_t^2$ is a white noise (not i.i.d. in general) and identically distributed under the strict stationary assumption of y_t. According to Hannan and Rissanen (1982), the LS estimators for an ARMA(1,1) model are obtained as follows: (a) fit a high-order autoregressive model of order m, AR(m), with $m > 1$, to the data by Yule-Walker method to obtain $\hat{\nu}_t$. (b) A linear regression of y_t^2 onto y_{t-1}^2 is fitted to estimate the parameter vector $\phi = (\omega, (\alpha + \beta), -\beta)'$. In more detail, let $y_t^2 - \alpha y_{t-1}^2 = \nu_t + \beta\nu_{t-1}$ be the ARMA(1,1) representation of the underlying GARCH(1,1) process, where $\{\nu_t\} \sim wn(0, \sigma^2)$. Then, in step (a), an AR(m) model is fitted to the data to obtain $\hat{\nu}_t$ such that $\hat{\nu}_t = y_t^2 - \hat{\alpha}_{m1}y_{t-1}^2 - \ldots - \hat{\alpha}_{mm}y_{t-m}^2$ for $t = m+1, \ldots, n$. In step (b), a linear regression $y_t^2 = \omega + (\alpha + \beta)y_{t-1}^2 - \beta\hat{\nu}_{t-1} + (\nu_t - \beta(\nu_{t-1} - \hat{\nu}_{t-1}))$ is fitted to obtain the LS estimator of ϕ, where the term given in bracket, $\xi = (\nu_t - \beta(\nu_{t-1} - \hat{\nu}_{t-1}))$, is the error term. In matrix notations, let $\mathbf{Z}_n$ and $\mathbf{X}$ are as follows:

$$\mathbf{Z}_n = \begin{bmatrix} y_{m+1}^2 \\ \vdots \\ y_n^2 \end{bmatrix}$$

and

$$\mathbf{X} = \begin{bmatrix} 1 & y_m^2 & \hat{\nu}_m \\ \vdots & \vdots & \vdots \\ 1 & y_{n-1}^2 & \hat{\nu}_{n-1} \end{bmatrix}.$$

Then, the LS estimator $\hat{\phi} = (\hat{\omega}, \widehat{(\alpha + \beta)}, -\hat{\beta})'$ is obtained as

$$\hat{\phi} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Z}_n, \tag{4.1}$$

given $\boldsymbol{X'X}$ is non-singular. The corresponding $\hat{\alpha}$ is calculated as $\hat{\alpha} = \widehat{(\alpha + \beta)} - \hat{\beta}$.

Based on the above, the complete algorithm of the ONBB prediction intervals for future returns and volatilities is as follows.

Step 1 For a realization of GARCH(1,1) process, $\{y_0, y_1, \ldots, y_T\}$, calculate the LS estimates of ARMA coefficients as in Eq. 4.1.

Step 2 For $t = 1, \ldots, T$, calculate the residuals $\hat{\epsilon}_t = y_t/\hat{\sigma}_t$ where $\hat{\sigma}_t^2 = \hat{\omega} + \hat{\alpha}y_{t-1}^2 + \hat{\beta}\hat{\sigma}_{t-1}^2$ and $\hat{\sigma}_0^2 = \hat{\omega}/(1 - (\hat{\alpha} + \hat{\beta}))$. Let $\hat{F}_\epsilon$ be the empirical distribution function of the centered and rescaled residuals.

Step 3 Obtain ONBB observations from the ARMA representation of GARCH process.

Step 4 Compute ONBB estimators of ARMA coefficients as

$$\hat{\phi}^* = (\mathbf{X}^{*\prime}\mathbf{X}^*)^{-1}\mathbf{X}^{*\prime}\mathbf{Z}_n^* = (\hat{\omega}^*, \widehat{(\alpha+\beta)}^*, -\hat{\beta}^*)'$$

and calculate the corresponding $\hat{\alpha}^*$ as $\hat{\alpha}^* = \widehat{(\alpha+\beta)}^* - \hat{\beta}^*$.

Step 5 Obtain ONBB volatilities as $\hat{\sigma}_t^{2*} = \hat{\omega}^* + \hat{\alpha}^* y_{t-1}^{2*} + \hat{\beta}^* \hat{\sigma}_{t-1}^{2*}$ with $\hat{\sigma}_0^{2*} = \hat{\omega}/(1 - (\hat{\alpha} + \hat{\beta}))$.

Step 6 Calculate $h = 1, 2, \ldots$ steps ahead ONBB future returns and volatilities with the following recursion:

$$\begin{aligned} \hat{\sigma}_{T+h}^{2*} &= \hat{\omega}^* + \hat{\alpha}^* y_{T+h-1}^{2*} + \hat{\beta}^* \hat{\sigma}_{T+h-1}^{2*} \\ y_{T+h}^* &= \hat{\sigma}_{T+h}^{2*} \hat{\epsilon}_{T+h}^*, \end{aligned}$$

where $y_{T+h}^* = y_{T+h}$ for $h \leq 0$ and $\hat{\epsilon}_{T+h}^*$ is randomly drawn from $\hat{F}_\epsilon$.

Step 7 Repeat Steps 3–6 B times to obtain bootstrap replicates of returns and volatilities $\{y_{T+h}^{*,1}, \ldots, y_{T+h}^{*,B}\}$ and $\{\hat{\sigma}_{T+h}^{2*,1}, \ldots, \hat{\sigma}_{T+h}^{2*,B}\}$ for each h.

As noted in Pascual et al. (2006), the one-step conditional variance is perfectly predictable if the model parameters are known, and the only uncertainty, which is caused by the parameter estimation, is associated with the prediction of σ_{T+1}^2. On the other hand, there are further uncertainties about future errors when predicting two or more step ahead variances. Thus, it is more interesting issue to have prediction intervals for future volatilities. Now, let $G_y^*(h) = P(y_{T+h}^* \leq h)$ and $G_{\sigma^2}^*(h) = P(\hat{\sigma}_{T+h}^{2*} \leq h)$ be the ONBB distribution functions of unknown distribution functions of y_{T+h} and σ_{T+h}^2, respectively, for $h = 1, 2, \ldots$. Also, let $G_{y,B}^*(h) = \#(y_{T+h}^{*,b} \leq h)/B$ and $G_{\sigma^2,B}^*(h) = \#(\hat{\sigma}_{T+h}^{2*,b} \leq h)/B$, for $b = 1, \ldots, B$, be the corresponding Monte Carlo estimates. Then, a $100(1-\gamma)\%$ bootstrap prediction intervals for y_{T+h} and σ_{T+h}^2, respectively, are given by

$$\begin{aligned} \left[L_{y,B}^*(y), U_{y,B}^*(y)\right] &= \left[Q_{y,B}^*(\gamma/2), Q_{y,B}^*(1-\gamma/2)\right], \\ \left[L_{\sigma^2,B}^*(y), U_{\sigma^2,B}^*(y)\right] &= \left[Q_{\sigma^2,B}^*(\gamma/2), Q_{\sigma^2,B}^*(1-\gamma/2)\right], \end{aligned}$$

where $Q_{y,B}^* = G_y^{*-1}(h)$ and $Q_{\sigma^2,B}^* = G_{\sigma^2}^{*-1}(h)$.

We have the following proposition which shows the large sample validity of the ONBB prediction intervals.

Proposition 1.

$$(i) \qquad \sup_x |P^*(\sqrt{n}[\hat{\phi}^* - \hat{\phi}] \leq x) - P(\sqrt{n}[\hat{\phi} - \phi] \leq x)| \xrightarrow{p} 0.$$

Hence, $\hat{\phi}^* \xrightarrow{p^*} \phi$.

$$(ii) \qquad y_{T+h}^* \xrightarrow{d^*} y_{T+h} \quad and \quad \hat{\sigma}_{T+h}^{2*} \xrightarrow{d^*} \sigma_{T+h}^2 \quad as \quad n \to \infty.$$

$$\begin{aligned} (ii) \qquad & \lim_{n\to\infty} \lim_{B\to\infty} P\left[L_{y,B}^* \leq Y_{T+h} \leq U_{y,B}^*\right] = 1 - \gamma \\ & \lim_{n\to\infty} \lim_{B\to\infty} P\left[L_{\sigma^2,B}^* \leq \sigma_{T+h}^2 \leq U_{\sigma^2,B}^*\right] = 1 - \gamma, \end{aligned}$$

where, for the random variables X_n and X, $X_n \xrightarrow{p} X$, $X_n \xrightarrow{p^*} X$, and $X_n \xrightarrow{d^*} X$ represent the convergence in probability, (conditional) convergence in probability, and (conditional) convergence in distribution conditional on a given sample $y = \{y_0, \ldots, y_T\}$, respectively.

5 Numerical Results

To investigate the performance of our proposed ONBB prediction intervals, we conduct a simulation study under GARCH(1,1) model given in Eq. 5.1 below, and we compare our results with the method proposed by Pascual et al. (2006) (abbreviated as "PRR") by means of coverage probabilities and length of prediction intervals. It is worth to mention that we also compared the performance of our proposed method with other existing block bootstrap methods mentioned in Section 1, e.g., NBB, MBB, CBB, and SB. Our method performed considerably better compared to them, and therefore to save space, we only report the comparative study with ONBB and PRR. Roughly, we observed the coverage probabilities of other block bootstrap methods range in between 90 and 94% for future returns while those range only in between 25 and 60% for future volatilities.

To discuss the numerical study we present here, let us start with the following GARCH(1,1) model.

$$\begin{aligned} y_t &= \sigma_t \epsilon_t \\ \sigma_t^2 &= 0.05 + 0.1 y_{t-1}^2 + 0.85 \sigma_{t-1}^2, \end{aligned} \tag{5.1}$$

where ϵ_t follows a $N(0,1)$ distribution. The significance level γ is set to 0.05 to obtain 95% prediction intervals for future returns and volatilities. Since the block bootstrap methods are sensitive to the choice of the block length ℓ, we choose three different block lengths in our simulation study: $n^{1/3}$, $n^{1/4}$, and $n^{1/5}$ as proposed by Hall et al. (1995). Let $h = 1, 2, \ldots, s$, $s \geq 1$, be defined as the lead time. We obtain the prediction intervals for next $s = 20$ observations. The experimental design is similar to those of Pascual et al. (2006) which is as follows:

Step 1 Simulate a GARCH(1,1) series with the parameters given in Eq. 5.1 and generate $R = 1000$ future values y_{T+h} and σ_{T+h}^2 for $h = 1, \ldots, s$.

Step 2 Calculate bootstrap future values $y_{T+h}^{*,b}$ and $\sigma_{T+h}^{2*,b}$ for $h = 1, \ldots, s$ and $b = 1, \ldots, B$. Then, estimate the coverage probabilities (C^*) of bootstrap prediction intervals for y_{T+h}^* and σ_{T+h}^{2*} as

$$\begin{aligned} C_{y_{T+h}}^{*,i} &= \frac{1}{R} \sum_{r=1}^{R} \mathbf{1}\{Q_{y_{T+h}}^{*,i}(\gamma/2) \leq y_{T+h}^{*,r} \leq Q_{y_{T+h}}^{*,i}(1-\gamma/2)\}. \\ C_{\sigma_{T+h}^2}^{*,i} &= \frac{1}{R} \sum_{r=1}^{R} \mathbf{1}\{Q_{\sigma_{T+h}^2}^{*,i}(\gamma/2) \leq \sigma_{T+h}^{2*,r} \leq Q_{\sigma_{T+h}^2}^{*,i}(1-\gamma/2)\}, \end{aligned}$$

Table 3: Prediction intervals for returns and volatilities of GARCH(1,1) model

Lead time	Sample size	Method	Coverage for return (SE)	Average length for return (SE)	Coverage for volatility (SE)	Average length for volatility (SE)
1	T	Empirical	0.95	3.814	0.95	–
	300	PRR	0.945(0.021)	3.748(0.874)	0.904(0.295)	0.649(0.520)
		$\ell = n^{1/3}$	0.947(0.022)	3.804(0.819)	0.922(0.268)	0.743(0.565)
	ONBB	$\ell = n^{1/4}$	0.946(0.020)	3.853(0.954)	0.958(0.200)	0.779(0.752)
		$\ell = n^{1/5}$	0.945(0.022)	3.821(0.961)	0.948(0.222)	0.781(0.755)
	1500	PRR	0.946(0.013)	3.695(0.748)	0.928(0.259)	0.236(0.181)
		$\ell = n^{1/3}$	0.949(0.018)	3.838(0.804)	0.928(0.258)	0.724(0.842)
	ONBB	$\ell = n^{1/4}$	0.948(0.016)	3.853(0.838)	0.962(0.191)	0.703(0.590)
		$\ell = n^{1/5}$	0.946(0.016)	3.893(0.886)	0.950(0.218)	0.738(0.641)
	3000	PRR	0.946(0.011)	3.800(0.863)	0.952(0.214)	0.181(0.194)
		$\ell = n^{1/3}$	0.950(0.018)	3.838(0.718)	0.888(0.315)	0.718(0.698)
	ONBB	$\ell = n^{1/4}$	0.947(0.016)	3.865(0.845)	0.958(0.200)	0.719(0.590)
		$\ell = n^{1/5}$	0.948(0.015)	3.819(0.879)	0.974(0.159)	0.713(0.633)
10	T	Empirical	0.95	3.946	0.95	1.389
	300	PRR	0.943(0.026)	3.846(0.712)	0.902(0.117)	1.564(1.387)
		$\ell = n^{1/3}$	0.945(0.021)	3.881(0.659)	0.905(0.094)	1.410(0.851)
	ONBB	$\ell = n^{1/4}$	0.945(0.020)	3.941(0.789)	0.927(0.078)	1.638(1.281)
		$\ell = n^{1/5}$	0.946(0.020)	3.926(0.697)	0.941(0.077)	1.753(1.381)
	1500	PRR	0.946(0.014)	3.806(0.527)	0.930(0.056)	1.302(0.689)
		$\ell = n^{1/3}$	0.946(0.016)	3.870(0.617)	0.906(0.061)	1.271(0.871)
	ONBB	$\ell = n^{1/4}$	0.947(0.015)	3.908(0.627)	0.941(0.043)	1.434(0.782)
		$\ell = n^{1/5}$	0.945(0.014)	3.926(0.647)	0.948(0.041)	1.526(0.943)
	3000	PRR	0.946(0.012)	3.875(0.604)	0.941(0.036)	1.354(0.653)
		$\ell = n^{1/3}$	0.946(0.015)	3.866(0.563)	0.911(0.056)	1.287(0.833)
	ONBB	$\ell = n^{1/4}$	0.946(0.014)	3.909(0.645)	0.940(0.040)	1.411(0.786)
		$\ell = n^{1/5}$	0.947(0.012)	3.882(0.644)	0.959(0.027)	1.495(0.864)
20	T	Empirical	0.95	3.948	0.95	1.661
	300	PRR	0.940(0.026)	3.876(0.647)	0.881(0.122)	1.771(1.515)
		$\ell = n^{1/3}$	0.942(0.023)	3.896(0.593)	0.882(0.097)	1.582(0.925)
	ONBB	$\ell = n^{1/4}$	0.944(0.022)	3.964(0.706)	0.901(0.090)	1.847(1.438)
		$\ell = n^{1/5}$	0.944(0.021)	3.947(0.603)	0.920(0.086)	1.928(1.278)
	1500	PRR	0.946(0.015)	3.859(0.399)	0.925(0.059)	1.569(0.705)
		$\ell = n^{1/3}$	0.944(0.015)	3.898(0.519)	0.900(0.060)	1.464(0.879)
	ONBB	$\ell = n^{1/4}$	0.946(0.014)	3.929(0.465)	0.933(0.043)	1.666(0.798)
		$\ell = n^{1/5}$	0.945(0.015)	3.937(0.503)	0.940(0.040)	1.757(0.981)

Table 3: (continued)

Lead time	Sample size	Method	Coverage for return (SE)	Average length for return (SE)	Coverage for volatility (SE)	Average length for volatility (SE)
	3000	PRR	0.946(0.012)	3.907(0.444)	0.940(0.033)	1.634(0.627)
		$\ell = n^{1/3}$	0.946(0.015)	3.903(0.470)	0.911(0.049)	1.471(0.796)
	ONBB	$\ell = n^{1/4}$	0.946(0.014)	3.934(0.505)	0.935(0.036)	1.645(0.747)
		$\ell = n^{1/5}$	0.946(0.013)	3.914(0.504)	0.951(0.029)	1.736(0.855)

where **1** represents the indicator function. The corresponding interval lengths (L^*) are calculated by

$$
\begin{aligned}
L^{*,i}_{y_{T+h}} &= Q^{*,i}_{y_{T+h}}(1-\gamma/2) - Q^{*,i}_{y_{T+h}}(\gamma/2) \\
L^{*,i}_{\sigma^2_{T+h}} &= Q^{*,i}_{\sigma^2_{T+h}}(1-\gamma/2) - Q^{*,i}_{\sigma^2_{T+h}}(\gamma/2).
\end{aligned}
$$

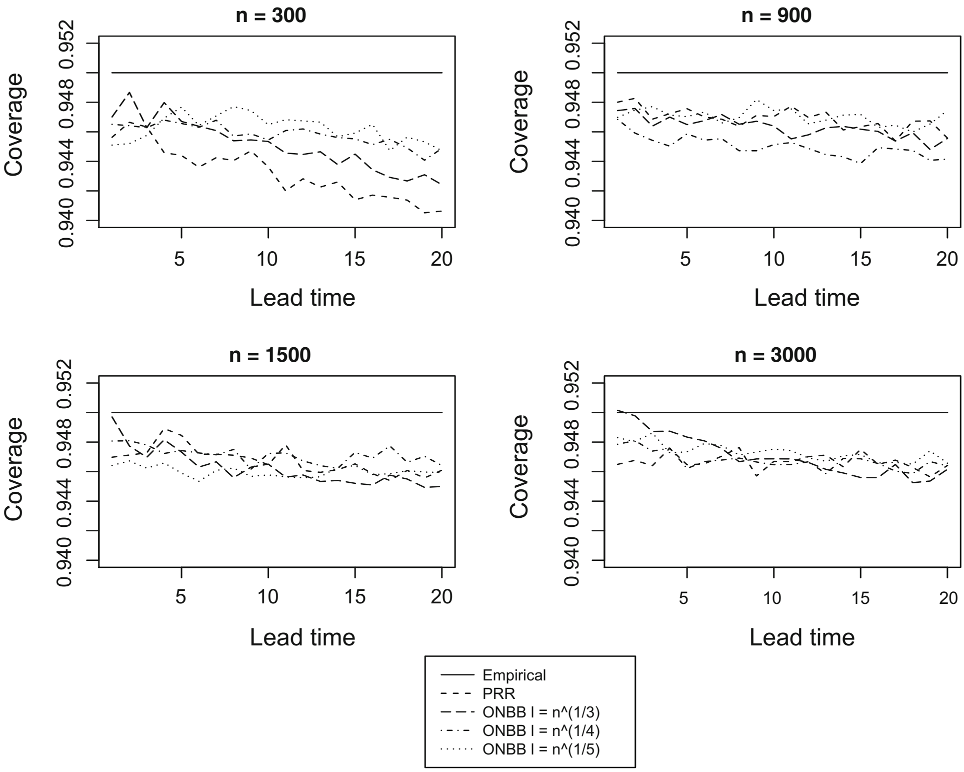

Figure 5: Estimated coverage probabilities of returns using PRR and ONBB

Step 3 Repeat Steps 1–2, MC = 1000 times to calculate the average values of $C^*_{y_{T+h}}$, $C^*_{\sigma^2_{T+h}}$, $L^*_{y_{T+h}}$, and $L^*_{\sigma^2_{T+h}}$ as

$$\begin{aligned} \text{ave}(C^*_{y_{T+h}}) &= \sum_{i=1}^{\text{MC}} \frac{C^{*,i}_{y_{T+h}}}{\text{MC}}, \qquad \text{ave}(C^*_{\sigma^2_{T+h}}) = \sum_{i=1}^{\text{MC}} \frac{C^{*,i}_{\sigma^2_{T+h}}}{\text{MC}} \\ \text{ave}(L^*_{y_{T+h}}) &= \sum_{i=1}^{\text{MC}} \frac{L^{*,i}_{y_{T+h}}}{\text{MC}}, \qquad \text{ave}(L^*_{\sigma^2_{T+h}}) = \sum_{i=1}^{\text{MC}} \frac{L^{*,i}_{\sigma^2_{T+h}}}{\text{MC}}. \end{aligned}$$

Also, calculate the standard errors of the estimated coverage probabilities and interval lengths by

$$\begin{aligned} s.e(C^*_{y_{T+h}}) &= \left\{ \sum_{i=1}^{\text{MC}} \left[C^{*,i}_{y_{T+h}} - \text{ave}(C^*_{y_{T+h}}) \right]^2 / \text{MC} \right\}^{1/2} \\ s.e(C^*_{\sigma^2_{T+h}}) &= \left\{ \sum_{i=1}^{\text{MC}} \left[C^{*,i}_{\sigma^2_{T+h}} - \text{ave}(C^*_{\sigma^2_{T+h}}) \right]^2 / \text{MC} \right\}^{1/2} \end{aligned}$$

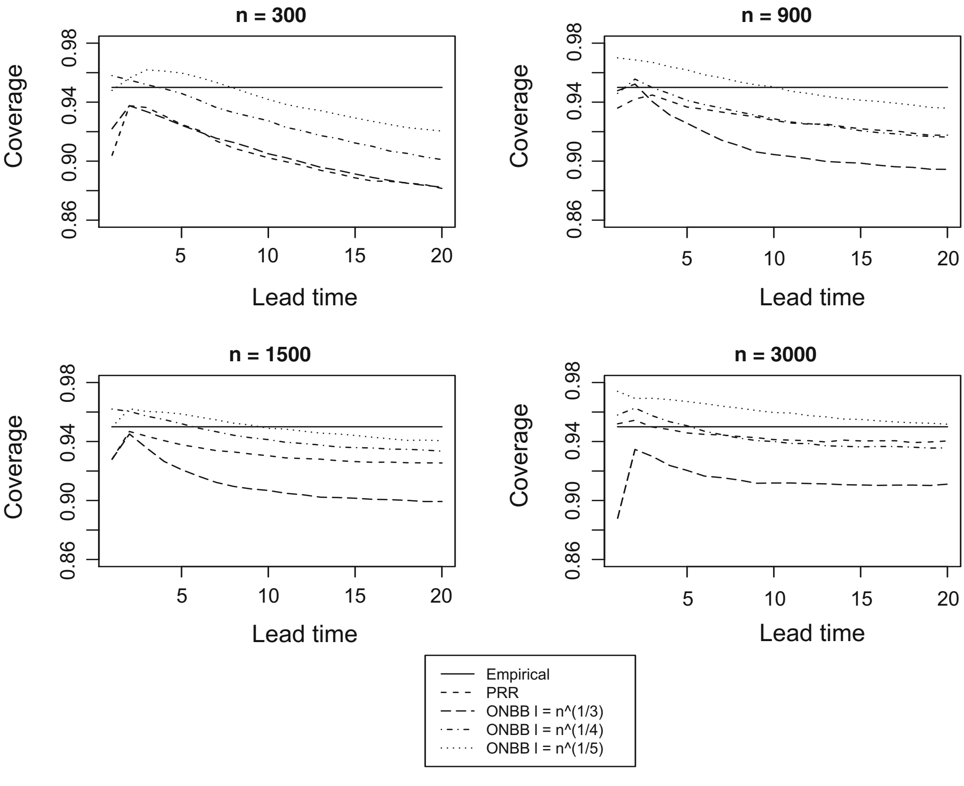

Figure 6: Estimated coverage probabilities of volatilities using PRR and ONBB

$$s.e(L^*_{y_{T+h}}) = \left\{ \sum_{i=1}^{MC} \left[L^{*,i}_{y_{T+h}} - \text{ave}(L^*_{y_{T+h}}) \right]^2 / \text{MC} \right\}^{1/2}$$

$$s.e(L^*_{\sigma^2_{T+h}}) = \left\{ \sum_{i=1}^{MC} \left[L^{*,i}_{\sigma^2_{T+h}} - \text{ave}(L^*_{\sigma^2_{T+h}}) \right]^2 / \text{MC} \right\}^{1/2}.$$

A short summary of the simulation results is given in Table 3. More detailed results are presented in Figs. 5, 6, 7, and 8. Our findings show that ONBB outperforms PRR in general. For the prediction intervals of future returns (see Fig. 5), the performances of both methods are almost same. Also, ONBB provides competitive interval lengths for returns (see Fig. 7). The accuracy of the prediction intervals for volatilities obtained by ONBB is sensitive to the choice of block length parameter ℓ, and the higher coverage probabilities are obtained when $\ell = n^{1/4}$ and $n^{1/5}$ are used. The performance of our proposed method is always better than PRR in small sample sizes, and it outperforms PRR also in large samples especially for long-term forecasts as it is shown in Table 3 and Fig. 6. Furthermore, in general, ONBB has less standard errors for coverage probabilities compared to PRR. Based on our

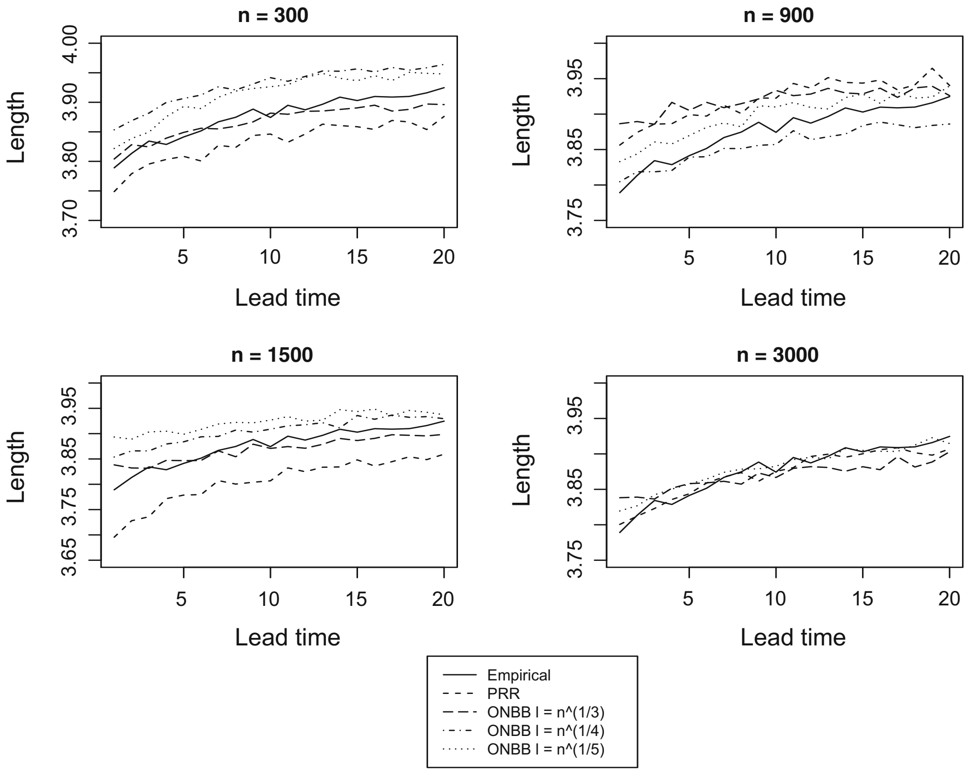

Figure 7: Estimated lengths of prediction intervals of returns using PRR and ONBB

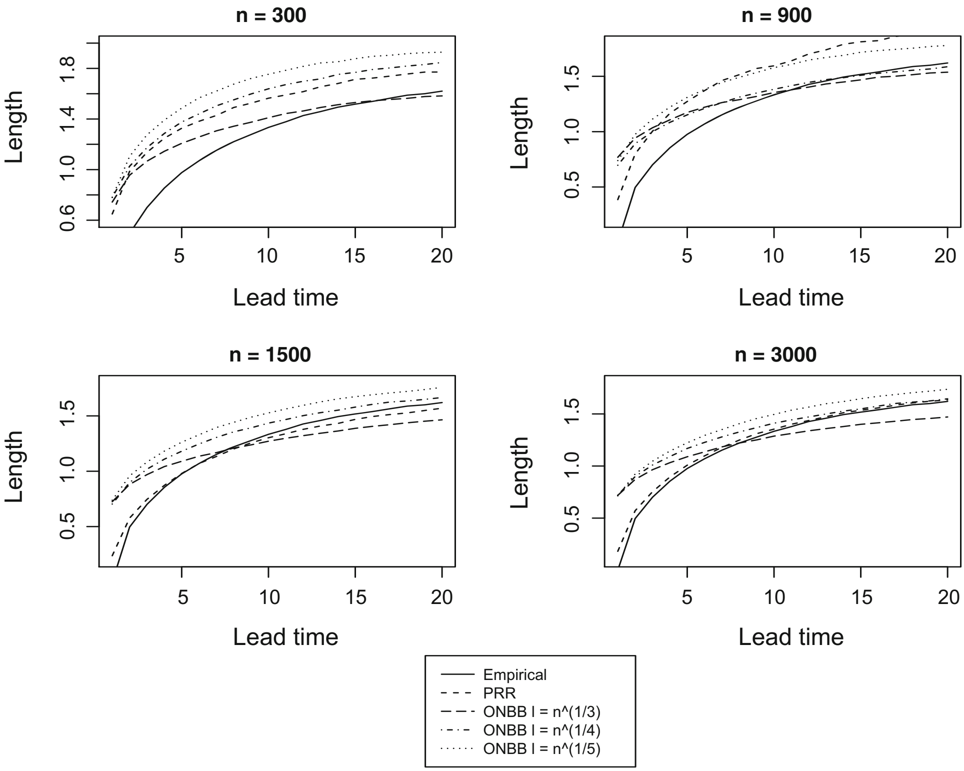

Figure 8: Estimated lengths of prediction intervals of volatilities using PRR and ONBB

findings, the proposed method achieves superior performance with $\ell = n^{1/5}$ for the prediction intervals of future returns. For the prediction intervals of volatilities, by taking into account the coverage probabilities and length of intervals, $\ell = n^{1/5}$ and $n^{1/4}$ seem to be the optimal choices for short-term and long-term forecasts, respectively. We also compare the ONBB and PRR in terms of their computing times, and Fig. 9 represents the approximate computing times for various sample sizes based on $B = 1000$ bootstrap replications and only one Monte Carlo simulation. As presented in Fig. 9, ONBB has considerably less computational time as PRR requires about 4–6 times more computing time than ONBB.

Moreover, we also compared our method with the one proposed by Chen et al. (2011) (hereafter referred to as the CGBA method) through a simulation study. Like PRR, the CGBA method is also based on residuals. The main difference is that PRR uses quasi-maximum likelihood method to estimate the parameters and then, uses residual-based resampling to construct intervals, whereas the CGBA method utilizes the ARMA representation of a GARCH process to first estimate the parameters of the original data and then, uses the sieve bootstrap to obtain prediction

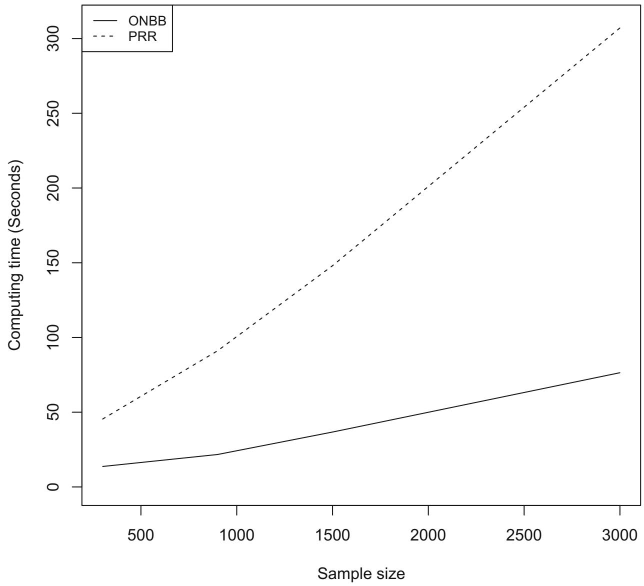

Figure 9: Estimated computing times for PRR and ONBB

intervals. The CGBA method requires approximately 2.5 more computing time than ours, and the coverage performance of our proposed method is significantly better than that of CGBA method. To save space, we omit the numerical details in this paper.

Finally, to compare PRR and ONBB methods in terms of having more representative resamples, we present the plot of fitted unstandardized residuals obtained by both methods (see Fig. 10) when the sample size $n = 300$. It is clear that the residuals obtained by ONBB have more similar fluctuations with the original residuals compared to PRR's residuals. We also conduct a simulation study when $B = 1000$ and $\mathrm{MC} = 1000$, and for each simulation, we recorded the sum of squared difference between original and bootstrap-standardized residuals for both methods. The result is 1.822 for the PRR while it is 0.886, 0.895, and 0.888 for the ONBB when the block size is $\ell = n^{1/3}, n^{1/4}$, and $n^{1/5}$, respectively. The results show that the ONBB has more similar resamples than PRR. Moreover, we performed a small simulation study to compare the robustness of the PRR and ONBB. To this end, we generated time series data as $(1-\alpha)\ GARCH(1,1) + \alpha\ AR(1)$ with AR parameter -0.9 and sample size $n = 1000$, where $0 < \alpha < 1/2$. This generated observations are "nearly GARCH," and we computed sum of squared difference between true residuals (unstandardized) and the residuals obtained by the bootstrap methods. The method having smaller difference values can be considered more robust than the other method. The results are presented in Table 4 for different α values. Consequently, we can say that the ONBB is more robust than the PRR.

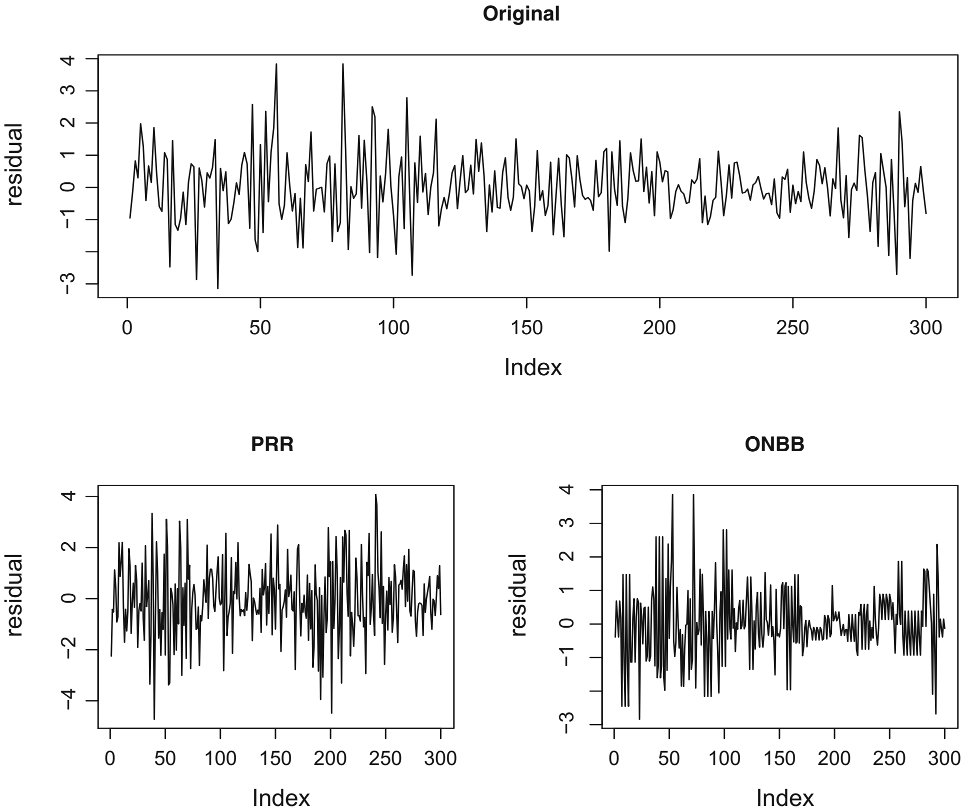

Figure 10: Unstandardized residual plots of PRR and ONBB

6 Case Study

To obtain out-of-sample prediction intervals for the real data described in Section 2, we divide the full data into the following two parts: the model is constructed based on the observations from 29th July, 2011 to 21st September, 2015 (1040 observations in total) to calculate 30 steps ahead predictions from 22nd September to 3rd November, 2015, and compare with the actual values. The fitted models for PRR and ONBB are obtained as in Eqs. 6.1 and 6.2, respectively.

$$\hat{\sigma}_t^2 = 0.0028 + 0.0549 y_{t-1}^2 + 0.9412 \hat{\sigma}_{t-1}^2, \tag{6.1}$$

Table 4: Sum of squared residuals of PRR and ONBB for different α values

Method	α				
	0.1	0.2	0.3	0.4	0.49
PRR	65086.006	1919.690	2321.617	11947.420	13409.390
ONBB	1432.617	1398.811	1555.436	1922.087	2522.636

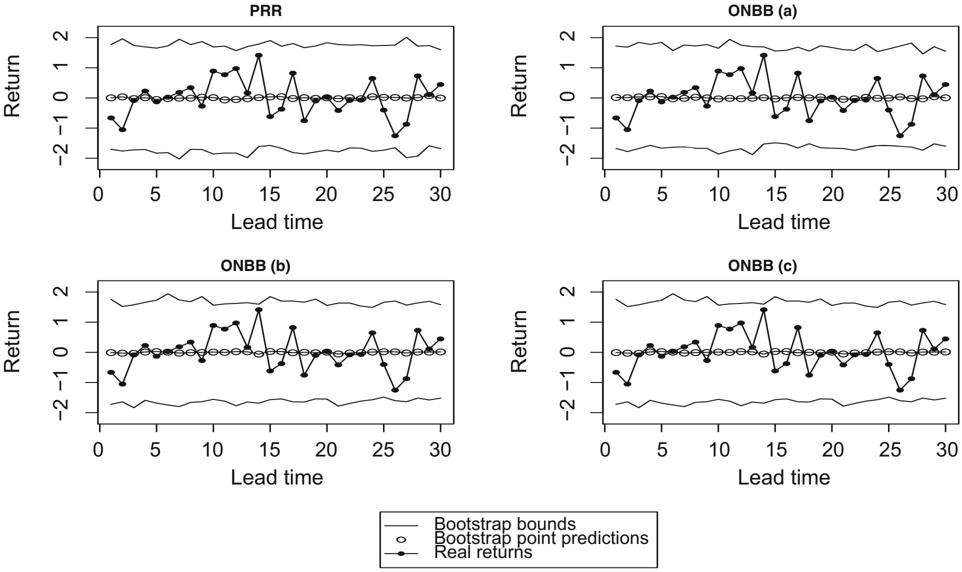

Figure 11: 95% prediction intervals of returns from 22nd September, 2015 to 3rd November, 2015, where (a), (b), and (c) denote results obtained by choosing block lengths $\ell = n^{1/3}, n^{1/4}$, and $n^{1/5}$, respectively

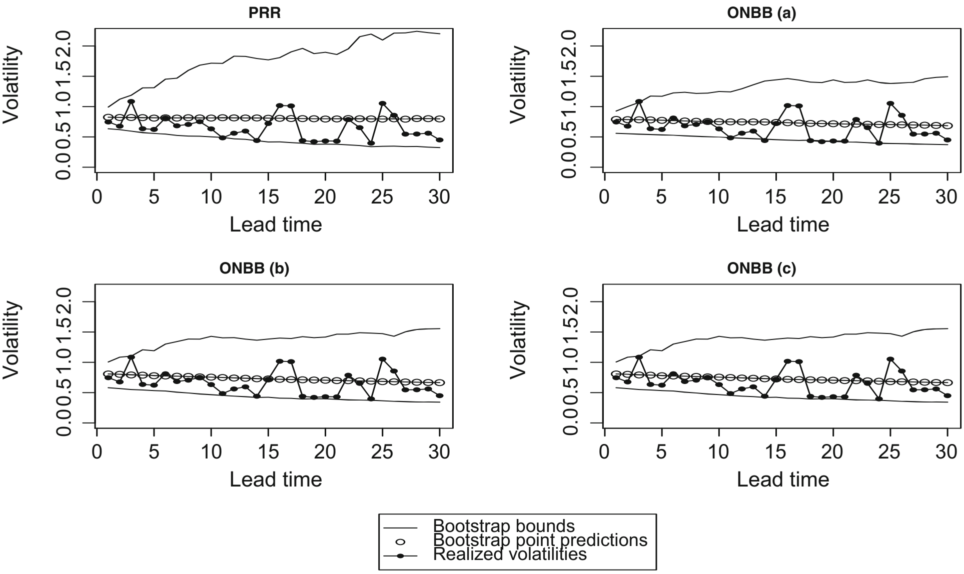

Figure 12: 95% prediction intervals of volatilities from 22nd September, 2015 to 3rd November, 2015, where (a), (b), and (c) denote the results obtained choosing block lengths $\ell = n^{1/3}, n^{1/4}$, and $n^{1/5}$, respectively

$$y_t^2 = 0.0078 + 0.9845 y_{t-1}^2 + \nu_t - 0.9408 \nu_{t-1}, \tag{6.2}$$

where $\hat{\omega} = 0.0078$, $\hat{\alpha}_1 = 0.0436$, and $\hat{\beta}_1 = 0.9408$ for the model estimated by Eq. 4.1. The 30 step ahead prediction intervals of the PRR and ONBB for returns y_{T+h} based on the models given in Eqs. 6.1 and 6.2, together with the true returns, are presented in Fig. 11. The intervals obtained using both methods are similar and they include all of the true values of returns. Note that the ONBB prediction interval for $\ell = n^{1/5}$ is slightly narrower than the others.

Figure 12 shows the predicted intervals for 30 step ahead volatilities σ_{T+h}^2. The true values of the volatilities can not be observed directly. We calculate the realized volatility by summing squared returns at day t, $\sigma_t^2 = y_{t,1}^2 + \ldots + y_{t,n}^2$, where n is the number of observations recorded during day t as proposed by Andersen and Bollerslev (1998). Since our data is from 24-h-open trading market, the realized volatilities are computed by using 1-min returns based on tick-by-tick prices such that $n = 1440$ approximately. Figure 12 indicates that the ONBB prediction intervals are significantly narrower than the PRR's for all block lengths. Moreover, by looking at this figure carefully, it can be seen that the point forecasts of the volatilities obtained by ONBB are closer to the realized values than the results based on the PRR method. This result clearly explains the supremacy of the ONBB-based prediction intervals over the existing ones. Additionally, we perform an extra simulation study from a GARCH(1,1) model with the parameters as in Eq. 6.1 and

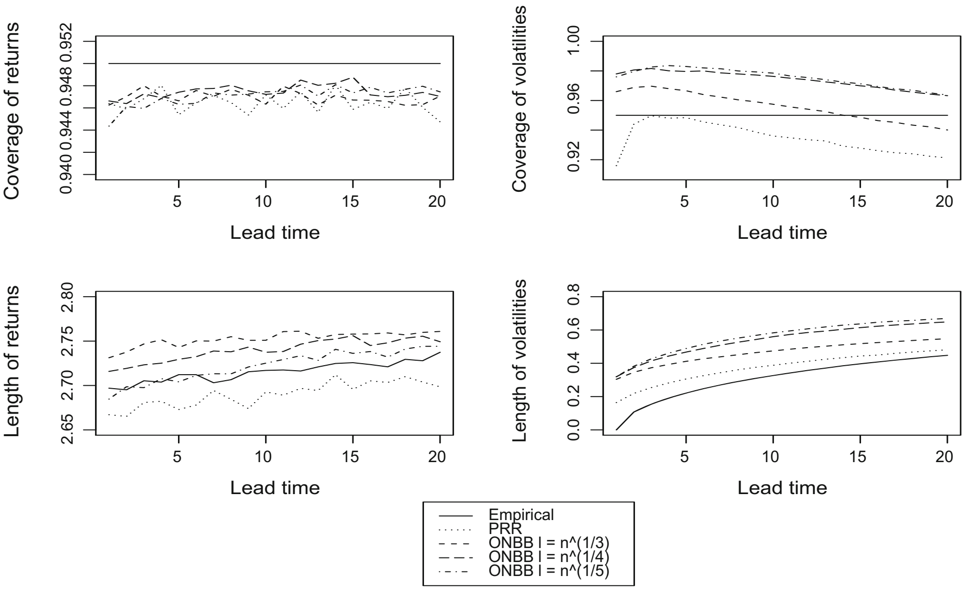

Figure 13: Simulation results from the GARCH(1,1) model fitted to the exchange rate data

sample size $n = 1040$ to compare the performances of the PRR and ONBB, and the results are presented in Fig. 13. The results for the coverage probabilities of the returns are consistent with the simulation results given in Section 5. For coverage probabilities of the volatilities, the ONBB outperforms PRR when $\ell = n^{1/3}$ while the other block lengths ONBB overestimates the coverage probability for all lead times. PRR provides narrower intervals than the ONBB but the difference is not too significant especially when $\ell = n^{1/3}$.

7 Conclusion

In this study, we examine recently proposed ONBB method in detail, and we show its superiority over the traditional block bootstrap methods by Spearman's rank correlation coefficient. Our DTW simulation results show that the ONBB resamples are more similar to the original time series on time axis compared to other block bootstrap methods. Therefore, the ONBB method comprises of more dependency structure of the original series and produces more reliable results among the others.

We also propose a novel, computationally efficient resampling algorithm to obtain better prediction intervals for returns and volatilities under GARCH models by using ONBB method, and we compare the performance of our method with the existing PRR method by both simulations and a case study. The important result produced by our proposed algorithm is that the short-term and long-term forecasting can be done with considerably narrower intervals especially for future volatilities. In financial contexts, the proposed method in this paper can be a good guide to the international investors and traders for their decisions to manage risks accurately.

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Appendix

PROOF OF THEOREM 1. We examine Theorem 1 under three cases given below, but it can be generalized to all other cases using a similar logic.

Case 1: Suppose that all the bootstrapped blocks are untied and in a dual order of j. In this case, for $j = 2, \ldots b-1$, $k_{(j)} < k_{(j-1)}$, and $R_{k_{(j)}} < R_{k_{(j-1)}}$ for the NBB while it is $k_{(j)} > k_{(j-1)}$ and $R_{k_{(j)}} > R_{k_{(j-1)}}$ for the ONBB. Note that $m_k = 1$ for $k = 1, \ldots, b$. Thus,

$$\rho_{\text{Original,NBB}} = -1$$

since $\sum_{j=1}^{b} d_j^2 = (b^3 - b)/3$, and

$$\rho_{\text{Original,ONBB}} = 1$$

holds since $\sum_{j=1}^{b} d_j^2 = 0$ and $j = R_{k_{(j)}}$ for all $j = 1, \ldots, b$ and $k_{(j)} = 1, \ldots, b$.

Case 2: Suppose that all the bootstrapped blocks are untied but in the same order of j. In this case, $k_{(j)} > k_{(j-1)}$, $R_{k_{(j)}} > R_{k_{(j-1)}}$, and $\sum_{j=1}^{b} d_j^2 = 0$ for both methods. So,

$$\rho_{\text{Original,NBB}} = \rho_{\text{Original,ONBB}} = 1.$$

On the other hand, suppose we change the positions of two blocks, and let t and z represent the blocks whose positions are changed. Let $s_{t,z} = |z - t|$ denotes the distance between two positions where $t, z = 1, \ldots, b$. It is clear that $\sum_{j=1}^{b} d_j^2 = 2\sum_{t \neq z} s_{t,z}^2$. So,

$$\rho_{\text{Original,ONBB}} - \rho_{\text{Original,NBB}} = (12 \sum_{t \neq z} s_{t,z}^2)/(b^3 - b) > 0.$$

Case 3: Suppose that all the bootstrapped blocks are tied so that $k_{(j)} = k_{(j-1)}$ and $R_{k_{(j)}} = (b+1)/2$. Note that $m_k = b$. In this case, $\sum_{j=1}^{b} d_j^2 = (1 - (b+1)/2)^2 + \ldots, +(b - (b+1)/2)^2 = (b^3 - b)/12$. So

$$\rho_{\text{Original,NBB}} = \rho_{\text{Original,ONBB}} = 0.5.$$

Let us consider a more general case given below where only the positions of label groups 1 and 2 are different for NBB and ONBB, while the other block labels have the same positions and frequencies.

$$\begin{aligned} \text{NBB block labels} &= 2, \ldots, 2, 1, \ldots, 1, \ldots, 3, \ldots, b \\ \text{ONBB block labels} &= 1, \ldots, 1, 2, \ldots, 2, \ldots, 3, \ldots, b. \end{aligned}$$

In this case, $\rho_{\text{Original,ONBB}} - \rho_{\text{Original,NBB}}$ depends on $\sum_{j=1}^{b} d_j^2$. Let $\sum_{j=1}^{b} d_{j_{\text{NBB}}}^2$ and $\sum_{j=1}^{b} d_{j_{\text{ONBB}}}^2$ be the $\sum_{j=1}^{b} d_j^2$ values obtained by NBB and ONBB, respectively. Then we have

$$\begin{aligned}
\sum_{j=1}^{b} d_{j_{\text{NBB}}}^2 - \sum_{j=1}^{b} d_{j_{\text{ONBB}}}^2 &= ((1 - R_{2_{(1)}})^2 - (1 - R_{1_{(1)}})^2) + \dots \\
&+ ((m_2 - R_{2_{(m_2)}})^2 - (m_1 - R_{1_{(m_1)}})^2) + \dots \\
&+ (((m_1 + m_2) - R_{1_{(m_1+m_2)}})^2 \\
&- ((m_1 + m_2) - R_{2_{(m_1+m_2)}})^2) = 2m^3.
\end{aligned}$$

Thus, $\rho_{\text{Original,ONBB}} - \rho_{\text{Original,NBB}} = (12m^3)/(b^3 - b) > 0$ when $m_1 = m_2 = m$. For the case where $m_1 \neq m_2$, we have

$$\sum_{j=1}^{b} d_{j_{\text{NBB}}}^2 - \sum_{j=1}^{b} d_{j_{\text{ONBB}}}^2 = (m_1 m_2)(m_1 + m_2).$$

Therefore, $\rho_{\text{Original,ONBB}} - \rho_{\text{Original,NBB}} = (6(m_1 m_2)(m_1 + m_2))/(b^3 - b) > 0$.

Theorem 1 follows directly combining the above three cases.

Proof of Corollary 1. The proof follows as a direct consequence of the above derivation.

Proof of Proposition 1. The LS estimator of an ARMA model $\hat{\phi}$ satisfies

$$\sqrt{n}[\hat{\phi} - \phi] = \left(\frac{\mathbf{X}'\mathbf{X}}{n}\right)^{-1} \frac{1}{\sqrt{n}} \mathbf{X}'\xi \xrightarrow{d} N(\mathbf{0}, \mathbf{V}_\phi),$$

where the covariance matrix $\mathbf{V}_\phi$ is given by $\mathbf{V}_\phi = \mathbf{D}^{-1}\Gamma\mathbf{D}^{-1}$ such that

$$\begin{aligned}
\left(\frac{\mathbf{X}'\mathbf{X}}{n}\right)^{-1} &\xrightarrow{p} \mathbf{D}^{-1} \\
\frac{1}{\sqrt{n}} \mathbf{X}'\xi &\xrightarrow{d} N(0, \Gamma),
\end{aligned}$$

where the non-diagonal matrix Γ is related to the covariance matrix of the moving average model. The bootstrap estimate $\hat{\phi}^*$ can be written as

$$\sqrt{n}[\hat{\phi}^* - \hat{\phi}] = \left(\frac{\mathbf{X}^{*\prime}\mathbf{X}^*}{n}\right)^{-1} \frac{1}{\sqrt{n}} \mathbf{X}^{*\prime}\hat{\xi}^*.$$

To prove part (i), it is suffice to show that $\mathbf{X}^{*\prime}\mathbf{X}^*/n$ converges in probability to $\mathbf{X}\prime\mathbf{X}/n$, and $\mathbf{X}^{*\prime}\hat{\xi}^*/\sqrt{n}$ convergence in distribution to $\mathbf{X}\prime\xi/\sqrt{n}$. We write $\mathbf{X}\prime\mathbf{X} =$

$\sum_{i=1}^{n} \mathbf{X}_i'\mathbf{X}_i$ and $\mathbf{X}^{*\prime}\mathbf{X}^* = \sum_{i=1}^{n} \mathbf{X}_i^{*\prime}\mathbf{X}_i^*$. For simplicity, we assume that $n = b\ell$. Let $\mathbf{U}_{k,\ell}$ be the mean of $\mathbf{X}_{nj}'\mathbf{X}_{nj}$ in block B_k such that $B_k = \{\mathbf{X}_{nj} : (k-1)\ell + 1 \le j \le k\ell\}$, for $k = 1, \cdots, b$. That is

$$\mathbf{U}_{k,\ell} = \sum_{j=(k-1)\ell+1}^{k\ell} \mathbf{X}_{nj}'\mathbf{X}_{nj}/\ell, \quad k = 1, \cdots, b.$$

Let

$$\mathbf{U}_{k,\ell}^* = \sum_{j=(k-1)\ell+1}^{k\ell} \mathbf{X}_{nj}^{*\prime}\mathbf{X}_{nj}^*/\ell, \quad k = 1, \cdots, b$$

be the NBB version of $\mathbf{U}_{k,\ell}$. Then, we have

$$\begin{aligned} & \left| \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i^{*\prime}\mathbf{X}_i^* - \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i'\mathbf{X}_i \right| \\ \le & \left| \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i^{*\prime}\mathbf{X}_i^* - E^*(\mathbf{U}_{k,\ell}^*) \right| + \left| E^*(\mathbf{U}_{k,\ell}^*) - \frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i'\mathbf{X}_i \right|, \end{aligned} \tag{7.1}$$

where E^* denotes the conditional expectation under the bootstrap distribution. The first term of the right hand side of Eq. 7.1 tends to 0 by the weak law of large numbers. Since $P^*((X_1^*, \cdots, X_\ell^* = X_{(j-1)\ell+1}, \cdots, X_{b\ell})) = 1/b$, for $j = 1, \cdots, b$, for the second term, we have

$$E^*(\mathbf{U}_{k,\ell}^*) = \frac{1}{b}\sum_{k=1}^{b}\left(\frac{1}{\ell}\sum_{i=1}^{\ell} \mathbf{X}_{(k-1)\ell+i}'\mathbf{X}_{(k-1)\ell+i}\right) = \frac{1}{b\ell}\sum_{i=1}^{n} \mathbf{X}_i'\mathbf{X}_i.$$

Hence, the second term tends to 0 as $n \to \infty$, which proves that $\mathbf{X}^{*\prime}\mathbf{X}^*/n \xrightarrow{p^*} \mathbf{X}'\mathbf{X}/n$.

To show $\frac{1}{\sqrt{n}}\mathbf{X}^{*\prime}\hat{\xi}^* \xrightarrow{d^*} N(0, \Gamma)$, write $\mathbf{X}'\xi = \sum_{i=1}^{n} \mathbf{X}_i'\xi_i$ and $\mathbf{X}^{*\prime}\hat{\xi}^* = \sum_{i=1}^{n} \mathbf{X}_i^{*\prime}\hat{\xi}_i^*$. Let $\mathbf{V}_{k,\ell} = \sum_{j=(k-1)\ell+1}^{k\ell} \mathbf{X}_{nj}'\hat{\xi}_{nj}/\ell$ and let $\mathbf{V}_{k,\ell}^*$ be the bootstrap version of $\mathbf{V}_{k,\ell}$. Then, we have

$$\begin{aligned} & \frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{X}^{*\prime}\hat{\xi}^* \\ = & \left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n} \mathbf{X}_i^{*\prime}\hat{\xi}_i^* - \sqrt{n}E^*(V_{k,l}^*)\right) + \sqrt{n}E^*(V_{k,l}^*) \\ = & \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{X}_i^{*\prime}\hat{\xi}_i^* - E^*(V_{k,l}^*)\right) + \sqrt{n}E^*(V_{k,l}^*). \end{aligned} \tag{7.2}$$

For the first term of Eq. 7.2, we have

$$\begin{aligned}
& \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i^{*\prime}\hat{\xi}_i^{*} - E^*(V_{k,l}^{*})\right) \\
\leq \quad & \sqrt{n}\left|\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i^{*\prime}\hat{\xi}_i^{*} - \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i^{\prime}\hat{\xi}_i\right| \\
\leq \quad & \sqrt{n}\left[\left|\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i^{*\prime}\hat{\xi}_i^{*} - E^*(V_{k,\ell}^{*})\right| + \left|E^*(V_{k,\ell}^{*}) - \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i^{\prime}\hat{\xi}_i\right|\right] \xrightarrow{p^*} 0 \qquad (7.3)
\end{aligned}$$

by weak law of large numbers and since $E^*(V_{k,\ell}^{*}) - \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i^{\prime}\hat{\xi}_i = o_p(1)$. For the second term,

$$\sqrt{n}E^*(V_{k,l}^{*}) = \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_i^{\prime}\hat{\xi}_i\right) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathbf{X}_i^{\prime}\hat{\xi}_i \xrightarrow{d^*} N(0,\Gamma)$$

holds by the Slutsky's theorem. Hence, $\hat{\phi}^* \xrightarrow{p^*} \phi$. By using the proof of part (i), the proofs of (ii) and (iii) directly follow from the proof of Theorem 3.1 of Thombs and Schucany (1990).

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