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Martingale Representations for Functionals of Lévy Processes

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Abstract

We describe the integrand in the martingale (or stochastic integral)representation of a square integrable functional F of a Lévy process in terms of (a derivative or difference operator acting on) a map β^F introduced in Rajeev and Fitzsimmons (*Stochastics* **81**, 5, 467–476, 2009). The kernels in the chaos expansion of F are also described in terms of the iterated derivative and difference operators.

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1 Introduction

The martingale representation theorem for Brownian motion originates with the (equivalent) chaos expansion proved by Itô in Itô (1951) & Wiener (1938); Itô also proved a chaos expansion for (time homogenous) Lévy processes in Itô (1956) using Poisson random measures. Applications of this result, and in particular its role in the theory of forward-backward stochastic differential equations, originating in control theory and finance are by now,well known. The chaos expansions have been well studied for Lévy processes (see, for example, Nualart and Schoutens, 2000; Solé et al., 2007; Di Nunno et al., 2009; Privault, 2009) and also for other processes like the Azema martingales (see Émery, 2006).

For a given square integrable functional F of Brownian motion, the Clark-Ocone formula provides a method for calculating the integrand in terms of the stochastic derivative of F. Since then a number of papers have been devoted to extending this result for Lévy processes using the techniques of stochastic analysis (and in particular chaos expansions)(see Nualart and Schoutens, 2000; Solé et al., 2007; Di Nunno et al., 2009; Privault, 2009) or by using white noise analysis (Di Nunno et al., 2004).

The central issue, as far as the martingale representation results - like the predictable representation property (PRP)or the weak predictable representation property (weak PRP, see He et al., 1992, chapter 13) - are concerned maybe formulated as follows : Given a functional F which has such a representation how does one calculate the integrands in the representation for F? There are two issues here : Firstly, one needs to obtain a representation. Secondly, there is the question of uniqueness. For the second question, see Remark 4.5, below. For existence of a representation, it is known that for time homogenous Lévy processes, the weak PRP holds (He et al., 1992, Theorem 13.49, Theorem 13.18). In this paper,we extend the explicit formula for the integrands proved in Rajeev and Fitzsimmons (2009) & Rajeev (2009) to general d - dimensional time homogenous Lévy processes. Given a square integrable functional F of a Lévy process (Y_t) with induced filtration $(\mathcal{F}_t^Y)$, we show the existence of a 'factorisation' of the conditional expectation $(E[F|\mathcal{F}_t^Y])$ in the form $\beta^F(t,\omega,Y_t(\omega))$ for an appropriately measurable functional $\beta^F(t,\omega,y)$. In other words,for each t,almost surely, $E[F|\mathcal{F}_t^Y] = \beta^F(t,\omega,Y_t(\omega))$. The martingale representation theorem then becomes a statement of the smoothness of the functional $\beta^F(t,\omega,y)$ in the variable $y = (y_1,\cdots,y_i,\cdots,y_d)$: it allows us to define the derivative $\partial_i\beta^F$ with respect to the variable y_i as a closed linear operator, initially on the dense subspace of smooth finite dimensional functionals (Lemma 3.3) and then on the whole of L^2 (Lemma 4.2, Proposition 4.3, Theorem 4.4). The difference operator $\delta\beta^F(= \beta_2^F$ in Lemma 3.3) that appears in the discontinuous part of the martingale representation,also arises from the factorisation of the conditional expectation mentioned above. In particular, $\delta\beta^F(t,\omega,y,z) := \beta^F(t,\omega,y+z) - \beta^F(t,\omega,y)$. A number of authors have introduced the notion of a 'stochastic derivative'for Levy processes via the chaos expansions (see Nualart and Schoutens, 2000; Solé et al., 2007; Di Nunno et al., 2009; Privault, 2009). Our definition,on the other hand, involving as it does the conditional expectations with respect to the underlying filtration $(\mathcal{F}_t)$ and the martingale representation theorem (rather than the chaos expansion), maybe considered to be that of an 'adapted derivative' (However, see Di Nunno, 2007, for a notion of adapted derivative in a somewhat different context). Once the martingale representation is obtained, it allows us, by repeated application, to obtain the chaos expansion in terms of iterated integrals,as in the case of Brownian motion.

The paper is organised as follows : In Section 2, we recall a few well known results on Lévy processes. Propositions 2.1 and 2.2, we believe are well known - we present proofs only for completeness. In Section 3, we introduce the map β and prove the main representation result for finite dimensional smooth functionals (Lemma 3.3). In Section 4, we present the main representational result as an isomorphism induced by the map β between L^2 and a Hilbert space of martingales $\mathcal{H}_1$ obtained by conditioning with respect to the natural filtration of the Lévy process. The derivative map $D = (\nabla, \delta)$ associated with β and mentioned above, is realised as a linear isomorphism between $\mathcal{H}_1$ and a Hilbert space $\mathcal{H}_0$ of processes, which are the integrands in the martingale representation of an element $F \in L^2$ (Proposition 4.3, Theorem 4.4). In Section 5, we define the iterated multiple stochastic integrals with respect to Brownian motion and Poisson random measures(see also Itô, 1956; Privault, 2009, p.234), derive the chaos expansion (Theorem 5.2), show equivalence of the chaos and martingale representations (Remark 5.3) and describe the kernels in the chaos expansion of a functional F in terms of the iterated derivatives of β^F (Remark 5.4), a result that extends the formula in Stroock (1987) to Lévy processes.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$be a probability space. For a measure space $(X, \mathcal{A}, \mu)$, $L^2(\mu)$ will denote (unless otherwise specified) the real Hilbert space of equivalence classes of real valued, $\mathcal{A}$ measurable functions, square integrable with respect to the measure μ. When X is $[0,\infty) \times \Omega$ or $[0,\infty) \times \Omega \times \mathbb{R}^d$, the sigma field $\mathcal{A}$ unless otherwise specified, will be the appropriate product sigma field viz. $\mathcal{B}[0,\infty) \times \mathcal{F}$ or $\mathcal{B}[0,\infty) \times \mathcal{F} \times \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}$ stands for the Borel sigma field. We use the notation $\mathbb{R}^d_0 := \mathbb{R}^d - \{0\}$.

Let (Y_t) be an $\mathbb{R}^d$ valued Lévy process i.e a process with stationary independent increments and whose trajectories $t \to Y_t(\omega)$ are right continuous and have left limits for all $\omega \in \Omega$ with $Y_0 = 0$. Let $\mathcal{F}^Y_t := \sigma\{Y_s, s \le t\}$ and $\mathcal{F}^Y_\infty := \sigma\{Y_s, s \ge 0\}$. Abusing notation, we will again denote by $(\mathcal{F}^Y_t)$ the corresponding right continuous and P -complete filtration.

We then have the Lévy-Itô decomposition (see Kallenberg, 2002, Thm 15.4, Cor 15.7) given as follows : For $A \subset \{y : 0 < \epsilon_1 < |y| < \epsilon_2\}$ and $t > 0$ the random measure associated with the jumps of (Y_t) is defined in the usual way:

$$N((0,t] \times A) := \#\{s \le t : \triangle Y_s \in A\}.$$

Further the compensated measure $\hat{N}$ is defined as

$$\hat{N}((0,t] \times A) := N((0,t] \times A) - t\nu(A)$$

where $\nu(.)$is the sigma finite measure on $\mathbb{R}_0^d := \mathbb{R}^d - \{0\}$ -a Lévy measure - for (Y_t) satisfying $EN((0,t] \times A) = t\nu(A)$. We then have the Lévy-Itô decomposition,

$$
\begin{aligned}
Y(t) = \bar{b}t + \sigma.B_t \quad &+ \quad \int_0^t \int_{\{y:0<|y|\leq 1\}} y\hat{N}(ds\ dy) \\
&+ \quad \int_0^t \int_{\{y:|y|>1\}} yN(ds\ dy) \qquad (2.1)
\end{aligned}
$$

where $\bar{b} = (\bar{b}_1, \cdots, \bar{b}_d) \in \mathbb{R}^d$; $\sigma = (\sigma_{ij})_{1\leq i,j\leq d}$ is a matrix which we shall assume is non singular ; (B_t) is a d dimensional standard Brownian motion and the Lévy measure ν satisfies

$$\int_{\mathbb{R}_0^d} |y|^2 \wedge 1\ \nu(dy) < \infty.$$

Proposition 2.1. *If $E|Y_t|^2 < \infty$ for all $t \geq 0$ then $\int_{\mathbb{R}_0^d} |y|^2\ \nu(dy) < \infty$ and we have*

$$Y_t = bt + \sigma.B_t + \int_0^t \int_{\mathbb{R}_0^d} y\ \widehat{N}(ds\ dy) \qquad (2.2)$$

for some $b = (b_1, \cdots, b_d)$ and σ, with (B_t) and $\widehat{N}(\cdot,\cdot)$ as above.

Proof. Equation (2.2) follows from the Lévy-Itô decomposition (2.1) as soon as we can show $\int_{\{|y|>1\}} |y|^2\ \nu(dy) < \infty$ which also suffices to prove the first statement. We will do the case $d = 1$, the general case being similar. Let, for $n \geq 1$,

$$Y_t^n := \int_0^t \int_{\{1<|y|<n\}} y\ N(ds\ dy).$$

Then (Y_t^n) is a square integrable Lévy process satisfying

$$\sup_n E(Y_t^n)^2 < \infty.$$

This follows from the fact that the means and variances of the sequence $\{Y_t^n\}$ are bounded. Let

$$\varphi_t^n(u) \quad := \quad E\ e^{iuY_t^n}$$

$$= \quad e^{t\{\int (e^{iuy}-1) I_{\{1<|y|<n\}}\nu(dy)\}}.$$

Then $\varphi_t^n(\cdot)$ is twice differentiable (at the origin), with $(\varphi_t^n)'(0) = E(iY_t^n)$ and $(\varphi_t^n)''(0) = -E(Y_t^n)^2$. In particular, we have

$$E(Y_t^n)^2 = -(\varphi_t^n)''(0) = t^2 \int y^2 I_{\{1<|y|<n\}}\nu(dy).$$

Since the sequence $\{Y_t^n\}$ is L^2-bounded and converges almost surely to $\int_0^t \int_{\{1<|y|<\infty\}} y \; N(ds\; dy) =: Y_t^\infty$ as n tends to infinity, we have

$$E(Y_t^n)^2 \to E(Y_t^\infty)^2.$$

Hence

$$\begin{aligned} \int_{(1,\infty)} y^2 \; \nu(dy) &= \lim_{n\to\infty} \int y^2 I_{\{1<|y|<n\}}\nu(dy) \\ &= \frac{1}{t^2} \lim_{n\to\infty} E(Y_t^n)^2 \\ &= \frac{1}{t^2} E(Y_t^\infty)^2 < \infty. \end{aligned}$$

□

Returning to the case of a general Lévy process, we have for every $\epsilon > 0$, the decomposition $Y_t = Y_t^\epsilon + Y_{\epsilon,t}$ where, almost surely,

$$Y_t^\epsilon = b^\epsilon t + \sigma . B_t + \int_0^t \int_{\{0<|y|\le\epsilon\}} y \; \widehat{N}(ds\; dy) \qquad (2.3)$$

and

$$Y_{\epsilon,t} = \int_0^t \int_{\{|y|>\epsilon\}} y \; N(ds\; dy)$$

for all $t \ge 0$. Let $\mathcal{F}_t^\epsilon = \mathcal{F}_t^{Y^\epsilon} = \sigma\{Y_s^\epsilon, 0 \le s \le t\} \subset \mathcal{F}_t^Y$ for $0 \le t \le \infty$ and $\epsilon > 0$. Note that (Y_t^ϵ) is a square integrable Lévy process with characteristics $(b^\epsilon, \sigma, \gamma^\epsilon)$ where $\gamma^\epsilon(dy) := 1_{\{0<|y|\le\epsilon\}}(y)\; \gamma(dy)$.

Let $C_0(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{R}, f$ continuous , $\lim_{|x|\to\infty} f(x) = 0\}$. Let $T_t : C_0(\mathbb{R}^d) \to C_0(\mathbb{R}^d)$ be the semi group corresponding to (Y_t) i.e for

$f \in C_0(\mathbb{R}^d), T_t f(y) := Ef(Y_t + y)$. Let $C_0^\infty(\mathbb{R}^d)$ denote the class of all infinitely differentiable functions f on $\mathbb{R}^d$ such that f and all its derivatives belong to $C_0(\mathbb{R}^d)$. Then the infinitesimal generator A of (T_t) maybe described as follows (see Kallenberg, 2002, Theorem 19.10): For $f \in C_0^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned} Af(y) &= \frac{1}{2}\sum_{ij=1}^{d}(\sigma\sigma^t)_{ij}\ \partial_{ij}^2 f(y) + \sum_{i=1}^{d}\bar{b}_i\ \partial_i g(y) \\ &+ \int\limits_{\mathbb{R}_0^d}\{f(y+x) - f(y) - \sum_{i=1}^{d}\partial_i f(y) x_i I_{\{x:|x|\le 1\}}\}\nu(dx) \end{aligned}$$

Using the representation given by (2.2) when (Y_t) is square integrable, the corresponding description of A is obtained from above as follows: For $f \in C_0^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned} Af(y) &= \frac{1}{2}\sum_{ij=1}^{d}(\sigma\sigma^t)_{ij}\partial_{ij}^2 f(y) + \sum_{i=1}^{d} b_i\partial_i g(y) \\ &+ \int\limits_{\mathbb{R}_0^d}\{f(y+x) - f(y) - \sum_{i=1}^{d}\partial_i f(y) x_i\}\nu(dx) \end{aligned} \tag{2.4}$$

$C^{1,\infty}\ ([0,\infty)\times\mathbb{R}^d) := \{f : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}, f(t,.) \in C_0^\infty(\mathbb{R}^d)\ \forall t \ge 0, f(.,y) \in C^1[0,\infty)\ \forall y \in \mathbb{R}^d\}$. Let $f \in C_0^\infty(\mathbb{R}^d)$ and define $g(t,y) := T_t f(y) \equiv E^y f(Y_t) \equiv E\ f(Y_t + y)$.

Proposition 2.2. *Let $g(t,y)$ be as defined above . Then $g \in C^{1,\infty}\ ([0,\infty) \times\mathbb{R}^d)$.*

Proof. Clearly for $t \ge 0$, $g(t,\cdot) \in C_0^\infty(\mathbb{R}^d)$ follows from the dominated convergence theorem. Moreover since $C_0^\infty \subset D(A)$ =: Domain of the infinitesimal generator A of (T_t) (see Kallenberg, 2002, Theorem 19.10) we have

$$g(t,y) = T_t f(y) = f(y) + \int\limits_0^t AT_s f(y)\ ds. \tag{2.5}$$

In particular,

$$\partial_t g(t,y) = AT_t f(y) = T_t Af(y)$$

which shows that $g(\cdot,y) \in C^1[0,\infty)$. The result follows.

□

3 Representations on a Dense Subspace

We will first obtain the integral representation of square integrable functionals coming from a dense subspace of L^2, which we now define.

$$\begin{aligned}\mathcal{C} \quad := \quad & \{F \in L^2 : F = f_1(Y_{t_1}) \dots f_n(Y_{t_n}), f_i \in C_0^\infty(\mathbb{R}^d), \\ & 0 \le t_1 < \dots < t_n < \infty, n \ge 1\}.\end{aligned}$$

The linear span of $\mathcal{C}$ will be denoted by $\mathcal{V}$.

Proposition 3.1. *$\mathcal{V}$ is dense in $L^2(\mathcal{F}_\infty^Y)$.*

Proof. This can be shown as in the case when (Y_t) is continuous (see for example Rajeev and Fitzsimmons, 2009, lemma 1.2), using the fact that the field of finite dimensional events generates $\mathcal{F}_\infty^Y$. □

Proposition 3.2. *Let $h \in C_0^\infty(\mathbb{R}^d)$ and define $g_t(s, y) := E[h(Y_{t-s} + y)], 0 \le s \le t, y \in \mathbb{R}$. Let (Y_t) be a square integrable Lévy process with representation given by (2.2) and the corresponding infinitesimal generator given by eqn.(2.4). Then for every $t > 0$, almost surely,*

$$\begin{aligned} h(Y(t)) \quad = \quad & E[h(Y(t))] + \int_0^t (\nabla g_t(s, Y_{s-}).\sigma).dB(s) \\ & + \int_0^t \int_{\mathbb{R}_0} \{g_t(s, Y_{s-} + z) - g_t(s, Y_{s-})\} \ \hat{N}(ds, dz). \qquad (3.1)\end{aligned}$$

Proof. Put $Z_s = g_t(s, Y(s)), 0 \le s \le t$. By Proposition 2.2, $g_t \in C^{1,\infty}$ $([0,\infty) \times \mathbb{R}^d)$. Then from Itô's formula we have,

$$\begin{aligned} Z(t) - Z(0) \quad = & \int_0^t \{\partial_s g_t(s, Y_s) + b.\nabla g_t(s, Y_s)\} \ ds + \int_0^t (\nabla g_t(s, Y_s).\sigma).dB(s) \\ & + \frac{1}{2} \sum_{i,j=1}^d \int_0^t (\sigma\sigma^t)_{ij} \ \partial_{ij}^2 g_t(s, Y_s) \ ds \\ & + \int_0^t \int_{\mathbb{R}_0^d} \{g_t(s, Y_{s-} + z) - g_t(s, Y_{s-})\} \hat{N}(ds, dz) \\ & + \int_0^t \int_{\mathbb{R}_0^d} \{g_t(s, Y_{s-} + z) - g_t(s, Y_{s-}) - z.\nabla g_t(s, Y_{s-})\} \nu(dz) ds \\ = \quad & \int_0^t \{\partial_s g_t(s, Y_s) + A g_t(s, Y_s)\} \, ds + \int_0^t (\nabla g_t(s, Y_s).\sigma).dB(s)\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t \int_{\mathbb{R}_0^d} \{g_t(s, Y_{s-}+z) - g_t(s, Y_{s-})\}\hat{N}(ds,dz) \\
= \quad & \int_0^t (\nabla g_t(s, Y_s).\sigma).dB(s) \\
&+\int_0^t \int_{\mathbb{R}_0^d} \{g_t(s, Y_{s-}+z) - g_t(s, Y_{s-})\}\hat{N}(ds,dz)
\end{aligned}
$$

where we have used (2.4) to obtain the middle equality; the equation $g_t(s,y) = T_{t-s}h(y)$ and the analogue of the semi-group relation given by eqn.(2.5) (with f replaced by h and t by $t-s$ respectively in the RHS of (2.5) and $g(t,y)$ in the LHS replaced by $g_t(s,y)$) to obtain the last equality . Now, using the fact that

$$
Z(0) \quad = \quad g_t(0, Y_0) = g(0,y) = E[h(Y_t + y)]
$$

and

$$
Z(t) \quad = \quad g_t(t, Y_t) = E[h(Y(0)+y)]_{y=Y_t} = h(Y_t).
$$

the proof of the proposition is complete. □

We revert back to a general Lévy process (Y_t). Let $F = f_1(Y_{t_1})\dots f_n(Y_{t_n}) \in \mathcal{C}$, where $0 < t_1 < \dots < t_n < \infty$. We define $\beta^F(t,\omega,y)$ as follows:

$$
\beta^F(t,\omega,y) := \begin{cases}
T_{t_1-t}(f_1 T_{t_2-t_1}(f_2(\cdots T_{t_n-t_{n-1}} f_n)\cdots)(y), & 0 \le t \le t_1, \\
\\
\left[f_1(Y_{t_1}(\omega))\dots f_{i-1}(Y_{t_{i-1}}(\omega))\times \right. & \\
\left. T_{t_i-t}(f_i T_{t_{i+1}-t_i}(f_{i+1}(\cdots T_{t_n-t_{n-1}} f_n)\cdots)(y)\right], & t_{i-1} < t \le t_i, \ 2 \le i \le n \\
\\
F(\omega), & t > t_n
\end{cases}
$$

We define functionals β_1^F, β_2^F as follows :

$$
\begin{aligned}
\beta_1^F(t,\omega,y) &= \nabla \beta^F(t,\omega,y)\cdot\ \sigma \\
\beta_2^F(t,\omega,y,z) &= \beta^F(t,\omega,y+z) - \beta^F(t,\omega,y)
\end{aligned}
$$

where ∇ in the right hand side of the first equality represents the gradient with respect to the y variable. Consequently $\beta_1^F(t,\omega,y) \equiv 0$ for $t > t_n$ which implies that the process $(\beta_{1i}^F(t, Y_{t-})) \in L^2(dt \times dP)$, $i = 1, \cdots, d$.

Similarly, by an application of the mean value theorem and Proposition 2.1 , we can conclude that for a square integrable Lévy process (Y_t), the process $(\beta_2^F(t, Y_{t-}, z)) \in L^2(dt \times dP \times d\nu)$.We extend the definition of $\beta^F, \beta_1^F, \beta_2^F$ to $F \in \mathcal{V}$ by linearity : If $F \in \mathcal{V},\ F = \sum_{i=1}^{n}\ \alpha_i F_i$ define

$$\begin{aligned}
\beta^F(t,\omega,y) &:= \sum_{i=1}^{n} \alpha_i \beta^{F_i}(t,\omega,y) \\
\beta_1^F(t,\omega,y) &:= \nabla\beta^F(t,\omega,y).\sigma \\
\beta_2^F(t,\omega,y,z) &:= \beta^F(t,\omega,y+z) - \beta^F(t,\omega,y) \\
&=: \delta\beta^F(t,\omega,y,z).
\end{aligned}$$

We sometimes (particularly in Section 5) use the notation $\beta_1^F(t)$ for the variable $\beta_1^F(t, Y_{t-})$ and similarly $\beta_2^F(t,z)$ or $\delta\beta^F(t,z)$ for the variable $\beta_2^F(t, Y_{t-}, z)$. The following Lemma provides the basic representation result for $F \in \mathcal{V}$.

LEMMA 3.3. Let (Y_t) be a Lévy process, with representation given by (2.1). Let $F \in \mathcal{V}, \beta^F, \beta_1^F$ and β_2^F be as above. Then

$$\begin{aligned}
F &= E[F] + \int_0^\infty \beta_1^F(s,w,Y_{s-}).dB_s \\
&\quad + \int_0^\infty \int_{\mathbb{R}_0^d} \beta_2^F(s,w,Y_{s-},z)\hat{N}(ds,dz). \qquad (3.2)
\end{aligned}$$

In particular, $(\beta_{1i}^F(t, Y_t)) \in L^2(dt \times dP)\ \ i = 1, \cdots, d$ and $(\beta_2^{\mathrm{F}}(\mathrm{t}, \mathrm{Y_t}, \mathrm{z})) \in \mathrm{L}^2(\mathrm{dt} \times \mathrm{dP} \times \mathrm{d}\nu)$.

PROOF. It suffices, by linearity, to prove (3.2) when $F \in \mathcal{C}$ and is of the form $F = f_1(Y_{t_1}) \dots f_m(Y_{t_m}) \in \mathcal{C}$, where $0 < t_1 < \cdots < t_m < \infty$. We further note that (3.2) follows for a general Lévy process, if we can show that it is true for a square integrable Lévy process. Indeed, let $F = f_1(Y_{t_1}) \dots f_m(Y_{t_m}) \in \mathcal{C}$ and let $F_n := f_1(Y_{t_1}^n) \dots f_m(Y_{t_m}^n)$, where $(Y_t^n), n \geq 1$ are square integrable Lévy processes with representation given by (2.3),with $\epsilon = n$. Since $Y_t = \lim_{n\to\infty} Y_t^n$ almost surely, it is easy to see , using the dominated convergence theorem, that almost surely,

$$\beta^F(t, Y_t) = \lim_{n\to\infty}\ \beta_n^{F_n}(t, Y_t^n)$$

where β_n is the functional β defined on finite dimensional functionals of the Lévy process (Y_t^n). In fact the convergence also holds in L^2 since

$\beta_n^{F_n}(t, Y_t^n) = E[F_n \mid \mathcal{F}_t^n] \to E[F \mid \mathcal{F}_t^Y]$ in L^2. It is also straight forward that

$$\partial_i \beta^F(t, Y_t) = \lim_{n\to\infty} \partial_i \beta_n^{F_n}(t, Y_t^n)$$

almost surely, for all $t \geq 0$ and $1 \leq i \leq d$, and that

$$\delta\beta^F(t, Y_{t-}, z) = \lim_{n\to\infty} \delta\beta_n^{F_n}(t, Y_{t-}^n, z)$$

almost surely, for all $t \geq 0$ and $z \in \mathbb{R}$. Using the above observations the statement of the theorem for $\beta^F(t, Y_t)$ can be derived by letting $n \to \infty$ in the corresponding statement for $\beta_n^{F_n}(t, Y_t^n)$.

Let now (Y_t) be a square integrable Lévy process.We prove the result by induction on m. Note that by Proposition 3.2 the result is true for $m = 1$.

Assume that (3.2) is true for all $F \in \mathcal{V}$ with no more than $m-1$ factors. Let $G := \prod_{k=1}^{m-1} f_k(Y_{t_k})$, $F_m := f_m(Y_{t_m})$ and define

$$Z_1(t) := \int_0^{t_{m-1}\wedge t} \beta_1^G(s, \omega, Y_{s-}).dB_s + \int_0^{t_{m-1}\wedge t} \int_{\mathbb{R}_0^d} \beta_2^G(s, \omega, Y_{s-}, z)\hat{N}(ds, dz)$$

and

$$Z_2(t) := \int_{t_{m-1}\wedge t}^{t_m\wedge t} \beta_1^{F_m}(s, \omega, Y_{s-}).dB_s + \int_{t_{m-1}\wedge t}^{t_m\wedge t} \int_{\mathbb{R}_0^d} \beta_2^{F_m}(s, \omega, Y_{s-}, z)\hat{N}(ds, dz).$$

By the induction hypothesis, if $s \geq t_{m-1}$ then

$$Z_1(s) = Z_1(t_{m-1}) = G - \mathbf{E}[G|\mathcal{F}_0].$$

Now we apply the case $m = 1$ to the process $Y_t^{t_{m-1}} := Y_{t+t_{m-1}}$, $t \geq 0$. It follows that for $s \geq t_m$

$$Z_2(s) = Z_2(t_m) = f_m(Y_{t_m}) - T_{t_m - t_{m-1}} f_m(Y_{t_{m-1}}).$$

Clearly $Z_2(s) = 0$ for $s \leq\ t_{m-1}$. We use the integration by parts formula

$$\begin{aligned} Z_1(t)Z_2(t) &= Z_1(0)Z_2(0) + \int_0^t Z_1(s-)dZ_2(s) + \int_0^t Z_2(s-)dZ_1(s) \\ &\quad + \langle Z_1, Z_2\rangle_c(t) + \sum_{s\leq t} \Delta Z_1(s)\Delta Z_2(s). \end{aligned}$$

We then get

$$Z_1(t_m)Z_2(t_m) = \int_{t_{m-1}}^{t_m} Z_1(s)\beta_1^{F_m}(s, \omega, Y_{s-}).dB_s$$

$$+\int_{t_{m-1}}^{t_m}\int_{\mathbb{R}_0^d} Z_1(s-)\beta_2^{F_m}(s,\omega,Y_{s-},z)\hat{N}(ds,dz)$$

$$= \int_0^\infty \{G-\mathbf{E}[G|\mathcal{F}_0]\}\,\beta_1^{F_m}(s,\omega,Y_{s-}).dB_s$$

$$+\int_0^\infty\int_{\mathbb{R}_0^d}\{G-\mathbf{E}[G|\mathcal{F}_0]\}\,\beta_2^{F_m}(s,\omega,Y_{s-},z)\hat{N}(ds,dz)$$

On the other hand, by the remarks above ,

$$\begin{aligned} Z_1(t_m)Z_2(t_m) &= \{G-\mathbf{E}[G|\mathcal{F}_0]\}\,.\left\{f_m(Y_{t_m})-T_{t_m-t_{m-1}}f_m(Y_{t_{m-1}})\right\} \\ &= F+\mathbf{E}[G|\mathcal{F}_0].T_{t_m-t_{m-1}}f_m(Y_{t_{m-1}}) \\ &\quad -\mathbf{E}[G|\mathcal{F}_0].f_m(Y_{t_m})-G.T_{t_m-t_{m-1}}f_m(Y_{t_{m-1}}). \end{aligned}$$

We shall examine the last three terms in the right hand side of the second equality separately. Applying the representation obtained for $m=1$ to $\hat{F}:=\hat{f}(Y_{t_{m-1}})$ where $\hat{f}(y)\equiv T_{t_m-t_{m-1}}f_m(y)$ we find that

$$\begin{aligned} & T_{t_m-t_{m-1}}f_m(Y_{t_{m-1}}) \\ =\ & T_{t_{m-1}}\left(T_{t_m-t_{m-1}}f_m\right)(Y_0)+\int_0^{t_{m-1}}\beta_1^{\hat{F}}(s,\omega,Y_{s-}).dB_s \\ & +\int_0^{t_{m-1}}\int_{\mathbb{R}_0^d}\beta_2^{\hat{F}}(s,\omega,Y_{s-},z)\hat{N}(ds,dz). \end{aligned}$$

Similarly,

$$\begin{aligned} \mathbf{E}[G|\mathcal{F}_0]f_m(Y_{t_m}) &= \mathbf{E}[G|\mathcal{F}_0].\Bigg\{T_{t_m}f_m(Y_0)+\int_0^{t_m}\beta_1^{F_m}(s,\omega,Y_{s-}).dB_s \\ &\quad +\int_0^{t_m}\int_{\mathbb{R}_0^d}\beta_2^{F_m}(s,\omega,Y_{s-},z)\hat{N}(ds,dz)\Bigg\}. \end{aligned}$$

Define $f^*_{m-1}(x) := f_{m-1}(x).T_{t_m-t_{m-1}}f_m(x)$ and $G^* := \left[\prod_{k=1}^{m-2} f_k(Y_{t_k})\right].$ $f^*_{m-1}(Y_{t_{m-1}})$. By the induction hypothesis,

$$\begin{aligned} G.T_{t_m-t_{m-1}}f_m(Y_{t_{m-1}}) &= G^* \\ &= E\left[G^*|\mathcal{F}_0\right] \\ &\quad +\int_0^\infty \beta_1^{G^*}(s,\omega,Y_{s-}).dB_s \end{aligned}$$

$$
\begin{aligned}
& + \int_0^\infty \int_{\mathbb{R}_0^d} \beta_2^{G^*}(s,\omega,Y_{s-},z)\hat{N}(ds,dz) \\
= \; & E\left[f_1(Y_{t_1})\cdots f_m(Y_{t_m})|\mathcal{F}_0\right] \\
& + \int_0^\infty \beta_1^{G^*}(s,\omega,Y_{s-}).dB_s \\
& + \int_0^\infty \int_{\mathbb{R}_0^d} \beta_2^{G^*}(s,\omega,Y_{s-},z)\hat{N}(ds,dz),
\end{aligned}
$$

where we have used the Markov property of (Y_t) in the last equality. Equating the (last) two expressions for $Z_1(t_m)Z_2(t_m)$, the statement of the theorem is established for any square integrable Lévy process and the proof of the lemma is complete. □

4 Representations on L^2

In this section we extend the representation obtained on $\mathcal{V}$ to the whole of $L^2(\mathcal{F}_\infty^Y)$.

Let $\mathcal{P}$ denote the previsible σ-field on $[0,\infty)\times\Omega$ generated by $(\mathcal{F}_t^Y)$ i.e. $\mathcal{P} = \sigma\{(g_t) : g_t(\omega)$ is left continuous on $(0,\infty)$ and $(\mathcal{F}_t^Y)$ adapted$\}$. Let $f : [0,\infty)\times\Omega\times\mathbb{R}_0^d \to \mathbb{R}$. We will say that f is previsible if f is $\mathcal{P}\otimes\mathcal{B}_0$ measurable where $\mathcal{B}_0$ is the Borel σ-field on $\mathbb{R}_0^d$. Note that if $f(t,\omega,z)$ is previsible then $f(t,\omega,Y_{t-})$ is previsible in (t,ω). For $F\in\mathcal{V}$, recall the map β^F defined in Section 3, following the proof of Proposition 3.2. We list some of the properties of β^F in the following proposition.

Proposition 4.1. *Let $F\in\mathcal{V}$. Then $\beta^F(t,\omega,y)$ has the following properties:*

1. *$y\to\beta^F(t,\omega,y)$ is a C^∞-map.*
2. *a.s., $t\to\beta^F(t,\omega,Y_t)$ is right continuous.*
3. *for all $t\geq 0$, $\beta^F(t,Y_t) = E[F|\mathcal{F}_t^Y]$ a.s.*
4. *The map $(t,\omega,y)\to\partial_i\beta^F(t,\omega,y)$ is $\mathcal{P}\otimes\mathcal{B}_0$ measurable. In particular, $(\partial_i\beta^F(t,\omega,Y_{t-}))$ is a previsible process.*
5. *If $\delta\beta^F(t,\omega,y,z) := \beta^F(t,\omega,y+z)-\beta^F(t,\omega,y)$, then $(\delta\beta^F(t,\omega,Y_{t-},z))$ is previsible.*

PROOF. 1) and 2) follow from the definition of β^F and the regularity of $T_t f(x)$ in t and x. 3) follows from the Markov property. As for 4), note that the first statement follows by inspection and the second follows from a monotone class argument. Lastly 5) follows from 4). $\square$

The following lemma proves a version of the integral representation property for $(\mathcal{F}_t^Y)$-martingales of a special form.

LEMMA 4.2. *Let $f : [0,\infty) \times \Omega \times \mathbb{R}^d \to \mathbb{R}$ be such that $(f(t, Y_t))$ is a right continuous, L^2-bounded $(\mathcal{F}_t^Y)$- martingale.Then there exists previsible processes $(f^i(s)) \in L^2(dt \times dP), 1 \le i \le d$ and a $\mathcal{P} \otimes \mathcal{B}_0$ previsible process $(f^{d+1}(s,z)) \in L^2(dt \times dP \times d\nu)$ such that for $0 \le t \le \infty$*

$$\begin{aligned} f(t, Y_t) &= E\ f(t, Y_t) + \sum_{i=1}^{d} \int_0^t f^i(s)\ dY_s^{c,i} \\ &\quad + \int_0^t \int_{\mathbb{R}_0^d} f^{d+1}(s,z) \hat{N}(ds\ dz) \end{aligned} \tag{4.1}$$

almost surely, where we use the notation $Y^{c,i} := (\sigma.B)_i$ for the continuous martingale part of (Y_t).

PROOF. Let $f(\infty) := \lim_{t\to\infty} f(t, Y_t)$. Since $f(\infty) \in L^2$ and $\mathcal{V}$ is dense in L^2, $\exists\ F^n \in \mathcal{V}$ such that $F^n \to f(\infty)$ in L^2. We then have from Lemma 3.3 ,

$$\begin{aligned} E[F^n|\mathcal{F}_t] &= \beta^{F^n}(t, Y_t) \\ &= EF^n + \sum_{i=1}^{d} \int_0^t \partial_i \beta^{F_n}(t, Y_{t-})\ dY_s^{c,i} \\ &\quad + \int_0^t \int_{\mathbb{R}_0^d} \delta\beta^{F_n}(s, Y_{s-}, z)\ \hat{N}(ds\ dz). \end{aligned}$$

Since $F^n \to f(\infty)$ in L^2, it follows from the properties of the stochastic integral that $\exists$ previsible processes $(f^i(s)), 1 \le i \le d$ and $(f^{d+1}(s,z))$ in $L^2(dt \times dP)$ and $L^2(dt \times dP \times d\nu)$ such that $(\sigma \cdot \partial_i \beta^{F^n}(s, Y_{s-})) \to (\sigma \cdot f^i(s))$ in $L^2(dt \times dP)$ and $(\delta\beta^{F_n}(s, Y_{s-}, z)) \to (f^{d+1}(s,z))$ in $L^2(dt \times dP \times d\nu)$. From the Itô isometry and the orthogonality of the martingales in the right hand side , it follows that the right hand side of the above equality converges in L^2 to the right hand side in the statement of the Lemma for each $t, 0 \leq t \leq \infty$.

□

We now define real Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_0$ as follows:

$$\begin{aligned}\mathcal{H}_1 \;&:=\; \{f : [0,\infty) \times \Omega \times \mathbb{R}^d \to \mathbb{R}, \text{jointly measurable such that } f(t, Y_t) \\ &\qquad \text{is a right continuous, } L^2 \text{ bounded, } (\mathcal{F}_t^Y) \text{ martingale}\}.\end{aligned}$$

Then $\mathcal{H}_1$ is a real Hilbert space with the inner product $\langle f, g\rangle_{\mathcal{H}_1} := E(f(\infty)g(\infty))$, where $f(\infty), g(\infty)$ are the L^2-limits of the martingale $f(t, Y_t), g(t, Y_t)$ respectively. Note that we identify $f(t, \omega, y)$ and $g(t, \omega, y)$ in $\mathcal{H}_1$ if for all $t \geq 0$, $f(t, Y_t) = g(t, Y_t)$ a.s. Under this identification an L^2-bounded, right continuous $(\mathcal{F}_t^Y)$ martingale (M_t) may be identified with the element f of $\mathcal{H}_1$ given by $f(t, \omega, y) := M_t(\omega) \otimes 1(y)$, where $1(y) = 1 \;\forall\; y$. For previsible processes $(f^i(s)), 1 \leq i \leq d$ and $(f^{d+1}(s, z))$ and $f := (f^1, \cdots, f^d, f^{d+1})$ we define

$$\begin{aligned}\|f\|^2_{\mathcal{H}_0} \;&:=\; \sum_{i,j=1}^{d} E \int_0^\infty f^i(s)\; f^j(s)\; d\langle Y^{c,i}, Y^{c,j}\rangle_s \\ &\qquad + E \int_0^\infty \int_{\mathbb{R}_0^d} (f^{d+1}(s, z))^2 \; ds\; \nu(dz),\end{aligned}$$

where $Y^{c,i} := (\sigma \cdot B)_i$. Note that since $\langle Y^{c,i}, Y^{c,j}\rangle_t = (\sigma\sigma^t)_{ij} t$ and $\sigma\sigma^t > 0$, it follows that $\|f\|_{\mathcal{H}_0} = 0$ implies $f^i(s, \omega) = 0$ for a.e. (s, ω) $ds \times dP$, $1 \leq i \leq d$ and $f^{d+1}(s, \omega, z) = 0$ for a.e. (s, ω, z) $ds \times dP \times d\nu$. Clearly with this identification, $\|\cdot\|_{\mathcal{H}_0}$ is a Hilbertian norm and we define the Hilbert space $\mathcal{H}_0$ as

$$\begin{aligned}\mathcal{H}_0 \;&:=\; \{f = (f^1, \cdots, f^{d+1}) : f^i \text{ is previsible } 1 \leq i \leq d+1 \\ &\qquad \text{and } \|f\|_{\mathcal{H}_0} < \infty\}.\end{aligned}$$

Note that the spaces $\mathcal{H}_0$ and $\mathcal{H}_1 \ominus \mathbb{R}$ are isomorphic via the representation given by Lemma 4.2. Note also that $\mathcal{H}_1$ is isomorphic to $\mathcal{M}_2$, the space of right continuous, L^2-bounded, $(\mathcal{F}_t^Y)$ martingales. We also have $\mathcal{M}_2 = \mathcal{M}_2^c \oplus \mathcal{M}_2^d$, a direct sum of the continuous and purely discontinuous spaces of martingales (see Dellacherie and Meyer, 1982,Chap.VIII, Sec.2). Accordingly we have $\mathcal{H}_1 = \mathbb{R} \oplus \mathcal{H}_{1,c} \oplus \mathcal{H}_{1,d}$ where

$$\mathcal{H}_{1,c} := \{f \in \mathcal{H}_1 : t \to f(t, Y_t) \text{ is continuous a.s.}, f(0,0) = 0 \; a.s.\}$$

and

$$\mathcal{H}_{1,d} \;:=\; \{f \in \mathcal{H}_1 : (f(t, Y_t)) \text{ is a purely discontinous martingale with}$$

$f(0,0) = 0 \ a.s.\}$.

The isomorphism between $\mathcal{H}_1 \ominus \mathbb{R}$ and $\mathcal{H}_0$ implies a similar decomposition for $\mathcal{H}_0 : \mathcal{H}_0 = \mathcal{H}_{0,c} \oplus \mathcal{H}_{0,d}$ where

$$\mathcal{H}_{0,c} := \{f \in \mathcal{H}_0 : f = (f^1, \cdots, f^{d+1}), f^{d+1} \equiv 0\}$$

and

$$\mathcal{H}_{0,d} := \{f \in \mathcal{H}_0 : f = (f^1, \cdots, f^{d+1}),\ f^i \equiv 0, 1 \leq i \leq d\}.$$

Note that the stochastic integral defines a linear map $I : \mathcal{H}_0 \to L^2 \ominus \mathbb{R}$ as follows : For $f = (f^1, \cdots, f^{d+1})$

$$I(f) := \sum_{i=1}^{d} \int_0^\infty f^i(s)\ dY_s^{c,i} + \int_0^\infty \int_{\mathbb{R}_0^d} f^{d+1}(s,z)\ \hat{N}(ds\ dz)$$

and is in fact an imbedding of $\mathcal{H}_0$ into $L^2 \ominus \mathbb{R}$. On the other hand we have the map $\beta : \mathcal{V} \subset L^2 \to \mathcal{H}_1$ given by the representation in Lemma 3.3 and we will denote by $\mathcal{H}_1^0$ the image $\beta(\mathcal{V})$ i.e.

$$\mathcal{H}_1^0 := \{f \in \mathcal{H}_1 : f = \beta^F \text{ for some } F \in \mathcal{V} \subset L^2\}.$$

Since for $F \in \mathcal{V}, \beta^F(t, \omega, y)$ is a smooth function in the variable y, we have the map $D : \mathcal{H}_1^0 \to \mathcal{H}_0$ defined as follows : For $f = \beta^F, F \in \mathcal{V}$, we define the $d+1$ dimensional vector DF as follows :

$$Df \equiv D\beta^F := (\nabla\beta^F, \delta\beta^F).$$

PROPOSITION 4.3. *The linear operator $D : \mathcal{H}_1^0 \to \mathcal{H}_0$ is closable and extends as a closed linear operator to $D : \mathcal{H}_1 \to \mathcal{H}_0$.*

PROOF. Suppose $F_n \in \mathcal{V}, \beta^{F_n} \to 0$ in $\mathcal{H}_1$ and $D\beta^{F_n} \to g = (g^1, \cdots, g^{d+1})$ in $\mathcal{H}_0$. We have from Lemma 3.3, with $Y^c = (Y^{c,1}, \cdots, Y^{c,d})$,

$$F_n = EF_n + \int_0^\infty \nabla\beta^{F_n}(s, Y_{s-}).dY_s^c + \int_0^\infty \int_{\mathbb{R}_0^d} \delta\beta^{F_n}(s, Y_{s-}, z)\ \hat{N}(ds\ dz).$$

Since $F_n \to 0$ in L^2 it follows that

$$\begin{aligned} \|D\beta^{F_n}\|_{\mathcal{H}_0} &= \|I(D\beta^{F_n})\|_{L^2} \\ &= \|F_n - EF_n\|_{L^2} \end{aligned}$$

$$\to \quad 0$$

as $n \to \infty$ since $\|F_n\|_{L^2} = \|\beta^{F_n}\|_{\mathcal{H}_1}$. It follows that $g = 0$.

From the denseness of $\mathcal{V}$ in L^2 and the definition of the norm in $\mathcal{H}_1^0$ it readily follows that $\mathcal{H}_1^0$ is dense in $\mathcal{H}_1$. For $f \in \mathcal{H}_1$, let $F^n \in L^2, F^n \to f(\infty)$ in L^2. Then $Df := \lim_{n\to\infty} D\beta^{F_n} = \lim_{n\to\infty} (\nabla\beta^{F_n}, \delta\beta^{F_n}) = (f^1, \cdots, f^{d+1})$ where the last equality follows from the representation for f given by Lemma 4.2. This gives the required closed extension of D to $\mathcal{H}_1$. □

Let $P_0 : \mathcal{H}_1 \to \mathbb{R}$ denote the orthogonal projection into the subspace $\mathbb{R}$ of $\mathcal{H}_1$ and $P_0^\perp$ the projection to its 'orthogonal complement' viz. $\mathcal{H}_1 \ominus \mathbb{R}$. The preceding formulations now yield the following result.

THEOREM 4.4. *The map $\beta : \mathcal{V} \to \mathcal{H}_1$ extends as an isometric isomorphism to the whole of L^2 with inverse given by*

$$\beta^{-1} = P_0 + I \circ D \circ P_0^\perp$$

where $D = (\nabla, \delta) : \mathcal{H}_1 \ominus \mathbb{R} \to \mathcal{H}_0$ and $I : \mathcal{H}_0 \to L^2 \ominus \mathbb{R}$ are isometric isomorphisms.

PROOF. For $F \in \mathcal{V}$, Lemma 3.3 and the properties of the map $\beta^F(t, \omega, y)$ show that $(\beta^F(t, Y_t))$ is a right continuous, L^2 bounded martingale. Hence $\beta^F \in \mathcal{H}_1$ and from the definition of the inner product in $\mathcal{H}_1$ it follows that if $F, G \in L^2 \cap \mathcal{V}$

$$\langle \beta^F, \beta^G \rangle_{\mathcal{H}_1} = E(FG) = \langle F, G \rangle_{L^2}.$$

For general $F \in L^2$ we define $\beta^F = f \in \mathcal{H}_1$ where $f(t, \omega, y) := M_t(\omega) \otimes 1(y)$ where (M_t) is a right continuous version of the martingale $(E[F \mid \mathcal{F}_t^Y])$. This gives the desired extension of $\beta : L^2 \to \mathcal{H}_1$ with $\beta(c) = c, c \in \mathbb{R}$. It is clear that $I : \mathcal{H}_0 \to L^2 \ominus \mathbb{R}$ is an isometric imbedding of $\mathcal{H}_0$ into $L^2 \ominus \mathbb{R}$. For $F \in L^2 \ominus \mathbb{R}$, the element $f \in \mathcal{H}_0$ such that $I(f) = F$ is given as $f = D\beta^F = D \circ P_0^\perp(\beta^F)$. It is also clear that for $F, G \in L^2 \ominus \mathbb{R}$,

$$\langle \beta^F, \beta^G \rangle_{\mathcal{H}_1} = \langle D\beta^F, D\beta^G \rangle_{\mathcal{H}_0}$$

first for $F, G \in \mathcal{V}$ and then for $F, G \in L^2$. The theorem follows. □

REMARK 4.5. Note that we have used the fact that σ is non-singular to ensure that the Hilbert space $\mathcal{H}_0$ as well as the stochastic gradient ∇ with respect to the martingale Y^c is well defined. However, we note that these and the results below work also when $\sigma = 0$.

5 The Chaos Expansion

In the decomposition $Y_t = Y_t^\epsilon + Y_{\epsilon,t}, \epsilon > 0$ we can write the square integrable Lévy process (Y_t^ϵ) as $Y_t^\epsilon = Y_t^{c,\epsilon} + Y_t^{d,\epsilon}$ where $(Y_t^{c,\epsilon})$ is the continuous semi-martingale part and $(Y_t^{d,\epsilon})$ the purely discontinuous martingale part of (Y_t^ϵ) , easily identified from eqn.(2.3). Let $u^{1i}(s), u^{2i}(s), i = 1, \cdots, d$, be deterministic simple functions of $s \geq 0$, $u^1 = (u^{11}, \cdots, u^{1d}), u^2 = (u^{21}, \cdots, u^{2d})$ and $u = (u^1, u^2)$. Define for $0 \leq t \leq \infty$,

$$Y_t^{\epsilon,u} := \int_0^t u^1(s).dY_s^{c,\epsilon} + \int_0^t u^2(s).dY_s^{d,\epsilon}. \tag{5.1}$$

The exponential martingales $(M_t^{\epsilon,u})$ are then defined as follows: for $0 \leq t \leq \infty$,

$$\begin{aligned} M_t^{\epsilon,u} &:= \exp\left\{ Y_t^{\epsilon,u} - \frac{1}{2}\sum_{i,j=1}^{d}\int_0^t u^{1i}(s)\ u^{1j}(s)\ d\langle Y^{c,\epsilon,i}, Y^{c,\epsilon,j}\rangle_s \right. \\ &\quad \left. -\int_0^t \int_{\{0<|y|\leq\epsilon\}} \{e^{u^2(s)\cdot y} - 1 - u^2(s).y\}\ \gamma(dy)\ ds \right\}. \end{aligned} \tag{5.2}$$

Lemma 5.1. *The set $\{M_\infty^{\epsilon,u} : \epsilon > 0, u = (u^{11}, \cdots, u^{2d})\}$ is total in L^2.*

Proof. Since $\mathcal{F}_\infty^\epsilon := \sigma\{Y_t^\epsilon : t \geq 0\} \uparrow \mathcal{F}_\infty^Y$ as $\epsilon \uparrow \infty$, it suffices to show for $\epsilon > 0$ fixed and $\varphi \in L^2(\mathcal{F}_\infty^\epsilon)$, $E(M_\infty^{\epsilon,u}\varphi) = 0$ for all $u = (u^1, u^2)$, u^{ij} simple functions, $i = 1, 2, j = 1, \cdots, d$, implies $\varphi = 0$ a.e. Since φ is $\mathcal{F}_\infty^\epsilon$ measurable, $\exists\ S \subset [0,\infty)$, $S = \{t_i,\ i = 1, 2, \cdots\}$ such that φ is measurable with respect to the σ-field $\sigma\{Y_t^{c,\epsilon},\ Y_t^{d,\epsilon}, t \in S\}$. Since moment generating functions uniquely determine the distributions of random variables it follows that $E[\varphi \mid \mathcal{G}_{s_0,\cdots,s_k}] = 0$ almost surely where $\mathcal{G}_{s_0,\cdots,s_k}$ is the σ-field generated by the random variables $\{Y_{s_i}^{c,\epsilon},\ Y_{s_i}^{d,\epsilon},\ 0 \leq s_0 < \cdots < s_k, s_i \in S\}$. Since the above is true for all $k \geq 1$, this implies $\varphi = 0$ almost surely. □

For $n \geq 1$, let $\alpha = (\alpha_1, \cdots, \alpha_n),\ \alpha_i \in D := \{1, \cdots, d+1\}$.

$$\begin{aligned} Z_n^\alpha &:= \{z : z = (z_1, \cdots, z_n),\ z_i = (s_i, y_i), s_i \geq 0,\ y_i = 0 \\ &\quad \text{if } \alpha_i \in \{1, \cdots, d\},\ y_i \in \mathbb{R}_0^d \text{ if } \alpha_i = d+1 \text{ and } s_i \leq s_j \text{ if } i < j\}. \end{aligned}$$

$\mathcal{B}(Z_n^\alpha)$ will denote the Borel σ-field on Z_n^α and μ_n^α will denote the restriction of the product measure $\bigotimes_{i=1}^n \mu_1^{\alpha_i}$ on $([0,\infty) \times \mathbb{R}^d)^n$, to Z_n^α where $\mu_1^{\alpha_i}(dz_i) =$

$ds_i\ \gamma^{\alpha_i}(dy_i)$, $\gamma^{\alpha_i}(dy_i) = \delta_0(dy_i)$ if $\alpha_i \in \{1, \cdots, d\}$ and $\gamma^{\alpha_i}(dy_i) = \nu(dy_i)\ |_{\mathbb{R}^d_0}$ if $\alpha_i = d+1$. For $n \geq 1$, $\mathcal{L}^\alpha_n$ will denote the Hilbert space $L^2(Z^\alpha_n,\ \mathcal{B}(Z^\alpha_n),\ \mu^\alpha_n)$ and $\mathcal{L}_n = \bigoplus_\alpha \mathcal{L}^\alpha_n$, the direct sum of $L^2(Z^\alpha_n,\ \mathcal{B}(Z^\alpha_n),\ \mu^\alpha_n)$. For $n = 0$, we set $\mathcal{L}_0 = \mathbb{R}$.

We now define an inner product $\langle\cdot,\cdot\rangle_n$ on $\mathcal{L}_n$ as follows: For $\alpha = (\alpha_1, \cdots, \alpha_n)$ and $\beta = (\beta_1, \cdots, \beta_n)$ define $a_{\alpha\beta} := \prod_{i=1}^{n} (\sigma\sigma^t)_{\alpha_i\beta_i}$. For $f = (f_\alpha), g = (g_\beta)$ in $\mathcal{L}_n$ we shall think of each f_α as being defined on all of $Z_n := ([0,\infty) \times \mathbb{R}^d) \times \cdots \times ([0,\infty) \times \mathbb{R}^d)$ - the n-fold cartesian product of $[0,\infty) \times \mathbb{R}^d$ - by taking f_α to be zero outside $Z^\alpha_n \subset Z_n$. We define the σ-finite measure μ_n on the Borel subsets of Z_n as follows:

$$\mu_n(A) := \sum_\alpha \mu^\alpha_n(A \cap Z^\alpha_n).$$

We define the inner product $\langle\cdot,\cdot\rangle_n$ on $\mathcal{L}_n$ as follows:

$$\langle f, g\rangle_n := \sum_{\alpha,\beta} \int_{Z_n} f_\alpha\ g_\beta\ a_{\alpha\beta}\ d\mu_n.$$

For $\alpha = (\alpha_1, \cdots, \alpha_n), k = 1, \cdots, d+1$, let $\tilde{\sigma}_{\alpha k} := \prod_{i=1}^{n} \sigma_{\alpha_i k}$. Then it is easy to see that

$$\|f\|^2_n\ =\ \int_{Z_n} \sum_{k=1}^{d+1} |\sum_\alpha f_\alpha \tilde{\sigma}_{\alpha k}|^2\ d\mu_n$$

Using the non singularity of σ, we can verify that $\|f\|^2_n\ = 0$ implies $f_\alpha = 0\ a.e.(\mu_n)$ for every α, and similarly that Cauchy sequences converge. Thus $(\mathcal{L}_n, \langle\cdot,\cdot\rangle_n)$ is a (real) Hilbert space.

Recall the notation $Y^{c,i} := (\sigma.B)_i$, $1 \leq i \leq d$. We will drop the superscript 'c' for convenience and write Y^i for $Y^{c,i}$; Y^{d+1} will denote the random measure $\widehat{N}(ds\ dy)$ on $[0,\infty) \times \mathbb{R}^d_0$. Integration with respect to $\widehat{N}(ds\ dy)$ will be denoted by dY^{d+1}_z.

Fix $n \geq 1$, $\alpha = (\alpha_1, \cdots, \alpha_n)$. Let $g \in \mathcal{L}^\alpha_n$. We shall define the iterated multiple stochastic integral $I_{n,\alpha}(g)$ by induction.For $z_n \in Z^{\alpha_n}_1$, define $\tilde{g}(\ ; z_n) : Z^{\tilde{\alpha}}_{n-1} \to \mathbb{R}$, where $\tilde{\alpha} = (\alpha^1, \cdots, \alpha^{n-1})$, by $\tilde{g}(z_1, \cdots, z_{n-1};\ z_n) = g(z_1, \cdots, z_n) I_{Z^\alpha_n}(z_1, \cdots, z_n)$, where I_G denotes indicator function of the set G. Then, for a.e. $z_n(\mu^{\alpha_n}_1), \tilde{g}(.; z_n) \in \mathcal{L}^{\tilde{\alpha}}_{n-1}$. By modifying $\tilde{g}(.; z_n)$ on a set of $\mu^{\alpha_n}_1$ measure zero, we will assume $\tilde{g}(., z_n) \in \mathcal{L}^{\tilde{\alpha}}_{n-1}$ for every $z_n \in Z^{\alpha_n}_1$. For $s > 0$ define $\phi(s_1, \cdots, s_{n-1}; s) := I_{\{0 \leq t_1 \leq \cdots \leq t_{n-1} \leq s\}}(s_1, \cdots, s_{n-1})$. Then

$\phi(.;s)\tilde{g}(.;z_n) \in \mathcal{L}_{n-1}^{\tilde{\alpha}}$ for every $s > 0, z_n \in Z_1^{\alpha_n}$. By the induction hypothesis, the $n-1$ fold iterated stochastic integrals $I_{n-1}(\phi(.;s)\tilde{g}(.,z_n))$ is defined and defines an adapted process in (s,ω) which is right continuous and has left limits for $s \in (0,\infty)$. For $z_n = (s_n, y_n)$, let $I_{n-1,\tilde{\alpha}}(\tilde{g}(.;z_n-))$ denote the left limit of $I_{n-1,\tilde{\alpha}}(\phi(.;s)\tilde{g}(.;z_n))$ as s increases to s_n. Then for each $\alpha = (\alpha_1, \cdots, \alpha_n)$ the n-fold iterated stochastic integral of g with respect to $Y^{\alpha_1} \cdots Y^{\alpha_n}$ is defined inductively as

$$I_{n,\alpha}(g) = \int_{Z_1^{\alpha_n}} I_{n-1,\tilde{\alpha}}(\tilde{g}(.;z_n-))\; dY_{z_n}^{\alpha_n}$$

where $I_{1,\alpha}(g),\ 1 \le \alpha \le d+1$ is the stochastic integral of $g : Z_1^{\alpha} \to \mathbb{R}$ with respect to Y^{α}. For $g = (g_\alpha) \in \mathcal{L}_n$ we define

$$I_n(g) := \sum_{\alpha} I_{n,\alpha}(g_\alpha).$$

We note that if $f \in \mathcal{L}_n, g \in \mathcal{L}_m$ then

$$E(I_n(f)\; I_m(g)) = \delta_{nm}\langle f, g\rangle_n.$$

The following theorem gives the chaos decomposition for $L^2(\mathcal{F}_\infty^Y)$.

THEOREM 5.2. *Let $F \in L^2(\mathcal{F}_\infty^Y)$. Then $\exists\ f_n \in \mathcal{L}_n, n \ge 1$ such that*

$$F = EF + \sum_{n=1}^{\infty} I_n(f_n).$$

In particular if $\mathcal{C}_n = \{I_n(f_n) : f_n \in \mathcal{L}_n\}$ then $\mathcal{C}_n$ are closed subspaces, isomorphic as a Hilbert space to $\mathcal{L}_n$, $C_n \perp C_m, m \ne n$ and $L^2 = \bigoplus_{n=0}^{\infty} \mathcal{C}_n$ where $\mathcal{C}_0 := \mathbb{R}$.

PROOF. The first part of the theorem follows from the continuity properties of $I_n(\cdot)$ as soon as we establish the expansion for a dense subspace of L^2. We do this in Lemma 5.5 below. The second statement follows from the first and the properties of I_n. □

REMARK 5.3. The chaos expansion given in the theorem implies the martingale representation for F in terms of the Y^i's : Clearly, from the definition of the iterated integrals $I_{n,\alpha}(f_n)$ we can write, using the notation introduced before the statement of Theorem 5.2 for $\tilde{f}_{n,\tilde{\alpha}}(.,z_n-)$,

$$I_n(f_n) \;\; := \sum_{\alpha=(\tilde{\alpha},\alpha_n)} I_{n,\alpha}(f_{n,\alpha}) = \sum_{\alpha_n=1}^{d+1} \sum_{\tilde{\alpha}} I_{1,\alpha_n}(I_{n-1,\tilde{\alpha}}(\tilde{f}_{n,\tilde{\alpha}}(.,z_n-)))$$

$$= \sum_{\alpha_n=1}^{d+1} I_{1,\alpha_n}(f_{n,\alpha_n}) \ ,$$

where the process (f_{n,α_n}) for $\alpha_n = 1, \cdots, d+1$ is given as

$$f_{n,\alpha_n}(z) := \sum_{\tilde{\alpha}} I_{n-1,\tilde{\alpha}}(\tilde{f}_{n,\tilde{\alpha}}(., z-)).$$

Introducing the $L^2(\Omega \to L^2(Z_1^{\alpha_n}, \mathcal{B}(Z_1^{\alpha_n}), \mu_1^{\alpha_n}))$ convergent sum $f_\alpha(z) := \sum_{n=1}^{\infty} f_{n,\alpha}(z), \ \alpha = 1, \cdots, d+1$ we get

$$\begin{aligned} F = EF + \sum_{n=1}^{\infty} I_n(f_n) &= EF + \sum_{n=1}^{\infty} \sum_{\alpha_n=1}^{d+1} I_{1,\alpha_n}(f_{n,\alpha_n}) \\ &= EF + \sum_{\alpha=1}^{d+1} I_{1,\alpha}(f_\alpha) \end{aligned}$$

Remark 5.4. For $F \in L^2(\mathcal{F}_\infty^Y)$ we can introduce iterated derivatives $\partial^\alpha \beta^F(z_1, \cdots, z_n, \omega)$ with $\alpha = (\alpha_1, \cdots, \alpha_n), \alpha_i \in \{1, \cdots, d{+}1\}$, $(z_1, \cdots, z_n) \in Z_n$ and $\partial^\alpha = \partial_{\alpha_1} \cdots \partial_{\alpha_n}$. Here for $\alpha_i \in \{1, \cdots, d\}$, ∂_{α_i} denotes the partial derivative with respect to the $\alpha_i th$ coordinate of y_i where $z_i = (s_i, y_i)$ and for $\alpha_i = d+1$, ∂_{α_i} will denote δ the difference operator. Let $\tilde{\alpha} = (\alpha_2, \cdots, \alpha_n)$. Then we can define inductively,

$$\partial^\alpha \beta^F(z_1, \cdots, z_n, \omega) := \partial^{\alpha_1} \beta^{G(z_2, \cdots, z_n)}(z_1, \omega).$$

where for fixed $(z_2, \cdots, z_n), G(z_2, \cdots, z_n, \omega) := \partial^{\tilde{\alpha}} \beta^F(z_2, \cdots, z_n, \omega) \in L^2$ $(\mathcal{F}_\infty^Y)$. Then the kernels $f_{n,\alpha}$ in the chaos expansion of F can be expressed as

$$f_{n,\alpha}(z_1, \cdots, z_n) = E \ \partial^\alpha \beta^F(z_1, \cdots, z_n)$$

To state the next result ,let $\epsilon > 0, u = (u^1, u^2), u^i = (u^{i1}, \cdots, u^{id}), i = 1, 2$ where $u^{ij}(s) = \sum_{k=1}^{m} u_k^{ij} I_{(t_{k-1}, t_k]}(s)$. Recall the processes $(Y_t^{c,\epsilon}), (Y_t^{d,\epsilon})$, $(Y_t^{\epsilon,u})$ defined at the beginning of Section 5 via (2.3) and (5.1); we now set $b^\epsilon = 0$ in the definition of $Y^{c,\epsilon}$. Consequently $Y_t^{c,\epsilon}$ is a continuous vector martingale, independent of ϵ and we denote it by $Y_t^c = (\sigma.B)_t = (Y_t^1, \cdots, Y_t^d)$. Let $F^{\epsilon,u} := M_\infty^{\epsilon,u}$ be defined as in (5.2). Note that Lemma 5.1 remains true in this case also. Let $f_n^{\epsilon,u} = (f_{n,\alpha}^{\epsilon,u})$ be given by $f_{n,\alpha}^{\epsilon,u}(z_1, \cdots, z_n) := \prod_{i=1}^{n} f_{\alpha_i}^{\epsilon,u}(z_i)$ where $f_{\alpha_i}^{\epsilon,u}(z_i) = u^{1j}(s_i)$ if $\alpha_i = j, 1 \le j \le d$ and $z_i = (s_i, 0)$ and $f_{\alpha_i}^{\epsilon,u}(z_i) = (e^{u^2(s_i)\cdot y_i} - 1) I_{\{0<|y_i| \le \epsilon\}}, \ \alpha_i = d+1, \ z_i = (s_i, y_i)$.

LEMMA 5.5. *Let $\epsilon > 0, u, F^{\epsilon,u}$ and $f_n^{\epsilon,u}$ be as above. Then*

$$F^{\epsilon,u} = EF^{\epsilon,u} + \sum_{n=1}^{\infty} I_n(f_n^{\epsilon,u})$$

.

PROOF. Let $0 < \epsilon' < \epsilon$. Let

$$\begin{aligned} X_t^{\epsilon',\epsilon,u} \quad := \quad & \int_0^t u^1(s).dY_s^c - \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t u^{1i}(s)u^{1j}(s)\ d\langle Y^i, Y^j\rangle_s \\ & + \int_0^t \int_{\{0<\epsilon'<|y|\leq\epsilon\}} u^2(s).y\ \widehat{N}(ds\ dy) \\ & - \int_0^t \int_{\{\epsilon'<|y|\leq\epsilon\}} \left(e^{u^2(s).y} - 1 - u^2(s).y\right)\ \gamma(dy)\ ds. \end{aligned}$$

Itô's formula applied to $f(X_t^{\epsilon',\epsilon,u}), f(x) := e^x$ and letting $\epsilon' \downarrow 0$ in the resulting equation yields the martingale representation,

$$F^{\epsilon,u} = 1 + \int_0^{\infty} M_{s-}^{\epsilon,u} u^1(s).dY_s^c + \int_0^{\infty} \int_{\{0<|y|\leq\epsilon\}} M_{s-}^{\epsilon,u} (e^{u^2(s)\cdot y} - 1)\ \widehat{N}(ds\ dy).$$

Repeated application of the above representation now yields for every $n \geq 1$,

$$F^{\epsilon,u} = 1 + \sum_{k=1}^{n} I_k(f_k) + \sum_{\alpha\in D^{n+1}} I_{n+1}(g_\alpha)$$

where D^n is the n-fold Cartesian product of D; the $f_k \equiv (f_{k,\alpha}) := (f_{k,\alpha}^{\epsilon,u})$ are as in the statement of the lemma; and

$$g_\alpha(z_1, \cdots, z_{n+1}) = \prod_{i=1}^{n+1} g^{\alpha_i}(z_i)$$

where $g^{\alpha_i}(z_i) = f_{\alpha_i}^{\epsilon,u}(z_i)\ M_{s_i-}^{\epsilon,u}$ where $z_i = (s_i, 0)$ or (s_i, y_i). Note that $I_{n+1}(g_\alpha)$ are orthogonal for different α and hence

$$E\left(\sum_{\alpha} I_{n+1}(g_\alpha)\right)^2 = \sum_{\alpha} E\left(I_{n+1}(g_\alpha)\right)^2 \leq \frac{C^{n+1}(d+1)^{n+1} t_m^{n+1}}{n!}$$

where $C = \max\{1, \int_{\{0<|y|\le\epsilon\}} |y|^2 \, \gamma(dy)\}$. The result follows on letting $n \to \infty$.

□

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