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Heuristic derivation of Zudilin's supercongruences for rational Ramanujan series

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Abstract
We derive, using a heuristic method, a p-adic mate of bilateral Ramanujan series. It has (among other consequences) Zudilin's supercongruences for rational Ramanujan series.

Keywords Hypergeometric series · Bilateral series · Fourier trigonometric series · Supercongruences · Linear diophantine equations · p-Adic analysis

Mathematics Subject Classification MSC2020 · 33C20 · 42A10 · 11A07 · 11D04

1 Rational Ramanujan series for π^{-m}

At the beginning of the twenty first century we discovered new families of Ramanujan-like series, but of greater degree [2], and proved several of them by the WZ (Wilf–Zeilberger) method [3].

We can write the rational Ramanujan-like series as

$$\sum_{n=0}^{\infty} R(n) = \sum_{n=0}^{\infty} \left(\prod_{i=0}^{2m} \frac{(s_i)_n}{(1)_n} \right) \sum_{k=0}^{m} a_k n^k z_0^n = \frac{\sqrt{(-1)^m \chi}}{\pi^m},$$

where z_0 is a rational such that $z_0 \neq 0$ and $z_0 \neq 1$, the parameters $a_0, a_1, ..., a_m$ are positive rationals, and χ the discriminant of a certain quadratic field (imaginary or real), which is an integer. In case that $|z_0| > 1$ we understand the series as its analytic

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continuation. An example is

$$\frac{1}{2048}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^7\left(\frac{1}{4}\right)_n\left(\frac{3}{4}\right)_n}{(1)_n^9}(43680n^4+20632n^3+4340n^2+466n+21)\left(\frac{1}{2^{12}}\right)^n = \frac{1}{\pi^4},$$

conjectured by Jim Cullen, and recently proved by Kam Cheong Au, using the WZ method [1].

2 Bilateral Ramanujan series

We define the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ in the following way:

$$f(x) = e^{-i\pi x}\prod_{s_k}\frac{\cos\pi x - \cos\pi s_k}{1-\cos\pi s_k}\sum_{n\in\mathbb{Z}}R(n+x).$$

Then there exist coefficients α_k and β_k (which we conjecture are rational) such that $f(x) = F(x)$, where

$$F(x) = \frac{\sqrt{(-1)^m\chi}}{\pi^m}\left(1-\sum_{k=1}^{m}(\alpha_k(\cos 2\pi kx - 1)+\beta_k\sin 2\pi kx)\right),$$

is the Fourier expansion of $f(x)$.

Proof The function $f(x)$ is 1-periodic because the product over s_k is 1-periodic as each $s_k = s$, has a companion $s_k = 1 - s$, and the sum over $\mathbb{Z}$ is clearly 1-periodic as well. In addition $f(x)$ is holomorphic because the zero of $\cos\pi x - \cos\pi s_k$ at $x = -s_k$ cancels the pole of $(s_k)_{n+x}$ at $x = -s_k$, and as $f(x)$ is periodic all the other poles are canceled as well. As $f(x)$ is holomorphic and periodic, it has a Fourier expansion. Finally, we can prove that $f(x) = \mathcal{O}(e^{(2m+1)\pi|\mathrm{Im}(x)|})$, and therefore the Fourier expansion terminates at $k = m$.

Example 2.1

$$\frac{1}{8}\sum_{n\in\mathbb{Z}}\frac{\left(\frac{1}{2}\right)_{n+x}^5}{(1)_{n+x}^5}(20(n+x)^2+8(n+x)+1)\left(\frac{-1}{4}\right)^{n+x} = e^{i\pi x}\frac{1-\frac{1}{2}(\cos 2\pi x - 1)+\frac{1}{2}(\cos 4\pi x - 1)}{\pi^2\cos^5\pi x}.$$

Example 2.2

$$\frac{1}{384}\sum_{n\in\mathbb{Z}}\frac{\left(\frac{1}{2}\right)_{n+x}\left(\frac{1}{3}\right)_{n+x}\left(\frac{2}{3}\right)_{n+x}\left(\frac{1}{6}\right)_{n+x}\left(\frac{5}{6}\right)_{n+x}}{(1)_{n+x}^5}\left(-\frac{3^6}{4^6}\right)^{n+x}$$
$$\times\left(1930(n+x)^2+549(n+x)+45\right)$$
$$=e^{i\pi x}\frac{3-14(\cos 2\pi x-1)+6(\cos 4\pi x-1)}{\pi^2\cos\pi x\,(4\cos^2\pi x-1)(4\cos^2\pi x-3)}.$$

Example 2.3

$$\frac{1}{32}\sum_{n\in\mathbb{Z}}\frac{\left(\frac{1}{2}\right)_{n+x}^3\left(\frac{1}{4}\right)_{n+x}\left(\frac{3}{4}\right)_{n+x}}{(1)_{n+x}^5}\left(\frac{1}{16}\right)^{n+x}(120(n+x)^2+34(n+x)+3)$$
$$=e^{i\pi x}\frac{1-\frac{7}{2}(\cos 2\pi x-1)+\frac{3}{2}(\cos 4\pi x-1)+\left(\frac{1}{2}\sin 2\pi x-\frac{1}{2}\sin 4\pi x\right)i}{\pi^2\cos^3\pi x\,(2\cos^2\pi x-1)}.$$

Example 2.4

$$\frac{1}{6}\sum_{n\in\mathbb{Z}}\frac{(\frac{1}{2})_{n+x}^3(\frac{1}{3})_{n+x}(\frac{2}{3})_{n+x}}{(1)_{n+x}^5}\left[28(n+x)^2+18(n+x)+3\right](-27)^{n+x}$$
$$=e^{i\pi x}\frac{3+(\cos 2\pi x-1)+\frac{3}{4}(\cos 4\pi x-1)}{\pi^2\cos^3\pi x(4\cos^2\pi x-1)}.$$

3 Series to the right and to the left

The series to the right hand side is

$$\sum_{n=0}^{\infty}R(n+x)=\sum_{n=0}^{\infty}\left(\prod_{i=0}^{2m}\frac{(s_i)_{n+x}}{(1)_{n+x}}\right)\sum_{k=0}^{m}a_k(n+x)^k\,z_0^{n+x},$$

extended by analytic continuation to all z_0 different from 0 and 1, and the series on the left hand side is

$$\sum_{n=1}^{\infty}R(-n+x)=\sum_{n=1}^{\infty}\left(\prod_{i=0}^{2m}\frac{(s_i)_{-n+x}}{(1)_{-n+x}}\right)\sum_{k=0}^{m}a_k(-n+x)^k\,z_0^{-n+x}$$
$$=x^{2m+1}\sum_{n=1}^{\infty}\left(\prod_{i=0}^{2m}\frac{(1)_{n-x}}{(s_i)_{n-x}}\right)\sum_{k=0}^{m}a_k(-n+x)^{k-2m-1}\,z_0^{-n+x}$$
$$=x^{2m+1}z_0^x\left(\prod_{i=0}^{2m}\frac{(s_i)_x}{(1)_x}\right)\sum_{n=1}^{\infty}\left(\prod_{i=0}^{2m}\frac{(1-x)_n}{(s_i-x)_n}\right)$$

$$\sum_{k=0}^{m} a_k (-n+x)^{k-2m-1} \, z_0^{-n},$$

extended by analytic continuation to all z_0 different from 0 and 1. We see that

$$\sum_{n=0}^{\infty} R(n) - \sum_{n=0}^{\infty} R(n+x) = \frac{\sqrt{(-1)^m \chi}}{\pi^m} - e^{i\pi x} \prod_{s_k} \frac{1-\cos \pi s_k}{\cos \pi x - \cos \pi s_k} \frac{\sqrt{(-1)^m \chi}}{\pi^m}$$
$$\left(1 - \sum_{k=1}^{m} (\alpha_k (\cos 2\pi k x - 1) + \beta_k \sin 2\pi k x)\right)$$
$$+ (A + Bx + Cx^2 + \cdots) x^{2m+1}, \quad |x| < 1,$$

where $(A + Bx + Cx^2 + \cdots)x^{2m+1}$ is the development of the series on the left hand side at $x = 0$, that is

$$z_0^x \left(\prod_{i=0}^{2m} \frac{(s_i)_x}{(1)_x} \right) \sum_{n=1}^{\infty} \left(\prod_{i=0}^{2m} \frac{(1-x)_n}{(s_i - x)_n} \right) \sum_{k=0}^{m} a_k (-n+x)^{k-2m-1} \, z_0^{-n} = A + Bx + Cx^2 + \cdots .$$

4 Heuristic derivation of a *p*-adic mate

Let

$$S(N) = \sum_{n=0}^{\infty} R(n) - \sum_{n=0}^{\infty} R(n+N) = \sum_{n=0}^{N-1} R(n).$$

As in a Ramanujan-like series each $s_k < 1/2$ has a companion $1 - s_k$, we notice that

$$e^{i\pi x} \prod_{s_k} \frac{1-\cos \pi s_k}{\cos \pi x - \cos \pi s_k} = e^{i\pi x} \prod_{s_k = \frac{1}{2}} \frac{1}{\cos \pi x} \prod_{s_k < \frac{1}{2}} \frac{1 - \cos^2 \pi s_k}{\cos^2 \pi x - \cos^2 \pi s_k}$$

tends to 1 as $x \to N$ because there is an odd number of factors when $s_k = 1/2$. Hence for $x \to N$, we formally have

$$S(x) = \sum_{n=0}^{\infty} R(n) - \sum_{n=0}^{\infty} R(n+x)$$
$$= \frac{\sqrt{(-1)^m \chi}}{\pi^m} \left(\sum_{k=1}^{m} (\alpha_k (\cos 2\pi k x - 1) + \beta_k \sin 2\pi k x) \right)$$
$$+ (A + Bx + Cx^2 + \cdots) x^{2m+1}.$$

Let

$$G(x)=\frac{\sqrt{(-1)^m\chi}}{\pi^m}\sum_{k=1}^{m}(\alpha_k(\cos 2\pi kx-1)+\beta_k\sin 2\pi kx)\,.$$

For obtaining the p-adic analogues $G_p(xp)$ and $G_p(x)$, we develop $G(xp)$ and $G(x)$ in powers of x. Then, replace the powers of π using values of Dirichlet L-functions, and the L-functions with the corresponding p-adic L-functions. Finally, the standard properties of the L_p-functions dictate turning even powers of π to 0 when $\chi>0$, or odd powers of π when $\chi<0$. After making the replacements, we see that

$$\lim_{x\to\nu}\frac{G_p(xp)}{G_p(x)}=p^m.$$

For $x=\nu$, where $\nu=1,2,3,\ldots$, we see that

$$z_0^{\nu}\left(\prod_{i=0}^{2m}\frac{(s_i)_\nu}{(1)_\nu}\right)\sum_{n=1}^{\infty}\left(\prod_{i=0}^{2m}\frac{(1-\nu)_n}{(s_i-\nu)_n}\right)\sum_{k=0}^{m}a_k(-n+\nu)^{k-2m-1}\,z_0^{-n}=A'+B'\nu+C'\nu^2+\cdots,$$

where

$$A'=z_0^{\nu}\left(\prod_{i=0}^{2m}\frac{(s_i)_\nu}{(1)_\nu}\right)A,\quad B'=z_0^{\nu}\left(\prod_{i=0}^{2m}\frac{(s_i)_\nu}{(1)_\nu}\right)B,\ldots.$$

On the other hand, we see that

$$\begin{aligned}S(\nu)&=(A'+B'\nu+C'\nu^2+\cdots)\nu^{2m+1}\\&=z_0^{\nu}\left(\prod_{i=0}^{2m}\frac{(s_i)_\nu}{(1)_\nu}\right)(A+B\nu+C\nu^2+\cdots)\nu^{2m+1}.\end{aligned}$$

To get the $p-$adic mate of $S(x)$ we must divide $S_p(\nu p)$ enter $S_p(\nu)$, taking into account that the contribution of $G(x)$ is $(\chi/p)p^m$, and the contribution of the left hand sum is given by

$$\begin{aligned}&\frac{(A_p+B_p\nu p+C_p\nu^2p^2+\cdots)p^{2m+1}\nu^{2m+1}}{(A+B\nu+C\nu^2+\cdots)_p\nu^{2m+1}}\\&=z_0^{\nu}\left(\prod_{i=0}^{2m}\frac{(s_i)_\nu}{(1)_\nu}\right)\frac{(A_p+B_p\nu p+C_p\nu^2p^2+\cdots)p^{2m+1}\nu^{2m+1}}{S_p(\nu)}.\end{aligned}$$

Associating (χ/p) to $S_p(\nu p)$ and noting that $\Gamma_p(1/2)^{4m}=1$ by the properties of the p-adic Γ-function, we have

$$S(\nu p)=S(\nu)\left(\frac{\chi}{p}\right)p^m+T(\nu)(A_p+B_p\nu p+C_p\nu^2p^2+\cdots)p^{2m+1}\nu^{2m+1},$$

where $A_p, B_p, C_p \dots$, are p-adic analogues of $A, B, C \dots$, and

$$T(\nu) = z_0^{\nu} \left(\prod_{i=0}^{2m} \frac{(s_i)_\nu}{(1)_\nu} \right).$$

Observe that taking positive integers values of ν we can eliminate some of the constants A_q, B_q,.., and obtain a new kind of supercongruences $\pmod{p^{2m+k}}$. For example, eliminating A_q and B_q, we obtain supercongruences $\pmod{p^{2m+3}}$ relating $S(p)$, $S(2p)$ and $S(3p)$.

We can apply a similar technique of bilateral series and p-adic mates to other kind of hypergeometric series, for example to those in [4].

5 Extended Zudilin's supercongruences

The above p-adic mate has (among other consequences) a generalization for positive integers ν of Zudilin's $\nu = 1$ supercongruences [7] and [5], namely

$$S(\nu p) = S(\nu) \left(\frac{\chi}{p} \right) p^m \pmod{p^{2m+1}},$$

except for very few values of p.

Example 5.1 See the Ramanujan-like series [2, Eq. (1–3)]. Let

$$S(N) = \sum_{n=0}^{N-1} \frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5} \left(\frac{-1}{4} \right)^n (20n^2 + 8n + 1)$$

If p is a prime number (except for very few of them), then

$$S(\nu p) \equiv S(\nu) \left(\frac{1}{p} \right) p^2 \pmod{p^5},$$

for positive integers ν. Observe that for all prime p we have $(1/p) = 1$.

Example 5.2 See the Ramanujan-like series [2, Eq. (4–1)]. Let

$$S(N) = \sum_{n=0}^{N-1} \frac{\left(\frac{1}{2}\right)^7}{(1)_n^7} \left(\frac{1}{64} \right)^n (168n^3 + 76n^2 + 14n + 1)$$

If p is a prime number (except for very few of them), then

$$S(\nu p) \equiv S(\nu) \left(\frac{-4}{p} \right) p^3 \pmod{p^7},$$

for positive integers ν.

Example 5.3 See the Ramanujan-like series [2, Eq. (2–4)]. Let

$$S(N)=\sum_{n=0}^{N-1}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n\left(\frac{1}{6}\right)_n\left(\frac{5}{6}\right)_n}{(1)_n^5}\left(\frac{-1}{80^3}\right)^n(5418n^2+693n+29)$$

If p is a prime number (except for very few of them), then

$$S(\nu p)\equiv S(\nu)\left(\frac{5}{p}\right)p^2 \pmod{p^5},$$

for positive integers ν.

6 Extended Zhao's supercongruences

By identifying numerical approximations, we conjecture that $A = rL(\chi, m+1)$, where r is a rational. The p-adic analogue of A is $A_p = rL_p(\chi, m+1)$. We have the following supercongruences:

$$S(\nu p)\equiv\left(\frac{\chi}{p}\right)S(\nu)p^m+rz_0^\nu\left(\prod_{i=0}^{2m}\frac{(s_i)_\nu}{(1)_\nu}\right)L_p(\chi,m+1)p^{2m+1} \pmod{p^{2m+2}}.$$

which generalizes for positive integers ν the Yue Zhao's supercongruences for $\nu = 1$ (author Y. Zhao at mathoverflow). To check these supercongruences use the following congruences

$$\begin{aligned} L_p(\chi, m+1) &\equiv L(\chi, 2+m-p) \pmod{p}, \\ \zeta_p(m+1) &\equiv \frac{\text{bernoulli}(p-m-1)}{m+1} \pmod{p}. \end{aligned}$$

Observe that $L(1, m+1) = \zeta(m+1)$ and $L_p(1, m+1) = \zeta_p(m+1)$. For Bernoulli numbers associated to χ see [6].

Example 6.1 See the Ramanujan-like series [2, Eq. (1–3)]. Let

$$S(N)=\sum_{n=0}^{N-1}\frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5}\left(\frac{-1}{4}\right)^n(20n^2+8n+1),\quad T(\nu)=\left(\frac{-1}{4}\right)^\nu\frac{\left(\frac{1}{2}\right)_\nu^5}{(1)_\nu^5}.$$

If p is a prime number (except for very few of them), then

$$S(\nu p)\equiv S(\nu)p^2+448T(\nu)\zeta_p(3)\nu^5p^5 \pmod{p^6},$$

for positive integers ν.

Example 6.2 See the Ramanujan-like series [2, Eq. (4–1)]. Let

$$S(N)=\sum_{n=0}^{N-1}\frac{\left(\frac{1}{2}\right)_n^7}{(1)_n^7}\left(\frac{1}{64}\right)^n(168n^3+76n^2+14n+1), \quad T(\nu)=\left(\frac{1}{64}\right)^\nu\frac{\left(\frac{1}{2}\right)_\nu^7}{(1)_\nu^7}.$$

If p is a prime number (except for very few of them), then

$$S(\nu p)\equiv S(\nu)\left(\frac{-4}{p}\right)p^3+1536T(\nu)L_p(-4,4)\nu^7p^7 \pmod{p^8},$$

for positive integers ν.

Example 6.3 See the Ramanujan-like series [2, Eq. (2–4)]. Let

$$S(N)=\sum_{n=0}^{N-1}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}\right)_n\left(\frac{1}{3}\right)_n\left(\frac{2}{3}\right)_n\left(\frac{1}{6}\right)_n\left(\frac{5}{6}\right)_n}{(1)_n^5}\left(\frac{-1}{80^3}\right)^n(5418n^2+693n+29),$$

and

$$T(\nu)=\left(\frac{-1}{80^3}\right)^\nu\frac{\left(\frac{1}{2}\right)_\nu\left(\frac{1}{3}\right)_\nu\left(\frac{2}{3}\right)_\nu\left(\frac{1}{6}\right)_\nu\left(\frac{5}{6}\right)_\nu}{(1)_\nu^5}.$$

If p is a prime number (except for very few of them), then

$$S(\nu p)\equiv S(\nu)\left(\frac{5}{p}\right)p^2+42000T(\nu)L_p(5,3)\nu^5p^5 \pmod{p^6},$$

for positive integers ν.

7 An application of the extended supercongruences

In next examples, we use the generalized Zudilin's supercongruences to obtain the rational parameters of the rational Ramanujan series. For that aim (except for a global rational factor) we just need taking a sufficiently large prime p and m values of ν. In addition, we can check that there is a rational r such that Zhao's supercongruences hold for that prime p and those m values of ν. Hence $A_p=rL_p(\chi,m+1)$, and we conclude that $A=rL(\chi,m+1)$. Finally, observe that if $|z_0|>1$ then the series for A is convergent.

Example 7.1 We want to see that there is a series of the following form:

$$\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^5}{(1)_n^5}\frac{(-1)^n}{4^n}(a_0+a_1n+a_2n^2)=t_0\frac{\sqrt{\chi}}{\pi^2}, \quad \chi=1,$$

where a_0, a_1, a_2, t_0 are positive integers. Indeed, using the Wilf–Zeilberger (WZ method) we proved that $a_0 = 1, a_1 = 8, a_2 = 20$. Here

$$S(\nu p) - S(\nu) p^2 \equiv 0 \pmod{p^5}, \quad \nu = 1, 2, 3, \dots,$$

and taking $p = 11$, and $\nu = 1, 2$, we get the linear system

$$\begin{aligned} 103175 a_0 + 126304 a_1 + 81213 a_2 &\equiv 0 \pmod{11^5}, \\ 23608 a_0 + 21777 a_1 + 22319 a_2 &\equiv 0 \pmod{11^5}. \end{aligned}$$

Let $a_0 = t$. From the above equations, we obtain

$$\begin{aligned} -66812987 t - 95491225 a_2 &\equiv 0 \pmod{11^4}, \\ -35044211 t - 95491225 a_1 &\equiv 0 \pmod{11^4}. \end{aligned}$$

Solving the equations taking into account that the inverse $\pmod{11^4}$ of 95491225 is 12252, we obtain

$$\begin{aligned} a_2 &= -14621 t \pmod{11^4} = 20t, \\ a_1 &= -14633 t \pmod{11^4} = 8t, \end{aligned}$$

Hence the solutions are of the following form:

$$a_0 = t, \quad a_1 = 8t, \quad a_2 = 20t.$$

Example 7.2 We want to know if there is a series of the following form:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{4}\right)_n \left(\frac{3}{4}\right)_n}{(1)_n^5} \frac{(-1)^n}{48^n} (a_0 + a_1 n + a_2 n^2) = t_0 \frac{\sqrt{\chi}}{\pi^2}, \quad \chi = 1,$$

and where a_0, a_1, a_2, t_0 are positive integers. Using the PSLQ algorithm we conjecture that $a_0 = 5, a_1 = 63, a_2 = 252$ and $t_0 = 48$. Here

$$S(\nu p) - S(\nu) p^2 \equiv 0 \pmod{p^5}, \quad \nu = 1, 2, 3, \dots,$$

and taking $p = 13$, and $\nu = 1, 2$, we get the linear system

$$\begin{aligned} 155250 a_1 + 1838 a_2 + 327490 a_0 &\equiv 0 \pmod{13^5}, \\ 304350 a_1 + 329224 a_2 + 67674 a_0 &\equiv 0 \pmod{13^5}. \end{aligned}$$

Let $a_0 = 5t$. From the above equations, we obtain

$$\begin{aligned} 26628 a_1 + 7535 t &\equiv 0 \pmod{13^4}, \\ 26628 a_2 + 1579 t &\equiv 0 \pmod{13^4}. \end{aligned}$$

As the inverse $\pmod{13^4}$ of 26628 is 9279, we obtain

$$\begin{aligned} a_2 &= -28309t \pmod{13^4} = 252t, \\ a_1 &= -28498t \pmod{13^4} = 63t, \end{aligned}$$

Hence the solutions are: $a_0 = 5t, \quad a_1 = 63t, \quad a_2 = 252t.$

Example 7.3 We want to know if there is a series of the following form:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n^7}{(1)_n^7} \left(\frac{1}{64}\right)^n (a_0 + a_1 n + a_2 n^2 + a_3 n^3) = t_0 \frac{\sqrt{-\chi}}{\pi^3}, \quad \chi = -4,$$

where a_0, a_1, a_2, a_3, t_0 are positive integers. Using the PSLQ algorithm, we conjecture that $a_0 = 1$, $a_1 = 14$, $a_2 = 76$, $a_3 = 168$ and $t_0 = 16$. Here

$$S(\nu p) - S(\nu) \left(\frac{-4}{p}\right) p^3 \equiv 0, \quad \pmod{p^7} \quad \nu = 1, 2, \ldots,$$

and taking $p = 11$, and $\nu = 1, 2, 3$, we get the equations

$$2078533a_1 + 9963171a_2 + 11695266a_3 + 16073136a_0 \equiv 0 \pmod{11^7},$$

$$\begin{aligned} 12453192a_1 + 988367a_2 + 3883033a_3 + 14086913a_0 &\equiv 0 \pmod{11^7}, \\ 17113786a_1 + 2247378a_2 + 4011161a_3 + 7012796a_0 &\equiv 0 \pmod{11^7}. \end{aligned}$$

Let $a_0 = t$. From the above equations, we obtain

$$\begin{aligned} 7854385a_1 + 3429250a_2 + 19159030t &\equiv 0 \pmod{11^4}, \\ 3851936a_1 + 8961898a_2 + 5481146t &\equiv 0 \pmod{11^4}. \end{aligned}$$

Solving the equations, we obtain

$$\begin{aligned} a_1 &= -11965t \pmod{11^4} = 14t, \\ a_2 &= -1255t \pmod{11^4} = 76t, \\ a_3 &= -14473t \pmod{11^4} = 168t. \end{aligned}$$

Example 7.4 We want to know if there is a series of the following form:

$$\sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{3}\right)_n \left(\frac{2}{3}\right)_n \left(\frac{1}{6}\right)_n \left(\frac{5}{6}\right)_n}{(1)_n^5} \left(\frac{-1}{80^3}\right)^n (a_0 + a_1 n + a_2 n^2) = t_0 \frac{\sqrt{\chi}}{\pi^2}, \quad \chi = 5,$$

and where a_0, a_1, a_2, t_0 are positive integers. Using the PSLQ algorithm we conjecture that $a_0 = 29$, $a_1 = 693$, $a_2 = 5418$ and $t_0 = 128$. Here

$$S(\nu p) - S(\nu)\left(\frac{5}{p}\right)p^2 \equiv 0 \pmod{p^5}, \quad \nu = 1, 2, 3, \ldots,$$

and taking $p = 41$, $a_0 = 29t$, and $\nu = 1, 2$, we get the linear system

$$\begin{aligned} 76877806a_2 + 113924268a_1 + 43501045t &\equiv 0 \pmod{41^5}, \\ 88965067a_2 + 84189111a_1 + 113390736t &\equiv 0 \pmod{41^5}. \end{aligned}$$

From the above equations, we obtain

$$\begin{aligned} 38939a_1 + 32305t &\equiv 0 \pmod{41^3}, \\ 29982a_2 + 4321t &\equiv 0 \pmod{41^3}. \end{aligned}$$

As the inverse of 38939 $\pmod{41^3}$ is 55540, we obtain

$$\begin{aligned} a_2 &= -63503t \pmod{41^3} = 5418t, \\ a_1 &= -68228t \pmod{41^3} = 693t, \end{aligned}$$

Hence the solutions are: $a_0 = 29t, \quad a_1 = 693t, \quad a_2 = 5418t.$

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