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Blow-up phenomenon to the semilinear heat equation for unbounded Laplacians on graphs

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Abstract

Let $G = (V, E)$ be an infinite graph. The purpose of this paper is to investigate the nonexistence of global solutions for the following semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + u^{1+\alpha}, & t > 0, x \in V, \\ u(0, x) = u_0(x), & x \in V, \end{cases}$$

where Δ is an unbounded Laplacian on G, α is a positive parameter and u_0 is a nonnegative and nontrivial initial value. Using on-diagonal lower heat kernel bounds, we prove that the semilinear heat equation admits the blow-up solutions, which is viewed as a discrete analog of that of Fujita (J Fac Sci Univ Tokyo 13:109–124, 1966) and had been generalized to locally finite graphs with bounded Laplacians by Lin and Wu (Calc Var Partial Diff Equ 56(4):22, 2017). In this paper, new techniques have been developed to deal with unbounded graph Laplacians.

Keywords Unbounded graph Laplacians · On-diagonal lower heat kernel estimate · Semilinear heat equation · Global solution

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1 Introduction and main results

Recently, there have been increasingly more studies on non-linear partial differential equations on graphs, especially the existence and nonexistence of solutions for such equations. For example, Grigor'yan, Lin and Yang [7, 8] used the variation method to get the existence results for Yamabe type equations and Kazdan-Warner equations on graphs. Using the Nehari method, Zhang and Zhao [20] proved that the nonlinear Schrödinger equation has a nontrivial ground state solution, and the ground state solution is convergent. Lin and Yang [13] proposed a heat flow for the mean field equation and obtained the existence of a unique solution on graphs.

On $\mathbb{R}^n$, Fujita [4] considered the following Cauchy problem of the semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + u^{1+\alpha}, & t > 0, x \in \mathbb{R}^n, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\alpha > 0$ and u_0 is initial value. Fujita showed that if $\alpha < \frac{n}{2}$, then for any nonnegative and nontrivial initial value, the solution of Eq. (1.1) blows up in finite time. The discretization of this problem has attracted lots of attentions. For example, Lin and Wu [15, 16] studied blow-up phenomenon of the semilinear heat equation on locally finite graphs. In [19], Wu established a new heat kernel lower estimate on graphs, and used it to improve the blow-up result that was done in [15]. Based on [16], Lenz, Schmidt and Zimmermann [12] presented a sufficient condition for blow-up of solutions to the abstract semilinear heat equation on measure spaces, including graphs. Chung and Choi [1] developed a new condition for blow-up solution to the semilinear heat equation on networks. Liu [11] considered the semilinear heat equation on a finite subgraph and investigated the conditions of blow-up phenomenon.

The common thing of the existing conclusions about blow-up phenomenon of the semilinear heat equation on graphs is for bounded Laplacians. However, in the case of manifolds, the geometry does not need to be assumed to be bounded, and it was able to work with for unbounded Laplacians. Furthermore, in the studies of the existence of solutions to partial differential equations on graphs, there are very few of them concerned with unbounded Laplacians. Recently, by assuming that the weights of the graph have a positive lower bound and the distance function of the graph belongs to L^p, which allows an unbounded Laplcian, Lin and Yang [14] derived the existence of solutions to Schrördinger equation, Mean field equation and Yamabe equation on locally finite graphs.

In this paper, we focus on the existence and non-existence of solutions to the semilinear heat equation on locally finite graphs with unbounded Laplcians. To deal with unbounded Laplcians, we introduce mild solution to avoid exchanging P_t (semigroup) and Δ, and then prove the existence of mild solution with small time to the semilinear heat equation (Theorem 1.2). Furthermore, by introducing intrinsic metric to graphs, we obtain on-diagonal lower heat kernel bounds (Theorem 1.1) and then get the nonexistence of solutions to the semilinear heat equation for unbounded Laplcians (Theorem 1.3).

Let us recall some basic notations and definitions for weighted graphs. Let V be a countable discrete space as the set of vertices of a graph, $\omega : V \times V \to [0, \infty)$ be an edge weight function that satisfies $\omega_{xy} = \omega_{yx}$ for all $x, y \in V$ and $\omega_{xy} > 0$ if and only if xy is an edge (which is denoted by $x \sim y$), and $m : V \to (0, \infty)$ be a positive measure on V. These induce a weighted graph $G = (V, \omega, m)$. We assume that the measure is non-degenerate,

$$m_0 := \inf_{x \in V} m(x) > 0.$$

The graph is called locally finite if the sets $\{y \in V : w_{xy} > 0\}$ are finite for all $x \in V$. In this paper, we are interested in connected and locally finite graphs.

The space of real valued functions on V is denoted by $C(V)$ and the space of finitely supported functions is denoted by $C_0(V)$. $C_0(V)$ is dense in $\ell^p(V, m)$, $p \in [1, \infty]$, the ℓ^p spaces of functions on V with respect to the measure m.

On $C(V)$, we define the Laplacian as

$$\Delta f(x) = \frac{1}{m(x)} \sum_{y \sim x} \omega_{xy}(f(y) - f(x)).$$

As well known, the Laplacian Δ is bounded in $\ell^2(V, m)$ if and only if

$$D := \frac{\sum_{y \sim x} \omega_{xy}}{m(x)} < \infty.$$

In this paper, we consider unbounded Laplacians. We denote by $P_t = e^{t\Delta}$ the semigroup associated to Δ. The heat kernel $p(t, x, y)$ with $t > 0, x, y \in V$ is the smallest non-negative fundamental solution for the discrete heat equation $\partial_t u = \Delta u$. Moreover, for any $f \in \ell^p(V, m)$ with $p \in [1, \infty]$ and $x \in V$,

$$P_t f(x) = \sum_{y \in V} m(y) p(t, x, y) f(y).$$

In order to adapt to unbounded Laplacians, we use the intrinsic metric on graphs. For manifolds, the intrinsic metric naturally arises from the energy form as well as from the geometry, which was introduced on graphs by Keller and Lenz [10] and improved by Frank et al. [5].

Definition 1.1 [Intrinsic metric] An intrinsic metric $\rho : V \times V \to [0, \infty)$ is a distance function satisfying for any $x \in V$

$$\sum_{y \in V} \omega_{xy} \rho^2(x, y) \leq m(x).$$

We say ρ has finite jump, if there exists a positive constant j such that the jump size

$$\sup\{\rho(x, y) : x, y \in V, x \sim y\} \leq j.$$

Fixed a point $x \in V$, the ball on graphs with the intrinsic metric ρ is defined by

$$B_\rho(x, r) = \{y \in V : \rho(x, y) \leq r\},$$

where $r \geq 0$. We say $B_\rho(x, r)$ is finite, if it is of finite cardinality, namely $\sharp B_\rho(x, r) < \infty$. The volume of balls $B_\rho(x, r)$ is

$$V_\rho(x, r) = \sum_{y \in B_\rho(x,r)} m(y).$$

Definition 1.2 [Polynomial volume growth] G has polynomial volume growth of degree $n \in \mathbb{N}_+$, if there exist constants $C > 0$ and $r_0 > 0$, such that for all $x \in V$, $r \geq r_0$,

$$V_\rho(x, r) \leq Cr^n. \tag{1.2}$$

In this paper, we investigate the non-existence of global solutions for the following semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + u^{1+\alpha}, & t > 0, x \in V, \\ u(0, x) = u_0(x), & x \in V, \end{cases} \tag{1.3}$$

on locally finite connected weighted graphs, where Δ is an unbounded graph Laplacian, α is a positive parameter and u_0 is an initial value. We will study non-negative mild solution of Eq. (1.3), which is motivated by the following observation. Let $u : [0, T) \times V \to \mathbb{R}$ be continuously differentiable with respect to t and satisfying $\partial_t u = \Delta u + u^{1+\alpha}$. If we further assume that Δ is a bounded Laplacian, then u satisfies the following integral equation

$$u(t, x) = P_t u_0(x) + \int_0^t P_{t-s}(u(s, \cdot)^{1+\alpha})(x) ds, \tag{1.4}$$

for any $t \in [0, T)$, which depends on the exchangeability of integral and limit, as well as $\Delta P_t = P_t \Delta$. In the case of unbounded Laplacians, they are not always true.

Definition 1.3 [Mild solution and Global solution] A function $u : [0, T) \times V \to \mathbb{R}$ is a mild solution of Eq. (1.3) on $[0, T)$ if u is continuously differentiable on $(0, T)$, uniformly bounded and Eq. (1.4) is satisfied on $[0, T) \times V$. If $T = \infty$, then we call u a global solution of Eq. (1.3).

Now, we are ready to introduce our main results.

Theorem 1.1 *Let $G = (V, \omega, m)$ be a weighted graph with an intrinsic metric ρ with finite jump size and finite balls. Suppose that the graph has polynomial volume growth property with power n. Then, for all sufficiently large t and all $x \in V$,*

$$p(t, x, x) \geq \frac{1}{4V_\rho(x, c\sqrt{t}\log t)},$$

where $c > \sqrt{\frac{n}{2}}$.

Theorem 1.2 *Suppose that $G = (V, \omega, m)$ be a weighted graph. Then there exists a unique mild solution of Eq. (1.3) for any bounded initial value on $[0, T)$ with some $T > 0$.*

Theorem 1.3 *Let $G = (V, \omega, m)$ be a weighted graph with an intrinsic metric ρ with finite jump size and finite balls. Suppose that the graph has polynomial volume growth property with power n. If $0 < n\alpha < 2$, then there is no non-negative global solution of Eq. (1.3) for any bounded, non-negative and non-trivial given initial value.*

2 Upper bounds of heat kernel

In this section, we introduce the on-diagonal upper bounds of heat kernel. First, we recall some important properties about the heat kernel from spectral theory [17, 18].

Lemma 2.1 *For $t, s > 0$ and any $x, y \in V$, we have*

(1) $p(t, x, y)$ *is continuous with respect to* t*;*
(2) $p(t, x, y) = p(t, y, x)$;
(3) $p(t, x, y) \geq 0$*;*
(4) $\sum_{y \in V} m(y)p(t, x, y) \leq 1$*;*
(5) $\partial_t p(t, x, y) = \Delta_x p(t, x, y) = \Delta_y p(t, x, y)$*;*
(6) $\sum_{z \in V} m(z)p(t, x, z)p(s, z, y) = p(t + s, x, y)$*;*
(7) $p(t, x, x)$ *is non-increasing with respect to* t.

The weighted graph $G = (V, \omega, m)$ is called *stochastically complete* if

$$\sum_{y \in V} m(y)p(t, x, y) = 1$$

for all $t > 0$ and $x \in V$. Using the intrinsic metric, Huang et.al established the volume growth criterion of stochastically complete for graphs, which is the exact discrete analogue to the works of Grigor'yan [6] on manifolds and can be found in [9, Theorem 1.1], as follows.

Lemma 2.2 *Let* $\log^{\sharp} = \max\{\log, 1\}$. *Let $G = (V, \omega, m)$ be a weighted graph with an intrinsic metric ρ with finite distance balls $B_\rho(r)$. If*

$$\int^{\infty} \frac{r}{\log^{\sharp}(V_\rho(r))} dr = \infty,$$

then the graph is stochastically complete.

Remark 1 The use of $\log^{\sharp}$ instead of log is to deal with the case when the measure m is small. And the actual value of the constant 1 can be replaced by any positive number.

Remark 2 Under the same assumptions as Lemma 2.2, if G satisfies the polynomial volume growth of degree n for large r, then G is stochastically complete. Indeed, when the measure of the graph is finite, the conclusion is obvious from the definition of $\log^\sharp$. If $m(V) = \infty$, we have

$$V_\rho(r) \le Cr^n \le Cr^{2n} \le Cn!e^{r^2}$$

for large r, which implies that

$$\int^\infty \frac{r}{\log(V_\rho(r))}dr \ge \int^\infty \frac{r}{\log(Cn!e^{r^2})}dr = \int^\infty \frac{r}{\log(Cn!) + r^2}dr = \infty.$$

These two cases yield the conclusion.

The following upper bounds of the heat kernel comes from Davies [2].

Lemma 2.3 *Let $\mathbb{H}$ be the set of all positive functions ϕ on V such that $\phi^{\pm 1} \in \ell^\infty(V)$. Then, for any $x, y \in V$ and all $t \ge 0$, we have*

$$p(t,x,y) \le \big(m(x)m(y)\big)^{-\frac{1}{2}} \inf_{\phi \in \mathbb{H}} \{\phi(x)^{-1}\phi(y)e^{c(\phi)t}\},$$

where $c(\phi) := \sup_{x\in V} b(\phi, x) - \lambda$,

$$b(\phi, x) := \frac{1}{2m(x)} \sum_{y\sim x} \omega_{xy} \left(\frac{\phi(y)}{\phi(x)} + \frac{\phi(x)}{\phi(y)} - 2\right),$$

and $\lambda \ge 0$ is the bottom of the ℓ^2 spectrum of $-\Delta$.

For given $x, y \in V$, we choose $\phi = e^{-s\psi}$, where $s > 0$ and

$$\psi(z) = \min\{\rho^2(x,z), \rho^2(x,y)\},$$

to yield the following upper bounds of the heat kernel.

Proposition 2.1 *Let $G = (V, \omega, m)$ be a weighted graph with an intrinsic metric ρ with finite jump size $j > 0$. For any $s > 0$, the inequality*

$$p(t,x,y) \le \big(m(x)m(y)\big)^{-\frac{1}{2}} \exp\left(-s\rho^2(x,y) + \frac{1}{2}(se^{j^2 s} - se^{-j^2 s})t - \lambda t\right) \tag{2.1}$$

holds for all $t > 0$ and $x, y \in V$.

Proof It is easy to see that

$$\phi(x) = 1, \;\; \phi(y) = e^{-s\rho^2(x,y)}.$$

From the mean value theorem, we obtain there exists $\xi \in (0, \rho^2(x,y)) \subset (0, j^2]$ such that

$$\frac{\phi(y)}{\phi(x)} = e^{-s\rho^2(x,y)} = -se^{-s\xi}\rho^2(x,y) + 1 \le -se^{-j^2 s}\rho^2(x,y) + 1.$$

Similarly,

$$\frac{\phi(x)}{\phi(y)} = e^{s\rho^2(x,y)} \le se^{j^2 s}\rho^2(x,y) + 1.$$

Together with the definition of ρ, it yields

$$b(\phi, x) \le \frac{1}{2}(se^{j^2 s} - se^{-j^2 s}).$$

Thus, applying Lemma 2.3 to complete the proof. □

Remark 3 For the graph with bounded Laplacians, Davies deduced the upper bounds of the heat kernel using the combinatorial distance [2, Theorem 10]. Moreover, Folz obtained another non-Gaussian upper bound of the heat kernel for the intrinsic metrics with finite jump size [3, Theorem 2.1].

3 Proof of main theorems

Proof of Theorem 1.1. Notice that $m_0 := \inf_{x\in V} m(x) > 0$. For any $x \in V$, summing the inequality (2.1) in Proposition 2.1 with respect to y on $B_\rho(x,r)^c$ with $r > 0$, we obtain for all $s > 0$, $t > 0$,

$$\sum_{y\in B_\rho(x,r)^c} m(y)p(t,x,y) \le \frac{1}{m_0}\sum_{y\in B_\rho(x,r)^c} m(y)\exp\left(-s\rho^2(x,y) + \frac{1}{2}(se^{j^2 s} - se^{-j^2 s})t - \lambda t\right).$$

Since $B_\rho(x,r)^c$ can be decomposed into the union of $B_\rho(x,2^{k+1}r)\backslash B_\rho(x,2^k r)$ from $k = 0$ to ∞, we obtain

$$\begin{aligned}
&\sum_{y\in B_\rho(x,r)^c} m(y)p(t,x,y)\\
&\le \frac{1}{m_0}\sum_{k=0}^{\infty}\sum_{y\in B_\rho(x,2^{k+1}r)\backslash B_\rho(x,2^k r)} m(y)\exp\left(-s\rho^2(x,y) + \frac{1}{2}(se^{j^2 s} - se^{-j^2 s})t - \lambda t\right)\\
&\le \frac{1}{m_0}\sum_{k=0}^{\infty}\sum_{y\in B_\rho(x,2^{k+1}r)\backslash B_\rho(x,2^k r)} m(y)\exp\left(-4^k sr^2 + \frac{1}{2}(se^{j^2 s} - se^{-j^2 s})t - \lambda t\right)\\
&\le \frac{1}{m_0}\sum_{k=0}^{\infty}\exp\left(-4^k sr^2 + \frac{1}{2}(se^{j^2 s} - se^{-j^2 s})t - \lambda t\right)\sum_{y\in B_\rho(x,2^{k+1}r)} m(y)\\
&= \frac{1}{m_0}\sum_{k=0}^{\infty} V_\rho(x,2^{k+1}r)\exp\left(-4^k sr^2 + \frac{1}{2}(se^{j^2 s} - se^{-j^2 s})t - \lambda t\right)
\end{aligned}$$

Together with the polynomial volume growth (1.2), we get for $r \geq r_0$ and $s > 0$,

$$\sum_{y \in B_\rho(x,r)^c} m(y)p(t,x,y) \leq \frac{C}{m_0} \sum_{k=0}^{\infty} \left(2^{k+1} r\right)^n \exp\left(-4^k sr^2 + \frac{1}{2}(se^{j^2 s} - se^{-j^2 s})t - \lambda t\right). \quad (3.1)$$

Let

$$a_k := \left(2^{k+1} r\right)^n \exp\left(-4^k sr^2 + \frac{1}{2}(se^{j^2 s} - se^{-j^2 s})t - \lambda t\right), \quad k = 0, 1, \cdots.$$

It is easy to see that

$$\frac{a_{k+1}}{a_k} = 2^n \exp\left(-4^{k+1} sr^2 + 4^k sr^2\right) \leq 2^n \exp\left(-3sr^2\right).$$

Here, we choose

$$r = c\sqrt{t} \log t \qquad \text{and} \qquad s = \frac{1}{t},$$

where $t > 1$, $c > \sqrt{\frac{n}{2}}$. Notice that there exists a $T_1 \geq e$ such that, for any $t \geq T_1$,

$$3sr^2 = 3c^2 (\log t)^2 \geq 3c^2 \log t \geq n.$$

Thus,

$$\frac{a_{k+1}}{a_k} \leq \left(\frac{2}{e}\right)^n < 1,$$

which implies

$$\sum_{k=0}^{\infty} a_k \leq \frac{a_0}{1 - \left(\frac{2}{e}\right)^n}. \quad (3.2)$$

Applying (3.2) to (3.1), we obtain for $r \geq r_0$ and $t \geq T_1$,

$$\sum_{y \in B_\rho(x,r)^c} m(y)p(t,x,y) \leq \beta r^n \exp\left(-sr^2 + \frac{1}{2}(se^{j^2 s} - se^{-j^2 s})t - \lambda t\right), \quad (3.3)$$

where $\beta = \frac{2^n C}{m_0\left(1 - \left(\frac{2}{e}\right)^n\right)}$, $r = c\sqrt{t} \log t$ and $s = \frac{1}{t}$. By combining the above conditions $r \geq r_0$ and $t \geq T_1$, there exists a $T_2 \geq T_1$ such that the inequality (3.3) holds for any $t \geq T_2$. More precisely, for any $t \geq T_2$,

$$\begin{aligned}
&\sum_{y\in B_\rho(x,r)^c} m(y)p(t,x,y)\\
&\leq \beta\left(c\sqrt{t}\log t\right)^n \exp\left(-c^2(\log t)^2+\frac{1}{2}\left(\frac{1}{t}e^{\frac{j^2}{t}}-\frac{1}{t}e^{-\frac{j^2}{t}}\right)t-\lambda t\right)\\
&\leq \beta\left(c\sqrt{t}\log t\right)^n \exp\left(-c^2\log t+\frac{1}{2}\left(e^{\frac{j^2}{t}}-e^{-\frac{j^2}{t}}\right)-\lambda t\right)\\
&= c^n\beta\left(\frac{\log t}{t^{\frac{c^2}{n}-\frac{1}{2}}}\right)^n e^{-\lambda t}\exp\left(\frac{1}{2}\left(e^{\frac{j^2}{t}}-e^{-\frac{j^2}{t}}\right)\right),
\end{aligned}\tag{3.4}$$

where $c^2 > \frac{n}{2}$.

We *claim* that the right-hand side of (3.4) approaches to 0 as $t \to +\infty$. Indeed, observe that $e^{-\lambda t} \leq 1$ for any $t \geq T_2$ since $\lambda \geq 0$, and

$$e^{\frac{j^2}{t}} - e^{-\frac{j^2}{t}} \to 0, \quad t \to +\infty.$$

Moreover, since $c^2 > \frac{n}{2}$, one has

$$\lim_{t\to+\infty}\left(\frac{\log t}{t^{\frac{c^2}{n}-\frac{1}{2}}}\right)^n = \lim_{t\to+\infty}\left(\frac{1}{(\frac{c^2}{n}-\frac{1}{2})t^{\frac{c^2}{n}-\frac{1}{2}}}\right)^n = 0.$$

The proof of the claim is complete. Thus, there exists a real number $T \geq T_2$ such that for any $t \geq T$,

$$\sum_{y\in B_\rho(x,r)^c} \mu(y)p(t,x,y) \leq \frac{1}{2},\tag{3.5}$$

where $r = c\sqrt{t}\log t$ with $c > \sqrt{\frac{n}{2}}$.

Using the properties of the heat kernel described in Lemma 2.1, stochastic completeness in Remark 2 along with Cauchy-Schwarz inequality, we obtain for any $t > 0$ and all $x \in V$,

$$\begin{aligned}
p(t,x,x) &\geq p(2t,x,x)\\
&= \sum_{y\in V} m(y)p^2(t,x,y)\\
&\geq \sum_{y\in B_\rho(x,r)} m(y)p^2(t,x,y)\\
&\geq \frac{1}{V_\rho(x,r)}\left(\sum_{y\in B_\rho(x,r)} m(y)p(t,x,y)\right)^2\\
&= \frac{1}{V_\rho(x,r)}\left(1-\sum_{y\in B_\rho(x,r)^c} m(y)p(t,x,y)\right)^2.
\end{aligned}\tag{3.6}$$

Set $r = c\sqrt{t}\log t$ with $c > \sqrt{\frac{n}{2}}$ and $t \geq T$ in (3.6). Combining (3.5) to complete the proof of Theorem 1.1. □

Proof of Theorem 1.2. Set

$\mathcal{F}_T = \{f : [0,T] \times V \to \mathbb{R} |\ f(t,x)$ is bounded and is continous with respect to t $\}$.

The norm in $\mathcal{F}_T$ is defined by

$$\|f\|_{\mathcal{F}_T} = \sup_{[0,T]\times V} |f|.$$

Obviously, $\mathcal{F}_T$ is a Banach space.

For any $u \in \mathcal{F}_T$, we consider

$$\psi(u)(t,x) = \sum_{x\in V} m(y)p(t,x,y)u(0,y) + \phi(u)(t,x),$$

where $\phi(u)(t,x) = \int_0^t \sum_{y\in V} m(y)p(t-s,x,y)u(s,y)^{1+\alpha}ds$. The integral is well-defined, since $u \in \mathcal{F}_T$. Next, we show that $\psi(u)$ is a contraction on the small closed ball

$$B(u_0, A\|u_0\|_{\mathcal{F}_T}) := \{u \in \mathcal{F}_T : \|u-u_0\|_{\mathcal{F}_T} \leq (A-1)\|u_0\|_{\mathcal{F}_T}\}$$

into itself, where $A \geq \frac{1}{\alpha} + 1$. Indeed, for any $t_1, t_2 \in [0,T]$ and any $x \in V$, from the fact that $\sum_{y\in V} m(y)p(t,x,y) \leq 1$ for any $t > 0$, we have

$$\left| \int_{t_1}^{t_2} \sum_{y\in V} m(y)p(t-s,x,y)u(s,y)^{1+\alpha}ds \right| \leq \|u\|_{\mathcal{F}_T}^{1+\alpha}|t_1-t_2| \leq A^{1+\alpha}\|u_0\|_{\mathcal{F}_T}^{1+\alpha}|t_1-t_2|. \tag{3.7}$$

Combining the continuousness of $\sum_{x\in V} m(y)p(t,x,y)u(0,y)$ with respect to t, it follows that $\psi(u)(t,x)$ is continuous with respect to t. Moreover, let

$$T < \frac{1}{A^\alpha(1+\alpha)\|u_0\|_{\mathcal{F}_T}^\alpha}, \tag{3.8}$$

for any $(t,x) \in [0,T] \times V$,

$$|\psi(u)(t,x)| \leq \|u_0\|_{\mathcal{F}_T} + \|u\|_{\mathcal{F}_T}^{1+\alpha}T < \left(1 + \frac{A}{1+\alpha}\right)\|u_0\|_{\mathcal{F}_T} \leq A\|u_0\|_{\mathcal{F}_T}, \tag{3.9}$$

which means that $\psi(u) \in B(u_0, A\|u_0\|_{\mathcal{F}_T})$. Furthermore, for any $u, v \in B(u_0, A\|u_0\|_{\mathcal{F}_T})$ with $u(0,\cdot) \equiv v(0,\cdot)$, from the mean value theorem, we have

$$|u^{1+\alpha} - v^{1+\alpha}| \leq (1+\alpha)\max\{\|u\|_{\mathcal{F}_T}^\alpha, \|v\|_{\mathcal{F}_T}^\alpha\}|u-v| \leq A^\alpha(1+\alpha)\|u_0\|_{\mathcal{F}_T}^\alpha\|u-v\|_{\mathcal{F}_T}.$$

Then, for any $(t, x) \in [0, T] \times V$,

$$|\psi(u) - \psi(v)|(t, x) \leq A^{\alpha}(1 + \alpha)\|u_0\|_{\mathcal{F}_T}^{\alpha} T \|u - v\|_{\mathcal{F}_T}. \tag{3.10}$$

Therefore, from the contraction mapping principle, there exists a unique solution $u \in B(u_0, A\|u_0\|_{\mathcal{F}_T})$ of the following integral equation

$$\begin{cases} u(t, x) = \sum\limits_{x \in V} m(y)p(t, x, y)u(0, y) + \phi(u)(t, x), & (t, x) \in (0, T] \times V, \\ u(0, x) = u_0(x), & x \in V, \end{cases} \tag{3.11}$$

when $A^{\alpha}(1 + \alpha)\|u_0\|_{\mathcal{F}_T}^{\alpha} T < 1$, namely T satisfies the condition (3.8). □

Given a mild solution u of Eq. (1.3) and $t \in (0, T)$, denote

$$J_t(s, x) = P_{t-s}(u(s, \cdot))(x)$$

and

$$K_t(s, x) = P_{t-s}(u(s, \cdot)^{1+\alpha})(x)$$

for any $0 < s < t$ and $x \in V$. The following lemma is the crucial ingredient of proving Theorem 1.3, which is the discrete version of [4, Lemma 2.1], and it can be founded in [16, Lemma 4.1] for bounded Laplacians.

Lemma 3.1 *Let $G = (V, \omega, m)$ be a weighted graph with an intrinsic metric ρ with finite balls. Suppose that the initial value function u_0 is positive at a given vertex x_0. Let $T > 0$, if $u = u(t, x)$ is a non-negative bounded mild solution of Eq.* (1.3) *in $[0, T) \times V$, then we have*

$$J_t(0, x_0)^{-\alpha} - u(t, x_0)^{-\alpha} \geq \alpha t, \quad t \in [0, T).$$

Proof First, we *claim* that for any $h \in \mathbb{R}$, we have

$$J_t(s + h, x) - J_t(s, x) = \int_s^{s+h} K_t(\tau, x) d\tau.$$

To end the claim, let $\eta_k = 1_{B_\rho(x_0, k)}$ be the characteristic function with 1 on $B_\rho(x_0, k)$ and 0 otherwise. Note that $\eta_k \in C_0(V)$, we have

$$P_{t-s}\left(\int_0^s P_{s-\tau}\left(u(\tau, \cdot)^{1+\alpha}\right) d\tau \cdot \eta_k\right) = \int_0^s P_{t-s}(P_{s-\tau}(u(\tau, \cdot)^{1+\alpha}) \cdot \eta_k) d\tau.$$

By virtual of

$$P_{t-s}(P_{s-\tau}(u(\tau, \cdot)^{1+\alpha}) \cdot \eta_k) \leq P_{t-s}(P_{s-\tau}(u(\tau, \cdot)^{1+\alpha}) \cdot \eta_{k+1})$$

and $P_{t-s}(P_{s-\tau}(u(\tau,\cdot)^{1+\alpha})\cdot\eta_k) \le P_{t-\tau}(u(\tau,\cdot)^{1+\alpha}) \le \|u^{1+\alpha}\|_{\mathcal{F}_T}$. Monotone convergence theorem yields that

$$P_{t-s}(P_{s-\tau}(u(\tau,\cdot)^{1+\alpha})\cdot\eta_k) \to P_{t-\tau}(u(\tau,\cdot)^{1+\alpha}),\quad k\to\infty.$$

Similarly,

$$P_{t-s}\left(\int_0^s P_{s-\tau}\left(u(\tau,\cdot)^{1+\alpha}\right)d\tau\cdot\eta_k\right) \to P_{t-s}\left(\int_0^s P_{s-\tau}\left(u(\tau,\cdot)^{1+\alpha}\right)d\tau\right),\quad k\to\infty.$$

From Dominated convergence theorem,

$$\int_0^s P_{t-s}(P_{s-\tau}(u(\tau,\cdot)^{1+\alpha})\cdot\eta_k)d\tau \to \int_0^s P_{t-\tau}(u(\tau,\cdot)^{1+\alpha})d\tau$$

as $k\to\infty$. These imply

$$P_{t-s}\left(\int_0^s P_{s-\tau}\left(u(\tau,\cdot)^{1+\alpha}\right)d\tau\right) = \int_0^s P_{t-\tau}(u(\tau,\cdot)^{1+\alpha})d\tau.$$

Therefore,

$$J_t(s,x) = P_t(u_0)(x) + \int_0^s K_t(\tau,x)d\tau.$$

which complete the claim. From the claim, we have for any $0<s<t$ and $x\in V$,

$$\partial_s J_t(s,x) = K_t(s,x). \tag{3.12}$$

Since u is a mild solution, we have $u(s,x)>0$ on $(0,t)\times V$ since $p(s,x_0,x_0)>0$ with $s>0$ and $u_0(x_0)>0$. It follows that $J_t(s,x)>0$ for all $(s,x)\in[0,t]\times V$. From (3.12), Jensen's inequality and stochastically completeness of G, we obtain

$$\partial_s J_t(s) \ge J_t(s)^{1+\alpha},\quad s\in[0,t],$$

which yields

$$J_t(0,x_0)^{-\alpha} - J_t(t,x_0)^{-\alpha} \ge \alpha t.$$

That ends the proof. □

Proof of Theorem 1.3. Suppose that there exists a non-negative global solution $u = u(t,x)$ of Eq.(1.3). From Theorem 1.1 and the polynomial volume growth property, we have that there exists a $T'>1$ such that for any $t>T'$ and $c^2>\frac{n}{2}$,

$$p(t,x_0,x_0) \ge \frac{1}{4V_\rho(x_0,c\sqrt{t}\log t)} \ge \frac{1}{4Cc^n}\left(\sqrt{t}\log t\right)^{-n}.$$

Hence, for any $t > T'$, we have

$$\begin{aligned} J_t(0, x_0) &= \sum_{x \in V} m(x) p(t, x_0, x) u_0(x) \\ &\geq m(x_0) u_0(x_0) p(t, x_0, x_0) \\ &\geq C' \left(\sqrt{t} \log t\right)^{-n}, \end{aligned} \tag{3.13}$$

where $C' = \frac{m(x_0)u_0(x_0)}{4Cc^n}$.

On the other hand, it follows from Lemma 3.1 that for any $t > 0$,

$$J_t(0, x_0)^{-\alpha} \geq u(t, x_0)^{-\alpha} + \alpha t \geq \alpha t. \tag{3.14}$$

Combining (3.13) and (3.14) yields for any $t > T'$,

$$\frac{\log t}{t^{\frac{2-n\alpha}{2n\alpha}}} \geq C'', \tag{3.15}$$

where $C'' = (\alpha(C')^{\alpha})^{\frac{1}{n\alpha}}$. When $0 < n\alpha < 2$, we have

$$\lim_{t \to +\infty} \frac{\log t}{t^{\frac{2-n\alpha}{2n\alpha}}} = 0,$$

which gets a contradiction with (3.15) and ends the proof of the theorem. □

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Declarations

Conflict of interest All authors declared no potential Conflict of interest.

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