Mixed Hodge structures on character varieties of nilpotent groups

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Abstract

Let $\mathsf{Hom}^0(\Gamma, G)$ be the connected component of the identity of the variety of representations of a finitely generated nilpotent group Γ into a connected reductive complex affine algebraic group G. We determine the mixed Hodge structure on the representation variety $\mathsf{Hom}^0(\Gamma, G)$ and on the character variety $\mathsf{Hom}^0(\Gamma, G) /\!\!/ G$. We obtain explicit formulae (both closed and recursive) for the mixed Hodge polynomial of these representation and character varieties.

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1 Introduction

Let K be a connected compact Lie group, and Γ be a finitely generated nilpotent group. The topology of the space of representations $\mathsf{Hom}(\Gamma, K)$ and of its conjugation quotient space $\mathsf{Hom}(\Gamma, K)/K$ was considered by Ramras and Stafa in [29, 35], who obtained explicit formulae for the Poincaré polynomials of their identity components $\mathsf{Hom}^0(\Gamma, K)$ and $\mathsf{Hom}^0(\Gamma, K)/K$.

Let G be the complexification of K, and consider now the affine algebraic varieties $\mathcal{R}_\Gamma G := \mathsf{Hom}(\Gamma, G)$ and the geometric invariant theoretic quotient by conjugation $\mathcal{M}_\Gamma G := \mathcal{R}_\Gamma G /\!\!/ G$. In this article we determine the mixed Hodge structures on the identity components $\mathcal{R}^0_\Gamma G \subset \mathcal{R}_\Gamma G$ and $\mathcal{M}^0_\Gamma G \subset \mathcal{M}_\Gamma G$ and compute their mixed Hodge polynomials, generalizing the formulas obtained in [29] and in [19].

We now describe more precisely our main results. A finitely generated nilpotent group Γ is said to have abelian rank r if the torsion free part of $\Gamma_{Ab} := \Gamma/[\Gamma, \Gamma]$ has rank r. A connected reductive complex affine algebraic group G will be called a *reductive $\mathbb{C}$-group*, and T, W will stand, respectively, for a fixed maximal torus and the Weyl group of G.

Recall that the mixed Hodge numbers $h^{k,p,q}(X)$ of a quasi-projective variety X are the dimensions of the (p, q)-summands of the natural mixed Hodge structure (MHS) on $H^k(X, \mathbb{C})$. We say that X is of *Hodge-Tate type* if $h^{k,p,q}(X) = 0$ unless $p = q$.

Theorem 1.1 *Let Γ be a finitely generated nilpotent group with abelian rank $r \geq 1$, and G a reductive $\mathbb{C}$-group. Then, both $\mathcal{M}^0_\Gamma G$ and $\mathcal{R}^0_\Gamma G$ are of Hodge-Tate type. More concretely, the MHS on $\mathcal{M}^0_\Gamma G$ coincides with the one of T^r/W, where W acts diagonally, and the MHS of $\mathcal{R}^0_\Gamma G$ coincides with that of $(G/T) \times_W T^r$.*

Remark 1.2 In Theorem 1.1, W acts on $(G/T) \times T^r$ via the standard action on the homogeneous space G/T and by simultaneous conjugation on T^r. The MHS on G/T is the natural one coming from the full flag variety G/B, where $B \subset G$ is a Borel subgroup. We also note that the condition $r \geq 1$ in Theorem 1.1 is not vacuous since finite nilpotent groups have abelian rank 0. In fact, a nilpotent group N has abelian rank $r \geq 1$ if and only if $|N| = \infty$.

Now, consider the mixed Hodge polynomial of the algebraic variety X, defined as:

$$\mu_X(t, u, v) = \sum_{k,p,q \geq 0} h^{k,p,q}(X)\, t^k u^p v^q \in \mathbb{Z}[t, u, v].$$

Knowing the MHS of $\mathcal{M}_\Gamma^0 G$ and $\mathcal{R}_\Gamma^0 G$ allows for the explicit computation of their mixed Hodge polynomials, as follows. Let $\mathfrak{t}$ denote the Lie algebra of the maximal torus T, and recall that W acts naturally on its dual $\mathfrak{t}^*$, as a reflection group.

Theorem 1.3 *Let Γ be a finitely generated nilpotent group with abelian rank $r \geq 1$, and G a reductive $\mathbb{C}$-group of rank m. Then, we have:*

$$\mu_{\mathcal{R}_\Gamma^0 G}(t, u, v) = \frac{1}{|W|} \prod_{i=1}^{m} (1 - (t^2 uv)^{d_i}) \sum_{g \in W} \frac{\det\left(I + tuv\, A_g\right)^r}{\det\left(I - t^2 uv\, A_g\right)} \qquad (1.1)$$

and

$$\mu_{\mathcal{M}_\Gamma^0 G}(t, u, v) = \frac{1}{|W|} \sum_{g \in W} \det\left(I + tuv\, A_g\right)^r,$$

where $d_1, \ldots, d_m$ are the characteristic degrees of W (see definition in Sect. 4.1), and A_g is the action of $g \in W$ on $H^1(T, \mathbb{C}) \cong \mathfrak{t}^*$.

We now outline the proofs of these theorems. Using the main results of [9], we start by considering $\mathbb{Z}^r$, the free part of Γ_{Ab}. Let K be a maximal compact subgroup of G. Considering the deformation retractions obtained in [14] for $\mathcal{M}_{\mathbb{Z}^r} G$, and in [28] for $\mathcal{R}_{\mathbb{Z}^r} G$, we are then reduced to describing the cohomology of the compact spaces $\mathcal{R}_{\mathbb{Z}^r}^0 K := \mathsf{Hom}^0(\Gamma, K)$ and $\mathsf{Hom}^0(\Gamma, K)/K$.

A priori, there is no reason for these compact spaces to have MHSs on their cohomology groups. In [3] the rational cohomology ring of $\mathcal{R}_{\mathbb{Z}^r}^0 K$ is shown to be the Weyl group invariants of $(K/T_K) \times T_K^r$ where $T_K = T \cap K \subset K$ is a maximal torus. $(K/T_K) \times_W T_K^r$ is a desingularization of $\mathcal{R}_{\mathbb{Z}^r}^0 K$, and is homotopic to the space $(G/T) \times_W T^r$. Given the natural MHSs on G/T, T, and on the classifying space BT, in the context of equivariant cohomology, we conclude that both $\mathcal{R}_{\mathbb{Z}^r}^0 G$ and $\mathcal{M}_{\mathbb{Z}^r}^0 G$ are of Hodge-Tate type.

The formula for $\mathcal{M}_\Gamma^0 G$ then follows from the one in [19]. To get the formula for $\mathcal{R}_\Gamma^0 G$ we observe, as in [29, Section 5], that the graded cohomology ring of $\mathcal{R}_{\mathbb{Z}^r} K$ is a regrading of the cohomology ring of the torus T^r. Using representation theory, analogous to what is done in [24], we determine the regrading explicitly to obtain Formula (1.1).

The main results are proved in Sects. 4 and 5, after a brief review of relevant facts about mixed Hodge structures and character varieties in Sect. 2. In Sect. 3, we show that although the path-component of the identity is a union of algebraic components and the mixed Hodge structure is determined by the torus component (irreducible by [34]), there is in fact only one irreducible component through the identity. This follows

by closely analyzing the main proof in [14]. Moreover, we give a description of the singular locus of these moduli spaces $\mathcal{M}_{\mathbb{Z}^r}^0 G$ in the cases of classical G (expanding on work of [34]). We also obtain interesting number-theoretic results in Sect. 5.3. In particular, we show that $\mathcal{M}_{\mathbb{Z}^r}^0 G$ are "polynomial count" and compute the number of $\mathbb{F}_q$-points of these varieties where $\mathbb{F}_q$ is a field of order q. The last section (Sect. 6) applies our results to examples of character and representation varieties with "exotic components" considered in [1]; here, the group G is of the form $\mathrm{SL}(p, \mathbb{C})^m / \mathbb{Z}_p$ for a prime p.

2 Character varieties and mixed Hodge structures

2.1 Character varieties

Let G be a connected reductive complex affine algebraic group. As mentioned earlier, we will say G is a *reductive $\mathbb{C}$-group* in abbreviation. Let Γ be a finitely generated group. The set of homomorphisms $\rho : \Gamma \to G$ has the structure of an affine algebraic variety over $\mathbb{C}$ (not necessarily irreducible); the generators of Γ are translated into elements of G satisfying algebraic relations determined by the relations of Γ. Since G admits a faithful representation $G \hookrightarrow \mathrm{GL}(n, \mathbb{C})$ for some n, we will sometimes refer to ρ as a G-representation of Γ, or simply a representation of Γ when the context is clear.

We have two naturally defined varieties: the G-representation variety of Γ,

$$\mathcal{R}_\Gamma G := \mathsf{Hom}(\Gamma, G),$$

and the G-character variety of Γ,

$$\mathcal{M}_\Gamma G := \mathsf{Hom}\,(\Gamma, G) \,/\!\!/ G,$$

which is the affine geometric invariant theoretic (GIT) quotient under the conjugation action of G on $\mathcal{R}_\Gamma G$.

We endow $\mathcal{R}_\Gamma G$ with the Euclidean topology coming from a choice of r generators of Γ and the natural embedding $\mathsf{Hom}(\Gamma, G) \hookrightarrow G^r \subset \mathbb{C}^{rn^2}$, for appropriate n. Hence, $\mathcal{M}_\Gamma G$ is naturally endowed with a Hausdorff topology, as the GIT quotient identifies orbits whose closures intersect (see [14, Theorem 2.1] for a precise statement). However, when speaking of irreducible components we refer to the Zariski topology.

We note that $\mathcal{M}_\Gamma G$ is homotopic to the non-Hausdorff (conjugation) quotient space $\mathcal{R}_\Gamma G/G$ by [17, Proposition 3.4], and so any homotopy invariant property of either $\mathcal{M}_\Gamma(G)$ or $\mathcal{R}_\Gamma G/G$ applies to the other.

2.2 Mixed Hodge structures

In this subsection we summarize facts about mixed Hodge structures; details can be found in [10, 11, 27, 38]. The singular cohomology of a complex variety X is endowed

with a decreasing *Hodge filtration* $F_\bullet$:

$$H^k(X, \mathbb{C}) = F_0 \supseteq \cdots \supseteq F_{k+1} = 0$$

that generalizes the same named filtration for smooth complex projective varieties. In general, the graded pieces of this filtration do not constitute a pure Hodge structure. However, the rational cohomology of these varieties admits an increasing *Weight filtration*:

$$0 = W^{-1} \subseteq \cdots \subseteq W^{2k} = H^k(X, \mathbb{Q}),$$

and the Hodge filtration induces a pure Hodge structure on the graded pieces of its complexification, denoted $W_{\mathbb{C}}^\bullet$. The triple $\left(H^k(X, \mathbb{C}), F_\bullet, W_{\mathbb{C}}^\bullet\right)$ constitutes a *mixed Hodge structure* (MHS) over $\mathbb{C}$, and we denote the graded pieces of the associated decomposition by:

$$H^{k,p,q}(X, \mathbb{C}) = Gr_F^p Gr_{p+q}^{W_{\mathbb{C}}} H^k(X, \mathbb{C}).$$

Their dimensions, called *mixed Hodge numbers* $h^{k,p,q}(X) := \dim_{\mathbb{C}} H^{k,p,q}(X, \mathbb{C})$, are encoded in the polynomial:

$$\mu_X(t, u, v) = \sum_{k,p,q \geq 0} h^{k,p,q}(X) t^k u^p v^q \in \mathbb{Z}[t, u, v],$$

called the *mixed Hodge polynomial* of X. This polynomial reduces to the Poincaré polynomial of X, by setting $u = v = 1$. These constructions can also be reproduced for compactly supported cohomology, yielding a similar decomposition into pieces denoted $H_c^{k,p,q}(X, \mathbb{C})$.

When the variety X is smooth and projective the Hodge structure on $H^*(X, \mathbb{C})$ is pure, that is: $h^{k,p,q} \neq 0 \implies k = p + q$. We are also interested in two other types of MHS that can be read from its Hodge numbers. We say that X is *balanced* (see [26]) or of *Hodge-Tate type* if $h^{k,p,q} \neq 0 \implies p = q$. For those varieties that further satisfy $h^{k,p,q} \neq 0 \implies k = p = q$ we call them *round* (see [19]).

Recall that MHSs satisfy the Künneth theorem, so that, for the cartesian product $X \times Y$ of varieties, we have:

$$\mu_{X \times Y} = \mu_X \mu_Y. \tag{2.1}$$

Also important for this paper is the behavior of these structures under an algebraic action of a finite group. If F is a finite group acting algebraically on a complex variety X, the induced action on the cohomology respects the mixed Hodge decomposition. Moreover, one can recover the mixed Hodge structure on the quotient by:

$$H^{k,p,q}(X/F, \mathbb{C}) \cong H^{k,p,q}(X, \mathbb{C})^F. \tag{2.2}$$

Then, the types of mixed Hodge structures on the quotient X/F have similar properties to that of X. In particular, if X is pure, balanced or round, respectively, so is X/F. The situation is even easier when G is an algebraic group and F is a finite subgroup acting by left translation.

Lemma 2.1 *Let G be an algebraic group and F a finite subgroup. Then the MHS on G and on G/F coincide.*

Proof This follows from the fact that the F-action on the MHS of G is trivial, as shown in [12, Section 6] (see also [26]). Intuitively, the idea is that the action of F extends to the action of a connected group. $\qquad\square$

Another important invariant related to the MHS of X is the E-polynomial, obtained by specializing μ_X to $t = -1$: $E_X(u, v) := \mu_X(-1, u, v)$. Then the Euler characteristic of X is obtained as $\chi(X) = \mu_X(-1, 1, 1)$. We will also consider the compactly supported version of E_X, also called the *Serre polynomial*:

$$E_X^c(u, v) := \sum_{k, p, q \geq 0} (-1)^k \, h_c^{k, p, q}(X) \, u^p v^q \in \mathbb{Z}[t, u, v],$$

where $h_c^{k, p, q}(X) = \dim H_c^{k, p, q}(X, \mathbb{C})$.

Let $K(\mathcal{V}ar_{\mathbb{C}})$ be the *Grothendieck ring of varieties* over $\mathbb{C}$. Additively, this is a ring generated by isomorphism classes of algebraic varieties modulo the excision relation: if $Y \hookrightarrow X$ is a closed subvariety, then in $K(\mathcal{V}ar_{\mathbb{C}})$ we identify:

$$[X] = [Y] + [X \backslash Y].$$

The product in $K(\mathcal{V}ar_{\mathbb{C}})$ is given by cartesian product: $[X] \cdot [Y] := [X \times Y]$. The Serre polynomial and the Euler characteristic are examples of *motivic measures*, that is, maps from the objects of $\mathcal{V}ar_{\mathbb{C}}$ to a ring that factors through the Grothendieck ring of varieties.

3 Irreducible components

For many groups Γ, $\mathcal{R}_\Gamma G$ is *not irreducible* and/or *not path-connected*, and so the same happens with $\mathcal{M}_\Gamma G$. Recall that path-connected algebraic varieties need not be irreducible, and that irreducible algebraic varieties (over $\mathbb{C}$) are necessarily path-connected.

Path-components of $\mathcal{R}_\Gamma G$ are sometimes related to path-components of $\mathcal{R}_\Gamma K$ for a maximal compact subgroup $K \subset G$. For example, for a finitely generated free group F_r, $\mathcal{R}_{F_r} G \cong G^r$ and $\mathcal{R}_{F_r} K \cong K^r$ and so there is a π_0-bijection by the (topological) polar decomposition: $G \cong K \times \mathbb{R}^n$, for $n = \dim_{\mathbb{R}} K$. Much more non-trivially, there is a strong deformation retraction from $\mathcal{R}_\Gamma G$ to $\mathcal{R}_\Gamma K$ for Γ finitely generated and nilpotent by [5]; see [28] for the abelian case. And thus, there is a bijection between path-components in these cases as well.

Example 3.1 Let $\Gamma = \mathbb{Z}^2$ and $K = \mathrm{SO}(3)$. Suppose $\rho \in \mathsf{Hom}(\Gamma, K)$ is given by the pair of commuting matrices $\mathrm{diag}(1, -1, -1)$ and $\mathrm{diag}(-1, -1, 1)$. Then, these matrices cannot be simultaneously conjugated, within K, to the same maximal torus of $\mathrm{SO}(3)$. This implies that $\mathsf{Hom}(\Gamma, K)$ is not path-connected, since the collection of pairs that can be simultaneously conjugated into a given maximal torus forms a disjoint path-component. Thus, by the discussion above, $\mathsf{Hom}(\mathbb{Z}^2, \mathrm{PGL}(2, \mathbb{C}))$ is also not connected, as $\mathrm{PGL}(2, \mathbb{C}) \cong \mathrm{SO}(3, \mathbb{C})$ is the complexification of $\mathrm{SO}(3)$.

Let us denote by

$$\mathcal{R}_\Gamma^0 G := \mathsf{Hom}^0(\Gamma, G),$$

and by

$$\mathcal{M}_\Gamma^0 G := \mathsf{Hom}^0(\Gamma, G) /\!/ G,$$

the path-connected components of the identity representation in $\mathcal{R}_\Gamma G$ and in $\mathcal{M}_\Gamma G$, respectively. In some cases, $\mathcal{R}_\Gamma^0 G$ and $\mathcal{M}_\Gamma^0 G$ are irreducible varieties; but they are always a finite union of irreducible varieties.

3.1 The torus component

An interesting case is that of a finitely presentable group Γ whose abelianization is free, that is

$$\Gamma_{Ab} := \Gamma/[\Gamma, \Gamma] \cong \mathbb{Z}^r,$$

for some $r \in \mathbb{N}$. Examples in this class of groups include "exponent canceling groups" (see [25]) which are those that admit presentations such that in all relations the exponents of each generator add up to zero; such as right angled Artin groups (abbreviated RAAGs), and fundamental groups of closed orientable surfaces.

For these groups, since $\Gamma \to \Gamma_{Ab} \cong \mathbb{Z}^r$ is surjective, we can consider the following sequence:

$$T^r \cong \mathsf{Hom}(\mathbb{Z}^r, T) \hookrightarrow \mathsf{Hom}(\Gamma_{Ab}, G) \hookrightarrow \mathsf{Hom}(\Gamma, G) \twoheadrightarrow \mathcal{M}_\Gamma G.$$

Let us denote by $\mathcal{M}_\Gamma^T G \subset \mathcal{M}_\Gamma G$ the image of the composition above and call it the *torus component*. It follows that $\mathcal{M}_\Gamma^T G$ is an irreducible subvariety of $\mathcal{M}_\Gamma G$, being the image of the irreducible variety T^r under a morphism. In the case when $\Gamma = \mathbb{Z}^r$, $\mathcal{M}_\Gamma^T G$ is an irreducible component of $\mathcal{M}_\Gamma G$ by [34, Theorem 2.1].

Obviously, the identity representation ($\rho(\gamma) = e$ for all $\gamma \in \Gamma$, $e \in G$ being the identity) belongs to $\mathcal{M}_\Gamma^T G$ since it comes from the identity representation in $\mathsf{Hom}(\Gamma, T)$. Since $\mathcal{M}_\Gamma^T G$ is path-connected (being irreducible over $\mathbb{C}$), we conclude that $\mathcal{M}_\Gamma^T G \subset \mathcal{M}_\Gamma^0 G$. We observe that there are pairs (Γ, G) where the varieties $\mathcal{M}_\Gamma^T G$ and $\mathcal{M}_\Gamma^0 G$ agree, and others where they do not.

Example 3.2 When G is abelian (we always assume connected), it is clear that $\mathcal{M}_\Gamma^T G = \mathcal{M}_\Gamma^0 G$. For an example where they disagree, let $\Gamma = \Gamma_\angle$ be the "angle RAAG" associated with a path graph with 3 vertices, considered in [16]. Then, even for a low dimensional group such as $G = \mathrm{SL}(2, \mathbb{C})$ we have that $\mathcal{M}_\Gamma^0 G$ has 3 irreducible components, one being $\mathcal{M}_\Gamma^T G$ and 2 extra ones. Moreover, for the case $G = \mathrm{SL}(3, \mathbb{C})$ there are components in $\mathcal{M}_\Gamma^0 G$ which have higher dimension than the dimension of $\mathcal{M}_\Gamma^T G$.

Remark 3.3 One can also ask if the identity representation is contained in a single irreducible component of $\mathcal{M}_\Gamma^0 G$. This also fails for $\mathcal{M}_{\Gamma_\angle}^0 (\mathrm{SL}(2, \mathbb{C}))$, as shown in [16].

3.2 The free Abelian case

As seen in Examples 3.1, 3.2 and Remark 3.3, the comparison between the varieties $\mathcal{M}_\Gamma G$, $\mathcal{M}_\Gamma^0 G$ and $\mathcal{M}_\Gamma^T G$ for general Γ and G is far from being trivial.

We now show that $\mathcal{M}_\Gamma^0 G = \mathcal{M}_\Gamma^T G$ when Γ is free abelian, for all G. In this situation, we are dealing with representations defined by elements of G that pairwise commute. The following theorem generalizes Remark 2.4 in [34], and completely answers a question raised in [14, Problem 5.7].

Theorem 3.4 *For every $r \in \mathbb{N}$ and reductive $\mathbb{C}$-group G, $\mathcal{M}_{\mathbb{Z}^r}^T G = \mathcal{M}_{\mathbb{Z}^r}^0 G$.*

Proof Let K be a fixed maximal compact subgroup of G, and let $T_K = T \cap K$. Then $T_K \subset K$ is a maximal torus in K. Tom Baird [3] considered the *compact character variety*

$$\mathcal{N}_{\mathbb{Z}^r} K := \mathrm{Hom}(\mathbb{Z}^r, K)/K,$$

and showed that:

(1) The (path) connected component of the identity $\mathcal{N}_{\mathbb{Z}^r}^0 K \subset \mathcal{N}_{\mathbb{Z}^r} K$ coincides with the space of conjugation classes of representations

$$\mathrm{Hom}^{T_K}(\mathbb{Z}^r, K)/K,$$

where $\mathrm{Hom}^{T_K}(\mathbb{Z}^r, K)$ denotes the representations ρ whose r evaluations $\rho(\gamma_i)$ can be simultaneously conjugated into the maximal torus T_K.
(2) $\mathcal{N}_{\mathbb{Z}^r}^0 K$ is homeomorphic to the quotient T_K^r/W, where $W = N_K(T_K)/T_K$ is the Weyl group associated to T_K.

By [14, Theorem 1.1], there is a strong deformation retraction from $\mathcal{M}_{\mathbb{Z}^r} G$ to $\mathcal{N}_{\mathbb{Z}^r} K$ which (by continuity) restricts to one from $\mathcal{M}_{\mathbb{Z}^r}^0 G$ to $\mathcal{N}_{\mathbb{Z}^r}^0 K$.

Let $[\rho] \in \mathcal{M}_{\mathbb{Z}^r}^0 G$. Then there exists a commuting tuple $(g_1, \ldots, g_r)$ in G^r such that $[\rho] = [(g_1, ..., g_r)]$. By [14, Proposition 3.1] we can assume that each element $g_i \in G$ is semisimple. The strong deformation retraction (SDR), which is G-conjugate equivariant, provides a path ρ_t from this tuple to a commuting tuple in G_K^r where $G_K = \{gkg^{-1} \mid g \in G, \ k \in K\}$. With respect to an embedding $G \hookrightarrow \mathrm{SL}(n, \mathbb{C})$,

which preserves semisimplicity, this SDR is given by the eigenvalue retraction defined by deforming $\ell e^{i\theta}$ to $e^{i\theta}$ by sending ℓ to 1. By [14, Lemma 3.4], there exists a single element g_0 in G that will conjugate the resulting r-tuple in G_K^r to be in K^r. And by Baird's result, and the fact that we remain in the identity component by continuity, we know there is a single element k_0 in K that we can conjugate the resulting tuple in K^r so it is in T_K^r. Let $h_0 := k_0 g_0$, and consider the conjugated reverse path $\psi_t := h_0 \rho_{1-t} h_0^{-1}$. This path begins in T_K^r (by definition) and ends in T^r since eigenvalue retraction deforms T to T_K and hence the reverse path takes elements in T_K and maps them to T. Since $\psi_1 = h_0 \rho_0 h_0^{-1} = (h_0 g_1 h_0^{-1}, \ldots, h_0 g_r h_0^{-1})$, we conclude that

$$[\rho] = [(g_1, ..., g_r)] = [(h_0 g_1 h_0^{-1}, \ldots, h_0 g_r h_0^{-1})] = [\psi_1]$$

is in T^r/W, as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ □

4 Mixed Hodge structure on $\mathsf{Hom}^0(\Gamma, G)$

Here we prove the statements in Theorems 1.1 and 1.3 that concern the connected component $\mathcal{R}_\Gamma^0 G$ of the trivial representation in the representation variety $\mathcal{R}_\Gamma G = \mathsf{Hom}(\Gamma, G)$.

Let us first describe the situation for $\Gamma \cong \mathbb{Z}^r$. Consider, as in the proof of Theorem 3.4, the compact character variety

$$\mathcal{N}_{\mathbb{Z}^r} K = \mathsf{Hom}(\mathbb{Z}^r, K)/K,$$

where K is a fixed maximal compact subgroup of G. Recall our convention that $T_K = T \cap K$ where T is a maximal torus in G. Baird [3] showed that the isomorphism $\mathcal{N}_{\mathbb{Z}^r}^0 K \cong T_K^r/W$, is part of a natural K-equivariant commutative diagram:

$$
\begin{array}{ccc}
(K/T_K) \times_W T_K^r & \xrightarrow{\ \varphi_K\ } & \mathsf{Hom}^0(\mathbb{Z}^r, K) \\
{\scriptstyle \pi_K}\big\downarrow & & \big\downarrow{\scriptstyle \pi_K} \\
T_K^r/W & \xrightarrow[\ \cong\ ]{} & \mathcal{N}_{\mathbb{Z}^r}^0 K,
\end{array}
$$

where φ_K is a desingularization of $\mathsf{Hom}^0(\mathbb{Z}^r, K)$ which induces an isomorphism in cohomology, and the vertical maps are the quotient maps by K-conjugation.

Passing to the complexification, there is an analogous G-equivariant commutative diagram:

$$
\begin{array}{ccc}
(G/T) \times_W T^r & \xrightarrow{\ \varphi_G\ } & \mathsf{Hom}^0(\mathbb{Z}^r, G) \\
{\scriptstyle \pi_G}\big\downarrow & & \big\downarrow{\scriptstyle \pi_G} \\
T^r/W & \xrightarrow[\ \chi\ ]{} & \mathcal{M}_{\mathbb{Z}^r}^0 G.
\end{array}
$$

There are some notable differences from the compact case:

(1) χ is bijective, birational, and a normalization map (Corollary 5.1),
(2) χ is not generally known to be an isomorphism, although it is when G is of classical type (Corollary 5.2),
(3) φ_G is not even surjective, hence not a desingularization morphism (we will say more about this below),
(4) π_G is the GIT quotient map (with respect to the G-conjugation action).

Despite these differences, we will show that the mixed Hodge structures of T^r/W and $\mathcal{M}^0_{\mathbb{Z}^r} G$ coincide, as do those of $(G/T) \times_W T^r$ and $\mathsf{Hom}^0(\mathbb{Z}^r, G)$.

4.1 Mixed Hodge structures on a smooth model of $\mathcal{R}^0_{\mathbb{Z}^r} G$

The above discussion suggests to consider the smooth irreducible algebraic variety:

$$S_r G := (G/T) \times_W T^r,$$

whose MHS we now determine. The natural MHS on G/T is the one of the full flag variety G/B, where B is a Borel subgroup, which has well-known cohomology. Indeed, it is a classical fact that there is an identification $K/T_K \cong G/B$. On the other hand, $K/T_K \hookrightarrow G/T$ is a strong deformation retraction (see for example [6, Theorem 10]), which provides isomorphisms of cohomology spaces:

$$H^*(G/B) \cong H^*(K/T_K) \cong H^*(G/T).$$

Since T is contained in a certain Borel subgroup, there is a surjective algebraic map $\varphi : G/T \to G/B$ which upgrades the above isomorphism to an isomorphism of MHSs, $H^*(G/B) \cong H^*(G/T)$, since the restriction of φ to K/T_K is the map that induces the isomorphism $K/T_K \cong G/B$.

Recall that the Weyl group W acts on $\mathfrak{t}^*$, the dual of the Lie algebra of T. By the Shephard-Todd theorem [33], the ring of W-invariants $\mathbb{C}[\mathfrak{t}^*]^W$ is a polynomial ring generated by homogeneous generators of degrees $d_1, \ldots, d_m$ called the *characteristic degrees* of W ($m = \dim \mathfrak{t}$). These are well-known for all Weyl groups of simple G (isomorphic to the Weyl groups of simple K), and can be consulted in [29, Table 1] or in [24, Page 7].

Theorem 4.1 *Let m be the rank of G and $d_1, \ldots, d_m$ be the characteristic degrees of W. The variety $S_r G$ is of Hodge-Tate type and its mixed Hodge polynomial is given by:*

$$\mu_{S_r G}(t, u, v) = \frac{1}{|W|} \prod_{i=1}^m (1 - (t^2 uv)^{d_i}) \sum_{g \in W} \frac{\det\left(I + tuv\, A_g\right)^r}{\det\left(I - t^2 uv\, A_g\right)}.$$

Proof Since mixed Hodge structures respect the Künneth formula, from $H^*(G/B) \cong H^*(G/T)$, we get an isomorphism:

$$H^*(S_r G) \cong [H^*(G/T) \otimes H^*(T^r)]^W \cong [H^*(G/B) \otimes H^*(T^r)]^W,$$

of mixed Hodge structures, where the superscript means that we are considering the W-invariant subspace. Since the full flag variety G/B is smooth and projective, its cohomology has a pure Hodge structure. Moreover, there is an isomorphism

$$H^*(G/B) \cong H^*(BT)_W$$

where $BT \cong (BS^1)^m$ is the classifying space of T, and $H^*(BT)_W$ is the algebra of co-invariants under the W-action on $H^*(BT)$. Also, $H^*(BT)$ is a polynomial ring $\mathbb{C}[x_1, \ldots, x_m]$ where each x_i has triple grading $(2, 1, 1)$, since BT can be identified with $(\mathbb{C}P^\infty)^m$ (in particular, it has pure cohomology). By a classical theorem of Borel (see [30] for a modern treatment), there is an isomorphism:

$$H^*(BT)_W \cong \mathbb{C}[x_1, \ldots, x_m]/(\sigma_1, \ldots, \sigma_m),$$

where the σ_i are the homogeneous generators of the ring of W-invariants $H^*(BT)^W$, with degrees $(2d_i, d_i, d_i)$.

From the above, and the fact that $\sigma_1, \ldots, \sigma_m$ are W-invariants, we obtain:

$$H^*(S_r G) \cong [H^*(G/B) \otimes H^*(T^r)]^W \cong [H^*(BT) \otimes H^*(T^r)]^W/(\sigma_1, \ldots, \sigma_m).$$

Now, the mixed Hodge polynomial $\mu_X(t, u, v)$ of a variety X is the Hilbert series of its cohomology $H^*(X)$ with the triple grading given by its mixed Hodge structure. Denote by $\mathfrak{H}(A)$ the Hilbert series of a graded algebra A, in the variable x. It is a standard result that, if $a \in A$ is not a zero divisor, then

$$\mathfrak{H}(A/(a)) = \mathfrak{H}(A)\,(1 - x^d),$$

where d is the degree of a. Applied to our case, and since $\sigma_1, \ldots, \sigma_m$ form a regular sequence (see [24]), we get the equality of Hilbert series in the three variables t, u, v:

$$\mathfrak{H}(H^*(S_r G)) = \mathfrak{H}([H^*(BT) \otimes H^*(T^r)]^W) \prod_{i=1}^{m}(1 - (t^2 uv)^{d_i}).$$

The result thus follows from:

$$\mathfrak{H}([H^*(BT) \otimes H^*(T^r)]^W) = \frac{1}{|W|} \sum_{g \in W} \frac{\det\left(I + tuv\, A_g\right)^r}{\det\left(I - t^2 uv\, A_g\right)}. \tag{4.1}$$

Formula (4.1) is obtained by applying Corollary 4.3 below with $V_0 = H^{2,1,1}(BT)$ and $V_1 = \cdots = V_r = H^{1,1,1}(T)$, since $H^*(BT) = S^\bullet V_0$ and T^r has round cohomology generated in degrees $(1, 1, 1)$: $H^*(T^r) = \wedge^\bullet H^{1,1,1}(T^r) \cong (\wedge^\bullet H^{1,1,1}(T))^{\otimes r}$. $\qquad \square$

Recall some definitions and facts from the theory of representations of finite groups. If $V = \oplus_{k\geq 0} V^k$ is a graded $\mathbb{C}$-vector space (possibly infinite dimensional, but with finite dimensional summands), and $g : V \to V$ is a linear map that preserves the grading, define the *graded-character* of g by:

$$\chi_g(V) := \sum_{k\geq 0} \mathrm{tr}(g|_{V^k})\, x^k \in \mathbb{C}[[x]].$$

It is additive and multiplicative, under direct sums and tensor products, respectively:

$$\chi_g(V_1 \oplus V_2) = \chi_g(V_1) + \chi_g(V_2), \qquad \chi_g(V_1 \otimes V_2) = \chi_g(V_1)\chi_g(V_2).$$

A linear map $g : V \to V$ induces linear maps on the direct sums of all symmetric powers $S^\bullet V := \oplus_{j\geq 0} S^j V$, and of all exterior powers $\wedge^\bullet V := \oplus_{j\geq 0}^{\dim V} \wedge^j V$. Note that $S^\bullet V$ is graded, with elements of $S^j V$ and $\wedge^j V$ having degree $j\delta$, when V is pure of degree δ. We need to consider two important cases, whose proofs are standard (see, for instance, [31, page 69]).

Lemma 4.2 *Let V be a vector space, whose elements are all of degree δ, and $g : V \to V$ a linear map. Then, we have:*

$$\chi_g(S^\bullet V) = \frac{1}{\det(I - x^\delta g)}, \qquad \chi_g(\wedge^\bullet V) = \det(I + x^\delta g).$$

Now, suppose that a finite group F acts on V preserving the grading. Recall that the Hilbert-Poincaré series of the graded vector space V^F, of F-invariants in V, can be computed as:

$$\mathfrak{H}(V^F) = \frac{1}{|F|} \sum_{g\in F} \chi_g(V). \tag{4.2}$$

Since all the above constructions are valid for triply graded vector spaces, and characters are multiplicative under tensor products, the following is an immediate consequence of Lemma 4.2.

Corollary 4.3 *If V_i, $i = 0, \ldots, r$ are finite dimensional representations of a finite group F, then the triply graded Hilbert-Poincaré series in t, u, v is:*

$$\mathfrak{H}([S^\bullet V_0 \otimes \wedge^\bullet V_1 \otimes \cdots \otimes \wedge^\bullet V_r]^F) = \frac{1}{|F|} \sum_{g\in F} \frac{\prod_{i=1}^r \det(I + t^{a_i} u^{b_i} v^{c_i} g)}{\det(I - t^{a_0} u^{b_0} v^{c_0} g)},$$

where each V_i has pure triple degree (a_i, b_i, c_i), $i = 0, \ldots, r$.

4.2 Mixed Hodge structure on $\mathcal{R}^0_\Gamma G$ for Nilpotent Γ

The lower central series of a group Γ is defined inductively by $\Gamma_1 := \Gamma$, and $\Gamma_{i+1} := [\Gamma, \Gamma_i]$ for $i > 1$. A group Γ is *nilpotent* if the lower central series terminates to the trivial group.

Let Γ be a finitely generated nilpotent group. It is a general theorem that all finitely generated nilpotent groups are finitely presentable (and residually finite); see [23].

Recall that the abelianization of Γ:

$$\Gamma_{Ab} := \Gamma/[\Gamma, \Gamma],$$

can be written as $Ab(\Gamma) \simeq \mathbb{Z}^r \oplus F$ where $r \in \mathbb{N}_{\geq 0}$ is the abelian rank of Γ, and F is a finite abelian group. We now generalize some of the previous results to nilpotent groups.

Theorem 4.4 *Let Γ be a finitely generated nilpotent group with abelian rank $r \geq 1$. Then, the algebraic variety $\mathcal{R}^0_\Gamma G = \mathsf{Hom}^0(\Gamma, G)$ has dimension $\dim G + (r-1) \dim T$ and its MHS coincides with the MHS on $(G/T) \times_W T^r$.*

Proof By [9], we have $\mathsf{Hom}^0(\Gamma_{Ab}, K) \cong \mathsf{Hom}^0(\Gamma, K)$ and consequently, from [5], $\mathsf{Hom}^0(\Gamma_{Ab}, G)$ is homotopic to $\mathsf{Hom}^0(\Gamma, G)$. From [3], we know that

$$\varphi_K : (K/T_K) \times_W T_K^r \to \mathsf{Hom}^0(\Gamma_{Ab}, K)$$

is a birational surjection (in fact, a desingularization) that induces an isomorphism in cohomology. This map is defined by $[(gT, t_1, ..., t_r)]_W \mapsto (gt_1 g^{-1}, ..., gt_r g^{-1})$, and we can likewise define φ_G in the complex situation.

These maps come together to form the following commutative diagram:

$$
\begin{array}{ccccc}
(G/T) \times_W T^r & \xrightarrow{\varphi_G} & \mathsf{Hom}^0(\Gamma_{Ab}, G) & \longrightarrow & \mathsf{Hom}^0(\Gamma, G) \\
\uparrow & & \uparrow & & \uparrow \\
(K/T_K) \times_W T_K^r & \xrightarrow{\varphi_K} & \mathsf{Hom}^0(\Gamma_{Ab}, K) & \xrightarrow{\cong} & \mathsf{Hom}^0(\Gamma, K).
\end{array}
\qquad (4.3)
$$

Since the bottom row induces isomorphisms in cohomology, by commutativity, all maps induce isomorphisms in cohomology. Since the upper row is formed by algebraic maps, these induce isomorphisms of mixed Hodge structures of the respective cohomologies. The dimension formula is clear since $\dim G/T = \dim G - \dim T$. □

Remark 4.5 Let $\Gamma_{Ab} \cong \mathbb{Z}^r$ with free abelian generators $\gamma_1, ..., \gamma_r$. We note some properties of the map $\varphi_G : (G/T) \times_W T^r \to \mathsf{Hom}^0(\mathbb{Z}^r, G)$. Let G_{ss} be the set of semisimple elements of G (elements in G with closed conjugation orbits), and $\mathsf{Hom}^0(\mathbb{Z}^r, G_{ss}) := \{\rho \in \mathsf{Hom}^0(\mathbb{Z}^r, G) \mid \rho(\gamma_i) \in G_{ss}, 1 \leq i \leq r\}$. It is shown in [14] that $\mathsf{Hom}^0(\mathbb{Z}^r, G_{ss})$ is exactly the set of representations with closed conjugation orbits. In the identity component, these are exactly the representations whose image can be conjugated to a fixed maximal torus. Hence, the image of φ_G is exactly

the set $\text{Hom}^0(\mathbb{Z}^r, G_{ss})$. So we see that φ_G is not surjective and $\text{Hom}^0(\mathbb{Z}^r, G_{ss})$ is a constructible set (not obvious a priori). Now from [28] we know that $\text{Hom}^0(\mathbb{Z}^r, G)$ is homotopic to $\text{Hom}^0(\mathbb{Z}^r, G_{ss})$, and since G_{ss} is dense in G we deduce that $\text{Hom}^0(\mathbb{Z}^r, G_{ss})$ is dense in $\text{Hom}^0(\mathbb{Z}^r, G)$. Thus, φ_G is dominant, homotopically surjective, and induces a cohomological isomorphism (from Diagram (4.3)). Since W acts freely, $(G/T) \times_W T^r$ is smooth although $\text{Hom}^0(\mathbb{Z}^r, G)$ is generally singular. By Remark 5.6 below, the Zariski dense representations in $\text{Hom}^0(\mathbb{Z}^r, G)$ are smooth points. It is easy to see that $\varphi_G^{-1}(\rho)$ is a point if ρ is Zariski dense (a generic condition). Hence, φ_G is birational, although, unlike its compact analogue φ_K, it is not a desingularization.

Corollary 4.6 *Let G be a reductive $\mathbb{C}$-group of rank m, whose Weyl group has characteristic degrees $d_1, \ldots, d_m$. Let Γ be a finitely generated nilpotent group of abelian rank $r \geq 1$. The variety $\mathcal{R}_\Gamma^0 G$ is of Hodge-Tate type and its mixed Hodge polynomial is given by:*

$$\mu_{\mathcal{R}_\Gamma^0 G}(t, u, v) = \frac{1}{|W|} \prod_{i=1}^m (1 - (t^2 uv)^{d_i}) \sum_{g \in W} \frac{\det (I + tuv \, A_g)^r}{\det (I - t^2 uv \, A_g)}. \tag{4.4}$$

Proof This follows immediately from Theorems 4.1 and 4.4. □

Corollaries 4.6 and 5.10 together establish Theorem 1.3 from the Introduction.

Corollary 4.7 *For every finitely generated nilpotent group Γ, and reductive $\mathbb{C}$-group G, the Poincaré polynomial and E-polynomial of $\mathcal{R}_\Gamma^0 G$ are given, respectively, by:*

$$P_t \left(\mathcal{R}_\Gamma^0 G \right) = \frac{1}{|W|} \prod_{i=1}^m (1 - t^{2d_i}) \sum_{g \in W} \frac{\det (I + t \, A_g)^r}{\det (I - t^2 \, A_g)}$$

$$E_{\mathcal{R}_\Gamma^0 G}(u, v) = \frac{1}{|W|} \prod_{i=1}^m (1 - (uv)^{d_i}) \sum_{g \in W} \det (I - uv \, A_g)^{r-1}$$

and the Euler characteristic of $\mathcal{R}_\Gamma^0 G$ vanishes. If the abelian rank of Γ is 2 and G is simply-connected, then the E-polynomial simplifies to $E_{\mathcal{R}_\Gamma^0 G}(u, v) = \prod_{i=1}^m (1 - (uv)^{d_i})$.

Proof This follows by evaluating Formula (4.4) at $u = v = 1$ for the Poincaré polynomial, and at $t = -1$ for the E-polynomial. Then, the Euler characteristic is obtained as $\chi(\mathcal{R}_\Gamma^0 G) = E_{\mathcal{R}_\Gamma^0 G}(1, 1) = 0$, as $r \geq 1$.

Finally, when $r = 2$ the term $(1/|W|) \sum_{g \in W} \det(I - uv A_g)$ is equal to $\mu_{\mathcal{M}_\mathbb{Z}^0 G}(-1, u, v)$ by Theorem 5.8 below. On the other hand, since G is simply-connected, $\mathcal{M}_\mathbb{Z}^0 G \cong T/W \cong \mathbb{C}^{\dim T}$, by a result of Steinberg in [37]. Being affine space, its E-polynomial equals 1 (see [19, Example 2.6]), as wanted. □

4.3 Some computations for classical groups

For certain classes of groups, such as $G = \mathrm{SL}(n, \mathbb{C})$ and $G = \mathrm{GL}(n, \mathbb{C})$, the above formulas can be made more explicit. These cases have Weyl group S_n, the symmetric group on n letters. In the $\mathrm{GL}(n, \mathbb{C})$ case, the action of a permutation $\sigma \in S_n$ on the dual of the Cartan subalgebra of $\mathfrak{gl}_n$ can be identified with the action on $\mathbb{C}^n$ by permuting the canonical basis vectors. Therefore, $\det(I - \lambda A_\sigma) = \prod_{j=1}^n (1 - \lambda^j)^{\sigma_j}$, where $\sigma \in S_n$ is a permutation with exactly $\sigma_j \geq 0$ cycles of size $j \in \{1, \ldots, n\}$ (see, for example, [19, Thm. 5.13]). The collection $(\sigma_1, \sigma_2 \ldots, \sigma_n)$ defines a partition of n, one with exactly σ_j parts of length j, and the number of permutations $\sigma \in S_n$ with this cycle pattern is (see [36, 1.3.2]):

$$m_\sigma = n! \, (\textstyle\prod_{j=1}^n \sigma_j! \, j^{\sigma_j})^{-1}.$$

Since the characteristic degrees of the Weyl group of $\mathrm{GL}(n, \mathbb{C})$ are exactly $1, 2, \ldots, n$, this leads to the following explicit formula:

$$\mu_{\mathcal{R}_\Gamma^0 \mathrm{GL}(n,\mathbb{C})}(t, u, v) = \prod_{i=1}^m (1 - (t^2 u v)^i) \sum_{\pi \vdash n} \prod_{j=1}^n \frac{(1 - (-tuv)^j)^{\pi_j r}}{\pi_j! \, j^{\pi_j} (1 - (t^2 u v)^j)^{\pi_j}},$$

where $\pi \vdash n$ denotes a partition of n with π_j parts of size j.

Moreover, in the $\mathrm{GL}(n, \mathbb{C})$ case, we can also derive a recursion relation, which completely avoids the determination of partitions or permutations. Since $\mu_{\mathcal{R}_\Gamma^0 \mathrm{GL}(n,\mathbb{C})}$ depends only on tuv and $t^2 uv$, we use the substitutions $x = tuv$, and $w = tx = t^2 uv$.

Proposition 4.8 *Let $G = \mathrm{GL}(n, \mathbb{C})$ and write $\mu_n^r(x, w) := \mu_{\mathcal{R}_\Gamma^0 G}(t, u, v)$ for a nilpotent group Γ, of abelian rank $r \geq 1$. Then, we have the recursion relation:*

$$\mu_n^r(x, w) = \frac{1}{n} \sum_{k=1}^n f((-x)^k, w^k) \, c_k(w) \, \mu_{n-k}^r(x, w), \qquad (4.5)$$

with $f(x, w) := \frac{(1-x)^r}{1-w}$ and $c_k(w) := \prod_{i=0}^{k-1}(1 - w^{n-i})$.

Proof For fixed $r \in \mathbb{N}$, let $\phi_n(z, w)$ be the rational function in variables z, w, defined by:

$$\phi_n(z, w) := \frac{1}{n!} \sum_{g \in S_n} \frac{\det(I - z A_g)^r}{\det(I - w A_g)},$$

with $\phi_0(z, w) \equiv 1$. By [13, Thm 3.1], the generating series for $\phi_n(z, w)$ is a so-called plethystic exponential:

$$1 + \sum_{n \geq 1} \phi_n(z, w) \, y^n = \mathrm{PE}(f(z, w)\, y) := \exp\left(\sum_{k \geq 1} f(z^k, w^k) \frac{y^k}{k} \right)$$

with $f(z, w) = \frac{(1-z)^r}{1-w}$. Differentiating the above identity with respect to y we get:

$$\sum_{n\geq 1} n\phi_n(z, w)\, y^{n-1} = \left(1 + \sum_{m\geq 1} \phi_m(z, w)\, y^m\right)\left(\sum_{k\geq 1} f(z^k, w^k)\, y^{k-1}\right),$$

which, by picking the coefficient of y^n, leads to the recurrence:

$$\phi_n(z, w) = \frac{1}{n}\sum_{k=1}^{n} f(z^k, w^k)\, \phi_{n-k}(z, w). \tag{4.6}$$

To apply this to $\mu_n^r(x, w)$ we use Eq. (4.4) in the form:

$$\phi_n(-x, w) = \frac{\mu_n^r(x, w)}{\prod_{i=1}^{n}(1 - w^i)},$$

so the wanted recurrence follows by replacing $z = -x$ in Eq. (4.6). $\qquad\square$

In the $SL(n, \mathbb{C})$ case, also with Weyl group S_n, the action is the same permutation action, but restricted to the vector subspace of $\mathbb{C}^n$ whose coordinates add up to zero. Hence, the formula for $\det(I - \lambda A_\pi)$ acting on dual of $\mathfrak{sl}_n$ is now:

$$\det(I - \lambda A_\pi) := \frac{1}{1-\lambda}\prod_{j=1}^{n}(1 - \lambda^j)^{\pi_j}, \tag{4.7}$$

for a permutation $\sigma \in S_n$ with σ_j cycles of size j. Recalling that $SL(n, \mathbb{C})$ is a group of rank $n-1$ whose Weyl group has characteristic degrees $2, 3, \ldots, n$, we derive the following formula, reflecting the fact that the $\mathcal{R}_\Gamma^0 GL(n, \mathbb{C})$ and $\mathcal{R}_\Gamma^0 SL(n, \mathbb{C})$ cases only differ by a torus.

Corollary 4.9 *Let $G = SL(n, \mathbb{C})$ and Γ be a finitely generated nilpotent group of abelian rank $r \geq 1$. Then:*

$$\mu_{\mathcal{R}_\Gamma^0 SL(n,\mathbb{C})}(x, w) = \frac{1}{(1+x)^r}\, \mu_{\mathcal{R}_\Gamma^0 GL(n,\mathbb{C})}(x, w). \tag{4.8}$$

Remark 4.10 The recursion formulae in (4.5) and (4.8) have been implemented in a *Mathematica* notebook available on [18].

Example 4.11 From (4.5) and (4.8) we can quickly write down the first few cases for $SL(n, \mathbb{C})$. To obtain $\mu_{\mathcal{R}_\Gamma^0 SL(n,\mathbb{C})}(t, u, v)$ one just needs to substitute $x = tuv$ and

$w = t^2 uv.$

$$\mu_{\mathcal{R}^0_\Gamma \mathrm{SL}(2,\mathbb{C})} = \tfrac{1}{2}\left((1+w)(1+x)^r + (1-w)(1-x)^r\right).$$

$$\mu_{\mathcal{R}^0_\Gamma \mathrm{SL}(3,\mathbb{C})} = \tfrac{1}{6}(1+2w+2w^2+w^3)(1+x)^{2r} + \tfrac{1}{2}(1-w^3)(1-x^2)^r$$
$$+ \tfrac{1}{3}(1-w-w^2+w^3)(1-x+x^2)^r.$$

$$\mu_{\mathcal{R}^0_\Gamma \mathrm{SL}(4,\mathbb{C})} = \tfrac{1}{24}(1+w)(1+w+w^2)(1+w+w^2+w^3)(1+x)^{3r}$$
$$+ \tfrac{1}{4}(1+w+w^2)(1-w^4)(1+x)^r(1-x^2)^r$$
$$+ \tfrac{1}{8}(1-w^3)(1-w+w^2-w^3)(1-x)^r(1-x^2)^{2r}$$
$$+ \tfrac{1}{3}(1-w^2)(1-w^4)(1+x^3)^r$$
$$+ \tfrac{1}{4}(1-w)(1-w^2)(1-w^3)(1+x+x^2+x^3)^r.$$

Putting $x = t$ and $w = t^2$ we recover the expressions for the Poincaré polynomial in [3] and [29].[1] Note that with $x = -1$, $w = 1$ we confirm the vanishing of the Euler characteristic. With $w = -x$ we get formulas for the E-polynomial, and with $x = w = 1$ (that is, $t = u = v = 1$) we get:

$$\mu_{\mathcal{R}^0_\Gamma \mathrm{SL}(n,\mathbb{C})}(1,1,1) = 2^{(n-1)r},$$

the dimension of the total cohomology of T^r, confirming that $H^*(\mathcal{R}^0_\Gamma \mathrm{SL}(n,\mathbb{C}))$ is a regrading of $H^*(T^r)$.

Example 4.12 Consider now the group $G = \mathrm{Sp}(2n,\mathbb{C})$ which has rank n and dimension $n(2n+1)$. Its Weyl group is the so-called *hyperoctahedral group*: the group of symmetries of the hypercube of dimension n, denoted C_n, of order $|C_n| = 2^n n!$. It can be described as the subgroup of permutations of the set $S_{\pm n} := \{-n,\dots,-1,1,\dots,n\}$ satisfying:

$$\sigma \in C_n \subset S_{\pm n} \quad \Longleftrightarrow \quad \sigma(-i) = -\sigma(i) \quad \forall 1 \le i \le n.$$

The action of $g \in C_n$ on the dual of the Lie algebra $\mathfrak{sp}_{2n} \cong \mathbb{C}^n$ is the following natural action. If we denote by $e_1,\dots,e_n$ the standard basis of $\mathbb{C}^n$, and let $e_{-i} := -e_i$, then $g \cdot e_i = e_{\sigma(i)}$, for all $1 \le i \le n$, where $g \in C_n$ corresponds to the permutation $\sigma \in S_{\pm n}$.

Given that $\mathrm{Sp}(2,\mathbb{C}) \cong \mathrm{SL}(2,\mathbb{C})$, we consider the next case: $n = 2$. $\mathrm{Sp}(4,\mathbb{C})$ has complex dimension 10, and its Weyl group is C_2, which is known to be isomorphic to the dihedral group of order 8 (the symmetries of the square):

$$C_2 = \{e, a, a^2, a^3, ba, ba^2, ba^3\},$$

where a acts by counter-clockwise rotation of $\frac{\pi}{2}$ (that is $e_1 \mapsto e_2 \mapsto -e_1 \mapsto -e_2 \mapsto e_1$) and b is the reflection along the first coordinate axis ($e_1 \mapsto e_1$ and $e_2 \mapsto -e_2$).

---

[1] Note that our first term for $n = 3$ corrects the corresponding term in [3, pg. 749].

Then, we have:

$$a = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and simple computations give the following table, with $p_g(\lambda) = \det(I - \lambda A_g)$.

$g \in C_2$	$p_g(\lambda)$
e	$(1 - \lambda)^2$
a, a^3	$1 + \lambda^2$
a^2	$(1 + \lambda)^2$
b, ba, ba^2, ba^3	$1 - \lambda^2$

From this, since the characteristic degrees of C_2 are 2, 4, we compute, using again $x = tuv$ and $w = t^2 uv$:

$$\mu_{\mathcal{R}_\Gamma^0 \mathrm{Sp}(4,\mathbb{C})} = \frac{1}{2^2 2!}(1 - w^2)(1 - w^4) \sum_{g \in C_2} \frac{p_g(-x)^r}{p_g(w)}$$

$$= \tfrac{1}{8}(1 - w^2)(1 - w^4)\left(\frac{(1+x)^{2r}}{(1-w)^2} + 2\frac{(1+x^2)^r}{1+w^2} + \frac{(1-x)^{2r}}{(1+w)^2} + 4\frac{(1-x^2)^r}{1-w^2}\right)$$

$$= \tfrac{1}{8}(1 + w)(1 + w + w^2 + w^3)(1 + x)^{2r} + \tfrac{1}{4}(1 - w^2)^2(1 + x^2)^r$$

$$+ \tfrac{1}{8}(1 - w)(1 - w + w^2 - w^3)(1 - x)^{2r} + \tfrac{1}{2}(1 - w^4)(1 - x^2)^r.$$

Again, we note that with $x = t$ and $w = t^2$ we obtain the Poincaré polynomial. Setting $w = x = 1$ we obtain 2^{2r}, and setting $w = 1 = -x$ we get zero, both as expected. The above formula gives a new result even for $\mathcal{R}_{\mathbb{Z}^2}\mathrm{Sp}(4, \mathbb{C})$, the 12 dimensional variety of pairs of commuting $\mathrm{Sp}(4, \mathbb{C})$ matrices. Indeed, we obtain the following Poincaré polynomial

$$P_{\mathcal{R}_{\mathbb{Z}^2}\mathrm{Sp}(4,\mathbb{C})}(t) = 1 + t^2 + t^4 + 2(t^3 + t^5 + t^6 + t^7 + t^9) + 3t^{10},$$

and E-polynomial $E_{\mathcal{R}_{\mathbb{Z}^2}\mathrm{Sp}(4,\mathbb{C})}(u, v) = (1 - (uv)^2)(1 - (uv)^4)$, as expected from Corollary 4.7

4.4 G-equivariant cohomology of $\mathcal{R}_\Gamma^0(G)$

For a Lie group G, denote G-equivariant cohomology (over $\mathbb{C}$) by H_G. We now resume our main setup: G is a reductive $\mathbb{C}$-group, K is a maximal compact subgroup of G, T is a maximal torus in G and T_K is a compatible maximal torus in K (so $T_K = T \cap K$). Again, let Γ be a finitely generated nilpotent group of abelian rank $r \geq 1$, so the torsion free part of its abelianization is $\mathbb{Z}^r$.

Since G and K are homotopic, as are $\mathcal{R}_\Gamma^0(G)$ and $\mathcal{R}_\Gamma^0(K)$, we conclude there is an isomorphism in equivariant cohomology:

$$H_G^*(\mathcal{R}_\Gamma^0(G)) \cong H_K^*(\mathcal{R}_\Gamma^0(K)).$$

Then, from Baird's thesis [2], precisely pages 39 and 55, and Corollary 7.4.4, the G-equivariant and K-equivariant maps in Diagram (4.3) imply we have the following isomorphisms:

$$
\begin{aligned}
H_K^*(\mathcal{R}_\Gamma^0(K)) &\cong H_K^*(\mathcal{R}_{\mathbb{Z}^r}^0(K)) \\
&\cong H_K^*((K/T_K) \times T_K^r)^W \\
&\cong H_{T_K}^*(T_K^r)^W \\
&\cong [H^*(T_K^r) \otimes H^*(BT_K)]^W \\
&\cong [H^*(T^r) \otimes H^*(BT)]^W.
\end{aligned}
$$

We have already computed the Hilbert series of this latter ring in Eq. (4.1). Thus we conclude:

Corollary 4.13 *There is a MHS on the G-equivariant cohomology of $\mathcal{R}_\Gamma^0(G)$ and the G-equivariant mixed Hodge series is:*

$$\mu_{\mathcal{R}_\Gamma^0(G)}^G = \frac{1}{|W|} \sum_{g \in W} \frac{\det\left(I + tuv\, A_g\right)^r}{\det\left(I - t^2 uv\, A_g\right)}.$$

5 Mixed Hodge structure on $\mathsf{Hom}^0(\Gamma, G) /\!/ G$

Now we prove the statements in Theorems 1.1 and 1.3 on the connected component $\mathcal{M}_\Gamma^0 G$ of the trivial representation of the character variety $\mathcal{M}_\Gamma G = \mathsf{Hom}(\Gamma, G)/\!/G$.

We start with the free abelian case, $\Gamma \cong \mathbb{Z}^r$, noting a number of corollaries to Theorem 3.4.

Corollary 5.1 *$\mathcal{M}_{\mathbb{Z}^r}^0 G$ is irreducible, and there exists a birational bijective morphism*

$$\chi : T^r/W \to \mathcal{M}_{\mathbb{Z}^r}^0 G$$

which is the normalization map. In particular, we have equality of Grothendieck classes: $[T^r/W] = [\mathcal{M}_{\mathbb{Z}^r}^0 G]$.

Proof As noted earlier, $\mathcal{M}_{\mathbb{Z}^r}^T G$ is irreducible, and we have shown that $\mathcal{M}_{\mathbb{Z}^r}^0 G = \mathcal{M}_{\mathbb{Z}^r}^T G$. We also know from [34] that there is a bijective birational morphism $T^r/W \to \mathcal{M}_{\mathbb{Z}^r}^T G$. The first sentence follows since T^r/W is normal (since the GIT quotient of a normal variety is normal). Since χ is a bijective map, the statement on Grothendieck classes follows from [4, Page 115] (see also [20]). $\qquad\square$

We will say that a reductive $\mathbb{C}$-group G is of *classical type* if its derived subgroup DG admits a central isogeny by a product of groups of type $SL(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$, or $SO(n, \mathbb{C})$ for varying n (not necessarily all the same n within the product).

Corollary 5.2 *If G is of classical type, then $\mathcal{M}^0_{\mathbb{Z}^r} G$ is normal and $\chi : T^r / W \to \mathcal{M}^0_{\mathbb{Z}^r} G$ is an isomorphism.*

Proof Given $\mathcal{M}^0_{\mathbb{Z}^r} G = \mathcal{M}^T_{\mathbb{Z}^r} G$ this follows from [19]. Here is a sketch of the result in [19]. Sikora showed the result for $SL(n, \mathbb{C})$, $Sp(2n, \mathbb{C})$, or $SO(n, \mathbb{C})$ in [34]. It is trivially true for tori. In general, $\mathcal{M}^0_{\mathbb{Z}^r}(G \times H) \cong \mathcal{M}^0_{\mathbb{Z}^r} G \times \mathcal{M}^0_{\mathbb{Z}^r} H$ and also $\mathcal{M}^0_{\mathbb{Z}^r}(G/F) \cong (\mathcal{M}^0_{\mathbb{Z}^r} G)/F^r$, for finite central subgroups F. The result then follows from the central isogeny theorem for reductive $\mathbb{C}$-groups and the facts that GIT quotients of normal varieties are normal, and cartesian products of normal varieties are normal. □

Since $\mathcal{M}^0_{\mathbb{Z}^r} G = \mathcal{M}^T_{\mathbb{Z}^r} G$ we know for any $[\rho] \in \mathcal{M}^0_{\mathbb{Z}^r} G$ its image is contained in some maximal torus which we may assume is T. We will say such a representation is *Zariski dense* if its image is Zariski dense in T. We note that every representation in the identity component is *reducible*; that is, its image is contained in a proper parabolic subgroup of G. For many choices of Γ, reducible representations are singular points; see for example [15, 21]. The next corollary is in contrast to this.

Corollary 5.3 *Assume $r \geq 2$, and that $[\rho] \in \mathcal{M}^0_{\mathbb{Z}^r} G$ is Zariski dense. Then*

(1) *$[\rho]$ is a smooth point, and*
(2) *the map $\chi : T^r / W \to \mathcal{M}^0_{\mathbb{Z}^r} G$ is étale at $[\rho]$.*

Proof Since $\mathcal{M}^0_{\mathbb{Z}^r} G = \mathcal{M}^T_{\mathbb{Z}^r} G$, and [34, Theorem 4.1] shows that if $[\rho] \in \mathcal{M}^T_{\mathbb{Z}^r} G$ and is Zariski dense then (1) holds on $\mathcal{M}^T_{\mathbb{Z}^r} G$, (1) is also true for $\mathcal{M}^0_{\mathbb{Z}^r} G$. For (2), [34, Theorem 4.1] shows that the map induces an isomorphism of tangent spaces on the torus component when ρ is Zariski dense; this implies the map is étale at $[\rho]$ by (1). □

Remark 5.4 If $r = 1$, then we have $G /\!\!/ G \cong T/W$ and is smooth if DG is simply-connected by [37] and [7, Proposition 3.1]. The converse is not true however, since $PSL(2, \mathbb{C}) /\!\!/ PSL(2, \mathbb{C}) \cong \mathbb{C}$ is smooth.

The map $\chi : T^r / W \to \mathcal{M}^0_{\mathbb{Z}^r} G$ is the normalization map in general and it is an open question whether or not it is an isomorphism in general [34]. We note that χ is an isomorphism if and only if χ is étale and that holds if and only if $\mathcal{M}^0_{\mathbb{Z}^r} G$ is normal.

Corollary 5.5 *Let G be of classical type. Then the singular locus of $\mathcal{M}^0_{\mathbb{Z}^r} G$ is of of orbifold type; that is, consists only of finite quotient singularities.*

Proof In the case that G is of classical type we know that χ is an isomorphism since $\mathcal{M}^0_{\mathbb{Z}^r} G$ is normal. Thus, the singular locus of $\mathcal{M}^0_{\mathbb{Z}^r} G$ is exactly the singular locus of T^r / W. Since T^r / W is the finite quotient of a manifold, the result follows. □

Remark 5.6 From this point-of-view, we can see easily why the Zariski dense representations are smooth. The Zariski dense representations are tuples $(t_1, ..., t_r)$ that

generate a Zariski dense subgroup of T (most t_i's do this by themselves). If $w \cdot \rho = \rho$ then $w \cdot \rho(\gamma) = \rho(\gamma)$ for all γ. Since ρ is Zariski dense we conclude that $w \cdot t = t$ for all $t \in T$. We conclude $w = 1$ and so W acts freely on the set of Zariski dense representations. This shows they are smooth points and the singular locus is contained in the non-Zariski dense representations.

Remark 5.7 Assume $r \geq 2$. If ρ is not Zariski dense, then the identity component of $A := \overline{\rho(\mathbb{Z}^r)}$ is, up to conjugation, a proper subtorus of T. It seems reasonable to suppose that A is contained in the fixed locus of a non-trivial $w \in W$. The fixed loci $(T^r)^w$ for $w \neq 1$ are of codimension greater than 1 since $r \geq 2$ and $(T^w)^0$ is a proper subtorus. So, in light of the Shephard-Todd Theorem [33], it appears likely that the non-Zariski dense representations are exactly the singular locus (for $r \geq 2$).

5.1 The MHS on $\mathcal{M}_\Gamma^0 G$

Given an isomorphism of groups $\varphi : \Gamma_1 \to \Gamma_2$ there exists a (contravariant) biregular morphism $\varphi^* : \mathcal{M}_{\Gamma_2} G \to \mathcal{M}_{\Gamma_1} G$ given by $\varphi^*([\rho]) = [\rho \circ \varphi]$ with inverse $(\varphi^{-1})^*$. Consequently, the topology of $\mathcal{M}_\Gamma G$ and its mixed Hodge structure (MHS) are independent of the presentation of Γ. Hence, the same holds for $\mathcal{M}_\Gamma^0 G$, for any Γ.

Let us start with the free abelian case, $\Gamma \cong \mathbb{Z}^r$, where we know that $\mathcal{M}_{\mathbb{Z}^r}^0 G = \mathcal{M}_{\mathbb{Z}^r}^T G$.

Theorem 5.8 *Let G be a reductive $\mathbb{C}$-group, T a maximal torus, and W the Weyl group. Then, the MHS of $\mathcal{M}_{\mathbb{Z}^r}^0 G$ coincides with the one of T^r/W and its mixed Hodge polynomial is given by:*

$$\mu_{\mathcal{M}_{\mathbb{Z}^r}^0 G}(t, u, v) = \frac{1}{|W|} \sum_{g \in W} \left[\det \left(I + tuv \, A_g \right) \right]^r, \qquad (4.1)$$

where A_g is the automorphism induced on $H^1(T, \mathbb{C})$ by $g \in W$, and I is the identity automorphism.

Proof We have the following commutative diagram with vertical arrows being strong deformation retractions from [14]:

$$
\begin{array}{ccc}
T_K^r/W & \overset{\cong}{\longrightarrow} & \mathcal{N}_{\mathbb{Z}^r}^0 K \\
\big\downarrow & & \big\downarrow \\
T^r/W & \overset{\chi}{\longrightarrow} & \mathcal{M}_{\mathbb{Z}^r}^0 G.
\end{array}
$$

Thus, χ induces isomorphisms in cohomology and since it is an algebraic map, these isomorphisms preserve mixed Hodge structures. Thus, the MHS on T^r/W and on $\mathcal{M}_{\mathbb{Z}^r}^0 G$ coincide. The formula then follows immediately from [19]. $\qquad \square$

Theorem 5.9 *Let Γ be a finitely generated nilpotent group of abelian rank $r \geq 1$. The MHS on $\mathcal{M}_\Gamma^0 G$ coincides with the MHS on T^r/W.*

Proof This follows from [9, Corollary 1.4], where they prove isomorphisms in cohomology given by algebraic maps. □

Corollary 5.10 *Let Γ be a finitely generated nilpotent group of abelian rank $r \geq 1$. Then, for all reductive $\mathbb{C}$-groups G we have:*

$$\mu_{\mathcal{M}_\Gamma^0 G}(t, u, v) = \frac{1}{|W|} \sum_{g \in W} \left[\det \left(I + tuv\, A_g \right) \right]^r.$$

Proof This follows directly from Theorems 5.8 and 5.9. □

Note that the G-equivariant cohomology of the moduli space $\mathcal{M}_\Gamma^0(G)$ is the usual cohomology since the G-action is trivial on $\mathcal{M}_\Gamma^0(G)$.

Corollary 5.11 *The compactly supported mixed Hodge polynomial of $\mathcal{M}_{\mathbb{Z}^r}^0(G)$ is:*

$$\mu^c_{\mathcal{M}_{\mathbb{Z}^r}^0(G)}(t, u, v) = \frac{t^{r \dim T}}{|W|} \sum_{g \in W} \left[\det(tuv I + A_g) \right]^r.$$

Proof If a variety X of (complex) dimension d satisfies Poincaré duality, then μ_X and μ_X^c are related by: $\mu_X^c(t, u, v) = (t^2 uv)^d \mu_X(t^{-1}, u^{-1}, v^{-1})$; see [19, Remark 3.10(1)]. From Theorem 5.8, the mixed Hodge structures and polynomials of $\mathcal{M}_{\mathbb{Z}^r}^0(G)$ and T^r/W coincide. Since T^r/W is an orbifold, it satisfies Poincaré duality for MHSs (see [19, Section 4.2]). Hence, using $d = \dim T$, we have:

$$\begin{aligned}
\mu^c_{T^r/W}(t, u, v) &= (t^2 uv)^{rd} \mu_{T^r/W}(t^{-1}, u^{-1}, v^{-1}) \\
&= (t^2 uv)^{rd} \frac{1}{|W|} \sum_{g \in W} \left[\det \left(I + \frac{1}{tuv} A_g \right) \right]^r \\
&= \frac{t^{rd}}{|W|} \sum_{g \in W} \left[(tuv)^d \det \left(I + \frac{1}{tuv} A_g \right) \right]^r
\end{aligned}$$

as claimed, since A_g are automorphisms of the Lie algebra of T. □

5.2 Examples for classical groups

As in the case of representation varieties, the character varieties for $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{C})$ also allow closed expressions in terms of partitions π of n. In [19, Thm 5.13], it was shown that $\mathcal{M}_{\mathbb{Z}^r} \mathrm{GL}(n, \mathbb{C})$ has round cohomology, so that $x = tuv$ is the only relevant variable, and that:

$$\mu_{\mathcal{M}_{\mathbb{Z}^r} \mathrm{GL}(n, \mathbb{C})}(x) = \mu_{\mathcal{M}_{\mathbb{Z}^r} \mathrm{SL}(n, \mathbb{C})}(x)\, (1 + x)^r.$$

From the present analysis, the same formulas work also for the identity components of the character varieties of any nilpotent group Γ with abelianization $\mathbb{Z}^r$. Moreover, we can also obtain a recurrence relation as follows.

Proposition 5.12 *Let $G = \mathrm{GL}(n, \mathbb{C})$ and write $v_n^r(x) := \mu_{\mathcal{M}_\Gamma^0 G}(t, u, v)$ for a nilpotent group Γ, of abelian rank r. Then, with $h(x) := (1 - x)^r$, we have:*

$$v_n^r(x) = \frac{1}{n} \sum_{k=1}^{n} h((-x)^k) \, v_{n-k}^r(x). \tag{4.2}$$

Proof As in Proposition 4.8, define $\psi_n(z)$ to be the rational function of z:

$$\psi_n(z) := \frac{1}{n!} \sum_{g \in S_n} \det \left(I - z \, A_g \right)^r = v_n^r(-z),$$

with $\psi_0(z) \equiv 1$. By [13, Thm 3.1], the generating series for the $\psi_n(z)$ is now the plethystic exponential $1 + \sum_{n \geq 1} \psi_n(z) \, y^n = \mathrm{PE}(h(z) \, y)$ with $h(z) = (1 - z)^r$. As before, the derivative with respect to y now gives:

$$\sum_{n \geq 1} n \psi_n(z) \, y^{n-1} = \left(1 + \sum_{m \geq 1} \psi_m(z) \, y^m \right) \left(\sum_{k \geq 1} h(z^k) \, y^{k-1} \right),$$

which, by picking the coefficient of y^n, leads to the recurrence:

$$\psi_n(z) = \frac{1}{n} \sum_{k=1}^{n} h(z^k) \, \psi_{n-k}(z), \tag{4.3}$$

so the proposition follows by replacing $z = -x$ in Eq. (4.3). $\qquad\square$

5.3 Point count over finite fields and compactly supported *E*-polynomials

In this subsection, we show that our formulae for mixed Hodge polynomials also compute the number of points of the identity component of character varieties of free abelian groups over finite fields.

Let X be a separated scheme of finite type over $\mathbb{Z}$. We say that X is *polynomial count*, with counting polynomial $\mathcal{P}_X(x) \in \mathbb{Z}[x]$ if for all but finitely many primes p, and finite fields $\mathbb{F}_q$ with $q = p^k$, its number of $\mathbb{F}_q$-points is given by $\#X(\mathbb{F}_q) = \mathcal{P}_X(q)$. By extension of scalars, we can consider the varieties $X(\mathbb{C})$ and $X(\overline{\mathbb{F}}_q)$, respectively, over $\mathbb{C}$ and over the algebraic closure $\overline{\mathbb{F}}_q$.

Next, consider the k-th compactly supported l-adic cohomology of $X(\overline{\mathbb{F}}_q)$, denoted $H_c^k(X(\overline{\mathbb{F}}_q), \overline{\mathbb{Q}}_l)$ for a prime l with $\gcd(l, q) = 1$, and the Frobenius morphism

$$F : X(\overline{\mathbb{F}}_q) \to X(\overline{\mathbb{F}}_q),$$

whose fixed points are precisely the $\mathbb{F}_q$ points of X.

Following Dimca-Lehrer, we say that X is *minimally pure* if X is irreducible of dimension n, and F acts on $H_c^k(X(\overline{\mathbb{F}}_q), \overline{\mathbb{Q}}_l)$ with all eigenvalues equal to q^{k-n} (see [12, Definition 5.1]) (This notion is the analogue of round, for X smooth over $\mathbb{C}$).

Now consider the $\mathbb{Z}$-scheme $X = Spec(\mathbb{Z}[x_i, y_i]/(x_i y_i - 1))$, with $2n$ variables x_i and y_i, whose complex variety is a torus $X_{\mathbb{C}} = (\mathbb{C}^*)^n$. According to [12, Thm. 5.4], X is minimally pure.

Theorem 5.13 *Fix $r \in \mathbb{N}$, and a prime power q, and let $T_{\mathbb{Z}}$ be a $\mathbb{Z}$-scheme such that $T_{\mathbb{C}}$ is a maximal torus of a reductive $\mathbb{C}$-group with Weyl group W. Then, the quotient of $T_{\mathbb{Z}}^r$ by the diagonal action of W, denoted $T_{\mathbb{Z}}^r/W$, is polynomial count and we have:*

$$\#(T_{\mathbb{Z}}^r/W)(\mathbb{F}_q) = \frac{1}{|W|} \sum_{g \in W} \left[\det\left(qI - A_g\right) \right]^r.$$

Proof We apply the following general result. If H is a finite group acting on a $\mathbb{Z}$-scheme Y, we have:

$$H_c^k((Y/H)(\overline{\mathbb{F}}_q), \overline{\mathbb{Q}}_l) \cong H_c^k(Y(\overline{\mathbb{F}}_q), \overline{\mathbb{Q}}_l)^H,$$

as in [12, proof of Prop. 5.5]. Now let $Y = T_{\mathbb{Z}}^r$ and $H = W$. Since $T_{\mathbb{Z}}^r$ is minimally pure, the Frobenius morphism acts on $V_q^k := H_c^k(T_{\mathbb{Z}}^r(\overline{\mathbb{F}}_q), \overline{\mathbb{Q}}_l)^W$ with all eigenvalues equal to q^{k-d}, $d = \dim T_{\mathbb{Z}}^r$.

Since $(T_{\mathbb{Z}}^r/W)(\mathbb{F}_q)$ consists precisely of the Frobenius fixed points of $(T_{\mathbb{Z}}^r/W)(\overline{\mathbb{F}}_q)$, we apply Grothendieck's fixed point formula (see for example [12, Equation (5.3.1)]), to obtain:

$$\#(T_{\mathbb{Z}}^r/W)(\mathbb{F}_q) = \sum_{k=0}^{2d} (-1)^k \mathrm{Tr}(F, V_q^k).$$

This is a polynomial in q since $\mathrm{Tr}(F, V_q^k)$, the trace of F on V_q^k, is a sum of powers of q. Hence, by Katz's theorem [22] and [32], the compactly supported E-polynomial of $(T_{\mathbb{Z}}^r/W)_{\mathbb{C}} = T_{\mathbb{C}}^r/W$ coincides with the counting polynomial, with $q = uv$. The required formula then comes from the compactly supported mixed Hodge polynomial in Corollary 5.11, setting $t = -1$ and $q = uv$. $\qquad\square$

Corollary 5.14 *The counting polynomial of $\mathcal{M}_{\mathbb{Z}^r}^0(G)$ is*

$$\mathcal{P}_{\mathcal{M}_{\mathbb{Z}^r}^0(G)}(x) = \frac{1}{|W|} \sum_{g \in W} \left[\det(xI - A_g) \right]^r.$$

Proof By Corollary 5.1, $\mathcal{M}_{\mathbb{Z}^r}^0(G)$ is bijective to T^r/W via the normalization map χ. Moreover, χ is induced by the inclusion $T^r \hookrightarrow \mathcal{R}_{\mathbb{Z}^r}^0(G)$ and so is defined over the ring of integers (with the possible inversion of a finite number of primes). Thus the counting functions of these two varieties are equal and so the result follows from Theorem 5.13. $\qquad\square$

Remark 5.15 The smooth model for $\mathcal{R}^0_{\mathbb{Z}^r}(G)$, namely $(G/T) \times_W T^r$, is also polynomial count since W acts freely on $(G/T) \times T^r$, the product of polynomial count varieties is polynomial count, and T^r and G/T are polynomial count by [8]. However, this does not show $\mathcal{R}^0_{\mathbb{Z}^r}(G)$ is polynomial count since the map relating $\mathcal{R}^0_{\mathbb{Z}^r}(G)$ and $(G/T) \times_W T^r$, although a cohomological isomorphism, is neither injective nor surjective.

6 Exotic components

In this last section we describe the MHS on the full character varieties of $\mathbb{Z}^r$ (not only the identity component) in special cases described in [1]. Let p be a prime integer, and $\mathbb{Z}_p$ be the cyclic group or order p. We will use the same notation for center of $\mathrm{SU}(p)$ which is realized as the subgroup of scalar matrices with values p-th roots of unity. Let $\Delta(p)$ be the diagonal of $\mathbb{Z}_p$ in $(\mathbb{Z}_p)^m = Z(\mathrm{SU}(p)^m)$. Let $K_{m,p} := \mathrm{SU}(p)^m/\Delta(p)$. For example, $K_{1,p} = \mathrm{PU}(p)$, the projective unitary group.

In [1], all the components in $\mathsf{Hom}(\mathbb{Z}^r, K_{m,p})$ and $\mathcal{M}_{\mathbb{Z}^r} K_{m,p}$ are described. In particular, $\mathcal{M}_{\mathbb{Z}^r} K_{m,p}$ consists of the identity component

$$\mathcal{M}^0_{\mathbb{Z}^r} K_{m,p} \cong (S^1)^{(p-1)rm}/(S_p)^m$$

where S_p is the symmetric group on p letters, and

$$N(p,m) := \frac{p^{(m-1)(r-2)}(p^r - 1)(p^{r-1} - 1)}{p^2 - 1}$$

discrete points.

There is a one-to-one correspondence between the isolated points in $\mathcal{M}_{\mathbb{Z}^r} K_{m,p}$ and non-identity path-components in $\mathsf{Hom}(\mathbb{Z}^r, K_{m,p})$. Each such path-component is isomorphic to the homogeneous space $\mathrm{SU}(p)^m/(\mathbb{Z}_p^{m-1} \times E_p)$ where $E_p \subset \mathrm{SU}(p)$ is isomorphic to the quaternion group Q_8 if p is even and the group of triangular 3×3 matrices over the $\mathbb{Z}_p$, with 1's on the diagonal when p is odd (either way it is "extra-special" of order p^3).

Corollary 6.1 *Let $G_{m,p}$ be the complexification of $K_{m,p}$: $G_{m,p} \cong \mathrm{SL}(p, \mathbb{C})^m/\Delta(p)$. Then: $\mu_{\mathcal{R}_{\mathbb{Z}^r} G_{m,p}}(t, u, v) =$*

$$= \left(\frac{1}{p!} \prod_{i=2}^p (1 - (t^2 u v)^i) \sum_{g \in S_p} \frac{\det(I + t u v A_g)^r}{\det(I - t^2 u v A_g)} \right)^m + N(p,m) \prod_{j=2}^p \left(1 + t^{2j-1} u^j v^j \right)^m.$$

Proof The Weyl group W of $G_{m,p}$ is a direct product of m copies of S_p: the Weyl group of $\mathrm{SL}(p, \mathbb{C})$, and its action on the (dual of the) Lie algebra of maximal torus, is the product action. Therefore, using again $x = t u v$ and $w = t^2 u v$, for the identity

component $\mathcal{R}^0_{\mathbb{Z}^r} G_{m,p}$ we have:

$$\mu_{\mathcal{R}^0_{\mathbb{Z}^r} G_{m,p}} (t, u, v)$$

$$= \left(\frac{1}{p!} \prod_{i=2}^{p} (1 - w^i) \right)^m \sum_{(g_1, \cdots, g_m) \in W} \frac{\det(I_p + x A_{g_1})^r \cdots \det(I_p + x A_{g_m})^r}{\det(I_p - w A_{g_1}) \cdots \det(I - w A_{g_m})}$$

$$= \left(\frac{1}{p!} \prod_{i=2}^{p} (1 - w^i) \right)^m \left(\sum_{g \in S_p} \frac{\det(I + x A_g)^r}{\det(I - w A_g)} \right)^m.$$

Now, since the MHS on $\mathrm{SL}(p, \mathbb{C})^m / (\mathbb{Z}_p^{m-1} \times E_p)$ coincides with that of $\mathrm{SL}(p, \mathbb{C})^m$ by Lemma 2.1, each of the $N(p, m)$ components, other than $\mathcal{R}^0_{\mathbb{Z}^r} G_{m,p}$, contributes $\mu(\mathrm{SL}(p, \mathbb{C})^m) = \mu(\mathrm{SL}(p, \mathbb{C}))^m = \prod_{j=2}^{p} \left(1 + t^{2j-1} u^j v^j \right)^m$. $\qquad\square$

For the character variety, from the fact that each isolated point adds a constant 1, we have the following corollary:

Corollary 6.2 *The mixed Hodge polynomial of $\mathcal{M}_{\mathbb{Z}^r} G_{m,p}$ is:*

$$\mu_{\mathcal{M}_{\mathbb{Z}^r} G_{m,p}} (t, u, v) = \left(\frac{1}{p!} \sum_{g \in S_p} \det \left(I + tuv A_g \right)^r \right)^m + N(p, m).$$

Proof The same argument of Corollary 6.1, implies that the identity component verifies: $\mu_{\mathcal{M}^0_{\mathbb{Z}^r} G_{m,p}} = (\mu_{\mathcal{M}_{\mathbb{Z}^r} \mathrm{SL}(p, \mathbb{C})})^m$, so the formula is clear. $\qquad\square$

Remark 6.3 Given two reductive groups G and H, both the $(G \times H)$-representation varieties and the $(G \times H)$-character varieties are cartesian products of the G and H varieties:

$$\mathcal{R}_\Gamma (G \times H) = \mathcal{R}_\Gamma G \times \mathcal{R}_\Gamma H, \qquad \mathcal{M}_\Gamma (G \times H) = \mathcal{M}_\Gamma G \times \mathcal{M}_\Gamma H.$$

From Corollaries 6.1 and 6.2, the mixed Hodge polynomial of the identity components of these $G_{m,p}$-character varieties behaves multiplicatively, even though $\mathcal{R}^0_{\mathbb{Z}^r} G_{m,p}$ and $\mathcal{M}^0_{\mathbb{Z}^r} G_{m,p}$ are not cartesian products.

Remark 6.4 Since G is connected, $\mathcal{R}^0_{\mathbb{Z}^r} G = \mathcal{R}_{\mathbb{Z}^r} G$ if and only if $\mathcal{M}^0_{\mathbb{Z}^r} G = \mathcal{M}_{\mathbb{Z}^r} G$. And by [14], $\mathcal{M}^0_{\mathbb{Z}^r} G = \mathcal{M}_{\mathbb{Z}^r} G$ if and only if (a) $r = 1$, or (b) $r = 2$ and G is simply-connected, or (c) $r \geq 3$ and G is a product of $\mathrm{SL}(n, \mathbb{C})$'s and $\mathrm{Sp}(n, \mathbb{C})$'s.

Declarations

References

1. Adem, A., Cohen, F.R., Gómez, J.: Commuting elements in central products of special unitary groups. Proc. Edinb. Math. Soc. (2) **56**(1), 1–12 (2013)
2. Baird, T.J.: The moduli space of flat G-bundles over a nonorientable surface. Ph.D. Thesis-University of Toronto, Canada, p 122 (2008). ISBN: 978-0494-39857-9, ProQuest LLC
3. Baird, T.J.: Cohomology of the space of commuting n-tuples in a compact Lie group. Algebraic Geometric Topol. **7**, 737–754 (2007)
4. Behrend, K., Bryan, J., Szendröi, B.: Motivic degree zero Donaldson–Thomas invariants. Invent. Math. **192**(1), 111–160 (2013)
5. Bergeron, M.: The topology of nilpotent representations in reductive groups and their maximal compact subgroups. Geom. Topol. **19**(3), 1383–1407 (2015)
6. Biswas, I., Florentino, C., Lawton, S., Logares, M.: The topology of parabolic character varieties of free groups. Geom. Dedicata **168**, 143–159 (2014)
7. Borel, A.: Sous-groupes commutatifs et torsion des groupes de Lie compacts connexes. Tohoku Math. J. (2) **13**, 216–240 (1961)
8. Brion, M., Peyre, E.: Counting points of homogeneous varieties over finite fields. J. Reine Angew. Math. **645**, 105–124 (2010)
9. Bergeron, M., Silberman, L.: A note on nilpotent representations. J. Group Theory **19**(1), 125–135 (2016)
10. Deligne, P.: Théorie de Hodge II. Publ. Math. I.H.E.S. **40**, 5–55 (1971)
11. Deligne, P.: Théorie de Hodge III. Publ. Math. I.H.E.S. **44**, 5–77 (1974)
12. Dimca, A., Lehrer, G.I.: Purity and equivariant weight polynomials. Algebraic Groups Lie Groups **9**, 161–181 (1997)
13. Florentino, C.: Plethystic exponential calculus and characteristic polynomials of permutations. Discrete Math. Lett. **8**, 22–29 (2022)
14. Florentino, C., Lawton, S.: Topology of character varieties of Abelian groups. Topol. Appl. **173**, 32–58 (2014)
15. Florentino, C., Lawton, S.: Singularities of free group character varieties. Pacific J. Math. **260**(1), 149–179 (2012)
16. Florentino, C., Lawton, S.: Flawed groups and the topology of character varieties. Topol. Appl. **341**, Paper No. 108756, 30 pp (2024)
17. Florentino, C., Lawton, S., Ramras, D.: Homotopy groups of free group character varieties. Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **17**(1), 143–185 (2017)
18. Florentino, C., Lawton, S.: MHS of Nilpotent GL(n, mathbbC) & SL(n, mathbbC)-Representation & Character Varieties. GitHub repository (2021). https://github.com/seanlawton/Mixed-Hodge-structures-on-character-varieties-of-nilpotent-groups
19. Florentino, C., Silva, J.: Hodge–Deligne polynomials of character varieties of free abelian groups. Open Math. **19**(1), 338–362 (2021)
20. González-Prieto, Á.: Pseudo-quotients of algebraic actions and their application to character varieties. Commun. Contemp. Math. (2023)

21. Guérin, C., Lawton, S., Ramras, D.: Bad representations and homotopy of character varieties. Ann. H. Lebesgue **5**, 93–140 (2022)
22. Hausel, T., Rodriguez-Villegas, F.: Mixed Hodge polynomials of character varieties. Invent. Math. **174**(3), 555–624 (2008). With an appendix by Nicholas M. Katz
23. Hirsch, K.A.: On infinite soluble groups, III. Proc. Lond. Math. Soc. **49**, 184–194 (1946)
24. Kishimoto, D., Takeda, M.: Spaces of commuting elements in the classical groups. Adv. Math. **386**, Paper No. 107809, 43 pp (2021)
25. Lawton, S., Ramras, D.: Covering spaces of character varieties. With an appendix by Nan-Kuo Ho and Chiu-Chu Melissa Liu. New York J. Math. **21**, 383–416 (2015)
26. Logares, M., Muñoz, V., Newstead, P.E.: Hodge polynomials of SL(2, ℂ)-character varieties for curves of small genus. Rev. Mat. Complut. **26**(2), 635–703 (2013)
27. Peters, C., Steenbrink, J.: Mixed Hodge Structures. Springer, Berlin (2008)
28. Pettet, A., Souto, J.: Commuting tuples in reductive groups and their maximal compact subgroups. Geom. Topol. **17**(5), 2513–2593 (2013)
29. Ramras, D., Stafa, M.: Hilbert-Poincaré series for spaces of commuting elements in Lie groups. Math. Z. **292**(1–2), 591–610 (2019)
30. Reeder, M.: On the cohomology of compact Lie groups. Enseign. Math. (2) **41**, 181–200 (1995)
31. Serre, J.P.: Linear Representations of Finite Groups, vol. 42. Springer, Berlin (2012)
32. Seshadri, C.S.: Geometric reductivity over arbitrary base. Adv. Math. **26**(3), 225–274 (1977)
33. Shephard, G.C., Todd, J.A.: Finite unitary reflection groups. Can. J. Math. **6**, 274–304 (1954)
34. Sikora, A.S.: Character varieties of abelian groups. Math. Z. **277**, 241–256 (2014)
35. Stafa, M.: Poincaré Series of Character Varieties for Nilpotent Groups. J. Group Theory **22**(3), 419–440 (2019)
36. Stanley, R.P.: Enumerative Combinatorics, vol. 1, 2nd edn. Cambridge University Press, Cambridge (2015)
37. Steinberg, R.: Regular elements of semisimple algebraic groups. Inst. Hautes Études Sci. Publ. Math. **25**, 49–80 (1965)
38. Voisin, C.: Hodge Theory and Complex Algebraic Geometry I, Cambridge Studies in Advanced Mathematics, vol. 76. Cambridge University Press, Cambridge (2002)