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Equivalent characterizations of martingale Hardy–Lorentz spaces with variable exponents

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Abstract
We prove that under the log-Hölder continuity condition of the variable exponent $p(\cdot)$, a new type of maximal operators, $U_{\gamma,s}$ is bounded from the variable martingale Hardy–Lorentz space $H_{p(\cdot),q}$ to $L_{p(\cdot),q}$, whenever $0 < p_- \le p_+ < \infty$, $0 < q \le \infty$, $0 < \gamma, s < \infty$ and $1/p_- - 1/p_+ < \gamma + s$. Moreover, the operator $U_{\gamma,s}$ generates equivalent quasi-norms on the Hardy–Lorentz spaces $H_{p(\cdot),q}$.

Keywords Variable exponent · Variable Hardy spaces · Variable Hardy–Lorentz spaces · Atomic decomposition · Maximal operators

Mathematics Subject Classification Primary 60G42; Secondary 42B25 · 42B30 · 46E30

1 Introduction

A measurable function $p(\cdot) : [0, 1) \to (0, \infty]$ is called a variable exponent. In this paper we suppose that

$$0 < p_- := \operatorname{ess} \inf_{x \in [0,1)} p(x) \le p_+ := \operatorname{ess} \sup_{x \in [0,1)} p(x) < \infty.$$

Variable Lebesgue spaces $L_{p(\cdot)}$ are investigated very intensively in the literature nowadays (see e.g. Cruz-Uribe and Fiorenza [5], Diening et al. [6], Kokilashvili et al. [15, 16], Nakai and Sawano [19, 25], Kempka and Vybíral [14], Jiao et al. [11–13], Yan et al. [36], Liu et al. [17, 18]). Interest in the variable Lebesgue spaces has increased since the 1990 s because of their use in a variety of applications (see the references in Jiao et al. [11]).

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As usual in this theory, we also suppose that $p(\cdot)$ satisfy the log-Hölder continuity condition, namely $p(\cdot) \in C^{\log}$. One of the most important results states that the classical Hardy-Littlewood maximal operator is bounded on the variable $L_{p(\cdot)}$ spaces under this condition (see for example Cruz-Uribe et al. [2], Nekvinda [20], Cruz-Uribe and Fiorenza [5] and Diening et al. [6]).

Nakai and Sawano [19] first introduced the variable Hardy spaces $H_{p(\cdot)}(\mathbb{R})$. Independently, Cruz-Uribe and Wang [4] also investigated the spaces $H_{p(\cdot)}(\mathbb{R})$. Cruz-Uribe et al. [3] proved the boundedness of fractional and singular integral operators on weighted and variable Hardy spaces. Sawano [25] improved the results in [19]. Ho [10] studied weighted Hardy spaces with variable exponents. Yan et al. [36] introduced the variable weak Hardy space $H_{p(\cdot),\infty}(\mathbb{R})$ and characterized these spaces via radial maximal functions. The Hardy–Lorentz spaces $H_{p(\cdot),q}(\mathbb{R})$ were investigated by Jiao et al. in [13]. Similar results for the anisotropic Hardy spaces $H_{p(\cdot)}(\mathbb{R})$ and $H_{p(\cdot),q}(\mathbb{R})$ can be found in Liu et al. [17, 18]. Martingale Musielak–Orlicz Hardy spaces were investigated in Xie et al. [33–35]. Recently, we [11] generalized these results for martingale Hardy spaces with variable exponent.

In this paper, we investigate the so called Vilenkin martingales defined as follows. Let $(p_n, n \in \mathbb{N})$ be a bounded sequence of natural numbers with entries at least 2. Introduce the notations $P_0 = 1$ and

$$P_{n+1} := \prod_{k=0}^{n} p_k \qquad (n \in \mathbb{N}).$$

We denote the set of natural numbers $\{0, 1, \ldots,\}$ by $\mathbb{N}$. By a Vilenkin interval, we mean one of the form $[kP_n^{-1}, (k+1)P_n^{-1})$ for some $k, n \in \mathbb{N}$, $0 \le k < P_n$, $k \in \mathbb{N}$. Let $\mathcal{F}_n$ be the σ-algebra

$$\mathcal{F}_n = \sigma\{[kP_n^{-1}, (k+1)P_n^{-1}) : 0 \le k < P_n, k \in \mathbb{N}\} \tag{1}$$

generated by the Vilenkin intervals. Martingales with respect to $(\mathcal{F}_n, n \in \mathbb{N})$ are called Vilenkin martingales. Vilenkin martingales were studied in a great number of papers, such as Gát and Goginava [7–9], Persson and Tephnadze [21–24] and Simon [26, 27].

For a fixed $x \in [0, 1)$ and $n \in \mathbb{N}$, let us denote the unique Vilenkin interval $[kP_n^{-1}, (k+1)P_n^{-1})$ which contains x by $I_n(x)$. Then the Doob maximal operator for Vilenkin martingales $f = (f_n, n \in \mathbb{N})$ can be rewritten as

$$M(f)(x) = \sup_{n\in\mathbb{N}} \frac{1}{\lambda(I_n(x))} \left| \int_{I_n(x)} f_n \, d\lambda \right|.$$

The boundedness of the Doob martingale maximal operator on the $L_{p(\cdot)}$ spaces was proved in Jiao et al. [11, 12]:

Theorem 1 *Suppose that* $p(\cdot) \in C^{\log}$ *and* $f \in L_{p(\cdot)}$. *If* $1 < p_- \le p_+ < \infty$, *then*

$$\|M(f)\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}. \tag{2}$$

If $1 \leq p_- \leq p_+ < \infty$, *then*

$$\sup_{\rho>0} \left\| \rho \chi_{\{M(f)>\rho\}} \right\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}. \tag{3}$$

Later we extended this result to $p_+ = \infty$ in [30]. In this paper the constants C are absolute constants and the constants $C_{p(\cdot)}$ are depending only on $p(\cdot)$ and may denote different constants in different contexts. For two positive numbers A and B, we use also the notation $A \lesssim B$, which means that there exists a constant C such that $A \leq CB$.

In [11, 29, 31, 32], we generalized the Doob maximal operator and introduced the operator $U_{\gamma,s}$ for Vilenkin martingales, where γ and s are two positive constants. These operators were the key point in the proof of the boundedness of the maximal Fejér operator of the Walsh- and Vilenkin-Fourier series from the variable Hardy space $H_{p(\cdot)}$ to $L_{p(\cdot)}$ (see [11, 31]). Recall the definition of $U_{\gamma,s}$. For a Vilenkin interval I with length P_n^{-1}, $i, j, n \in \mathbb{N}$, $l = 0, \ldots, p_j - 1$, let us use the notation

$$I^{l,j,i} := I \dot{+} [0, P_i^{-1}) \dot{+} l P_{j+1}^{-1}$$

for the translation of I, where $\dot{+}$ denotes the Vilenkin addition (see Sect. 3). Parallel, we denote $I_n(x)^{l,j,i} := (I_n(x))^{l,j,i}$. For a Vilenkin martingale $f = (f_n, n \in \mathbb{N})$ and $0 < \gamma, s < \infty$, let

$$U_{\gamma,s}(f)(x) := \sup_{n\in\mathbb{N}} \sum_{m=0}^{n} \sum_{j=0}^{m} \left(\frac{P_j}{P_n}\right)^{\gamma} \sum_{i=j}^{m} \left(\frac{P_j}{P_i}\right)^{s} \sum_{l=0}^{p_j-1} \frac{1}{\lambda(I_n(x)^{l,j,i})} \left| \int_{I_n(x)^{l,j,i}} f_n d\lambda \right|. \tag{4}$$

We will see later that $M(f) \leq U_{\gamma,s}(f)$ for all $0 < \gamma, s < \infty$. So the next theorem proved in [31, 32], generalizes (2).

Theorem 2 *Let* $p(\cdot) \in C^{\log}$, $1 < p_- \leq p_+ < \infty$ *and* $0 < \gamma, s < \infty$. *If*

$$\frac{1}{p_-} - \frac{1}{p_+} < \gamma + s, \tag{5}$$

then

$$\left\| U_{\gamma,s}(f) \right\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)} \qquad (f \in L_{p(\cdot)}).$$

Obviously, inequality (5) and Theorem 2 hold if $p_- > \max(1/(\gamma + s), 1)$. We proved in [31] that condition (5) is also necessary, the results are not true without this condition.

In [29, 32], we generalized Theorem 2 and, under the same conditions, we obtained also the boundedness of $U_{\gamma,s}$ from the martingale Hardy space $H_{p(\cdot)}$ to $L_{p(\cdot)}$ for $0 < p_- \leq p_+ < \infty$. In this paper, we generalize these results to variable Lorentz and Hardy–Lorentz spaces. We will prove that $U_{\gamma,s}$ is bounded from the martingale Hardy–Lorentz space $H_{p(\cdot),q}$ to $L_{p(\cdot),q}$, where $0 < q \leq \infty$. More exactly, we have

Theorem 3 *Let* $p(\cdot) \in C^{\log}$, $0 < p_- \le p_+ < \infty$, $0 < q \le \infty$ *and* $0 < \gamma, s < \infty$. *If* (5) *holds, then*

$$\|U_{\gamma,s}(f)\|_{p(\cdot),q} \lesssim \|f\|_{H_{p(\cdot),q}} \qquad (f \in H_{p(\cdot),q}).$$

As a corollary, we get $U_{\gamma,s}$ is bounded from the Lorentz space $L_{p(\cdot),q}$ to $L_{p(\cdot),q}$ and we generalize (3).

Corollary 1 *Let* $p(\cdot) \in C^{\log}$ *satisfy* (5), $0 < \gamma, s < \infty$. *If* $1 < p_- \le p_+ < \infty$, $0 < q \le \infty$ *and* $f \in L_{p(\cdot),q}$, *then*

$$\left\|U_{\gamma,s}(f)\right\|_{p(\cdot),q} \lesssim \|f\|_{p(\cdot),q}.$$

If $1 \le p_- \le p_+ < \infty$ *and* $f \in L_{p(\cdot)}$, *then*

$$\sup_{\rho>0} \left\|\rho \chi_{\{U_{\gamma,s}(f)>\rho\}}\right\|_{p(\cdot)} \lesssim \|f\|_{p(\cdot)}.$$

Moreover, we obtain an equivalent characterization of the martingale Hardy–Lorentz space $H_{p(\cdot),q}$, namely, we show that $\|U_{\gamma,s}(f)\|_{L_{p(\cdot),q}}$ is equivalent to $\|f\|_{H_{p(\cdot),q}}$:

Corollary 2 *Let* $p(\cdot) \in C^{\log}$, $0 < p_- \le p_+ < \infty$, $0 < q \le \infty$ *and* $0 < \gamma, s < \infty$. *If* (5) *holds and* $f \in H_{p(\cdot),q}$, *then*

$$\|f\|_{H_{p(\cdot),q}} \le \|U_{\gamma,s}(f)\|_{p(\cdot),q} \le C_{p(\cdot)}\|f\|_{H_{p(\cdot),q}}.$$

Finally, we note again that condition (5) is also necessary.

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2 Variable Lebesgue and Lorentz spaces

Let λ denote the Lebesgue measure on the unit interval $[0, 1)$. For a constant p, the L_p space is equipped with the quasi-norm

$$\|f\|_p := \left(\int_0^1 |f(x)|^p \, d\lambda(x)\right)^{1/p} \qquad (0 < p < \infty),$$

with the usual modification for $p = \infty$.

To introduce the variable Lebesgue spaces let

$$\rho(f) := \int_0^1 |f(x)|^{p(x)} d\lambda(x),$$

where $p(\cdot) : [0, 1) \to (0, \infty]$ is a variable exponent. The variable Lebesgue space $L_{p(\cdot)}$ is the collection of all measurable functions f for which there exists $\nu > 0$ such that

$$\rho(f/\nu) < \infty.$$

We equip $L_{p(\cdot)}$ with the quasi-norm

$$\|f\|_{p(\cdot)} := \inf\{\nu > 0 : \rho(f/\nu) \leq 1\}.$$

If $p(\cdot) = p$ is a constant, then we get back the definition of the usual L_p spaces. For any $f \in L_{p(\cdot)}$, we have $\rho(f) \leq 1$ if and only if $\|f\|_{p(\cdot)} \leq 1$. It is known that $\|\nu f\|_{p(\cdot)} = |\nu| \|f\|_{p(\cdot)}$ and

$$\left\| |f|^s \right\|_{p(\cdot)} = \|f\|_{sp(\cdot)}^s,$$

where $s \in (0, \infty)$ and $\nu \in \mathbb{C}$. Details can be found in the monographs Cruz-Uribe and Fiorenza [5] and Diening et al. [6]. Moreover, for

$$0 < b \leq \min\{p_-, 1\} =: \underline{p},$$

we have

$$\|f + g\|_{p(\cdot)}^b \leq \|f\|_{p(\cdot)}^b + \|g\|_{p(\cdot)}^b. \tag{6}$$

The variable exponent $p'(\cdot)$ is defined pointwise by

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1, \quad x \in [0, 1).$$

The next lemma is well known, see Cruz-Uribe and Fiorenza [5] or Diening et al. [6].

Lemma 1 *Let $1 \leq p_- \leq p_+ \leq \infty$. For all $f \in L_{p(\cdot)}$ and $g \in L_{p'(\cdot)}$,*

$$\int_0^1 |fg| \, d\lambda \leq C_{p(\cdot)} \|f\|_{p(\cdot)} \|g\|_{p'(\cdot)}.$$

Moreover,

$$\|f\|_{p(\cdot)} \sim \sup_{\|g\|_{p'(\cdot)} \leq 1} \left| \int_0^1 fg \, d\lambda \right|,$$

where $\sim$ denotes the equivalence of the numbers.

The variable Lorentz spaces were introduced and investigated by Kempka and Vybíral [14]. $L_{p(\cdot),q}$ is defined to be the space of all measurable functions f such that

$$\|f\|_{p(\cdot),q} := \begin{cases} \left(\int_0^\infty \rho^q \left\|\chi_{\{x\in[0,1):\,|f(x)|>\rho\}}\right\|_{p(\cdot)}^q \frac{d\rho}{\rho}\right)^{1/q}, & \text{if } 0<q<\infty; \\ \sup_{\rho\in(0,\infty)} \rho \left\|\chi_{\{x\in[0,1):\,|f(x)|>\rho\}}\right\|_{p(\cdot)}, & \text{if } q=\infty \end{cases}$$

is finite. If $p(\cdot)$ is a constant, we get back the classical Lorentz spaces (see Bergh and Löfström [1]). In contrary to the spaces with constant $p(\cdot)$, the variable Lorentz spaces $L_{p(\cdot),q}$ do not include the variable Lebesgue spaces $L_{p(\cdot)}$ as a special cases.

3 Maximal operators

We always suppose that the sequence (p_n) of natural numbers is bounded. Let

$$\widehat{p} := \sup_{n\in\mathbb{N}} p_n < \infty. \tag{7}$$

The conditional expectation operators relative to $\mathcal{F}_n$ are denoted by E_n, where $\mathcal{F}_n$ was defined in (1). An integrable sequence $f = (f_n)_{n\in\mathbb{N}}$ is said to be a Vilenkin martingale if f_n is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$ and $E_n f_m = f_n$ in case $n \le m$. It is easy to show (see e.g. Weisz [28]) that the sequence $(\mathcal{F}_n, n \in \mathbb{N})$ is regular, i.e., there exist a constant $R > 0$ such that $f_n \le R \cdot f_{n-1}$ for all non-negative Vilenkin martingales. We can see easily that $R \ge \widehat{p}$, where $\widehat{p}$ is defined in (7).

For a Vilenkin martingale $f = (f_n)_{n\in\mathbb{N}}$, the Doob maximal function is defined by

$$M(f) := \sup_{n\in\mathbb{N}} |f_n|.$$

If $f \in L_1$, then we can replace f_n by f in the integral.

In the literature the log-Hölder continuity condition is usually supposed. Under this condition, the Hardy-Littlewood maximal operator is bounded on $L_{p(\cdot)}$ if $1 < p_- \le p_+$. We denote by $C^{\log}$ the set of all variable exponents $p(\cdot)$ satisfying the so-called log-Hölder continuous condition, namely, there exists a positive constant $C_{\log}(p)$ such that, for any $x, y \in [0, 1)$,

$$|p(x) - p(y)| \le \frac{C_{\log}(p)}{\log(e + 1/|x-y|)}. \tag{8}$$

In [31, 32], we generalized the Doob martingale maximal operator as follows. Every point $x \in [0, 1)$ can be written in the following way:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{P_{k+1}} \qquad (0 \le x_k < p_k,\ x_k \in \mathbb{N}).$$

If there are two different forms, choose the one for which $\lim_{k\to\infty} x_k = 0$. The so called Vilenkin addition is defined by

$$x\dot{+}y = \sum_{k=0}^{\infty} \frac{z_k}{P_{k+1}}, \qquad \text{where } z_k := x_k + y_k \bmod p_k,\ (k \in \mathbb{N}).$$

We defined the maximal operator $U_{\gamma,s}$ in (4), where $0 < \gamma, s < \infty$. Of course, if $f \in L_1$, then we can write in the definition f instead of f_n. Let us define $I_{k,n} := [kP_n^{-1}, (k+1)P_n^{-1})$, where $0 \le k < P_n$, $n \in \mathbb{N}$. The definition of $U_{\gamma,s}(f)$ can be rewritten to

$$U_{\gamma,s}(f) = \sup_{n\in\mathbb{N}} \sum_{k=0}^{P_n-1} \chi_{I_{k,n}} \sum_{m=0}^{n} \sum_{j=0}^{m} \left(\frac{P_j}{P_n}\right)^{\gamma} \sum_{i=j}^{m} \left(\frac{P_j}{P_i}\right)^{s} \sum_{l=0}^{p_j-1} \frac{1}{\lambda(I_{k,n}^{l,j,i})} \left| \int_{I_{k,n}^{l,j,i}} f_n d\lambda \right|,$$

where $I_{k,n}^{l,j,i} := (I_{k,n})^{l,j,i}$. Now we point out four special cases of this operator.

If $j = i = n = m$, we obtain the first spacial case,

$$U_{\gamma,s}^{(1)}(f)(x) := \sup_{n\in\mathbb{N}} \sum_{l=0}^{p_n-1} \frac{1}{\lambda(I_n(x)^{l,n,n})} \left| \int_{I_n(x)^{l,n,n}} f_n d\lambda \right| = \sup_{n\in\mathbb{N}} \frac{p_n}{\lambda(I_n(x))} \left| \int_{I_n(x)} f_n d\lambda \right|,$$

which is basically $M(f)$. Note that $I_n(x)^{l,n,n} = I_n(x)$ $(n \in \mathbb{N}, l = 0, \dots, p_n - 1)$.

If $j = i = m$, we have

$$U_{\gamma,s}^{(2)}(f)(x) := \sup_{n\in\mathbb{N}} \sum_{m=0}^{n} \left(\frac{P_m}{P_n}\right)^{\gamma} \sum_{l=0}^{p_m-1} \frac{1}{\lambda(I_n(x)^{l,m,m})} \left| \int_{I_n(x)^{l,m,m}} f_n d\lambda \right| = \sup_{n\in\mathbb{N}} \sum_{m=0}^{n} \left(\frac{P_m}{P_n}\right)^{\gamma} \frac{p_m}{\lambda(I_m(x))} \left| \int_{I_m(x)} f_n d\lambda \right|.$$

Here $I_n(x)^{l,m,m} = I_n(x)\dot{+}[0, P_m^{-1})\dot{+}lP_{m+1}^{-1} = x\dot{+}[0, P_m^{-1}) = I_m(x)$. It is easy to see that

$$M(f) \le U_{\gamma,s}^{(1)}(f) \le U_{\gamma,s}^{(2)}(f) \le CM(f)$$

for all $0 < \gamma, s < \infty$ and so Theorem 1 holds also for these two operators.

If $m = n$ and $i = n$, we get that

$$U_{\gamma,s}^{(3)}(f)(x) := \sup_{n\in\mathbb{N}} \sum_{j=0}^{n} \left(\frac{P_j}{P_n}\right)^{\gamma+s} \sum_{l=0}^{p_j-1} \frac{1}{\lambda(I_n(x)^{l,j,n})} \left| \int_{I_n(x)^{l,j,n}} f_n d\lambda \right|.$$

Note that $I_n(x)^{l,j,n} = I_n(x)\dot{+}[0, P_n^{-1})\dot{+}lP_{j+1}^{-1} = I_n(x)\dot{+}lP_{j+1}^{-1}$.

If $m = n$, we obtain the last special case,

$$U_{\gamma,s}^{(4)}(f)(x) := \sup_{n\in\mathbb{N}} \sum_{j=0}^{n} \left(\frac{P_j}{P_n}\right)^{\gamma} \sum_{i=j}^{n} \left(\frac{P_j}{P_i}\right)^{s} \sum_{l=0}^{p_j-1} \frac{1}{\lambda(I_n(x)^{l,j,i})} \left| \int_{I_n(x)^{l,j,i}} f_n d\lambda \right|.$$

The maximal operators $U_{\gamma,s}^{(3)}(f)$ and $U_{\gamma,s}^{(4)}(f)$ as well as $U_{\gamma,s}(f)$ cannot be estimated by $M(f)$ from above pointwise. In [31], we investigated the operators $U_{\gamma,s}^{(3)}$ and $U_{\gamma,s}^{(4)}$. Their boundedness on $L_{p(\cdot)}$ was the key point in the proof of boundedness and convergence results for the Fejér means of the Vilenkin-Fourier series (see [31]).

It is easy to see that, for all $0 < \gamma, s < \infty$,

$$M(f) \le U_{\gamma,s}^{(j)}(f) \le U_{\gamma,s}(f) \qquad (j = 1, \ldots, 4). \tag{9}$$

4 Martingale Hardy–Lorentz spaces

Now we introduce the variable martingale Hardy–Lorentz spaces by

$$H_{p(\cdot),q} := \left\{ f = (f_n)_{n\in\mathbb{N}} : \|f\|_{H_{p(\cdot),q}} := \|M(f)\|_{p(\cdot),q} < \infty \right\}.$$

These spaces have several equivalent characterizations, for example an equivalent quasi-norm can be defined by the quadratic variation and by the conditional quadratic variation (see [11]). In this paper, we will give more equivalent characterizations of these Hardy–Lorentz spaces using the above maximal functions.

The atomic decomposition is a useful characterization of the Hardy–Lorentz spaces. First, we introduce the concept of stopping times (see e.g. [28]). A map $\tau : [0, 1) \longrightarrow \mathbb{N} \cup \{\infty\}$ is called a stopping time relative to $(\mathcal{F}_n, n \in \mathbb{N})$ if

$$\{x \in [0, 1) : \tau(x) = n\} =: \{\tau = n\} \in \mathcal{F}_n.$$

It is well known that the last condition is equivalent to the conditions

$$\{\tau \le n\} \in \mathcal{F}_n \qquad (n \in \mathbb{N})$$

and

$$\{\tau \ge n\} \in \mathcal{F}_{n-1} \qquad (n \in \mathbb{N}).$$

This implies that the sequence $(f_n^\tau, n \in \mathbb{N})$ defined by

$$f_n^\tau := \sum_{k=0}^{n} \chi_{\{\tau \ge m\}} (f_k - f_{k-1})$$

is again a martingale, called stopped martingale, whenever $(f_n, n \in \mathbb{N})$ is a martingale. This fact is used in the proof of Theorem 4.

A measurable function a is called a $p(\cdot)$-atom if there exists a stopping time τ such that

(i) $E_n(a)(\cdot) = 0$ for all $n \le \tau(\cdot)$,
(ii) $\|M(a)\|_\infty \le \left\|\chi_{\{\tau<\infty\}}\right\|_{p(\cdot)}^{-1}$.

This form of the atoms was used first in [28] for a constant p. The atomic decomposition of the spaces $H_{p(\cdot),q}$ were proved in Jiao et al. [11]. The classical case can be found in [28].

Theorem 4 *Let $p(\cdot) \in C^{\log}$, $0 < p_- \le p_+ < \infty$ and $0 < q \le \infty$. Then the martingale $f = (f_n)_{n\in\mathbb{N}} \in H_{p(\cdot),q}$ if and only if there exists a sequence $(a^k)_{k\in\mathbb{Z}}$ of $p(\cdot)$-atoms such that for every $n \in \mathbb{N}$,*

$$f_n = \sum_{k\in\mathbb{Z}} \mu_k E_n a^k \qquad \text{almost everywhere},$$

where $\mu_k = 3 \cdot 2^k \left\|\chi_{\{\tau_k<\infty\}}\right\|_{p(\cdot)}$ and τ_k is the stopping time associated with the $p(\cdot)$-atom a^k. Moreover,

$$\|f\|_{H_{p(\cdot),q}} \sim \inf \left(\sum_{k\in\mathbb{Z}} 2^{kq} \left\|\chi_{\{\tau_k<\infty\}}\right\|_{p(\cdot)}^q\right)^{1/q},$$

respectively, where the infimum is taken over all decompositions of f as above.

5 Proofs

Proof of Theorem 3 According to Theorem 4, we can write f as

$$f = \sum_{k\in\mathbb{Z}} \mu_k a^k = f_1 + f_2,$$

where $k_0 \in \mathbb{Z}$,

$$f_1 = \sum_{k=-\infty}^{k_0-1} \mu_k a^k, \qquad f_2 = \sum_{k=k_0}^{\infty} \mu_k a^k, \qquad \mu_k = 3 \cdot 2^k \left\|\chi_{\{\tau_k<\infty\}}\right\|_{p(\cdot)}$$

and τ_k is the stopping time associated with the $p(\cdot)$-atom a^k. Moreover,

$$\left(\sum_{k\in\mathbb{Z}} 2^{kq} \left\|\chi_{\{\tau_k<\infty\}}\right\|_{p(\cdot)}^q\right)^{1/q} \lesssim \|f\|_{H_{p(\cdot),q}}.$$

Since

$$U_{\gamma,s}(f) \le \sum_{m=0}^{n}\sum_{j=0}^{m}\left(\frac{P_j}{P_n}\right)^{\gamma}\sum_{i=j}^{m}\left(\frac{P_j}{P_i}\right)^{s}\sum_{l=0}^{p_j-1}\|f\|_\infty \tag{10}$$

$$\le \sum_{m=0}^{n}\sum_{j=0}^{m}2^{(j-n)\gamma}\sum_{i=j}^{m}2^{(j-i)s}p_j\|f\|_\infty \lesssim \|f\|_\infty, \tag{11}$$

$U_{\gamma,s}$ is bounded on L_∞. This implies that

$$\left\|U_{\gamma,s}(f_1)\right\|_\infty \le \sum_{k=-\infty}^{k_0-1}\mu_k\left\|U_{\gamma,s}(a^k)\right\|_\infty \le \sum_{k=-\infty}^{k_0-1}\mu_k\left\|a^k\right\|_\infty$$
$$\le \sum_{k=-\infty}^{k_0-1}\mu_k\left\|\chi_{\{\tau_k<\infty\}}\right\|_{p(\cdot)}^{-1} \le 3\cdot 2^{k_0}.$$

Thus

$$2^{k_0}\|\chi_{\{U_{\gamma,s}(f)>6\cdot 2^{k_0}\}}\|_{p(\cdot)} \le 2^{k_0}\|\chi_{\{U_{\gamma,s}(f_2)>3\cdot 2^{k_0}\}}\|_{p(\cdot)},$$

so we have to consider

$$U_{\gamma,s}(f_2) \le \sum_{k=k_0}^{\infty}\mu_k U_{\gamma,s}(a^k)\chi_{\{\tau_k<\infty\}} + \sum_{k=k_0}^{\infty}\mu_k U_{\gamma,s}(a^k)\chi_{\{\tau_k=\infty\}} =: A_1 + A_2. \tag{12}$$

Obviously,

$$\{A_1 > 3\cdot 2^{k_0-1}\} \subset \{A_1 > 0\} \subset \bigcup_{k=k_0}^{\infty}\{\tau_k < \infty\}.$$

Suppose that $0 < q < \infty$ and let us choose $0 < \varepsilon < \min(\underline{p}, q)$ and $0 < \delta < 1$. Applying (6), we have

$$\left\|\chi_{\{A_1>3\cdot 2^{k_0-1}\}}\right\|_{p(\cdot)} \le \left\|\sum_{k=k_0}^{\infty}\chi_{\{\tau_k<\infty\}}\right\|_{p(\cdot)} \le \left(\sum_{k=k_0}^{\infty}\left\|\chi_{\{\tau_k<\infty\}}\right\|_{p(\cdot)}^{\varepsilon}\right)^{1/\varepsilon}$$
$$= \left(\sum_{k=k_0}^{\infty}2^{-k\delta\varepsilon}2^{k\delta\varepsilon}\left\|\chi_{\{\tau_k<\infty\}}\right\|_{p(\cdot)}^{\varepsilon}\right)^{1/\varepsilon}.$$

Using Hölder's inequality for $\frac{q-\varepsilon}{q} + \frac{\varepsilon}{q} = 1$, we get

$$\begin{aligned}\left\|\chi_{\{A_1>3\cdot 2^{k_0-1}\}}\right\|_{p(\cdot)} &\leq \left(\sum_{k=k_0}^{\infty} 2^{-k\delta\varepsilon\frac{q}{q-\varepsilon}}\right)^{\frac{q-\varepsilon}{\varepsilon q}} \left(\sum_{k=k_0}^{\infty} 2^{k\delta q} \left\|\chi_{\{\tau_k<\infty\}}\right\|_{p(\cdot)}^{q}\right)^{1/q} \\ &\lesssim 2^{-k_0\delta} \left(\sum_{k=k_0}^{\infty} 2^{k\delta q} \left\|\chi_{\{\tau_k<\infty\}}\right\|_{p(\cdot)}^{q}\right)^{1/q}.\end{aligned}$$

Consequently,

$$\begin{aligned}\sum_{k_0=-\infty}^{\infty} 2^{k_0 q} \left\|\chi_{\{A_1>3\cdot 2^{k_0-1}\}}\right\|_{p(\cdot)}^{q} &\lesssim \sum_{k_0=-\infty}^{\infty} 2^{k_0(1-\delta)q} \sum_{k=k_0}^{\infty} 2^{k\delta q} \left\|\chi_{\{\tau_k<\infty\}}\right\|_{p(\cdot)}^{q} \\ &= \sum_{k=-\infty}^{\infty} 2^{k\delta q} \left\|\chi_{\{\tau_k<\infty\}}\right\|_{p(\cdot)}^{q} \sum_{k_0=-\infty}^{k} 2^{k_0(1-\delta)q} \\ &\lesssim \sum_{k=-\infty}^{\infty} 2^{kq} \left\|\chi_{\{\tau_k<\infty\}}\right\|_{p(\cdot)}^{q} \\ &\lesssim \|f\|_{H_{p(\cdot),q}}^{q}.\end{aligned}$$

Next, let us estimate the term A_2. For a fixed $k \in \mathbb{Z}$, the sets $\{\tau_k = K\}$ are disjoint and there exist disjoint Vilenkin intervals $I_{k,K,\mu} \in \mathcal{F}_K$ such that

$$\{\tau_k = K\} = \bigcup_{\mu} I_{k,K,\mu} \qquad (K \in \mathbb{N}),$$

where the union in μ is finite and $\lambda(I_{k,K,\mu}) = P_K^{-1}$. Thus

$$\{\tau_k < \infty\} = \bigcup_{K\in\mathbb{N}} \bigcup_{\mu} I_{k,K,\mu},$$

where the Vilenkin intervals $I_{k,K,\mu}$ are disjoint for a fixed $k \in \mathbb{Z}$. Then

$$a^k = \sum_{K\in\mathbb{N}} \sum_{\mu} a^k \chi_{I_{k,K,\mu}}.$$

The operator $U_{\gamma,s}$ can be written as

$$U_{\gamma,s}(a^k)(x) := \sup_{n\in\mathbb{N}} \sup_{x\in I} \sum_{m=0}^{n} \sum_{j=0}^{m} \left(\frac{P_j}{P_n}\right)^{\gamma} \sum_{i=j}^{m} \left(\frac{P_j}{P_i}\right)^{s} \sum_{l=0}^{p_j-1} \frac{1}{\lambda(I^{l,j,i})} \left|\int_{I^{l,j,i}} a^k d\lambda\right|,$$

where $I \in \mathcal{F}_n$ is a Vilenkin interval. Since $\int_{I_{k,K,\mu}} a^k \, d\lambda = 0$, we have

$$\int_{I^{l,j,i}} a^k \, d\lambda = 0$$

if $i \le K$. Thus we can suppose that $i > K$, and so $n \ge m > K$. If $x \notin I_{k,K,\mu}$, $x \in I$ and $j \ge K$, then $I^{l,j,i} \cap I_{k,K,\mu} = \emptyset$. Therefore we can suppose that $j < K$. Similarly, if

$$x \in I_{k,K,\mu} \dot{+} [l P_{j+1}^{-1}, (l+1) P_{j+1}^{-1}) \setminus (I_{k,K,\mu} \dot{+} l P_{j+1}^{-1}),$$

then $I^{l,j,i} \cap I_{k,K,\mu} = \emptyset$, so we may assume that $x \in I_{k,K,\mu} \dot{+} l P_{j+1}^{-1} = I_{k,K,\mu}^{l,j,K}$. Therefore, for $x \notin I_{k,K,\mu}$,

$$U_{\gamma,s}(a^k \chi_{I_{k,K,\mu}})(x) \le \sup_{n>K} \chi_I(x) \sum_{m=K+1}^{n} \sum_{j=0}^{K-1} \left(\frac{P_j}{P_n}\right)^{\gamma} \sum_{i=K+1}^{m} \left(\frac{P_j}{P_i}\right)^{s} \sum_{l=0}^{p_j-1} \frac{1}{\lambda(I^{l,j,i})} \left| \int_{I^{l,j,i}} a^k \, d\lambda \right| \chi_{I_{k,K,\mu}^{l,j,K}}(x).$$

It is easy to see that

$$\sum_{i=K+1}^{m} \left(\frac{1}{P_i}\right)^{s} = \sum_{i=K+1}^{m} \left(\frac{1}{P_K p_K \cdots p_{i-1}}\right)^{s} \le \sum_{i=K+1}^{m} \left(\frac{1}{P_K 2^{i-K}}\right)^{s} \le C_s \left(\frac{1}{P_K}\right)^{s}.$$

Hence

$$\begin{aligned} &U_{\gamma,s}(a^k \chi_{I_{k,K,\mu}})(x) \\ &\le \left\| \chi_{\{\tau_k < \infty\}} \right\|_{p(\cdot)}^{-1} \sup_{n>K} \chi_I(x) \sum_{m=K+1}^{n} \sum_{j=0}^{K-1} \left(\frac{P_j}{P_n}\right)^{\gamma} \sum_{i=K+1}^{\infty} \left(\frac{P_j}{P_i}\right)^{s} \sum_{l=0}^{p_j-1} \chi_{I_{k,K,\mu}^{l,j,K}}(x) \\ &\lesssim \left\| \chi_{\{\tau_k < \infty\}} \right\|_{p(\cdot)}^{-1} \sup_{n>K} (n-K) \left(\frac{P_K}{P_n}\right)^{\gamma} \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^{\gamma+s} \sum_{l=0}^{p_j-1} \chi_{I_{k,K,\mu}^{l,j,K}}(x) \\ &\lesssim \left\| \chi_{\{\tau_k < \infty\}} \right\|_{p(\cdot)}^{-1} \sup_{n>K} (n-K) 2^{(K-n)\gamma} \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^{\gamma+s} \sum_{l=0}^{p_j-1} \chi_{I_{k,K,\mu}^{l,j,K}}(x). \end{aligned}$$

Since the function $x \mapsto x 2^{-\gamma x}$ is bounded, we obtain that

$$U_{\gamma,s}(a^k \chi_{I_{k,K,\mu}})(x) \lesssim \left\| \chi_{\{\tau_k < \infty\}} \right\|_{p(\cdot)}^{-1} \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^{\gamma+s} \sum_{l=0}^{p_j-1} \chi_{I_{k,K,\mu}^{l,j,K}}(x).$$

From this it follows that, for $x \in \{\tau_k = \infty\}$,

$$U_{\gamma,s}(a^k)(x) \lesssim \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-1} \sum_{K\in\mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^{\gamma+s} \sum_{l=0}^{p_j-1} \chi_{I_{k,K,\mu}^{l,j,K}}(x). \tag{13}$$

Let us choose $0 < \beta < 1$ and $0 < \epsilon < \underline{p}$. By (13),

$$\begin{aligned}
&\left\| |U_{\gamma,s}(a^k)|^{\beta\epsilon} \chi_{\{\tau_k=\infty\}} \right\|_{p(\cdot)/\epsilon} \\
&\lesssim \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{-\beta\epsilon} \left\| \sum_{K\in\mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^{(\gamma+s)\beta\epsilon} \sum_{l=0}^{p_j-1} \chi_{I_{k,K,\mu}^{l,j,K}} \right\|_{p(\cdot)/\epsilon}.
\end{aligned}$$

Choose $\max(1, \beta p_+) < r < \infty$. By Lemma 1, there exists a function $g \in L_{(\frac{p(\cdot)}{\varepsilon})'}$ with $\|g\|_{(\frac{p(\cdot)}{\varepsilon})'} \le 1$ such that

$$\begin{aligned}
&\left\| |U_{\gamma,s}(a^k)|^{\beta\epsilon} \chi_{\{\tau_k=\infty\}} \right\|_{p(\cdot)/\epsilon} \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{\beta\epsilon} \\
&\lesssim \int_0^1 \sum_{K\in\mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^{(\gamma+s)\beta\epsilon} \sum_{l=0}^{p_j-1} \chi_{I_{k,K,\mu}^{l,j,K}}\, g\, d\lambda \\
&\le \sum_{K\in\mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^{(\gamma+s)\beta\epsilon} \sum_{l=0}^{p_j-1} \left\| \chi_{I_{k,K,\mu}^{l,j,K}} \right\|_{\frac{r}{\beta\epsilon}} \left\| \chi_{I_{k,K,\mu}^{l,j,K}}\, g \right\|_{(\frac{r}{\beta\epsilon})'} \\
&\lesssim \sum_{K\in\mathbb{N}} \sum_{\mu} \sum_{j=0}^{K-1} \left(\frac{P_j}{P_K}\right)^{(\gamma+s)\beta\epsilon} \\
&\quad \sum_{l=0}^{p_j-1} \int_0^1 \chi_{I_{k,K,\mu}}(x) \left(\frac{1}{\lambda(I_{k,K,\mu}^{l,j,K})} \int_{I_{k,K,\mu}^{l,j,K}} |g|^{(\frac{r}{\beta\epsilon})'}\, d\lambda \right)^{1/(\frac{r}{\beta\epsilon})'} dx.
\end{aligned}$$

We use Hölder's inequality to obtain

$$\begin{aligned}
&\left\| |U_{\gamma,s}(a^k)|^{\beta\epsilon} \chi_{\{\tau_k=\infty\}} \right\|_{p(\cdot)/\epsilon} \|\chi_{\{\tau_k<\infty\}}\|_{p(\cdot)}^{\beta\epsilon} \\
&\lesssim \int_0^1 \sum_{K\in\mathbb{N}} \sum_{\mu} \chi_{I_{k,K,\mu}}(x) \sum_{j=0}^{K-1} \sum_{l=0}^{p_j-1} \left(\frac{P_j}{P_K}\right)^{(\gamma+s)\beta\epsilon(1/(\frac{r}{\beta\epsilon})+1/(\frac{r}{\beta\epsilon})')} \\
&\quad \left(\frac{1}{\lambda(I_{k,K,\mu}^{l,j,K})} \int_{I_{k,K,\mu}^{l,j,K}} |g|^{(\frac{r}{\beta\epsilon})'}\, d\lambda \right)^{1/(\frac{r}{\beta\epsilon})'} dx
\end{aligned}$$

$$\lesssim \int_0^1 \sum_{K\in\mathbb{N}} \sum_{\mu} \chi_{I_{k,K,\mu}}(x) \left(\sum_{j=0}^{K-1} \sum_{l=0}^{p_j-1} \left(\frac{P_j}{P_K} \right)^{(\gamma+s)\beta\epsilon} \right)^{1/(\frac{r}{\beta\epsilon})}$$

$$\left(\sum_{j=0}^{K-1} \sum_{l=0}^{p_j-1} \left(\frac{P_j}{P_K} \right)^{(\gamma+s)\beta\epsilon} \frac{1}{\lambda(I_{k,K,\mu}^{j,K})} \int_{I_{k,K,\mu}^{j,K}} |g|^{(\frac{r}{\beta\epsilon})'} \, d\lambda \right)^{1/(\frac{r}{\beta\epsilon})'} dx$$

$$\leq \int_0^1 \sum_{K\in\mathbb{N}} \sum_{\mu} \chi_{I_{k,K,\mu}} \left(U^{(3)}_{\gamma\beta\epsilon,s\beta\epsilon} \left(|g|^{(\frac{r}{\beta\epsilon})'} \right) \right)^{1/(\frac{r}{\beta\epsilon})'} d\lambda$$

$$\leq \left\| \sum_{K\in\mathbb{N}} \sum_{\mu} \chi_{I_{k,K,\mu}} \right\|_{p(\cdot)/\epsilon} \left\| \left(U^{(3)}_{\gamma\beta\epsilon,s\beta\epsilon} \left(|g|^{(\frac{r}{\beta\epsilon})'} \right) \right)^{1/(\frac{r}{\beta\epsilon})'} \right\|_{(p(\cdot)/\epsilon)'}.$$

Inequality (5) is equivalent to

$$\frac{p_+ - \epsilon}{p_+} - \frac{p_- - \epsilon}{p_-} < (\gamma + s)\epsilon.$$

We can choose β near to 1 such that

$$\frac{p_+ - \epsilon}{p_+} - \frac{p_- - \epsilon}{p_-} < (\gamma + s)\beta\epsilon.$$

Next we can choose r so large that

$$\frac{1}{((p(\cdot)/\epsilon)'/(r/\beta\epsilon)')_-} - \frac{1}{((p(\cdot)/\epsilon)'/(r/\beta\epsilon)')_+} = \frac{r/(r-\beta\epsilon)}{p_+/(p_+-\epsilon)} - \frac{r/(r-\beta\epsilon)}{p_-/(p_--\epsilon)} < (\gamma + s)\beta\epsilon.$$

Since $(r/\beta\epsilon)' < (p(\cdot)/\epsilon)'$, we can apply Theorem 2 and conclude

$$\left\| |U_{\gamma,s}(a^k)|^{\beta\epsilon} \chi_{\{\tau_k=\infty\}} \right\|_{p(\cdot)/\epsilon} \left\| \chi_{\{\tau_k<\infty\}} \right\|_{p(\cdot)}^{\beta\epsilon}$$

$$\lesssim \left\| \chi_{\{\tau_k<\infty\}} \right\|_{p(\cdot)/\epsilon} \left\| U^{(3)}_{\gamma\beta\epsilon,s\beta\epsilon} \left(|g|^{(\frac{r}{\beta\epsilon})'} \right) \right\|_{\frac{(p(\cdot)/\epsilon)'}{(r/\beta\epsilon)'}}^{1/(r/\beta\epsilon)'}$$

$$\lesssim \left\| \chi_{\{\tau_k<\infty\}} \right\|_{p(\cdot)/\epsilon} \left\| |g|^{(\frac{r}{\beta\epsilon})'} \right\|_{\frac{(p(\cdot)/\epsilon)'}{(r/\beta\epsilon)'}}^{1/(r/\beta\epsilon)'}$$

$$\lesssim \left\| \chi_{\{\tau_k<\infty\}} \right\|_{p(\cdot)/\epsilon}.$$

From this it follows that

$$\left\| \chi_{\{A_2 > 3\cdot 2^{k_0-1}\}} \right\|_{p(\cdot)} \leq \left\| \frac{\sum_{k=k_0}^{\infty} \mu_k^{\beta} |U_{\gamma,s}(a^k)|^{\beta} \chi_{\{\tau_k=\infty\}}}{3^{\beta} 2^{\beta(k_0-1)}} \right\|_{p(\cdot)}$$

$$
\begin{aligned}
&\lesssim 2^{-\beta k_0} \left\| \sum_{k=k_0}^{\infty} \mu_k^{\beta\epsilon} |U_{\gamma,s}(a^k)|^{\beta\epsilon} \chi_{\{\tau_k=\infty\}} \right\|_{p(\cdot)/\epsilon}^{1/\epsilon} \\
&\lesssim 2^{-\beta k_0} \left(\sum_{k=k_0}^{\infty} \mu_k^{\beta\epsilon} \left\| |U_{\gamma,s}(a^k)|^{\beta\epsilon} \chi_{\{\tau_k=\infty\}} \right\|_{p(\cdot)/\epsilon} \right)^{1/\epsilon} \\
&\lesssim 2^{-\beta k_0} \left(\sum_{k=k_0}^{\infty} 2^{k\beta\epsilon} \left\| \chi_{\{\tau_k<\infty\}} \right\|_{p(\cdot)/\epsilon} \right)^{1/\epsilon} \\
&\leq 2^{-\beta k_0} \left(\sum_{k=k_0}^{\infty} 2^{k(\beta-\delta)\epsilon} 2^{k\delta\epsilon} \left\| \chi_{\{\tau_k<\infty\}} \right\|_{p(\cdot)}^{\epsilon} \right)^{1/\epsilon}, \qquad (14)
\end{aligned}
$$

where $\beta < \delta < 1$. Let us again use Hölder's inequality with $\frac{q-\varepsilon}{q} + \frac{\varepsilon}{q} = 1$:

$$
\begin{aligned}
\left\| \chi_{\{A_2>3\cdot 2^{k_0-1}\}} \right\|_{p(\cdot)} &\lesssim 2^{-\beta k_0} \left(\sum_{k=k_0}^{\infty} 2^{k(\beta-\delta)\epsilon \frac{q}{q-\epsilon}} \right)^{\frac{q-\epsilon}{\epsilon q}} \left(\sum_{k=k_0}^{\infty} 2^{k\delta q} \left\| \chi_{\{\tau_k<\infty\}} \right\|_{p(\cdot)}^{q} \right)^{1/q} \\
&\lesssim 2^{-k_0\delta} \left(\sum_{k=k_0}^{\infty} 2^{k\delta q} \left\| \chi_{\{\tau_k<\infty\}} \right\|_{p(\cdot)}^{q} \right)^{1/q}.
\end{aligned}
$$

By changing the order of the sums, we obtain

$$
\begin{aligned}
\sum_{k_0=-\infty}^{\infty} 2^{k_0 q} \left\| \chi_{\{A_2>3\cdot 2^{k_0-1}\}} \right\|_{p(\cdot)}^{q} &\lesssim \sum_{k_0=-\infty}^{\infty} 2^{k_0(1-\delta)q} \sum_{k=k_0}^{\infty} 2^{k\delta q} \left\| \chi_{\{\tau_k<\infty\}} \right\|_{p(\cdot)}^{q} \\
&= \sum_{k=-\infty}^{\infty} 2^{k\delta q} \left\| \chi_{\{\tau_k<\infty\}} \right\|_{p(\cdot)}^{q} \sum_{k_0=-\infty}^{k} 2^{k_0(1-\delta)q} \\
&\lesssim \sum_{k=-\infty}^{\infty} 2^{kq} \left\| \chi_{\{\tau_k<\infty\}} \right\|_{p(\cdot)}^{q} \\
&\lesssim \|f\|_{H_{p(\cdot),q}}^{q}.
\end{aligned}
$$

This finishes the proof of Theorem 3 when $0 < q < \infty$. The proof is very similar for $q = \infty$, so we omit it. □

Remark 1 Inequality (5) obviously holds if $1/(\gamma+s) \leq p_- \leq p_+ < \infty$. If $p_- < 1/(\gamma+s)$, then (5) is equivalent to

$$
p_+ < \frac{p_-}{1-(\gamma+s)p_-}.
$$

Proof of Corollary 1 Jiao et al. [11] proved that $H_{p(\cdot),q}$ is equivalent to $L_{p(\cdot),q}$, whenever $1 < p_- \leq p_+ < \infty$ and $0 < q \leq \infty$. Then the first inequality follows from Theorem 3. By Theorem 3 and (3),

$$\sup_{\rho>0} \left\| \rho \chi_{\{U_{\gamma,s}(f)>\rho\}} \right\|_{p(\cdot)} = \left\| U_{\gamma,s}(f) \right\|_{p(\cdot),\infty} \lesssim \|f\|_{H_{p(\cdot),\infty}}$$
$$= \|M(f)\|_{p(\cdot),\infty} \lesssim \|f\|_{p(\cdot)},$$

which proves the second inequality. □

Finally, besides Corollary 2, we give equivalent characterizations of the Hardy–Lorentz spaces with the help of the maximal operators defined above.

Corollary 3 *Let* $p(\cdot) \in C^{\log}$, $0 < p_- \leq p_+ < \infty$, $0 < q \leq \infty$ *and* $0 < \gamma, s < \infty$. *If* (5) *holds,* $f \in H_{p(\cdot),q}$ *and* $j = 1, \ldots, 4$, *then*

$$\|f\|_{H_{p(\cdot),q}} = \|M(f)\|_{p(\cdot),q} \leq \|U^{(j)}_{\gamma,s}(f)\|_{p(\cdot),q} \leq \|U_{\gamma,s}(f)\|_{p(\cdot),q} \leq C_{p(\cdot)} \|f\|_{H_{p(\cdot),q}}.$$

Proof The inequalities follow from (9) and Theorem 3. □

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