



A monotonicity result for the first Steklov–Dirichlet Laplacian eigenvalue

Nunzia Gavitone¹ · Gianpaolo Piscitelli¹ 

Received: 17 December 2021 / Accepted: 23 March 2023 / Published online: 30 April 2023
© The Author(s) 2023, corrected publication 2024

Abstract

In this paper, we consider the first Steklov–Dirichlet eigenvalue of the Laplace operator in annular domains with a spherical hole. We prove a monotonicity result with respect to the hole, when the outer region is centrally symmetric.

Keywords Laplacian eigenvalue · Steklov–Dirichlet boundary conditions · Shape derivative

Mathematics Subject Classification Primary 35J25 · 35P15; Secondary 28A75 · 49Q10

1 Introduction and main result

In recent years, the study of eigenvalue problems in holed domains has been the object of much interest. These kind of problems are usually defined in annular domains with the outer region and the hole satisfying suitable assumptions. Specifically, different boundary conditions can be imposed on the outer and inner boundary and hence several optimization problems can be studied (e.g. Robin–Neumann [18], Neumann–Robin [7], Dirichlet–Neumann [2, 3], Steklov–Dirichlet [11, 14, 17], Steklov–Robin [12]). At the mean time, the optimal placement of an obstacle has been studied, so as to maximize or minimize a prescribed functional (e.g. the Dirichlet heat content [15], the first Steklov eigenvalue [10]).

In this paper, we consider an eigenvalue problem for the Laplace operator in a suitable annular domain with outer Steklov and inner Dirichlet boundary conditions.

✉ Gianpaolo Piscitelli
gianpaolo.piscitelli@unina.it
Nunzia Gavitone
nunzia.gavitone@unina.it

¹ Dipartimento di Matematica e Applicazioni “R. Caccioppoli”, Università degli Studi di Napoli Federico II Via Cintia, Complesso Universitario Monte S. Angelo, 80126 Naples, Italy

More precisely, let $\Omega_0 \subset \mathbb{R}^n$, $n \geq 2$, be an open bounded set with Lipschitz boundary and $B_r(y)$ be the ball of radius $r > 0$, centered at y , such that $B_r(y) \subset \overset{\circ}{\Omega}_0$. We study the following Steklov–Dirichlet eigenvalue problem

$$\sigma(\Omega) = \min_{\substack{v \in H^1_{\partial B_r(y)}(\Omega) \\ v \neq 0}} \left\{ \int_{\Omega} |Dv|^2 \, dx, \|v\|_{L^2(\partial\Omega_0)} = 1 \right\}, \tag{1.1}$$

where $H^1_{\partial B_r(y)}(\Omega)$ denotes the set of Sobolev functions which vanish on the boundary of $B_r(y)$ (see Sect. 2 for the precise definition). If $u \in H^1_{\partial B_r(y)}(\Omega)$ is a minimizer of (1.1), then it satisfies:

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ \frac{\partial u}{\partial \nu} = \sigma u & \text{on } \partial\Omega_0 \\ u = 0 & \text{on } \partial B_r(y), \end{cases} \tag{1.2}$$

where Ω is the annular domain $\Omega = \Omega_0 \setminus \overline{B_r(y)}$, $\sigma \in \mathbb{R}$ and ν is the outer unit normal to $\partial\Omega_0$.

In [17] (see also [8, 14]), the authors prove that the minimum in (1.1) is achieved by a function $u \in H^1_{\partial B_r(y)}(\Omega)$, which is a weak solution to problem (1.2) with constant sign in Ω , and that $\sigma(\Omega)$ is simple. Furthermore, the authors prove that, keeping the measure of Ω and the radius of the inner ball r fixed, $\sigma(\Omega)$ is maximized, among quasi-spherical sets, when Ω is a spherical shell, that is when Ω_0 is a ball with the same center of the hole. On the other hand, in [11] the authors extend this result to a class of annular sets with a suitable convex outer domain Ω_0 .

More properties are known when Ω is an eccentric spherical shell, that is, when the outer domain Ω_0 is a ball not necessarily centered at the same point of the spherical hole. In [22], the authors study the optimal placement of the hole in eccentric spherical shell Ω so that $\sigma(\Omega)$ is maximized when the outer ball and the inner radius are fixed. If $n \geq 3$, they prove that $\sigma(\Omega)$ achieves the maximum when the two balls are concentric. Subsequently, this result has been also proved for any dimension $n \geq 2$ in [10], by using different proofs (see also [19] for an analogous result in two-points homogeneous spaces).

Moreover, by performing numerical experiments, the authors in [14] exhibit that $\sigma(\Omega)$ is monotone decreasing with respect to the distance between the centers of the two disks. Our aim is to prove that this monotonicity property holds in any dimension and in a more general setting.

Through this paper, we assume that the outer domain Ω_0 verifies the following hypotheses.

Definition 1.1 Let $\Omega_0 \subset \mathbb{R}^n$, $n \geq 2$, be an open, bounded set with Lipschitz boundary and centrally symmetric with respect to $x_0 \in \overset{\circ}{\Omega}_0$, that is there exists $x' \in \overline{\Omega}_0$ such that $\frac{1}{2}(x + x') = x_0$ for any $x \in \overline{\Omega}_0$.

Let $r > 0$ be fixed, our first question is

(\mathcal{Q}_1) Where we have to place the center of the spherical hole with fixed radius r in order to maximize $\sigma(\Omega)$?

To give an answer, let $w \in \mathbb{R}^n$ be a unit vector and let us consider the holes $B_r(t)$ with the centers on the w -direction:

$$B_r(t) = \{x(t) = x + tw, x \in B_r(x_0)\} \quad 0 \leq t < \rho_w(x_0, r) - r, \quad (1.3)$$

where

$$\rho_w = \rho_w(x_0, r) = \sup\{t > 0: B_r(x_0 + tw) \subset \Omega\}. \quad (1.4)$$

We stress that ρ_w is the distance between x_0 and the center of the farthest ball from x_0 , well contained in Ω_0 , in the direction w passing at x_0 . Then we consider the following type of annular domains

$$\Omega(t) = \Omega_0 \setminus \overline{B_r(t)}, \quad (1.5)$$

where Ω_0 verifies the assumption of Definition 1.1. We stress that when Ω_0 is a ball, then the sets $\Omega(t)$ are eccentric spherical shells. Furthermore, for any $0 \leq t < \rho_w(x_0, r) - r$, we denote by $\sigma(t)$ the first Steklov–Dirichlet eigenvalue of the Laplacian in $\Omega(t)$, that is

$$\sigma(t) = \min_{\substack{v \in H^1_{\partial B_r(t)}(\Omega(t)) \\ v \neq 0}} \left\{ \int_{\Omega(t)} |Dv|^2 dx, \|v\|_{L^2(\partial\Omega_0)} = 1 \right\}. \quad (1.6)$$

Our second questions is

(\mathcal{Q}_2) Is $\sigma(t)$ decreasing with respect to t ?

Our main result gives an answer to both questions (\mathcal{Q}_1) and (\mathcal{Q}_2).

Theorem 1.2 *Let Ω_0 be as in Definition 1.1, $w \in \mathbb{R}^n$ be a unit vector, $B_r(t)$, ρ_w , $\Omega(t)$ and $\sigma(t)$ be defined as in (1.3), (1.4), (1.5) and (1.6), respectively. Then, $\sigma(t)$ is strictly monotone decreasing with respect to $t \in [0, \rho_w - r)$.*

As an immediate consequence of our main result, we obtain that, in order to maximize $\sigma(t)$, the hole has to be centered at the symmetry point of Ω_0 , when the inner radius is fixed.

To prove Theorem 1.2, we use a shape derivative approach. In particular we compute the first and the second domain derivatives of the first Steklov–Dirichlet eigenvalue. We emphasize that our result implies that the monotonicity property holds also for eccentric spherical shell in any dimension and in particular in two dimensions, as suggested by the numerical computation contained in [14].

Finally, we describe the outline of the paper. In Sect. 2, we summarize some results about the first Steklov–Dirichlet eigenvalue problems. In Sect. 3, we compute the first

shape derivative of $\sigma(t)$, observing that a stationary set occurs when the center of the hole coincides with the center of symmetry of the outer region. Finally, in Sect. 4, we compute the second shape derivative of $\sigma(t)$ and prove the main result.

2 The Steklov–Dirichlet Laplacian eigenvalue problem

In this Section, we study the Steklov–Dirichlet Laplacian eigenvalue problem both on fixed domain Ω and on a parameter-dependent domain $\Omega(t)$, respectively.

2.1 Foundation of the problem

Let $\Omega_0 \subset \mathbb{R}^n$ be an open, bounded set with Lipschitz boundary and such that $B_r(y) \subset \overset{\circ}{\Omega}_0$, where $B_r(y)$ is the ball of radius $r > 0$ centered at y . Let us consider the annular domain $\Omega := \Omega_0 \setminus \overline{B_r(y)}$. In what follows we denote the set of Sobolev functions on Ω vanishing on $\partial B_r(y)$ by $H^1_{\partial B_r(y)}(\Omega)$, that is (see [9]) the closure in $H^1(\Omega)$ of the following set

$$C^\infty_{\partial B_r(y)}(\Omega) := \{u|_\Omega \mid u \in C^\infty_0(\mathbb{R}^n), \text{ spt}(u) \cap \partial B_r(y) = \emptyset\}.$$

It is known (see for instance [1, 8, 16]) that the spectrum of the Steklov–Dirichlet eigenvalue problem (1.2) for the Laplace operator in Ω is discrete and that the sequence of eigenvalues can be ordered as follows

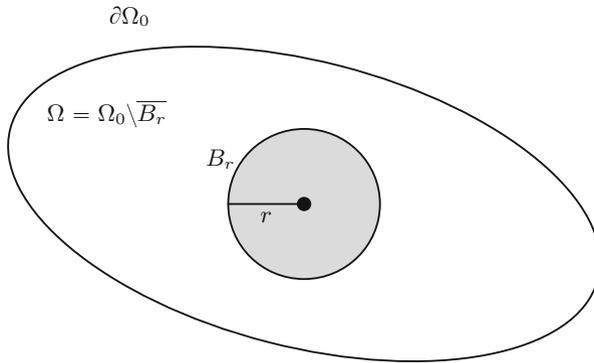
$$0 < \sigma(\Omega) \leq \sigma_2(\Omega) \leq \sigma_3(\Omega) \leq \dots \nearrow +\infty.$$

In particular, the first eigenvalue $\sigma(\Omega)$ has the variational characterization (1.1), see for instance [8, 14, 17]. Moreover in [17] (see also [14]), the authors prove the following result.

Proposition 2.1 *There exists a function $u \in H^1_{\partial B_r(y)}(\Omega)$ which achieves the minimum in (1.1) and is a weak solution to the problem (1.2). Moreover $\sigma(\Omega)$ is simple and the first eigenfunctions have constant sign in Ω .*

In order to prove our main result, we need to compute the first and the second shape derivative of $\sigma(\Omega)$ and, to study these derivatives, we need to consider a family of domains approaching to Ω . In our case, we get the desired monotonicity result by studying the behavior of the involved quantities on the family of domains obtained by moving the hole in a fixed direction.

In what follows we fix notation and recall some preliminary results. Let Ω_0 be as in Definition 1.1. For the reader convenience, from now on, we assume that $x_0 = 0$. In the sequel, we denote by B_r the ball centered at the origin with radius r and we set $\Omega = \Omega_0 \setminus \overline{B_r}$ (see the figure below)



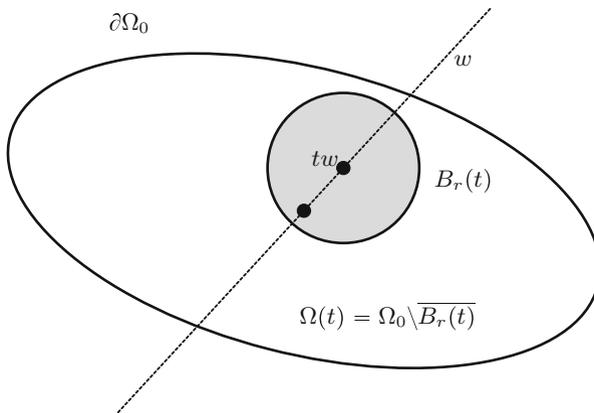
In order to study the Steklov–Dirichlet eigenvalue problem on the annular domain $\Omega(t)$ defined in (1.5), we define a suitable smooth vector field which move the hole B_r in a given direction w keeping the boundary of Ω_0 and the inner radius r fixed. Hence, the perturbed holes have the form described in (1.3). We consider the following variational field in \mathbb{R}^n :

$$V(x) = w\varphi(x), \tag{2.1}$$

where $\varphi \in C_0^\infty(\Omega_0)$ is a cut-off function such that $\varphi(x) = 1$ on $\overline{B_r}$. Consequently, the perturbed annular domains $\Omega(t)$, defined in (1.5), can be seen as

$$\Omega(t) = \{x(t) = x + tV(x), x \in \Omega\} \quad t \in [0, \rho_w - r),$$

where ρ_w is defined in (1.4). Let us observe that $\partial\Omega(t) = \partial\Omega_0 \cup \partial B_r(t)$ and that $\Omega(t)$ is centrally symmetric if and only if $t = 0$ and the center of symmetry is the origin. Also refer to the figure below.



For any $t \in [0, \rho_w - r)$, let $\sigma(t)$ be the first Steklov–Dirichlet eigenvalue (1.6) of the Laplacian in $\Omega(t)$ and u^t be the corresponding normalized and positive eigenfunction. The first eigenvalue admit the variational characterization (1.6) and the eigenfunction

$u^t \in H^1_{\partial B_r(t)}(\Omega(t))$ is a solution to the following eigenvalue problem

$$\begin{cases} \Delta u^t = 0 & \text{in } \Omega(t) \\ \frac{\partial u^t}{\partial \nu} = \sigma(t)u^t & \text{on } \partial\Omega_0 \\ u^t = 0 & \text{on } \partial B_r(t). \end{cases} \tag{2.2}$$

By a little abuse of notation, we also indicate by ν the outer unit normal to the boundary of the annular domain $\partial\Omega(t)$. Then, the variational characterization assures that $u^t \in H^1_{\partial B_r(t)}$ is a positive function such that

$$\sigma(t) = \int_{\Omega_t} |\nabla u^t|^2 dx,$$

and that verifies the following equality

$$\int_{\partial\Omega_0} (u^t)^2 d\mathcal{H}^{n-1} = 1. \tag{2.3}$$

2.2 Differentiability of $\sigma(t)$

Standard arguments on shape derivatives assure that u^t and $\sigma(t)$ are differentiable with respect to t . For the sake of completeness, we sketchily give the proof, that follows exactly the same arguments contained in [13]. The main tool is a general version of the implicit function theorem (see [13, 20] and also [6, Lem.2.1]) applied to the equation transferred onto the fixed domain Ω_0 (for the details, we refer to [13] and also to [6, Lem. 2.7] and [14, Th.1]).

Proposition 2.2 *Let Ω_0 be as in Definition 1.1, $w \in \mathbb{R}^n$ be a unit vector, $B_r(t)$, ρ_w , $\Omega(t)$ and $\sigma(t)$ be defined as in (1.3), (1.4), (1.5) and (1.6), respectively. Let u^t be the first normalized eigenfunction of $\sigma(t)$, then the functions*

$$t \in [0, \rho_w - r) \rightarrow \sigma(t), \quad t \in [0, \rho_w - r) \rightarrow u^t$$

are differentiable for any direction $w \in \mathbb{S}^{n-1}$ and for any $t \in [0, \rho_w - r)$.

Proof Let us fix $t \in [0, \rho_w - r)$ and let $s > 0$ be such that $t + s < \rho_w - r$. Therefore, we are able to consider $\sigma(t + s)$ and u^{t+s} and the following weak formulation for problem (1.6) holds:

$$\int_{\Omega_0} \nabla u^{t+s} \nabla \varphi dx = \sigma(t + s) \int_{\partial\Omega_0} u^{t+s} \varphi d\mathcal{H}^{n-1} \quad \forall \varphi \in H^1_{\partial B_r(t+s)}(\Omega). \tag{2.4}$$

Let V be as in (2.1) and let us define the following map

$$\Phi : (s, x) \in (-\rho_w + r - t, \rho_w - r - t) \times \Omega_0 \rightarrow x + sV(x) \in \mathbb{R}^n.$$

It is easily seen that, for any s , $\Phi(s, \Omega_0) = \Omega_0$, $u^{t+s}(\Phi(s, \cdot)) \in H^1_{\partial B_r(t)}(\Omega)$ and $D\Phi(0, \cdot) = I$, where $D\Phi$ denotes the Jacobian matrix of Φ and I the identity matrix of order n . Therefore, there exists a neighborhood U of 0 such that $\Phi(s, \cdot)$ is a diffeomorphism of Ω_0 and hence, by making a change of variables, (2.4) becomes

$$\int_{\Omega_0} \nabla(u^{t+s}(\Phi(s, \cdot))) (D\Phi(s, \cdot))^{-1} (\nabla(\varphi(\Phi(s, \cdot))) (D\Phi(s, \cdot))^{-1}) |D\Phi(s, \cdot)| dx = \sigma(t+s) \int_{\partial\Omega_0} (u^{t+s}(\Phi(s, \cdot))) (\varphi(\Phi(s, \cdot))) d\mathcal{H}^{n-1} \quad \forall \varphi \in H^1_{\partial B_r(t+s)}(\Omega),$$

and the normalization becomes

$$\int_{\partial\Omega_0} (u^{t+s}(\Phi(s, \cdot)))^2 d\mathcal{H}^{n-1} = 1.$$

Now, let us denote by $(H^1_{\partial B_r(t)}(\Omega))'$ the dual space of $H^1_{\partial B_r(t)}(\Omega)$ and let us define

$$f : (s, v, \sigma) \in \mathbb{R} \times H^1_{\partial B_r(t)}(\Omega) \times \mathbb{R} \rightarrow (f_1, f_2) \in (H^1_{\partial B_r(t)}(\Omega))' \times \mathbb{R},$$

where

$$\begin{cases} \langle f_1(s, v, \sigma), \Psi \rangle = \int_{\Omega_0} ((\nabla v)(D\Phi(s, \cdot))^{-1}) ((\nabla \Psi)(D\Phi(s, \cdot))^{-1}) |D\Phi(s, \cdot)| dx \\ \quad - \sigma \int_{\Omega_0} v \Psi d\mathcal{H}^{n-1} \\ \langle f_2(s, v, \sigma), \Psi \rangle = \int_{\partial\Omega_0} v^2 d\mathcal{H}^{n-1}, \end{cases}$$

for any $\Psi \in H^1_{\partial B_r(t)}(\Omega_0)$. If we consider the function

$$g : s \in U \rightarrow (u^{t+s}(\Phi(s, \cdot)), \sigma(t+s)) \in H^1_{\partial B_r(t)}(\Omega) \times \mathbb{R},$$

then $f(s, g(s)) = 0$ for any $s \in U$ and $g(0) = (u^t, \sigma(t))$. In order to obtain the claim, we have to prove that g is differentiable in $s = 0$, that follows by applying the implicit function Theorem. To do this, we have to prove that

$$\frac{\partial f}{\partial(v, \sigma)} \Big|_{0, u^t, \sigma(t)} : H^1_{\partial B_r(t)}(\Omega) \times \mathbb{R} \rightarrow (H^1_{\partial B_r(t)}(\Omega))' \times \mathbb{R}$$

is an isomorphism. This can be proved following line by line the same arguments of [6, Lem 2.7] and [14, Th. 1] (see also chapter 5 in [13]). □

Remark 2.3 We observe that being u^t harmonic for any $t \in [0, \rho_w - r)$, then $u^t \in C^\infty(\overline{\Omega(t)})$. Then by the general theory of the shape derivatives (see Chapter 5 of [13]), follows that u^t is C^∞ in a neighborhood of t .

Since, to reach our aims, we need to consider the total and partial derivative of u^t with respect to the parameter t , we observe that, by (1.3), we have

$$\frac{d}{dt}[u^t(t, x(t))] = (u^t)' + \langle \nabla u^t, w \rangle, \tag{2.5}$$

where $(u^t)' = \frac{\partial u^t}{\partial t}$ and $\langle \cdot, \cdot \rangle$ denotes the usual scalar product in \mathbb{R}^n . Recalling that the perturbed hole $B_r(t)$ is the zero-level set of the function u^t , then (2.5) implies that

$$(u^t)' = -\frac{\partial u^t}{\partial \nu} \langle \nu, w \rangle, \quad \text{on } \partial B_r(t), \tag{2.6}$$

for any $t \in [0, \rho_w - r)$. Moreover, we observe that it holds

$$\nu = -\frac{\nabla u^t}{|\nabla u^t|} \quad \text{on } \partial B_r(t) \tag{2.7}$$

and

$$(n - 1)H = \operatorname{div} \left(\frac{\nabla u^t}{|\nabla u^t|} \right) \quad \text{on } \partial B_r(t).$$

Therefore, we have

$$\begin{aligned} \frac{(n - 1)}{r} &= (n - 1)H = -\frac{1}{|\nabla u^t|^2} \langle \nabla (|\nabla u^t|), \nabla u^t \rangle \\ &= -\frac{1}{|\nabla u^t|^3} \langle \nabla u^t \cdot D^2(u^t), \nabla u^t \rangle \quad \text{on } \partial B_r(t), \end{aligned} \tag{2.8}$$

where $D^2(u^t)$ denotes the Hessian matrix of u^t . Furthermore, by using (2.6) and the fact that u^t is a solution to (2.2), we get that $(u^t)'$ is a weak solution to the following problem

$$\begin{cases} \Delta(u^t)' = 0 & \text{in } \Omega(t) \\ \frac{\partial(u^t)'}{\partial \nu} = \sigma'(t)u^t + \sigma(t)(u^t)' & \text{on } \partial\Omega_0 \\ (u^t)' = -\frac{\partial u^t}{\partial \nu} \langle \nu, w \rangle & \text{on } \partial B_r(t). \end{cases} \tag{2.9}$$

We observe that, if we derive the normalized condition (2.3), we get

$$\int_{\partial\Omega_0} (u^t)' u^t \, d\mathcal{H}^{n-1} = 0. \tag{2.10}$$

Finally, for every $t \in [0, \rho_w - r)$, it will be useful for the sequel to consider the harmonic extension $H(|\nabla u^t| \langle v, w \rangle)$ of the function with the same boundary value of (2.9) on $\partial B_r(t)$; that is

$$\Delta H(|\nabla u^t| \langle v, w \rangle) = 0 \text{ in } B_r(t), \quad H(|\nabla u^t| \langle v, w \rangle) = |\nabla u^t| \langle v, w \rangle \text{ on } \partial B_r(t). \tag{2.11}$$

3 The first order shape derivative of $\sigma(t)$

In this Section, we compute the first order derivative of the eigenvalue $\sigma(t)$ on $\Omega(t)$. Before doing this, we recall an Hadamard’s formula in the framework of the domain derivative (see for instance [5, 13, 21]).

Let $E \subset \mathbb{R}^n$ be an open bounded set with Lipschitz boundary and let $V(x)$ a vector field such that $V \in W^{1,\infty}(\mathbb{R}^n; \mathbb{R}^n)$. For any $t > 0$, let $E(t) = \{x(t) = x + tV(x), x \in E\}$, and $f(t, x(t))$ be such that $f(t, \cdot) \in W^{1,1}(\mathbb{R}^n)$ and differentiable at t . Then it holds:

$$\begin{aligned} \frac{d}{dt} \int_{E(t)} f(t, x) d\mathcal{H}^{n-1} &= \int_{E(t)} \frac{\partial}{\partial t} f(t, x) d\mathcal{H}^{n-1} + \\ &+ \int_{+\partial E(t)} f(t, x) \langle v, V(x) \rangle d\mathcal{H}^{n-1}, \end{aligned} \tag{3.1}$$

where v is the outer unit normal to the boundary of $E(t)$ and \mathcal{H}^{n-1} denotes the Hausdorff measure (see for instance the Chapter 5 of [13]). We use formula (3.1) to prove the following.

Theorem 3.1 *Let Ω_0 be as in Definition 1.1, $w \in \mathbb{R}^n$ be a unit vector, $B_r(t)$, ρ_w , $\Omega(t)$ and $\sigma(t)$ be defined as in (1.3), (1.4), (1.5) and (1.6), respectively. Then, for any $t \in [0, \rho_w - r)$, it holds*

$$\frac{d}{dt} \sigma(t) = - \int_{\partial B_r(t)} |\nabla u^t|^2 \langle w, v \rangle d\mathcal{H}^{n-1},$$

where u^t the is the normalized, positive eigenfunction corresponding to $\sigma(t)$.

Proof By using the Hadamard’s formula (3.1), by observing that the unit outer normals $v(x)$ to ∂B_r and $v(x(t))$ to $\partial B_r(t)$ coincide and by the fact that u^t is a solution of (2.2), we get

$$\begin{aligned}
 \frac{d}{dt}\sigma(t) &= \frac{d}{dt} \left(\int_{\Omega(t)} |\nabla u^t|^2 dx \right) \\
 &= 2 \int_{\Omega(t)} \langle \nabla u^t, \nabla (u^t)' \rangle dx + \int_{\partial B_r(t)} |\nabla u^t|^2 \langle w, \nu \rangle d\mathcal{H}^{n-1} \\
 &= 2 \left(\int_{\partial\Omega_0} (u^t)' \frac{\partial u}{\partial \nu} d\mathcal{H}^{n-1} + \int_{\partial B_r(t)} (u^t)' \frac{\partial u^t}{\partial \nu} d\mathcal{H}^{n-1} \right) + \\
 &\quad + \int_{\partial B_r(t)} |\nabla u^t|^2 \langle w, \nu \rangle d\mathcal{H}^{n-1} \\
 &= 2 \left(\sigma(t) \int_{\partial\Omega_0} (u^t)' u d\mathcal{H}^{n-1} - \int_{\partial B_r(t)} \left(\frac{\partial u^t}{\partial \nu} \right)^2 \langle w, \nu \rangle d\mathcal{H}^{n-1} \right) + \\
 &\quad + \int_{\partial B_r(t)} |\nabla u^t|^2 \langle w, \nu \rangle d\mathcal{H}^{n-1}.
 \end{aligned}$$

By taking into account the relation (2.10), we get the conclusion. □

In order to obtain our main result, we study the behavior of the first order derivative of $\sigma(t)$ for $t = 0$. We stress that the symmetry of Ω_0 has a key role in the proofs of these results.

Corollary 3.2 *Let Ω_0 be as in Definition 1.1, $w \in \mathbb{R}^n$ be a unit vector, $B_r(t)$, ρ_w , $\Omega(t)$ and $\sigma(t)$ be defined as in (1.3), (1.4), (1.5) and (1.6), respectively. Then it holds*

$$\frac{d}{dt}[\sigma(t)]_{t=0} = 0.$$

Proof Let u^t be the normalized, positive eigenfunctions corresponding to $\sigma(0) = \sigma(\Omega)$. We observe that, since Ω_0 is centrally symmetric with respect the origin (see Definition 1.1), then

$$u^t(x) = u^t(-x) \quad x \in \partial B_r. \tag{3.2}$$

If we denote by x' the symmetric point of x , then (3.2) immediately follows by taking $v(x) = u^t(x')$ as test function in (1.1).

Finally, by Theorem 3.1 and being $\nu = -\frac{x}{r}$, we have

$$\frac{d}{dt}[\sigma(t)]_{t=0} = - \int_{\partial B_r} |\nabla u^t|^2 \langle \nu, w \rangle d\mathcal{H}^{n-1} = 0.$$

□

4 The second order shape derivative of $\sigma(t)$

To prove the main result (Theorem 1.2), we need a stationary property of the first order derivative (analyzed in the previous Section) and a sign of the second order derivative.

We compute the second order domain derivative of $\sigma(t)$, by using the same notations of the previous Section and by recalling some useful definitions from [4].

For any bounded domain $E \in C^{2,\alpha}$ and for any $f \in C^1(\partial E)$, the tangential derivative of v is given by

$$\nabla^\tau f = \nabla f - \langle \nabla f, \nu \rangle \nu \quad \text{on } \partial E, \tag{4.1}$$

where ν is the unit outer normal. Furthermore, for any smooth vector field $\Psi : \bar{E} \mapsto \mathbb{R}^n$, the tangential divergence is defined as

$$\operatorname{div}_{\partial E}(\Psi) = \operatorname{div}(\Psi) - \langle \nu, D(\Psi)\nu \rangle \quad \text{on } \partial E, \tag{4.2}$$

where $D(\Psi)$ is the Jacobian matrix of Ψ . Then, the Gauss theorem on surfaces holds:

$$\int_{\partial E} f \operatorname{div}_{\partial E} \Psi d\mathcal{H}^{n-1} = - \int_{\partial E} \langle \Psi, \nabla^\tau f \rangle d\mathcal{H}^{n-1} + (n-1) \int_{\partial E} f H \langle \Psi, \nu \rangle d\mathcal{H}^{n-1} \tag{4.3}$$

The following Theorem gives the expression of the second order domain derivative of $\sigma(t)$.

Theorem 4.1 *Let Ω_0 be as in Definition 1.1, $w \in \mathbb{R}^n$ be a unit vector, $B_r(t)$, ρ_w , $\Omega(t)$ and $\sigma(t)$ be defined as in (1.3), (1.4), (1.5) and (1.6), respectively. Then, for any $t \in [0, \rho_w - r)$, it holds*

$$\begin{aligned} \frac{d^2}{dt^2} \sigma(t) - \frac{2(n-1)}{r} \frac{d}{dt} \sigma(t) &= -2 \int_{B_r(t)} |\nabla H(|\nabla u^t| \langle w, \nu \rangle)|^2 dx \\ &\quad - \frac{1}{r} \int_{\partial B_r(t)} |\nabla u^t|^2 d\mathcal{H}^{n-1} - \frac{n-2}{r} \int_{\partial B_r(t)} |\nabla u^t|^2 (\langle w, \nu \rangle)^2 d\mathcal{H}^{n-1}, \end{aligned}$$

where u^t is the normalized positive eigenfunction corresponding to $\sigma(t)$ and $H(|\nabla u^t| \langle w, \nu \rangle)$ is the harmonic extension of $|\nabla u^t| \langle w, \nu \rangle$ in $\bar{B}_r(t)$, defined in (2.11).

Proof In order to compute the second order derivative, we first observe that

$$\int_{\partial B_r(t)} |\nabla u^t|^2 \langle w, \nu \rangle d\mathcal{H}^{n-1} = \int_{\partial \Omega(t)} |\nabla u^t|^2 \langle V, \nu \rangle d\mathcal{H}^{n-1},$$

where V is defined in (2.1). Therefore, by using the divergence Theorem, we have

$$\sigma'(t) = - \int_{\Omega(t)} \operatorname{div}(|\nabla u^t|^2 V) dx.$$

By using the Hadamard’s formula (3.1) in the right-hand side, we compute the second order derivative of $\sigma(t)$:

$$\begin{aligned} \sigma''(t) &= -\frac{d}{dt} \int_{\Omega(t)} \operatorname{div}(|\nabla u^t|^2 V) \, dx = -\int_{\Omega(t)} \operatorname{div} \left(\frac{\partial}{\partial t} |\nabla u^t|^2 V \right) \, dx \\ &\quad - \int_{\partial\Omega(t)} \operatorname{div}(|\nabla u^t|^2 V) \langle V, \nu \rangle \, d\mathcal{H}^{n-1}. \end{aligned} \tag{4.4}$$

Let us consider the second term in (4.4), by using the tangential divergence (4.2), we have

$$\begin{aligned} -\int_{\partial\Omega(t)} \operatorname{div}(|\nabla u^t|^2 V) \langle V, \nu \rangle \, d\mathcal{H}^{n-1} &= -\int_{\partial B_r(t)} \operatorname{div}(|\nabla u^t|^2 w) \langle w, \nu \rangle \, d\mathcal{H}^{n-1} \\ &= -\int_{\partial B_r(t)} \operatorname{div}_{\partial B_r(t)}(|\nabla u^t|^2 w) \langle w, \nu \rangle \, d\mathcal{H}^{n-1} - \int_{\partial B_r(t)} \langle \nu, D(|\nabla u^t|^2 w) \nu \rangle \, d\mathcal{H}^{n-1}. \end{aligned} \tag{4.5}$$

Hence, by inserting (4.5) in (4.4), we have

$$\begin{aligned} \sigma''(t) &= -\int_{\Omega(t)} \operatorname{div} \left(\frac{\partial}{\partial t} |\nabla u^t|^2 V \right) \, dx - \int_{\partial B_r(t)} \operatorname{div}_{\partial B_r(t)}(|\nabla u^t|^2 w) \langle w, \nu \rangle \, d\mathcal{H}^{n-1} \\ &\quad - \int_{\partial B_r(t)} \langle \nu, D(|\nabla u^t|^2 w) \nu \rangle \, d\mathcal{H}^{n-1} := I + II + III. \end{aligned} \tag{4.6}$$

For reader’s convenience, we separately study the terms I , II and III in (4.6). Let us focus on I ; by using the divergence Theorem, we have

$$\begin{aligned} I &= -\int_{\Omega(t)} \operatorname{div} \left(\frac{\partial}{\partial t} |\nabla u^t|^2 V \right) \, dx = -2 \int_{\Omega(t)} \operatorname{div} (\langle \nabla u^t, \nabla(u^t)' \rangle V) \, dx \\ &= -2 \int_{\partial\Omega(t)} \langle \nabla u^t, \nabla(u^t)' \rangle \langle V, \nu \rangle \, d\mathcal{H}^{n-1} \\ &= -2 \int_{\partial B_r(t)} \langle \nabla u^t, \nabla(u^t)' \rangle \langle w, \nu \rangle \, d\mathcal{H}^{n-1} \\ &= -2 \int_{\partial B_r(t)} \frac{\partial(u^t)'}{\partial \nu} (u^t)' \, d\mathcal{H}^{n-1} \\ &= -2 \int_{B_r(t)} |\nabla H(|\nabla u^t| \langle w, \nu \rangle)|^2 \, dx. \end{aligned} \tag{4.7}$$

Let us remark that ν is also the inner unit normal to $\partial B_r(t)$.

Now let us consider II; by using the Gauss Theorem (4.3), we have

$$\begin{aligned}
 II &= - \int_{\partial B_r(t)} \operatorname{div}_{\partial B_r(t)}(|\nabla u^t|^2 w) \langle w, v \rangle d\mathcal{H}^{n-1} \\
 &= \int_{\partial B_r(t)} \langle |\nabla u^t|^2 w, \nabla^\tau(\langle w, v \rangle) \rangle d\mathcal{H}^{n-1} - \frac{n-1}{r} \int_{\partial B_r(t)} |\nabla u^t|^2 (\langle w, v \rangle)^2 d\mathcal{H}^{n-1} \\
 &= \int_{\partial B_r(t)} \langle |\nabla u^t|^2 w, \nabla(\langle w, v \rangle) - \langle \nabla(\langle w, v \rangle), v \rangle \rangle d\mathcal{H}^{n-1} \\
 &\quad - \frac{n-1}{r} \int_{\partial B_r(t)} |\nabla u^t|^2 (\langle w, v \rangle)^2 d\mathcal{H}^{n-1} \\
 &= \int_{\partial B_r(t)} \left\langle |\nabla u^t|^2 w, -\frac{w}{r} + \left\langle \frac{w}{r}, v \right\rangle v \right\rangle d\mathcal{H}^{n-1} \\
 &\quad - \frac{n-1}{r} \int_{\partial B_r(t)} |\nabla u^t|^2 (\langle w, v \rangle)^2 d\mathcal{H}^{n-1} \\
 &= -\frac{1}{r} \int_{\partial B_r(t)} |\nabla u^t|^2 d\mathcal{H}^{n-1} + \frac{1}{r} \int_{\partial B_r(t)} |\nabla u^t|^2 (\langle w, v \rangle)^2 d\mathcal{H}^{n-1} \\
 &\quad - \frac{n-1}{r} \int_{\partial B_r(t)} |\nabla u^t|^2 (\langle w, v \rangle)^2 d\mathcal{H}^{n-1} \\
 &= -\frac{1}{r} \int_{\partial B_r(t)} |\nabla u^t|^2 d\mathcal{H}^{n-1} - \frac{n-2}{r} \int_{\partial B_r(t)} |\nabla u^t|^2 (\langle w, v \rangle)^2 d\mathcal{H}^{n-1}, \tag{4.8}
 \end{aligned}$$

where in the third line we have used the tangential gradient (4.1).

Let us focus on III; we have

$$\begin{aligned}
 III &= - \int_{\partial B_r(t)} \langle v, D(|\nabla u^t|^2 w) v \rangle d\mathcal{H}^{n-1} \\
 &= -2 \int_{\partial B_r(t)} \langle v, D^2(u^t) \nabla u^t \rangle \langle w, v \rangle d\mathcal{H}^{n-1} \\
 &= 2 \int_{\partial B_r(t)} \frac{1}{|\nabla u^t|^3} \langle \nabla u^t, D^2(u^t) \nabla u^t \rangle |\nabla u^t|^2 \langle w, v \rangle d\mathcal{H}^{n-1} \\
 &= -\frac{2(n-1)}{r} \int_{\partial B_r(t)} |\nabla u^t|^2 \langle w, v \rangle d\mathcal{H}^{n-1}, \tag{4.9}
 \end{aligned}$$

where we have used the relations (2.7) and (2.8). The conclusion follows by using (4.7), (4.8) and (4.9) in (4.6). □

Finally, the proof of the main Theorem is a direct consequence of the computed expressions of the first and second order shape derivative of the first Steklov–Dirichlet eigenvalue.

Proof of the Theorem 1.2 Since $n \geq 2$, the claim is a direct consequence of the Theorems 3.1, Corollary 3.2 and Theorem 4.1 □

Remark 4.2 An immediate consequence of Theorem 1.2 is that $\sigma(t)$ is maximum when the hole is centered at the symmetry point of Ω_0 , when $t = 0$.

Remark 4.3 We stress that, when $\Omega_0 = B_R$, the authors in [14] prove the following estimate in two dimensions for $R > r$:

$$\liminf_{t \rightarrow (R-r)^-} \sigma(t) \geq \frac{r}{2R(R-r)}, \quad (4.10)$$

where $\sigma(t) = \sigma(\Omega(t))$ and $\Omega(t) = B_R \setminus B_r(t)$. By Theorem 1.2, the estimate (4.10) can be written as the following lower bound

$$\frac{1}{R \log(\frac{R}{r})} = \sigma(B_R \setminus B_r) \geq \sigma(t) \geq \frac{r}{2R(R-r)}, \quad \forall t \in [0, \rho_w - r). \quad (4.11)$$

Finally, we observe that the inequality (4.11) gives an upper and lower bound for $\sigma(t)$ in terms of the two radius of the eccentric annulus.

Acknowledgements This work has been partially supported by the MiUR-PRIN 2017 grant ‘‘Qualitative and quantitative aspects of nonlinear PDEs’’, by GNAMPA of INdAM and by FRA 2020 ‘‘Optimization problems in Geometric-functional inequalities and nonlinear PDEs’’ (OPTImIzE). We would like to thank the reviewer for his/her suggestions to improve the paper.

Funding Open access funding provided by Università degli Studi di Napoli Federico II within the CRUI-CARE Agreement.

Declarations

Conflict of interest This paper does not disclose of potential conflicts of interest.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Agranovich, M.S.: On a mixed Poincaré-Steklov type spectral problem in a Lipschitz domain. *Russ. J. Math. Phys.* **13**(3), 239–244 (2006)
2. Anoop, T. V., Ashok Kumar, K.: Domain variations of the first eigenvalue via a strict Faber-Krahn type inequality. pp 1–21 (2022)
3. Anoop, T.V., Ashok Kumar, K., Kesavan, S.: A shape variation result via the geometry of eigenfunctions. *J. Differ. Equ.* **298**, 430–462 (2021)
4. Bandle, C., Wagner, A.: Second domain variation for problems with Robin boundary conditions. *J. Optim. Theory Appl.* **167**(2), 430–463 (2015)
5. Bellettini, G.: *Lecture notes on mean curvature flow: barriers and singular perturbations*. Springer, p 327 (2014)

6. Bonder, J.F., Groisman, P., Rossi, J.D.: Optimization of the first Steklov eigenvalue in domains with holes: a shape derivative approach. *Ann. Mat. Pura Appl.* **186**(2), 341–358 (2007)
7. Della Pietra, F., Piscitelli, G.: An optimal bound for nonlinear eigenvalues and torsional rigidity on domains with holes. *Milan J. Math.* **88**(2), 373–384 (2020)
8. Dittmar, B.: Eigen Value Problems and Conformal Mapping, *Handbook of Complex Analysis: Geometric Function Theory*, vol. 2, pp. 669–686. Elsevier, Amsterdam (2005)
9. Egert, M., Tolksdorf, P.: Characterizations of Sobolev functions that vanish on a part of the boundary. *Discrete Contin. Dyn. Syst. Ser. S* **10**(4), 729–743 (2017)
10. Ftouhi, I.: Where to place a spherical obstacle so as to maximize the first Steklov eigenvalue. *ESAIM Control Optim. Calc. Var.* **28**(6), 1–21 (2022)
11. Gavitone, N., Paoli, G., Piscitelli, G., Sannipoli, R.: An Isoperimetric inequality for the first Steklov–Dirichlet Laplacian eigenvalue of convex sets with a spherical hole. *Pac. J. Math.* **320**(2), 241–259 (2022)
12. Gavitone, N., Sannipoli, R.: On a Steklov–Robin eigenvalue problem. *J. Math. Anal. Appl.* (in press)
13. Henrot, A., Pierre, M.: Shape variation and optimization. *EMS Tracts Math.* **28**, 401 (2018)
14. Hong, J., Lim, M., Seo, D.H.: On the first Steklov? Dirichlet eigenvalue for eccentric annuli. *Ann. Mat. Pur. Appl.* **201**(2), 769–799 (2022)
15. Li, L.: On the placement of an obstacle so as to optimize the Dirichlet heat content. *SIAM J. Math. Anal.* **54**(3), 3275–3291 (2022)
16. Pal'tsev, B.V.E.: Mixed problems with non-homogeneous boundary conditions in Lipschitz domains for second-order elliptic equations with a parameter. *Sb. Math.* **187**(4), 525 (1996)
17. Paoli, G., Piscitelli, G., Sannipoli, R.: A stability result for the Steklov Laplacian Eigenvalue Problem with a spherical obstacle. *Commun. Pure Appl. Anal.* **20**(1), 145–158 (2021)
18. Paoli, G., Piscitelli, G., Trani, L.: Sharp estimates for the first p -laplacian eigenvalue and for the p -torsional rigidity on convex sets with holes. *ESAIM Control Calc. Var.* **26**(111), 1–15 (2020)
19. Seo, D.H.: A shape optimization problem for the first mixed Steklov–Dirichlet eigenvalue. *Ann. Glob. Anal. Geom.* **59**(3), 345–365 (2021)
20. Schwartz, L.: *Cours d'analyse*, 2nd edn. Hermann, Paris (1981)
21. Sokolowski, J., Zolésio, P.: *Introduction to Shape Optimization*, pp. 5–12. Springer, Berlin (1992)
22. Verma, S., Santhanam, G.: On eigenvalue problems related to the laplacian in a class of doubly connected domains. *Monatsh. Math.* **193**, 879–899 (2020)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.