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Two theorems on the intersections of horospheres in asymptotically harmonic spaces

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Abstract
We use Busemann functions to construct volume preserving mappings in an asymptotically harmonic manifold. If the asymptotically harmonic manifold satisfies the visibility condition, we construct mappings which preserve distances in some directions. We also prove that some integrals on the intersection of horospheres are independent of the differences between the values of the corresponding Busemann functions and we establish an upper bound of the volume of the intersection of two horospheres which is independent of the difference between values of corresponding Busemann functions.

Keywords Asymptotically harmonic manifold · Busemann function · Horosphere · Visibility manifold

Mathematics Subject Classification 53C25 · 53C30

1 Introduction

Let (M, g) be a simply connected, complete Riemannian manifold without con- jugate points. Let $d(p, q)$ be the distance between $p, q \in M$. For each unit tangent vector v to M, the Busemann function $b_v : M \to \mathbb{R}$ on (M, g) is defined by

$$b_v(x) = \lim_{t\to\infty} (d(x, \gamma_v(t)) - t),$$

where $\gamma_v : [0, \infty) \to M$ is a geodesic ray such that $\gamma_v'(0) = v$. Busemann functions are convex and C^2 [7, 10]. The level hypersurfaces of Busemann functions are called

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horospheres. A Riemannian manifold is called harmonic, if, about any point, the geodesic spheres of sufficiently small radii are of constant mean curvature. A simply connected, complete Riemannian manifold (M, g) without conjugate points is called asymptotically harmonic if the mean curvatures of the horospheres are constant [9, 14]. Szabó [18] proved that a compact harmonic manifolds with a finite fundamental group is locally symmetric. Knieper [13] proved that a compact harmonic manifold with an infinite fundamental group is locally symmetric provided that the harmonic manifold has non-positive curvature or the fundamental group is Gromov hyperbolic. Ranjan and Shah [17] proved that a non-compact harmonic manifold with subexponential volume growth is flat. We refer to [14] for further results, including asymptotically harmonic manifolds. All harmonic manifolds are asymptotically harmonic, and all known asymptotically harmonic manifolds are harmonic and homogeneous [9]. It is also remarkable that the following question is remained open: Is a harmonic manifold homogeneous?

To consider the homogeneousness of a harmonic manifold and, more generally, of an asymptotically harmonic manifold, we construct a volume preserving mapping which maps one point to another point. For this construction, we use the mean curvature of horospheres and the variation of the volume density for the flow in the orthogonal direction to horospheres.

Let $C_c(M)$ be the set of continuous functions with compact support and T^1M be a unit tangent bundle. We obtain the following theorem.

Theorem 1.1 *Let (M, g) be an asymptotically harmonic manifold. Then, for all points $p \neq q \in M$, there exists a diffeomorphism $F : M \to M$ such that $F(p) = q$ and*

$$\int_M f(x)d\mu(x) = \int_M f(F(x))d\mu(x),$$

for all $f \in C_c(M)$, where $d\mu$ is the Riemannian measure on M.

Szabó [18] proved that a Riemannian manifold is a harmonic manifold if and only if the volume of the intersection of geodesic spheres depends only on the radii and the distance between the centers of the geodesic spheres. Csikós and M. Horváth [4, 5] proved that the intersections can be restricted to the cases with the same radii, and they also proved that the volume of a tubular neighborhood about a geodesic depends only on the length of the geodesic and the radius if and only if the Riemannian manifold is harmonic. In addition, some relations between integrals and measures on a harmonic manifold and its ideal boundary were found by Itoh and Satoh [11] and by Rouviére [16]. Knieper and Peyerimhoff [12] also considered the integrals and measures on harmonic manifolds to find a solution of the Dirichlet problem at infinity, and Biswas, Knieper, and Peyerimhoff [2] proved that there exists a Fourier transform between harmonic manifolds and its ideal boundary.

A Hadamard manifold (M, g) is a simply connected, complete Riemannian manifold of non-positive sectional curvature, and it is called a a visibility manifold (or satisfies the visibility condition) if, for any two different points v_1, v_2 at infinity, there is a geodesic $\gamma : \mathbb{R} \to M$ with $\gamma(\infty) = v_1$, $\gamma(-\infty) = v_2$ (see Sect. 2). All intersections of two horospheres in a Hadamard manifold are bounded if and only if the Hadamard manifold is a visibility manifold (see Definition 2.6 and Lemma 2.7). We note that the volume of an intersection of two horospheres in a visibility manifold is finite. In particular, for every harmonic manifold which satisfies the visibility condition, the volume of the intersection $b_{v_1}^{-1}(c_1) \cap b_{v_2}^{-1}(c_2)$ of two horospheres is independent of $c_1 - c_2$. For asymptotically harmonic manifolds which satisfy the visibility condition (i.e., asymptotically harmonic, visibility manifolds), in this article, we prove that some integrals on the intersection of two horospheres are independent of the difference between values of corresponding Busemann functions. We also obtain an upper bound of the volume of the intersection of two horospheres. Throughout this paper, we assume that the dimension of the manifold is $n \geq 2$. Our main theorem is the following:

Theorem 1.2 *Let (M, g) be an asymptotically harmonic, visibility manifold. Let $p \in M$, $v_1 \neq v_2 \in T_p^1 M$, and $c \in \mathbb{R}$. Then there exists a constant $C > 0$ such that the $(n-2)$-dimensional volume of the intersection $b_{v_1}^{-1}(c-t) \cap b_{v_2}^{-1}(c+t)$ is less than C, for all $t \in \mathbb{R}$.*

2 Preliminary

The integration of a function on a connected Riemannian manifold (M, g) can be computed in terms of the integrals on hypersurfaces:

Proposition 2.1 [6] *Let (M, g) be a connected Riemannian manifold. Let $\varphi : M \to \mathbb{R}$ be a C^1 function such that $\nabla\varphi$ is non-vanishing on M, and let S_t be the hypersuface defined by $S_t = \{x \in M : \varphi(x) = t\}$, for all $t \in \mathbb{R}$. Then, for all $f \in C_c(M)$ and $t \in \mathbb{R}$,*

$$\int_M f(x) d\mu(x) = \int_{\mathbb{R}} \int_{S_t} \frac{f(x)}{\|\nabla\varphi(x)\|} d\mu_t(x) dt,$$

where $d\mu$ is the Riemannian measure on M and $d\mu_t$ is the induced Riemannian measure on S_t.

We introduce an infinitesimal "volume preserving" mapping on (M, g) in [8]. We say that a vector field X on M is volume preserving if $L_X(d\mu) = 0$, where L_X is the Lie derivative with respect to X. If ϕ_t is the flow generated by X, then we call ϕ_t volume preserving if X is volume preserving. In that case, we have $(\phi_t)^*(d\mu) = d\mu$. This flows preserve integrals on a Riemannian manifold (M, g) since it preserves the Riemannian measure if it is a diffeomorphism.

Proposition 2.2 *Let ϕ_t be a flow on a Riemannian manifold (M, g) and suppose that ϕ_t is a diffeomorphism on M for all $t \in \mathbb{R}$. Then ϕ_t is volume preserving if and only if*

$$\int_M f(x) d\mu(x) = \int_M f(\phi_t(x)) d\mu(x), \tag{1}$$

for all $f \in C_c(M)$, where $d\mu$ is the Riemannian measure on M.

Proof If ϕ_t is volume preserving, then, clearly, Eq. (1) holds. Now, suppose that Eq. (1) holds for all $f \in C_c(M)$. Then

$$\int_M f(\phi_t(x))d\mu(x) = \int_M f(x)d\mu(x) = \int_M f(\phi_t(x))(\phi_t)^*(d\mu)(x) \tag{2}$$

for all $f \in C_c(M)$. We denote $(\phi_t)^*(d\mu) = f_0\, d\mu$ for some function f_0. Then, by Eq. (2), $f_0 = 1$ and $(\phi_t)^*(d\mu) = d\mu$. □

Now, we define asymptotic geodesic rays:

Definition 2.3 [3] Let (M, g) be a Hadamard manifold, and $\gamma_1, \gamma_2 : [0, \infty) \to M$ geodesic rays. γ_1, γ_2 are said to be asymptotic if there exists a constant $C > 0$ such that

$$d(\gamma_1(t), \gamma_2(t)) \leq C,$$

for all $t \geq 0$.

This gives an equivalence relation on geodesic rays: two geodesic rays are equivalent if and only if they are asymptotic. The set $\partial_\infty M$ of points at infinity is the set of equivalence classes of this relation. This is also called the ideal boundary of (M, g). Consequently, there exists a set of distance functions from each point and their limits. Let $C(M)$ be the space of continuous functions on M equipped with the topology of uniform convergence on bounded subsets. Let $C_*(M)$ denote the quotient of $C(M)$ by the 1-dimensional subspace of constant functions. For all $v \in T^1M$ and $t \in \mathbb{R}$,

$$\nabla b_v(\gamma(t)) = -\gamma'(t),$$

where γ is a geodesic ray in (M, g) asymptotic to γ_v. In particular, the image of Busemann functions in $C_*(M)$ can be associated to geodesic rays. The points at infinity also corresponds to the images of Busemann functions in $C_*(M)$.

Proposition 2.4 [3] *Let (M, g) be a Hadamard manifold. Then the Busemann functions associated to asymptotic geodesic rays in M are equal up to addition of a constant.*

Proposition 2.5 [3] *Let (M, g) be a Hadamard manifold, $\theta \in \partial_\infty M$, and $x \in M$. Then there exists a unique geodesic ray $\gamma \in \theta$ from x.*

By the propositions, points at infinity bijectively correspond to images of Busemann functions in $C_*(M)$.

For two elements of $\partial_\infty M$, there could exist a geodesic from one direction to another direction. If such geodesic exists for all pairs of distinct elements of $\partial_\infty M$, then we call such a Hadamard manifold a visibility manifold:

Definition 2.6 [1] A Hadamard manifold (M, g) is called a visibility manifold if, for all $p \in M$ and $v_1 \neq v_2 \in T_p^1 M$, there exists a geodesic ray γ such that γ is a asymptotic to γ_{v_1}, and the geodesic ray $t \mapsto \gamma(-t)$ is asymptotic to γ_{v_2}.

Such geodesic ray γ is said to be bi-asymptotic to v_1, v_2. There exist several equivalent conditions for the visibility, one of which is as follows.

Lemma 2.7 [1] *Let (M, g) be a Hadamard manifold. Then the following statements are equivalent:*

(i) (M, g) is a visibility manifold.
(ii) $b_{v_1}^{-1}((-\infty, c_1)) \cap b_{v_2}^{-1}((-\infty, c_2))$ is bounded for all $p \in M$, $v_1 \neq v_2 \in T_p^1 M$, and $c_1, c_2 \in \mathbb{R}$.

For example, if a Hadamard manifold (M, g) satisfies the curvature condition $K \leq -a^2 < 0$, for some $a \in \mathbb{R}$, then (M, g) is a visibility manifold [1]. In a Hadamard manifold, bi-asymptotic geodesics are normal geodesics of some intersection of horospheres of the form $b_v^{-1}(0) \cap b_{-v}^{-1}(0)$:

Proposition 2.8 [7] *Let (M, g) be a Hadamard manifold. Then, for every $v \in T^1 M$, $b_v^{-1}(0) \cap b_{-v}^{-1}(0)$ is connected,*

$$\nabla b_v(x) + \nabla b_{-v}(x) = 0,$$

for all $x \in b_v^{-1}(0) \cap b_{-v}^{-1}(0)$, and the geodesics which is asymptotic to γ_v and intersects $b_v^{-1}(0) \cap b_{-v}^{-1}(0)$ orthogonally at a point are bi-asymptotic to $v, -v$.

Consequently, for two distinct bi-asymptotic geodesics, there exists a 2-dimensional flat, totally geodesic embedding containing them:

Theorem 2.9 [7] *Let (M, g) be a Hadamard manifold. Then, for all $v \in T^1 M$, the Busemann function b_v is convex, and the set $b_v^{-1}(0) \cap b_{-v}^{-1}(0)$ is convex. If two geodesics γ_1, γ_2 are bi-asymptotic, there exists $a > 0$ and a totally geodesic, isometric embedding $F : [0, a] \times \mathbb{R} \to M$ such that $\gamma_1 = F|_{\{0\} \times \mathbb{R}}$ and $\gamma_2 = F|_{\{a\} \times \mathbb{R}}$.*

Let $p \in M$, $v_1 \neq v_2 \in T_p^1 M$, and

$$D = \{x \in M : \nabla b_{v_1}(x) + \nabla b_{v_2}(x) = 0\}.$$

Note that D is closed and D is nonempty if (M, g) is a visibility manifold. For every point $x \in D$, by Proposition 2.5, there exists a unique geodesic ray from x which is asymptotic to v_1, so it is bi-asymptotic to v_1, v_2. In particular, by Theorem 2.9, D is connected. Suppose that D is nonempty, so D contains a bi-asymptotic geodesic to v_1, v_2. Let c_0 be the constant value of $b_{v_1} + b_{v_2}$ on D. If $c_1, c_2 \in \mathbb{R}$ and $c_1 + c_2 = c_0$, then there exists a point x on each bi-asymptotic geodesic γ such that $b_{v_1}(x) = c_1$ and $b_{v_2}(x) = c_2$ since γ is contained in D which implies $b_{v_1}(\gamma(0)) + b_{v_2}(\gamma(0)) = c_0 = c_1 + c_2$, and $\gamma'(t) = -\nabla b_{v_1}(\gamma(t)) = \nabla b_{v_2}(\gamma(t))$ which implies

$$b_{v_1}(x) = b_{v_1}(\gamma(0)) - t = c_1, \quad b_{v_2}(x) = b_{v_2}(\gamma(0)) + t = c_2,$$

where $t = b_{v_1}(\gamma(0)) - c_1 = c_2 - b_{v_2}(\gamma(0))$ and $x = \gamma(t)$. Set $v = \nabla b_{v_1}(x)$. Then, since $v = -\gamma'(t)$ and γ is bi-asymptotic to v_1 and v_2, $-v$ and v are asymptotic to v_1 and v_2, respectively. By Proposition 2.4, the equations

$$b_{-v} = b_{v_1} - c_1, \quad b_v = b_{v_2} - c_2$$

hold. Consequently, by Proposition 2.8,

$$b_{v_1}^{-1}(c_1) \cap b_{v_2}^{-1}(c_2) = b_v^{-1}(0) \cap b_{-v}^{-1}(0) \subseteq D \tag{3}$$

holds.

3 Proof of Theorem 1.1

Let $p, q \in M$. Since (M, g) is connected and complete, there exists a unit-speed geodesic γ such that $\gamma(0) = p$, $\gamma(t_0) = q$ for some $t_0 > 0$. Set $v = -\gamma'(0)$. Let $S_t = b_v^{-1}(t)$ be the level set of b_v for all $t \in \mathbb{R}$. Consider a diffeomorphism $\phi : \mathbb{R} \times S_0 \to M$ defined by

$$\phi_t(x) = \phi(t, x) := \exp_x\left(t\nabla b_v(x)\right),$$

for all $(t, x) \in \mathbb{R} \times S_0$.

For all $w \in T^1M$, we denote the space of orthogonal tangent vectors to w by $w^\perp$. Define $U = U(t) : \nabla b_v(\phi_t(x))^\perp \to \nabla b_v(\phi_t(x))^\perp$ by

$$U(w) := \nabla_w \nabla b_v,$$

for all $t \in \mathbb{R}$ and $w \in \nabla b_v(\phi_t(x))^\perp$. Since (M, g) is asymptotically harmonic, $\operatorname{tr} U = -\Delta b_v = h$ for some constant $h \in \mathbb{R}$. Since b_v is convex, $h \geq 0$.

Lemma 3.1 *For every $t \in \mathbb{R}$,*

$$(\phi_t)^*(d\mu_t) = e^{ht} d\mu_0.$$

We note that Lemma 3.1 was given in [15]. Here, we provide the proof for the reader's convenience.

Proof For all $(t, x) \in \mathbb{R} \times S_0$,

$$(\phi_t)_*|_x \left(\frac{\partial}{\partial t}\right) = \nabla b_v(\phi_t(x)).$$

Then

$$(\phi_t)_*|_x (w) = \exp_*\big|_{t\nabla b_v(x)} (w, tU(w)). \tag{4}$$

Set $x \in S_0$ and $w \in \nabla b_v(x)^\perp$. Define

$$\Gamma(s, t) := \exp_{\sigma(s)}(t\nabla b_v(\sigma(s))),$$

for all $s \in (-\varepsilon, \varepsilon)$ and $t \in \mathbb{R}$, where $\sigma = \sigma(s)$ is the curve in S_0 such that $\sigma(0) = x$ and $\sigma'(0) = w$ and $\varepsilon > 0$.

We denote $\partial_t\Gamma = \Gamma_*\left(\frac{\partial}{\partial t}\right)$, $\partial_s\Gamma = \Gamma_*\left(\frac{\partial}{\partial s}\right)$, and $J(t) = \partial_s\Gamma(0,t)$, for all $t \in \mathbb{R}$. So, $(\phi_t)_*|_x\left(\frac{\partial}{\partial t}\right) = \nabla b_v(\phi_t(x))$ and $(\phi_t)_*|_x(w) = J(t)$, for all $t \in \mathbb{R}$. Then, from Eq. (4), we have

$$U(J(t)) = \nabla_{\partial_s\Gamma(0,t)}\partial_t\Gamma = \nabla_{\partial_t\Gamma(0,t)}\partial_s\Gamma = J'(t),$$

for all $t \in \mathbb{R}$. Thus, we get

$$U = A'A^{-1},$$

on $\nabla b_v(\phi_t(x))^\perp$, where $t \in \mathbb{R}$, R_t is the Jacobi operator along $t \mapsto \phi_t(x)$, and $A = A(t)$ is the $(1,1)$ tensor field on $\nabla b_v(\phi_t(x))^\perp$ such that $A'' + R_tA = 0$. Since (M,g) is asymptotically harmonic, $\operatorname{tr} U = h$ is constant. So,

$$(\ln(\det A(t)))' = \operatorname{tr} U(t) = h,$$

and $\det A(t) = e^{ht}$. Hence,

$$\begin{aligned}(\phi_t)^*(d\mu_t(\phi_t(x))) &= d\mu_t\left((\phi_t)_*|_x(w_1),\ldots,(\phi_t)_*|_x(w_{n-1})\right)d\mu_0(x)\\ &= \det A(t)\,d\mu_0(x) = e^{ht}d\mu_0(x),\end{aligned}$$

where $w_1,\ldots,w_{n-1}$ are orthonormal tangent vectors to S_0 at x. □

Define $F : M \to M$ by

$$F(\phi(t,x)) := \phi(\alpha(t),x),$$

for all $(t,x) \in \mathbb{R}\times S_0$, where $\alpha = \alpha(t)$ is a smooth function on $\mathbb{R}$ such that $\alpha(0) = t_0$.

Corollary 3.2 *For every $t \in \mathbb{R}$,*

$$F^*(d\mu_{\alpha(t)}) = e^{h\alpha(t)-ht}d\mu_t.$$

Proof For all points $x \in S_t$,

$$F(x) = \exp_x((\alpha(t)-t)\nabla b_v(x)),$$

So, similarly to Lemma 3.1, Corollary 3.2 holds. □

Therefore, by Proposition 2.1 and Corollary 3.2,

$$\begin{aligned}\int_M f(x)d\mu(x) &= \int_{\mathbb{R}}\int_{S_s} f(x)d\mu_s(x)ds\\ &= \int_{\mathbb{R}}\int_{F^{-1}(S_s)} f(F(x))F^*(d\mu_s)(x)ds\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} \int_{S_t} f(F(x))\alpha'(t)F^*(d\mu_{\alpha(t)})(x)dt \\
&= \int_{\mathbb{R}} \int_{S_t} f(F(x))\alpha'(t)e^{h\alpha(t)-ht}d\mu_t(x)dt \\
&= \int_{\mathbb{R}} \int_{S_t} f(F(x))d\mu_t(x)dt \\
&= \int_M f(F(x))d\mu(x),
\end{aligned}
$$

if α is strictly monotonic and

$$\alpha'(t)e^{h\alpha(t)-ht} = 1 \tag{5}$$

for all $t \in \mathbb{R}$. Consider two cases. First assume that $h = 0$. Then, Eq. (5) holds if $\alpha(t) = t + t_0$. Thus $F = \phi_{t_0}$. Now assume that $h > 0$. Then Eq. (5) holds if and only if

$$\alpha'(t)e^{h\alpha(t)} = e^{ht},$$

or, equivalently,

$$\frac{e^{h\alpha(t)} - e^{ht_0}}{h} = \frac{e^{ht} - 1}{h},$$

so

$$\alpha(t) = \frac{1}{h} \ln\left(e^{ht} + e^{ht_0} - 1\right).$$

Therefore,

$$F(\phi(t, x)) = \phi\left(\frac{1}{h} \ln\left(e^{ht} + e^{ht_0} - 1\right), x\right).$$

□

Remark If $h = 0$, then $F = \phi_{t_0}$ is just a translation in $\mathbb{R}^n$, so it is an isometry. On the other hand, if $h > 0$, then

$$F_*(\nabla b_v(\phi_t(x))) = \alpha'(t)\nabla b_v(\phi_{\alpha(t)}(x)),$$

so F is not an isometry since

$$\alpha'(t) = \frac{he^{ht}}{h(e^{ht} + e^{ht_0} - 1)} = \frac{e^{ht}}{e^{ht} + e^{ht_0} - 1} < 1.$$

Also, $\alpha(t) - t$ is a strictly monotone decreasing function on $\mathbb{R}$, and F is well-defined.

4 Proof of Theorem 1.2

For asymptotically harmonic manifolds, we can find a volume preserving mapping from two Busemann functions directly.

Lemma 4.1 *Let (M, g) be an asymptotically harmonic manifold, $p \in M$, and $v_1 \neq v_2 \in T_p^1 M$. Then the vector field $X = \nabla b_{v_1} - \nabla b_{v_2}$ is volume preserving.*

Proof Note that X is non-vanishing since $v_1 \neq v_2$ and

$$\operatorname{div} X = -\Delta b_{v_1} + \Delta b_{v_2}.$$

In particular, since (M, g) is asymptotically harmonic, $L_X(d\mu) = (\operatorname{div} X)d\mu = 0$. □

Consequently, this vector field derives a flow such that it preserves db_{v_1} and db_{v_2}, and the flow is not volume preserving in general.

Lemma 4.2 *Let (M, g) be an asymptotically harmonic manifold, $p \in M$, $v_1 \neq v_2 \in T_p^1 M$, and*

$$X = \frac{1}{\|\nabla b_{v_1} - \nabla b_{v_2}\|^2}(\nabla b_{v_1} - \nabla b_{v_2}).$$

Then the flow ϕ_t of X satisfies

$$(\phi_t)^*(db_{v_1}) = db_{v_1}, \quad (\phi_t)^*(db_{v_2}) = db_{v_2}, \tag{6}$$

for all $t \in \mathbb{R}$.

Proof We denote $\beta = g(\nabla b_{v_1}, \nabla b_{v_2})$ and set

$$X := \frac{1}{\|\nabla b_{v_1} - \nabla b_{v_2}\|^2}(\nabla b_{v_1} - \nabla b_{v_2}) = \frac{1}{2 - 2\beta}(\nabla b_{v_1} - \nabla b_{v_2}). \tag{7}$$

Then, since $g(X, \nabla b_{v_1} - \nabla b_{v_2}) = 1$ and $g(X, \nabla b_{v_1} + \nabla b_{v_2}) = 0$,

$$b_{v_1}(\phi_t(x)) = b_{v_1}(x) + \frac{t}{2}, \quad b_{v_2}(\phi_t(x)) = b_{v_2}(x) - \frac{t}{2}, \tag{8}$$

for all $t \in \mathbb{R}$ and $x \in M$, where ϕ_t is the flow of X. In particular,

$$(\phi_t)^*(db_{v_1}) = db_{v_1}, \quad (\phi_t)^*(db_{v_2}) = db_{v_2},$$

for all $t \in \mathbb{R}$. □

By Eq. (8), for all $c_1, c_2 \in \mathbb{R}$,

$$\phi_t(b_{v_1}^{-1}(c_1) \cap b_{v_2}^{-1}(c_2)) = b_{v_1}^{-1}\left(c_1 + \frac{t}{2}\right) \cap b_{v_2}^{-1}\left(c_2 - \frac{t}{2}\right).$$

It means that the flow in Lemma 4.2 maps one intersection of two horospheres onto another intersection of two horospheres, and the intersection $\phi_t(b_{v_1}^{-1}(c_1)\cap b_{v_2}^{-1}(c_2))$ of two horospheres is orthogonal to each gradient of the Busemann function, ∇b_{v_1},∇b_{v_2}, for all $c_1, c_2 \in \mathbb{R}$. Similarly, we consider the flow of $\frac{1}{2+2\beta}(\nabla b_{v_1} + \nabla b_{v_2})$:

Proposition 4.3 *Let (M, g) be a visibility manifold, $p \in M$, and $v_1 \neq v_2 \in T_p^1 M$. Then there exists a constant $c_0 \in \mathbb{R}$ such that, for every $x \in M$,*

$$b_{v_1}(x) + b_{v_2}(x) \geq c_0.$$

In addition, if $b_{v_1}(x) + b_{v_2}(x) = c_0$, then $\nabla b_{v_1}(x) + \nabla b_{v_2}(x) = 0$, for any $x \in M$.

Proof Let $v_1 \neq v_2 \in T_p^1 M$,

$$D := \{x \in M : \nabla b_{v_1}(x) + \nabla b_{v_2}(x) = 0\},$$

and let c_0 be the value of $b_{v_1} + b_{v_2}$ on D. Note that $D \neq \emptyset$ since (M, g) is a visibility manifold. Set

$$Y := \frac{1}{\|\nabla b_{v_1} + \nabla b_{v_2}\|^2}(\nabla b_{v_1} + \nabla b_{v_2}) = \frac{1}{2+2\beta}(\nabla b_{v_1} + \nabla b_{v_2}),$$

on $M - D$, where $\beta = g(\nabla b_{v_1}, \nabla b_{v_2})$. For all $x \in M$, if $b_{v_1}(x) + b_{v_2}(x) = c_0$, then, by Eq. (3), $x \in D$, so Y is not defined at x. Conversely, if $b_{v_1}(x) + b_{v_2}(x) \neq c_0$, then $x \notin D$, so Y is well-defined at x. For all $x \in M$ and $s \in \mathbb{R}$, $(b_{v_1} + b_{v_2})(\psi_{s'}(x)) = b_{v_1}(x) + b_{v_2}(x) + s' \neq c_0$ for all s' between 0 and s if and only if both $(b_{v_1} + b_{v_2})(x)$ and $(b_{v_1} + b_{v_2})(\psi_s(x))$ are larger than c_0, or both of them are smaller than c_0, which is equivalent to the following inequality

$$(b_{v_1}(x) + b_{v_2}(x) - c_0)(b_{v_1}(x) + b_{v_2}(x) + s - c_0) > 0.$$

Since $g(Y, \nabla b_{v_1} + \nabla b_{v_2}) = 1$ and $g(Y, \nabla b_{v_1} - \nabla b_{v_2}) = 0$, for all $s \in \mathbb{R}$ and $x \in M$ satisfying $(b_{v_1}(x) + b_{v_2}(x) - c_0)(b_{v_1}(x) + b_{v_2}(x) + s - c_0) > 0$, we have

$$b_{v_1}(\psi_s(x)) = b_{v_1}(x) + \frac{s}{2}, \quad b_{v_2}(\psi_s(x)) = b_{v_2}(x) + \frac{s}{2}, \tag{9}$$

where ψ_s is the flow of Y.

Now, suppose that $b_{v_1}(x) + b_{v_2}(x) = c < c_0$ for some $x \in M$. Let U_i be the $(0, 2)$-tensor field on M defined by

$$U_i(w_1, w_2) := g\left(\nabla_{w_1}\nabla b_{v_i}, w_2\right)$$

for all $x' \in M$ and $w_1, w_2 \in T_{x'}M$, where $i = 1, 2$. Since b_{v_i} is convex, U_i is positive semi-definite. Consequently,

$$Y[\beta] = \frac{(\nabla b_{v_1} + \nabla b_{v_2})\left[g\left(\nabla b_{v_1}, \nabla b_{v_2}\right)\right]}{2+2\beta}$$

$$= \frac{g\left(\nabla_{\nabla b_{v_1}} \nabla b_{v_2}, \nabla b_{v_1}\right) + g\left(\nabla_{\nabla b_{v_2}} \nabla b_{v_1}, \nabla b_{v_2}\right)}{2+2\beta}$$

$$= \frac{U_2(\nabla b_{v_1}, \nabla b_{v_1}) + U_1(\nabla b_{v_2}, \nabla b_{v_2})}{2+2\beta} \geq 0. \tag{10}$$

Hence, $s \mapsto \beta(\psi_s(x))$ is non-decreasing. By Lemma 2.7,

$$(b_{v_1} + b_{v_2})^{-1}((-\infty, c_0]) \cap (b_{v_1} - b_{v_2})^{-1}(t)$$

is compact and, since $(b_{v_1} + b_{v_2})(\psi_s(x)) = (b_{v_1} + b_{v_2})(x) + s = c + s < c_0$ and $(b_{v_1} - b_{v_2})(\psi_s(x)) = (b_{v_1} - b_{v_2})(x) = t$, it contains $\psi_s(x)$ for all $s \in (0, c_0 - c)$, where $t = b_{v_1}(x) - b_{v_2}(x)$. Thus, there exists a sequence $s_i \in (0, c_0 - c)$, $i = 1, 2, \ldots$, such that $\lim_{i\to\infty} s_i = c_0 - c$ and the limit $x_0 := \lim_{i\to\infty} \psi_{s_i}(x)$ exists. Since

$$(b_{v_1} + b_{v_2})(x_0) = (b_{v_1} + b_{v_2})(x) + c_0 - c = c_0,$$

by Eq. (3), we have $x_0 \in D$ so that $\beta(x_0) = -1$. Thus, since $s \mapsto \beta(\psi_s(x))$ is non-decreasing, we have

$$-1 \leq \beta(x) \leq \beta(x_0) = -1,$$

which implies $x \in D$ and $c = c_0$. It is a contradiction to $c < c_0$. □

To prove Theorem 1.2, we need the following theorem:

Theorem 4.4 *Let (M, g) be an asymptotically harmonic, visibility manifold. Let $p \in M$, $v_1 \neq v_2 \in T_p^1 M$, and $c_1, c_2 \in \mathbb{R}$. Let S be the intersection $b_{v_1}^{-1}(c_1) \cap b_{v_2}^{-1}(c_2)$ of horospheres and $\nabla b_{v_1}(x) + \nabla b_{v_2}(x) \neq 0$ for all points $x \in S$. Then the following integrals are independent of $c_1 - c_2$:*

$$\int_S \sqrt{\frac{1 - g(\nabla b_{v_1}, \nabla b_{v_2})}{1 + g(\nabla b_{v_1}, \nabla b_{v_2})}} d\mu', \quad \int_S \sqrt{\frac{1 + g(\nabla b_{v_1}, \nabla b_{v_2})}{1 - g(\nabla b_{v_1}, \nabla b_{v_2})}} d\mu',$$

where $d\mu'$ is the induced measure on the submanifold S of (M, g).

Proof We adopt the notations

$$\beta, X, Y, \phi_t, \psi_s, D, c_0, U_1, U_2$$

in the proof of Lemma 4.2 and Proposition 4.3.

Note that D is closed. If there exists a neighborhood V of $x \in D$ in M such that $V \subseteq D$, then, for every geodesic γ with $\gamma(0) \in V$ and $\gamma'(0) \perp \nabla b_{v_1}(\gamma(0))$, by Theorem 2.9, there exists a totally geodesic, isometric embedding $F : [0, a] \times \mathbb{R} \to M$ between γ_1, γ_2 where $a > 0$ is a sufficiently small constant for $\gamma([0, a]) \subseteq V$, γ_1 and γ_2 are unique asymptotic geodesics to v_1 with $\gamma_1(0) = \gamma(0)$ and $\gamma_2(0) =$

$\gamma(a)$, respectively. By the uniqueness of a geodesic joining two points in a Hadamard manifold, $F(s,0)=\gamma(s)$ for all $s\in[0,a]$. Since F is a totally geodesic, isometric embedding, $d(\gamma_1(t),F(s,t))=s$ for all $s\in[0,a]$, so the curve $t\in\mathbb{R}\mapsto F(s,t)$ is asymptotic to γ_1, so it is asymptotic to v_1, for all $s\in[0,a]$. Since F is isometric, γ is orthogonal to the unique asymptotic curve $t\in\mathbb{R}\mapsto F(s,t)$ to v_1 at $\gamma(s)$, for all $s\in[0,a]$. Hence, $\gamma([0,a])$ is contained in the level set of b_{v_1}. It means that the level set of b_{v_1} containing x in V is totally geodesic, thus $h=-\Delta b_v=0$, for all $v\in T^1M$. Let $v\in T^1M$ and let U be the $(0,2)$-tensor field defined by $U(w_1,w_2):=g(\nabla_{w_1}\nabla b_v,w_2)$ for all $x\in M$ and $w_1,w_2\in T_xM$. So, U is symmetric, and, since a Busemann function is convex, U is positive semi-definite. Also, since $\operatorname{tr}U=h=0$, $U=0$. By the Riccati equation for horospheres,

$$\nabla_{\nabla b_v}U+U^2+R_{\nabla b_v}=R_{\nabla b_v}=0,$$

where $R_{\nabla b_v}$ is the Jacobi operator along a geodesic with the velocity vector ∇b_v. Thus, $R_v=R_{\nabla b_{-v}(x)}=0$ for all $x\in M$ and $v\in T^1_xM$. Therefore, (M,g) is flat. Hence, (M,g) is not a visibility manifold which contradicts the assumption of Theorem 4.4.

Suppose that $h\neq 0$. We note that $b_{v_1}+b_{v_2}\geq c_0$ by Proposition 4.3. By Eqs. (8) and (9), the flows ϕ_t,ψ_s map one intersection $b_{v_1}^{-1}(c_1)\cap b_{v_2}^{-1}(c_2)$ of horospheres onto another intersection, where $c_1,c_2\in\mathbb{R}$ and $c_1+c_2>c_0$. Let $E_1,\ldots,E_n$ be a positively oriented, orthonormal frame on $M-D$, and $\theta_1,\ldots,\theta_n$ be the dual 1-forms of the frame field such that E_1,E_2 spans a subbundle containing ∇b_{v_1},∇b_{v_2}, and $db_{v_1}\wedge db_{v_2}=\sqrt{1-\beta^2}\,\theta_1\wedge\theta_2$. There always exists such frame $E_1,\ldots,E_n$ on $M-D$. For example, if $E_1=\nabla b_{v_1}$ and $E_2=\frac{1}{\|\nabla b_{v_2}-\beta\nabla b_{v_1}\|}(\nabla b_{v_2}-\beta\nabla b_{v_1})$, then, since $\|\nabla b_{v_2}-\beta\nabla b_{v_1}\|=\sqrt{1-\beta^2}$, $\theta_1=db_{v_1}$ and

$$\theta_2(w)=\frac{1}{\sqrt{1-\beta^2}}g(w,\nabla b_{v_2}-\beta\nabla b_{v_1})=\frac{1}{\sqrt{1-\beta^2}}(db_{v_2}-\beta db_{v_1})(w),$$

for all $x\in M-D$ and $w\in T_xM$. Thus, $\theta_2=\frac{1}{\sqrt{1-\beta^2}}(db_{v_2}-\beta db_{v_1})$ and $\theta_1\wedge\theta_2=\frac{1}{\sqrt{1-\beta^2}}db_{v_1}\wedge db_{v_2}$.

By Eq. (6), we have $(\phi_t)^*(db_{v_i})=db_{v_i}$ for $i=1,2$, so $(\phi_t)^*(db_{v_1}\wedge db_{v_2})=(\phi_t)^*(db_{v_1})\wedge(\phi_t)^*(db_{v_2})=db_{v_1}\wedge db_{v_2}$. Also, from Eq. (9), we have $(\psi_s)^*(db_{v_i})=db_{v_i}$ for $i=1,2$, so $(\psi_s)^*(db_{v_1}\wedge db_{v_2})=db_{v_1}\wedge db_{v_2}$. Thus, $db_{v_1}\wedge db_{v_2}$ is invariant under the pullback of ϕ_t and ψ_s, so that, for all $s>0$ and $t\in\mathbb{R}$,

$$\begin{aligned}(\phi_t)^*(\theta_1\wedge\theta_2)&=(\phi_t)^*\left(\frac{1}{\sqrt{1-\beta^2}}\,db_{v_1}\wedge db_{v_2}\right)\\&=\frac{1}{\sqrt{1-\beta^2(\phi_t)}}\,(\phi_t)^*(db_{v_1}\wedge db_{v_2})\\&=\frac{1}{\sqrt{1-\beta^2(\phi_t)}}\,db_{v_1}\wedge db_{v_2}\end{aligned}$$

$$= \sqrt{\frac{1-\beta^2}{1-\beta^2(\phi_t)}}\,\theta_1\wedge\theta_2, \tag{11}$$

$$(\psi_s)^*(\theta_1\wedge\theta_2) = \sqrt{\frac{1-\beta^2}{1-\beta^2(\psi_s)}}\,\theta_1\wedge\theta_2. \tag{12}$$

on $M-D$. Since (M,g) is asymptotically harmonic,

$$\operatorname{div}\left(\nabla b_{v_1}-\nabla b_{v_2}\right)=0,\quad \operatorname{div}\left(\nabla b_{v_1}+\nabla b_{v_2}\right)=2h.$$

Thus, we obtain

$$\begin{aligned}
\operatorname{div} X &= \operatorname{div}\left(\frac{1}{2-2\beta}\left(\nabla b_{v_1}-\nabla b_{v_2}\right)\right)\\
&= \left(\nabla b_{v_1}-\nabla b_{v_2}\right)\left[\frac{1}{2-2\beta}\right]+\frac{1}{2-2\beta}\operatorname{div}\left(\nabla b_{v_1}-\nabla b_{v_2}\right)\\
&= -\frac{\left(\nabla b_{v_1}-\nabla b_{v_2}\right)[2-2\beta]}{(2-2\beta)^2} = -\frac{X[2-2\beta]}{2-2\beta}\\
&= -X\left[\ln(2-2\beta)\right]\\
&= X\left[\ln\left(\frac{1}{1-\beta}\right)\right],
\end{aligned} \tag{13}$$

and

$$\begin{aligned}
\operatorname{div} Y &= \operatorname{div}\left(\frac{1}{2+2\beta}\left(\nabla b_{v_1}+\nabla b_{v_2}\right)\right)\\
&= \left(\nabla b_{v_1}+\nabla b_{v_2}\right)\left[\frac{1}{2+2\beta}\right]+\frac{1}{2+2\beta}\operatorname{div}\left(\nabla b_{v_1}+\nabla b_{v_2}\right)\\
&= -\frac{\left(\nabla b_{v_1}+\nabla b_{v_2}\right)[2+2\beta]}{(2+2\beta)^2} + \frac{2h}{2+2\beta}\\
&= -\frac{Y[2+2\beta]}{2+2\beta}+\frac{h}{1+\beta}\\
&= Y\left[\ln\left(\frac{1}{1+\beta}\right)\right]+\frac{h}{1+\beta}.
\end{aligned} \tag{14}$$

Consequently, for the induced Riemannian measure $d\mu$ on $M-D$, the following equations hold:

$$\begin{aligned}
\frac{\partial}{\partial t}(\phi_t)^*(d\mu) &= (\phi_t)^*(L_X(d\mu)) = (\phi_t)^*\left(\operatorname{div} X d\mu\right)\\
&= (\operatorname{div} X\circ\phi_t)\,(\phi_t)^*(d\mu),\\
\frac{\partial}{\partial s}(\psi_s)^*(d\mu) &= (\operatorname{div} Y\circ\psi_s)\,(\psi_s)^*(d\mu).
\end{aligned}$$

Set $(\phi_t)^*(d\mu) = f_t\, d\mu$ where $f_t \in C^\infty(M - D)$. Then

$$\begin{aligned} \frac{\partial}{\partial t} f_t\, d\mu &= (\operatorname{div} X \circ \phi_t)\, f_t\, d\mu, \\ \frac{\partial}{\partial t} \ln f_t &= \quad \operatorname{div} X \circ \phi_t. \end{aligned}$$

Note that $X[f](\phi_t) = \frac{\partial}{\partial t} f(\phi_t)$ for all $f \in C^\infty(M - D)$. Thus, since $f_0 = 1$, from Eq. (13), we get

$$f_t = \frac{1-\beta}{1-\beta(\phi_t)}.$$

Hence,

$$(\phi_t)^*(d\mu) = \frac{1-\beta}{1-\beta(\phi_t)} d\mu. \tag{15}$$

Similarly, when $(\psi_s)^*(d\mu) = g_s\, d\mu$ for some $g_s \in C^\infty(M - D)$, by Eq. (14),

$$\ln g_s = \ln\left(\frac{1+\beta}{1+\beta(\psi_s)}\right) + \int_0^s \frac{h}{1+\beta(\psi_k)} dk,$$

so

$$(\psi_s)^*(d\mu) = \exp\left(\int_0^s \frac{h}{1+\beta(\psi_k)} dk\right) \frac{1+\beta}{1+\beta(\psi_s)} d\mu. \tag{16}$$

Set $d\mu' := \theta_3 \wedge \cdots \wedge \theta_n$. Since E_1,E_2 span a subbundle containing ∇b_{v_1},∇b_{v_2}, $E_3, \ldots, E_n$ are tangent to the intersection $b_{v_1}^{-1}(c_1) \cap b_{v_2}^{-1}(c_2)$ of horospheres for all $c_1, c_2 \in \mathbb{R}$ such that $c_1 + c_2 > c_0$. That is, $E_3, \ldots, E_n$ is an orthonormal frame on the intersection of horospheres and $d\mu'$ is the induced Riemannian measure on the intersection of horospheres. Since ϕ_t maps the intersection $b_{v_1}^{-1}(c_1) \cap b_{v_2}^{-1}(c_2)$ of horospheres onto another intersection of horospheres which is orthogonal to ∇b_{v_1} and ∇b_{v_2} for all $c_1, c_2 \in \mathbb{R}$ such that $c_1 + c_2 > c_0$, $(\phi_t)^*(d\mu') = f\, d\mu'$ for some function $f \in C^\infty(M - D)$. By Eqs. (11) and (15),

$$\begin{aligned} \frac{1-\beta}{1-\beta(\phi_t)} d\mu = (\phi_t)^*(d\mu) &= (\phi_t)^*(\theta_1 \wedge \theta_2) \wedge (\phi_t)^*(d\mu') \\ &= \sqrt{\frac{1-\beta^2}{1-\beta^2(\phi_t)}} \theta_1 \wedge \theta_2 \wedge (\phi_t)^*(d\mu'), \end{aligned}$$

so

$$(\phi_t)^*(d\mu') = \sqrt{\frac{(1-\beta)(1+\beta(\phi_t))}{(1+\beta)(1-\beta(\phi_t))}} d\mu'.$$

Similarly, since ψ_s maps the intersection of horospheres onto another intersection of horospheres, by Eqs. (12) and (16), we have

$$(\psi_s)^*(d\mu') = \exp\left(\int_0^s \frac{h}{1+\beta(\psi_k)}dk\right)\sqrt{\frac{(1+\beta)(1-\beta(\psi_s))}{(1-\beta)(1+\beta(\psi_s))}}d\mu',$$

or, equivalently,

$$\sqrt{\frac{1-\beta(\phi_t)}{1+\beta(\phi_t)}}\,(\phi_t)^*(d\mu') = \sqrt{\frac{1-\beta}{1+\beta}}d\mu', \tag{17}$$

$$\sqrt{\frac{1+\beta(\psi_s)}{1-\beta(\psi_s)}}\,(\psi_s)^*(d\mu') = \exp\left(\int_0^s \frac{h}{1+\beta(\psi_k)}dk\right)\sqrt{\frac{1+\beta}{1-\beta}}d\mu'. \tag{18}$$

Now, let

$$S(s,t) := b_{v_1}^{-1}\left(\frac{s+c_0+t}{2}\right) \cap b_{v_2}^{-1}\left(\frac{s+c_0-t}{2}\right),$$

for all $s \geq 0$ and $t \in \mathbb{R}$. Note that

$$S(c_1+c_2-c_0, c_1-c_2) = b_{v_1}^{-1}(c_1) \cap b_{v_2}^{-1}(c_2),$$

for all $c_1, c_2 \in \mathbb{R}$ such that $c_1 + c_2 \geq c_0$. Set

$$V(s,t) := \int_{S(s,t)} \sqrt{\frac{1-\beta}{1+\beta}}d\mu',$$

$$W(s,t) := \int_{S(s,t)} \sqrt{\frac{1+\beta}{1-\beta}}d\mu',$$

for all $s > 0$ and $t \in \mathbb{R}$. Then, $x \in S(s,t)$ holds if and only if $b_{v_1}(x)+b_{v_2}(x) = s+c_0$ and $b_{v_1}(x) - b_{v_2}(x) = t$, for all $s > 0$, $t \in \mathbb{R}$, and $x \in M$. From Eq. (8), we have

$$(b_{v_1}+b_{v_2})(\phi_t(x)) = (b_{v_1}+b_{v_2})(x),$$
$$(b_{v_1}-b_{v_2})(\phi_t(x)) = (b_{v_1}-b_{v_2})(x)+t,$$

for all $t \in \mathbb{R}$ and $x \in M$. Thus, $\phi_t(S(s,t_0)) = S(s,t_0+t)$ holds for all $s > 0$ and $t_0, t \in \mathbb{R}$. Also, from Eq. (9),

$$(b_{v_1}+b_{v_2})(\psi_s(x)) = (b_{v_1}+b_{v_2})(x)+s,$$
$$(b_{v_1}-b_{v_2})(\psi_s(x)) = (b_{v_1}-b_{v_2})(x),$$

for all $s > 0$ and $x \in M - D$. Hence, $\psi_s(S(\varepsilon, t)) = S(s + \varepsilon, t)$ holds, for all $s, \varepsilon > 0$ and $t \in \mathbb{R}$. By Eq. (17),

$$\begin{aligned} V(s,t) &= \int_{\phi_t(S(s,0))} \sqrt{\frac{1-\beta}{1+\beta}} d\mu' \\ &= \int_{S(s,0)} \sqrt{\frac{1-\beta(\phi_t)}{1+\beta(\phi_t)}} (\phi_t)^*(d\mu') \\ &= \int_{S(s,0)} \sqrt{\frac{1-\beta}{1+\beta}} d\mu' \\ &= V(s,0), \end{aligned} \tag{19}$$

for all $s > 0$ and $t \in \mathbb{R}$. For all $s, \varepsilon > 0$ and $t \in \mathbb{R}$, from Eq. (18), we obtain

$$\begin{aligned} W(s+\varepsilon, t) &= \int_{\psi_s(S(\varepsilon,t))} \sqrt{\frac{1+\beta}{1-\beta}} d\mu' \\ &= \int_{S(\varepsilon,t)} \sqrt{\frac{1+\beta(\psi_s)}{1-\beta(\psi_s)}} (\psi_s)^*(d\mu') \\ &= \int_{S(\varepsilon,t)} \exp\left(\int_0^s \frac{h}{1+\beta(\psi_k)} dk \right) \sqrt{\frac{1+\beta}{1-\beta}} d\mu'. \end{aligned}$$

By differentiating it for s, we get

$$\begin{aligned} \frac{\partial}{\partial s} W(s+\varepsilon, t) &= \int_{S(\varepsilon,t)} \frac{h}{1+\beta(\psi_s)} \exp\left(\int_0^s \frac{h}{1+\beta(\psi_k)} dk \right) \sqrt{\frac{1+\beta}{1-\beta}} d\mu' \\ &= \int_{S(\varepsilon,t)} \frac{h}{1+\beta(\psi_s)} \sqrt{\frac{1+\beta(\psi_s)}{1-\beta(\psi_s)}} (\psi_s)^*(d\mu') \\ &= \int_{S(s+\varepsilon,t)} \frac{h}{\sqrt{1-\beta^2}} d\mu'. \end{aligned}$$

By Eq. (19), we have

$$\frac{\partial}{\partial t} V(s,t) = 0, \tag{20}$$

$$\begin{aligned} \frac{\partial}{\partial s} W(s,t) &= \int_{S(s,t)} \frac{h}{\sqrt{1-\beta^2}} d\mu' \\ &= \frac{h}{2} W(s,t) + \frac{h}{2} V(s,t). \end{aligned} \tag{21}$$

By differentiating Eq. (21) for t, from Eq. (20), the following equation holds:

$$\frac{\partial^2}{\partial s \partial t} W(s,t) = \frac{h}{2} \frac{\partial}{\partial t} W(s,t),$$

which implies, for all $s, \varepsilon > 0$ and $t \in \mathbb{R}$,

$$\frac{\partial}{\partial t} W(s+\varepsilon, t) = e^{hs/2} \frac{\partial}{\partial t} W(\varepsilon, t). \tag{22}$$

By Eq. (17),

$$\begin{aligned} W(s,t) &= \int_{\phi_t(S(s,0))} \sqrt{\frac{1+\beta}{1-\beta}} d\mu' \\ &= \int_{S(s,0)} \sqrt{\frac{1+\beta(\phi_t)}{1-\beta(\phi_t)}} \, (\phi_t)^*(d\mu') \\ &= \int_{S(s,0)} \frac{1+\beta(\phi_t)}{1-\beta(\phi_t)} \sqrt{\frac{1-\beta(\phi_t)}{1+\beta(\phi_t)}} \, (\phi_t)^*(d\mu') \\ &= \int_{S(s,0)} \frac{1+\beta(\phi_t)}{1-\beta(\phi_t)} \sqrt{\frac{1-\beta}{1+\beta}} d\mu' \\ &= \int_{S(s,0)} \left(-1 + \frac{2}{1-\beta(\phi_t)}\right) \sqrt{\frac{1-\beta}{1+\beta}} d\mu'. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \frac{\partial}{\partial t} W(s,t) &= \int_{S(s,0)} \frac{2X[\beta](\phi_t)}{(1-\beta(\phi_t))^2} \sqrt{\frac{1-\beta}{1+\beta}} d\mu' \\ &= \int_{S(s,0)} \frac{2X[\beta](\phi_t)}{(1-\beta(\phi_t))^2} \sqrt{\frac{1-\beta(\phi_t)}{1+\beta(\phi_t)}} \, (\phi_t)^*(d\mu') \\ &= \int_{S(s,t)} \frac{2X[\beta]}{(1-\beta)\sqrt{1-\beta^2}} d\mu'. \end{aligned}$$

Now, we get, from Eq. (7),

$$\begin{aligned} X[\beta] &= \frac{1}{2-2\beta} (\nabla b_{v_1} - \nabla b_{v_2}) \left[g(\nabla b_{v_1}, \nabla b_{v_2}) \right] \\ &= \frac{1}{2-2\beta} \left(U_2(\nabla b_{v_1}, \nabla b_{v_1}) - U_1(\nabla b_{v_2}, \nabla b_{v_2}) \right). \end{aligned} \tag{23}$$

Consequently, we have

$$\begin{aligned}\frac{\partial}{\partial t}W(s,t) &= \int_{S(s,t)} \frac{\sqrt{1-\beta^2}}{(1-\beta)^2}\frac{U_2(\nabla b_{v_1},\nabla b_{v_1}) - U_1(\nabla b_{v_2},\nabla b_{v_2})}{1-g(\nabla b_{v_1},\nabla b_{v_2})^2}d\mu' \\ &= \int_{S(s,t)} \frac{\sqrt{1-\beta^2}}{(1-\beta)^2}\frac{U_2(\nabla b_{v_1}-\beta\nabla b_{v_2},\nabla b_{v_1}-\beta\nabla b_{v_2})}{\|\nabla b_{v_1}-\beta\nabla b_{v_2}\|^2}d\mu' \\ &\quad - \int_{S(s,t)} \frac{\sqrt{1-\beta^2}}{(1-\beta)^2}\frac{U_1(\nabla b_{v_2}-\beta\nabla b_{v_1},\nabla b_{v_2}-\beta\nabla b_{v_1})}{\|\nabla b_{v_2}-\beta\nabla b_{v_1}\|^2}d\mu',\end{aligned}$$

since $\nabla_{\nabla b_{v_1}}\nabla b_{v_1} = 0 = \nabla_{\nabla b_{v_2}}\nabla b_{v_2}$,

$$\begin{aligned}U_1(\nabla b_{v_1}, w) = U_1(w, \nabla b_{v_1}) = g(\nabla_{\nabla b_{v_1}}\nabla b_{v_1}, w) = 0,\\ U_2(\nabla b_{v_2}, w) = U_2(w, \nabla b_{v_2}) = g(\nabla_{\nabla b_{v_2}}\nabla b_{v_2}, w) = 0,\end{aligned}$$

for all $w \in TM$,

$$\begin{aligned}U_1(\nabla b_{v_2},\nabla b_{v_2}) &= U_1(\nabla b_{v_2},\nabla b_{v_2}) - 2\beta U_1(\nabla b_{v_1},\nabla b_{v_2}) + \beta^2 U_1(\nabla b_{v_1},\nabla b_{v_1}) \\ &= U_1(\nabla b_{v_2}-\beta\nabla b_{v_1},\nabla b_{v_2}-\beta\nabla b_{v_1}), \\ U_2(\nabla b_{v_1},\nabla b_{v_1}) &= U_2(\nabla b_{v_1},\nabla b_{v_1}) - 2\beta U_2(\nabla b_{v_1},\nabla b_{v_2}) + \beta^2 U_2(\nabla b_{v_2},\nabla b_{v_2}) \\ &= U_2(\nabla b_{v_1}-\beta\nabla b_{v_2},\nabla b_{v_1}-\beta\nabla b_{v_2}),\end{aligned}$$

and $1-g(\nabla b_{v_1},\nabla b_{v_2})^2 = 1-\beta^2 = \|\nabla b_{v_1}-\beta\nabla b_{v_2}\|^2 = \|\nabla b_{v_2}-\beta\nabla b_{v_1}\|^2$.

Now, use the following lemma:

Lemma 4.5 *Let (M, g) be an asymptotically harmonic manifold. Then, for all $v, w \in T^1M$,*

$$g(\nabla_w\nabla b_v, w) \le h, \tag{24}$$

where $h = -\Delta b_v$.

Proof Let U be the $(0, 2)$-tensor field defined by $U(w_1, w_2) := g(\nabla_{w_1}\nabla b_v, w_2)$ for all $x \in M$ and $w_1, w_2 \in T_xM$. Then U is symmetric, positive semi-definite, and $\operatorname{tr} U = h$. Thus, every eigenvalue of U is real, non-negative, and less than or equal to h. In particular, Eq. (24) holds. □

By Lemma 4.5,

$$\begin{aligned}0 \le \frac{U_1(\nabla b_{v_2},\nabla b_{v_2})}{1-\beta^2} = \frac{U_1(\nabla b_{v_2}-\beta\nabla b_{v_1},\nabla b_{v_2}-\beta\nabla b_{v_1})}{\|\nabla b_{v_2}-\beta\nabla b_{v_1}\|^2} \le h,\\ 0 \le \frac{U_2(\nabla b_{v_1},\nabla b_{v_1})}{1-\beta^2} = \frac{U_2(\nabla b_{v_1}-\beta\nabla b_{v_2},\nabla b_{v_1}-\beta\nabla b_{v_2})}{\|\nabla b_{v_1}-\beta\nabla b_{v_2}\|^2} \le h.\end{aligned} \tag{25}$$

Hence, by Eqs. (23) and (25), $\frac{X[\beta]}{1+\beta}$ is bounded. By Lemma 2.7, $\bigcup_{\varepsilon\in[0,s]} S(\varepsilon, t)$ is compact for all $s > 0$. Fix $s > 0$. Then β has a maximum value on the compact set $\bigcup_{\varepsilon\in[0,s]} S(\varepsilon, t)$. In particular, there exists a constant $C > 0$ such that, for all $\varepsilon \in (0, s)$,

$$\left|\frac{\partial}{\partial t} W(\varepsilon, t)\right| \leq C \int_{S(\varepsilon,t)} \sqrt{1+\beta}\, d\mu'. \tag{26}$$

Now, we use the following lemma:

Lemma 4.6 *The $(n-2)$-dimensional volume $\mathrm{vol}_{n-2} S(s,t)$ of $S(s,t)$ is non-decreasing for $s > 0$ where $t \in \mathbb{R}$ is fixed, and, for all $s > 0$ and $x \in S(s,t)$,*

$$\beta(x) \leq -2e^{-hs} + 1.$$

Proof For all $s, \varepsilon > 0$ and $t \in \mathbb{R}$,

$$\begin{aligned}\mathrm{vol}_{n-2} S(s+\varepsilon, t) &= \int_{\psi_s(S(\varepsilon,t))} d\mu' = \int_{S(\varepsilon,t)} (\psi_s)^*(d\mu') \\ &= \int_{S(\varepsilon,t)} \exp\left(\int_0^s \frac{h}{1+\beta(\psi_k)} dk\right) \sqrt{\frac{(1+\beta)(1-\beta(\psi_s))}{(1-\beta)(1+\beta(\psi_s))}}\, d\mu'.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}\frac{\partial}{\partial s} \mathrm{vol}_{n-2} S(s+\varepsilon, t) &= \int_{S(\varepsilon,t)} \left(\frac{h}{1+\beta(\psi_s)} - \frac{Y[\beta](\psi_s)}{1-\beta^2(\psi_s)}\right) (\psi_s)^*(d\mu') \\ &= \int_{S(s+\varepsilon,t)} \left(\frac{h}{1+\beta} - \frac{Y[\beta]}{1-\beta^2}\right) d\mu' \\ &= \int_{S(s+\varepsilon,t)} \frac{1}{1+\beta} \left(h - \frac{Y[\beta]}{1-\beta}\right) d\mu'\end{aligned}$$

Hence, we have

$$\frac{\partial}{\partial s} \mathrm{vol}_{n-2} S(s,t) = \int_{S(s,t)} \frac{1}{1+\beta} \left(h - \frac{Y[\beta]}{1-\beta}\right) d\mu'. \tag{27}$$

By Eqs. (10) and (25),

$$Y\left[\ln\left(\frac{1}{1-\beta}\right)\right] = \frac{Y[\beta]}{1-\beta} \leq h, \tag{28}$$

and we obtain

$$\ln\left(\frac{1-\beta(x)}{1-\beta(\psi_s(x))}\right) \leq hs$$

and

$$\beta(\psi_s(x)) \le e^{-hs}\beta(x) + 1 - e^{-hs},$$

for all $s > 0$ and $x \in M - D$. Since, $\lim_{k\to s-} \beta(\psi_{-k}(x)) = -1$ for all $x \in S(s,t)$, by considering $\beta(x) = \beta(\psi_s(\psi_{-s}(x)))$, we obtain, for all $s > 0$ and $x \in S(s,t)$,

$$\beta(x) \le -2e^{-hs} + 1. \tag{29}$$

Thus, by Eqs. (27), (28), and (29),

$$\frac{\partial}{\partial s}\operatorname{vol}_{n-2} S(s,t) \ge \frac{1}{2(1-e^{-hs})}(h-h)\operatorname{vol}_{n-2} S(s,t) = 0. \tag{30}$$

Thus, $\operatorname{vol}_{n-2} S(s,t)$ is non-decreasing for $s > 0$. □

By Eq. (26) and Lemma 4.6, for small $\varepsilon > 0$, $\operatorname{vol}_{n-2} S(\varepsilon,t)$ is bounded, and for some constant $C > 0$

$$\left|\frac{\partial}{\partial t}W(\varepsilon,t)\right| \le C\sqrt{2-2e^{-h\varepsilon}}.$$

Then we obtain

$$\lim_{\varepsilon\to 0+}\frac{\partial}{\partial t}W(\varepsilon,t) = 0, \tag{31}$$

and, by Eqs. (22) and (31), we have

$$\frac{\partial}{\partial t}W(s,t) = 0 \tag{32}$$

By Eqs. (20) and (32), Theorem 4.4 is proved. □

Proof of Theorem 1.2 Since $\beta = -1$ on D, the notations X, ϕ_t can be extended to M, and, by Eq. (15), we have

$$\operatorname{vol}_{n-2} S(0,t) = \int_{S(0,0)} (\phi_t)^*(d\mu') = \int_{S(0,0)} d\mu' = \operatorname{vol}_{n-2} S(0,0).$$

By setting a constant $C = \operatorname{vol}_{n-2} S(0,0) + 1$, Theorem 1.2 is proved for $2c = c_0$. Now, suppose that $c_1 + c_2 \ne c_0$. The $(n-2)$-dimensional volume of the intersection $S = b_{v_1}^{-1}(c_1) \cap b_{v_2}^{-1}(c_2)$ satisfies

$$\operatorname{vol}_{n-2} S = \int_S d\mu' \le \int_S \frac{1}{\sqrt{1-\beta^2}}d\mu' = \frac{1}{2}(V(s,t) + W(s,t)),$$

where $s = c_1 + c_2 - c_0$, $t = c_1 - c_2$, and $V(s,t) + W(s,t)$ is independent of t by Theorem 4.4. By setting $C = \frac{1}{2}(V(s,t) + W(s,t) + 1)$, the proof is completed. Such C has a minimum value when ∇b_{v_1} and ∇b_{v_2} are orthogonal, and increases if the angle between ∇b_{v_1} and ∇b_{v_2} approaches to 0 or π. □

Now, we give an example which supports the main theorem.

Example Consider the Poincaré upper-half plane model $\{(x, y, z) \in \mathbb{R}^3 : z > 0\}$ with the metric:

$$g = \frac{dx^2 + dy^2 + dz^2}{z^2}.$$

The distance between two points (x_1, y_1, z_1), (x_2, y_2, z_2) is

$$d((x_1, y_1, z_1), (x_2, y_2, z_2)) = 2 \operatorname{arcsinh}\left(\frac{\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}}{2\sqrt{z_1 z_2}}\right),$$

We consider the volume of the intersections of horospheres for the case: $v_1 = \frac{\partial}{\partial x}\Big|_{(0,0,1)}$ and $v_2 = -\frac{\partial}{\partial x}\Big|_{(0,0,1)}$. The geodesic γ_{v_1} is the unit circle centered at $(0, 0, 0)$ and the equation of the geodesic sphere centered at $(\cos r, 0, \sin r)$ containing $(0, 0, 2)$ is

$$\frac{(x - \cos r)^2 + y^2 + (z - \sin r)^2}{4 \sin r\, z} = \frac{5 - 4 \sin r}{8 \sin r},$$

or equivalently,

$$(x - \cos r)^2 + y^2 + \left(z - \frac{5}{4}\right)^2 = \frac{9}{16}.$$

Every point (x, y, z) of the intersection of the geodesic spheres centered at $(\cos r, 0, \sin r)$ and $(-\cos r, 0, \sin r)$ containing $(0, 0, 2)$ satisfies

$$x = 0, \quad y^2 + \left(z - \frac{5}{4}\right)^2 = \frac{9}{16}.$$

It is also the equation of the intersection of horospheres. Write $y = \frac{3}{4}\cos t$ and $z = \frac{3}{4}\sin t + \frac{5}{4}$. Then the volume of the intersection of horospheres is

$$\int_0^{2\pi} \frac{3}{3 \sin t + 5} dt < 3\pi.$$

Remark Let (M, g) be an asymptotically harmonic, visibility manifold. Let $c_1, c_2 \in \mathbb{R}$ such that $c_1 + c_2 > c_0$ and $r > 0$. The set $b_{v_1}^{-1}([c_1, c_1 + r]) \cap b_{v_2}^{-1}([c_2, c_2 + r])$ is the union of countably many sets of the form

$$S = \{x \in M : a_1 + c_0 \le b_{v_1} + b_{v_2} \le a_2 + c_0,\ \ a_3 \le b_{v_1} - b_{v_2} \le a_4\},$$

where $a_1, a_2 \ge 0$ and $a_3, a_4 \in \mathbb{R}$. It can be obtained by taking the middle points of each sides of the square $[c_1, c_1 + r] \times [c_2, c_2 + r]$ repeatedly. The n-dimensional volume of a set of the form equals

$$\begin{aligned}\int_S d\mu &= \int_S \theta_1 \wedge \theta_2 \wedge d\mu' \\ &= \int_S db_{v_1} \wedge db_{v_2} \wedge \left(\frac{1}{\sqrt{1-\beta^2}} d\mu' \right) \\ &= \frac{1}{2} \int_{a_1}^{a_2} \int_{a_3}^{a_4} \int_{S(s,t)} \frac{1}{\sqrt{1-\beta^2}} d\mu'\, dt\, ds.\end{aligned}$$

By Theorem 4.4,

$$\int_S d\mu = \frac{a_4 - a_3}{2} \int_{a_1}^{a_2} \int_{S(s,0)} \frac{1}{\sqrt{1-\beta^2}} d\mu'\, ds.$$

Therefore, the n-dimensional volume of $b_{v_1}^{-1}([c_1, c_1 + r]) \cap b_{v_2}^{-1}([c_2, c_2 + r])$ is independent of $c_1 - c_2$.

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Ballmann, W., Gromov, M., Schroeder, V.: Manifolds of nonpositive curvature, Progress in Mathematics, 61, Birkhäuser Boston, Inc., Boston, MA, pp. vi+263 (1985)
2. Biswas, K., Knieper, G., Peyerimhoff, N.: The Fourier transform on harmonic manifolds of purely exponential volume growth. J. Geom. Anal. **31**(1), 126–163 (2021)
3. Bridson, M.R., Haefliger, A.: Metric spaces of non-positive curvature, Grundlehren der Mathematischen Wissenschaften, 319, Springer-Verlag, Berlin, pp. xxii+643 (1999)
4. Csikós, B., Horváth, M.: A characterization of harmonic spaces. J. Differ. Geom. **90**(3), 383–389 (2012)
5. Csikós, B., Horváth, M.: Harmonic manifolds and tubes. J. Geom. Anal. **28**(4), 3458–3476 (2018)
6. Chavel, I.: Riemannian geometry, Second edition, Cambridge Studies in Advanced Mathematics, 98, Cambridge University Press, Cambridge, pp. xvi+471 (2006)
7. Eschenburg, J.-H.: Horospheres and the stable part of the geodesic flow. Math. Z. **153**(3), 237–251 (1977)
8. Haller, G., Mezć, I.: Reduction of three-dimensional, volume-preserving flows with symmetry. Nonlinearity **11**(2), 319–339 (1998)

9. Heber, J.: On harmonic and asymptotically harmonic homogeneous spaces. Geom. Funct. Anal. **16**(4), 869–890 (2006)
10. Heintze, E., Im Hof, H.C.: Geometry of horospheres. J. Diff. Geom. **12**, 481–491 (1977)
11. Itoh, M., Satoh, H.: Information geometry of Busemann-barycenter for probability measures. Int. J. Math. **26**(6), 1541007 (2015)
12. Knieper, G., Peyerimhoff, N.: Harmonic functions on rank one asymptotically harmonic manifolds. J. Geom. Anal. **26**(2), 750–781 (2016)
13. Knieper, G.: New results on harmonic manifolds. Comment. Math. Helv. **87**(3), 669–703 (2012)
14. Knieper, G.: A survey on noncompact harmonic and asymptotically harmonic manifolds. Geometry, topology, and dynamics in negative curvature, 146–197, London Math. Soc. Lecture Note Ser., 425, Cambridge Univ. Press, Cambridge (2016)
15. Peyerimhoff, N., Samiou, E.: Integral geometric properties of non-compact harmonic spaces. J. Geom. Anal. **25**, 122–148 (2015)
16. Rouviére, F.: Radon transform on a harmonic manifold. J. Geom. Anal. **31**(6), 6365–6385 (2021)
17. Ranjan, A., Shah, H.: Harmonic manifolds with minimal horospheres. J. Geom. Anal. **12**(4), 683–694 (2002)
18. Szabò, Z.I.: The Lichnerowicz conjecture on harmonic manifolds. J. Differ. Geom. **31**, 1–28 (1990)