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Existence and multiplicity of solutions to p-Laplacian equations on graphs

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Abstract

In this paper, we investigate the existence of multiple solutions to the nonlinear p-Laplacian equation

$$-\Delta_p u + h(x)|u|^{p-2}u = f(x,u) + g(x)$$

on the locally finite graph G, where Δ_p is the discrete p-Laplacian on graphs, $p \geq 2$. Under more general conditions, we prove that the p-Laplacian equation admits at least two nontrivial different solutions by using the variational methods and the new analytical techniques. Our results extend some related works.

Keywords p-Laplacian equation · Locally finite graph · Multiple solutions · Variational methods

Mathematics Subject Classification 35R02 · 35Q55 · 35A15 · 35J92

1 Introduction

Recently, differential equations on discrete graphs has attracted much attention from many researchers, due to its strong application background, such as neural network [1], image processing [2] and so on. In this paper, we study the existence of multiple solutions to the following p-Laplacian equation

$$-\Delta_p u + h(x)|u|^{p-2}u = f(x,u) + g(x) \tag{S_p}$$

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on the locally finite graph $G = (V, E)$, where $p \geq 2$, Δ_p is the discrete p-Laplacian on graphs. In Euclidean space, the p-Laplacian $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ arises in non-Newtonian fluids, flow through porous media, nonlinear elasticity, and other physical phenomena. Problems like (S_p) has been extensively studied in the Euclidean space; see for examples Alves and Figueiredo [3], Cao and Zhou [4], Ding and Ni [5]. Especially, Jeanjean [6] studied the existence of two positive solutions for the equation

$$-\Delta u + u = f(x, u) + g(x), \quad x \in \mathbb{R}^n,$$

where the perturbation term $g \in H^{-1}(\mathbb{R}^n)$, $g \not\equiv 0$ and f satisfies some growth conditions. For more interesting results, we refer the reader to [7–10].

When $p = 2$ in (S_p), for any $0 < \varepsilon < \varepsilon_0$, Grigor'yan, Lin and Yang [11] proved that the perturbed equation

$$-\Delta u + hu = f(x, u) + \varepsilon g(x) \tag{1}$$

has two distinct strictly positive solutions on a locally finite graph, where g satisfies the following condition:
(g_1) *$g \in \mathcal{H}^{-1}$, $g \geq 0$ and $g \not\equiv 0$, where $\mathcal{H}^{-1}$ is the dual space of $\mathcal{H} := \{u \in W^{1,2}(V) : \int_V hu^2 d\mu < \infty\}$.*
In particular, when $g \equiv 0$, they [11] also established existence of positive solutions of (1) on locally finite graphs.

The discrete p-Laplacian on graphs was introduced in [12] and has been well studied ever since, mostly in the context of nonlinear potential and spectral theory, cf. [13, 14] for historical overviews. The problem (S_p) can be regarded as a perturbation problem of the following problem

$$-\Delta_p u + h(x)|u|^{p-2}u = f(x, u), \quad x \in V. \tag{2}$$

In [15], Grigor'yan, Lin and Yang studied the existence of nontrivial solutions to the equation (2) on a finite graph. In addition, there were many interesting and important papers of p-Laplacian equations on graphs. For example, Ge [16] studied a p-th Kazdan-Warner equation on a finite graph. Han and Shao [17] studied the convergence of ground state solutions to a p-Laplacian equation on a locally finite graph. For other related works, we refer the reader to [18–21] and the references therein.

However, the multiple solutions of p-Laplacian equations on graphs have been investigated less extensively. Moreover, most of the current research to perturbation problems on graphs were in the case of $p = 2$ and without considering the case of $p > 2$.

Motivated by [6, 11, 15], we focus on the p-Laplacian equation (S_p) with perturbation term g on the locally finite G. Specifically, we obtain the existence of positive energy and negative energy solutions to the equation (S_p) respectively. Throughout this paper, we assume that G, h and f satisfy the following assumptions.

(G_1) *G is a locally finite and connected graph and its measure $\mu(x) \geq \mu_{\min} > 0$ for all $x \in V$.*

(G_2) *For any $xy \in E$, the weight satisfies*

$$\omega_{xy} = \omega_{yx} > 0 \text{ and } M := \sup_{x \in V} \frac{\deg_x}{\mu(x)} < +\infty,$$

where $\deg_x := \sum_{y \in V} \omega_{xy}$.

(h_1) *There exists a constant $h_0 > 0$ such that $h(x) \geq h_0$ for all $x \in V$.*

(h_2) $\frac{1}{h} \in L^{\frac{1}{p-1}}(V)$.

(f_1) *For any $x \in V$, $f(x, s)$ is continuous in $s \in \mathbb{R}$ and there exist some $q > p \geq 2$ and $C > 0$ such that $|f(x, s)| \leq C(1 + |s|^{q-1})$ uniformly in $x \in V$.*

(f_2) *There exists some $\alpha > p$ such that for any $s \in \mathbb{R} \setminus \{0\}$ there holds*

$$0 < \alpha F(x, s) \leq s f(x, s)$$

for all $x \in V$, where $F(x, s) := \int_0^s f(x, t)dt$.

(f_3) $f(x, s) = o(|s|^{p-1})$ *as $s \to 0$ uniformly in $x \in V$.*

Our main result is the following theorem:

Theorem 1.1 *Assume that (G_1), (G_2), (h_1), (h_2), $(f_1) - (f_3)$ hold and $g \in L^{\frac{p}{p-1}}(V)$, $g \not\equiv 0$. Then there exists a constant $\delta > 0$ such that the equation (S_p) admits at least two nontrivial different solutions, provided that $\|g\|_{L^{\frac{p}{p-1}}(V)} \leq \delta$.*

Example 1.1 For any $q > p \geq 2$ the function $|u|^{q-2}u$ is a typical example of f which satisfies $(f_1) - (f_3)$.

Remark 1.1 The condition $M := \sup_{x \in V}(\frac{1}{\mu(x)} \sum_{y \in V} \omega_{xy}) < +\infty$ in (G_2) is an essential assumption, which ensures the reflexivity of the Sobolev space $W^{1,p}(V)$ (see Corollary 5.8 in [17]).

Remark 1.2 Our argument is based on variational method and critical point theory. Though this idea has been used in the Euclidean space case, the Sobolev space and Sobolev embedding in our setting are quite different from those cases. If $g \equiv 0$ in (S_p), then we have the same conclusion as Grigor'yan-Lin-Yang in [15] and our results extend their work from finite graphs to locally finite graphs. The method in [11] is not applicable to this paper, because $g \in L^{\frac{p}{p-1}}(V)$ in the present paper does not satisfy (g_1) in [11] when $p = 2$. Moreover, we generalized their results in [11] for $p = 2$ to any $p > 1$ and enrich the existing results.

This paper is organized as follows. In Sect. 2, we introduce several preliminaries and functional settings on graphs. In Sect. 3, we prove that the equation (S_p) admits at least two nontrivial different solutions.

2 Preliminaries and functional settings

In this section, we introduce some preliminaries and basic functional settings. First we give some definitions and notations on graphs.

Let $G = (V, E)$ be a graph, where V denotes the set of vertices and E denotes the set of edges, $\omega_{xy}\colon V \times V \to \mathbb{R}^+$ be an edge weight function and $\mu\colon V \to \mathbb{R}^+$ be a positive measure on $G = (V, E)$. $y \sim x$ stands for any vertex y connected with x by an edge $xy \in E$. The distance $d(x, y)$ of two vertices $x, y \in V$ is defined by the minimal number of edges which connect these two vertices.

For any function $u : V \to \mathbb{R}$, the μ-Laplacian of u is defined as

$$\Delta u(x) := \frac{1}{\mu(x)}\sum_{y\sim x}\omega_{xy}(u(y) - u(x)).$$

The associated gradient form is defined by

$$\Gamma(u, v)(x) := \frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(u(y) - u(x))(v(y) - v(x)).$$

The length of $\Gamma(u)$ at $x \in V$ is denoted by

$$|\nabla u|(x) := \sqrt{\Gamma(u)(x)}.$$

With respect to the vertex weight μ, the integral of u over V is defined by

$$\int_V u\mathrm{d}\mu = \sum_{x\in V}\mu(x)u(x).$$

The p-Laplacian of u is defined by

$$\Delta_p u(x) := \frac{1}{2\mu(x)}\sum_{y\sim x}(|\nabla u|^{p-2}(y) + |\nabla u|^{p-2}(x))\omega_{xy}(u(y) - u(x)).$$

Let $C(V)$ be the set of real functions on V. For any $1 \le s < \infty$, we denote by

$$L^s(V) = \left\{u \in C(V) : \int_V |u|^s \mathrm{d}\mu = \sum_{x\in V}\mu(x)|u(x)|^s < \infty\right\}$$

the set of integrable functions on V with the respect to the measure μ. For $s = \infty$, let

$$L^\infty(V) = \left\{u \in C(V) : \sup_{x\in V}|u(x)| < \infty\right\}.$$

Define

$$W^{1,p}(V) := \{u : V \to \mathbb{R} : \int_V (|\nabla u|^p + |u|^p) d\mu < +\infty\}, \quad p \geq 2, \tag{3}$$

where

$$\|u\|_{W^{1,p}(V)} = \left(\int_V (|\nabla u|^p + |u|^p) d\mu \right)^{\frac{1}{p}}.$$

Let $C_c(V)$ be a set of all functions with finite support, then $W^{1,p}(V)$ is the completion of $C_c(V)$ under the norm $\|\cdot\|_{W^{1,p}(V)}$ (see Proposition 5.7 in [17]). We define a subspace of $W^{1,p}(V)$, which is also a reflexive Banach space, namely

$$X := \{u \in W^{1,p}(V) : \int_V h(x)|u|^p d\mu < +\infty\}$$

with the norm

$$\|u\|_X = \left(\int_V (|\nabla u|^p + h(x)|u|^p) \mathrm{d}\mu \right)^{\frac{1}{p}}.$$

Clearly, X is a Banach space and also a reflexive space.

The functional related to (S_p) is

$$J(u) = \Phi(u) - \Psi(u), \quad u \in X, \tag{4}$$

where

$$\Phi(u) = \frac{1}{p} \int_V (|\nabla u|^p + h(x)|u|^p) \mathrm{d}\mu \tag{5}$$

and

$$\Psi(u) = \int_V F(x, u) \mathrm{d}\mu + \int_V g(x) u \mathrm{d}\mu.$$

A *weak solution* to (S_p) is a function $u \in X$ satisfying

$$\int_V (|\nabla u|^{p-2} \Gamma(u, \phi) + h|u|^{p-2} u\phi) \mathrm{d}\mu = \int_V f(x, u)\phi \mathrm{d}\mu + \int_V g(x)\phi \mathrm{d}\mu,$$

for any $\phi \in X$, and corresponds to a critical point of the energy functional J. Obviously, $J \in C^1(X, \mathbb{R})$ and

$$(J'(u), v) = (\Phi'(u), v) - (\Psi'(u), v), \quad \forall v \in X, \tag{6}$$

where

$$(\Phi'(u), v) = \int_V (|\nabla u|^{p-2}\Gamma(u, v) + h(x)|u|^{p-2}uv)\mathrm{d}\mu$$

and

$$(\Psi'(u), v) = \int_V f(x, u)v d\mu + \int_V g(x)v\mathrm{d}\mu.$$

Now, we present an important property of the space $L^s(V)$, which will be used later in Lemma 3.10.

Lemma 2.1 *$C_c(V)$ is dense in $L^s(V)$, $s \in [1, +\infty)$, where $C_c(V)$ be a set of all functions with finite support.*

Proof We only need to prove that for any $u \in L^s(V)$, there exist $u_k \in C_c(V)$ such that $\|u_k - u\|_s \to 0$ as $k \to \infty$. Fix a base point $x_0 \in V$ and define $\eta_k : V \to \mathbb{R}$ as

$$\eta_k(x) = \begin{cases} 1, & \mathrm{d}_x \leq k, \\ (2k - \mathrm{d}_x)/k, & k < \mathrm{d}_x < 2k, \\ 0, & \mathrm{d}_x \geq 2k, \end{cases}$$

where $\mathrm{d}_x := \mathrm{d}(x, x_0)$. Obviously, $\{\eta_k\}$ is a nondecreasing sequence of finitely supported functions which satisfies $0 \leq \eta_k \leq 1$ and $\lim\limits_{k\to\infty} \eta_k = 1$.

Let $u_k = u\eta_k \in C_c(V)$. It suffices to show that $\|u_k - u\|_s^s = \int_V |u_k - u|^s \mathrm{d}\mu \to 0$ as $k \to \infty$. Since $\int_V |u|^s d\mu < +\infty$ and $|\frac{k-d_x}{k}| < 1$, we have

$$\begin{aligned} \int_V |u_k - u|^s d\mu &= \sum_{x\in V, d_x \leq k} |u_k - u|^s(x)\mu(x) + \sum_{x\in V, k<d_x<2k} |u_k - u|^s(x)\mu(x) \\ &\quad + \sum_{x\in V, d_x\geq 2k} |u_k - u|^s(x)\mu(x) \\ &= \sum_{x\in V, k<d_x<2k} |u(x)|^s \left|\frac{k - d_x}{k}\right|^s \mu(x) + \sum_{x\in V, d_x\geq 2k} |u(x)|^s \mu(x) \\ &\leq \sum_{x\in V, k<d_x<2k} |u(x)|^s \mu(x) + \sum_{x\in V, d_x\geq 2k} |u(x)|^s \mu(x) \\ &= \sum_{x\in V, d_x>k} |u(x)|^s \mu(x) \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

□

Lemma 2.2 *([17, Lemma 2.1]). Assume that $u \in W^{1,p}(V)$. Then for any $v \in C_c(V)$ we have*

$$\int_V |\nabla u|^{p-2}\Gamma(u,v)d\mu = -\int_V (\Delta_p u)v d\mu.$$

Lemma 2.3 *Assume that $h(x)$ satisfies (h_1) and (h_2). Then X is continuously embedded into $L^s(V)$ for any $s \in [1,+\infty]$. Namely there exists a constant ξ depending on s, p, $\mu_{\min}$, h_0 and $||h^{-1}||_{\frac{1}{p-1}}$ such that for any $u \in X$, $||u||_s \le \xi ||u||_X$. Moreover, for any bounded sequence $\{u_k\} \subset X$, there exists $u \in X$ such that, up to subsequence,*

$$\begin{cases} u_k \rightharpoonup u & in\ \ X, \\ u_k(x) \to u(x) & \forall x \in V, \\ u_k \to u & in\ \ L^s(V), \forall s \in [1,+\infty]. \end{cases}$$

Proof The proof is similar to Lemma 2.6 in [17] and we include it here for completeness. Suppose $u \in X$. At any vertex $x_0 \in V$, we have

$$||u||_E^p \ge \mu_{\min} h_0 |u(x_0)|^p,$$

which gives

$$u(x_0) \le \left(\frac{1}{\mu_{\min} h_0}\right)^{\frac{1}{p}} ||u||_X. \tag{7}$$

Therefore, $X \hookrightarrow L^\infty(V)$ continuously. Thus $X \hookrightarrow L^s(V)$ continuously for any $p \le s \le \infty$. In fact, for any $u \in X$, we have $u \in L^p(V)$. Then for any $p \le s$,

$$\begin{aligned} \int_V |u|^s d\mu &= \int_V |u|^p |u|^{s-p} d\mu \\ &\le (\mu_{\min} h_0)^{\frac{p-s}{p}} ||u||_X^{s-p} \int_V |u|^p d\mu \\ &\le (\mu_{\min} h_0)^{\frac{p-s}{p}} ||u||_X^{s-p} \int_V \frac{h}{h_0} |u|^p d\mu \\ &\le (\mu_{\min} h_0)^{\frac{p-s}{p}} \frac{1}{h_0} ||u||_X^{s-p} \int_V h|u|^p d\mu \\ &\le (\mu_{\min})^{\frac{p-s}{p}} h_0^{-\frac{s}{p}} ||u||_X^s < +\infty, \end{aligned}$$

which implies that $u \in L^s(V)$ and for any $p \le s$,

$$\|u\|_s = \left(\int_V |u|^s d\mu\right)^{\frac{1}{s}} \le (\mu_{\min})^{\frac{p-s}{ps}} h_0^{-\frac{1}{p}} ||u||_X. \tag{8}$$

Next, we prove that $E \hookrightarrow L^s(V)$ continuously for any $1 \le s < p$. Indeed, (h_2) implies that

$$h^{-1} \in L^{\frac{1}{p-1}}(V). \tag{9}$$

Then for any $u \in X$,

$$\begin{aligned}\int_V |u| d\mu &= \int_V h^{-\frac{1}{p}} h^{\frac{1}{p}} |u| d\mu \\ &\le \left(\int_V h^{-\frac{1}{p}\cdot\frac{p}{p-1}} d\mu\right)^{\frac{p-1}{p}} \left(\int_V h|u|^p d\mu\right)^{\frac{1}{p}} \\ &= ||h^{-1}||_{\frac{1}{p-1}}^{\frac{1}{p}} \left(\int_V h|u|^p d\mu\right)^{\frac{1}{p}} \\ &\le ||h^{-1}||_{\frac{1}{p-1}}^{\frac{1}{p}} ||u||_X < +\infty, \end{aligned} \tag{10}$$

which implies that $u \in L^1(V)$. And it follows from

$$||u||_{L^\infty(V)} \le \frac{1}{\mu_{\min}} \int_V |u| d\mu$$

that

$$\int_V |u|^s d\mu = \int_V |u|^{s-1}|u| d\mu \le \frac{1}{{\mu_{\min}}^{s-1}} \left(\int_V |u| d\mu\right)^s \le \frac{1}{{\mu_{\min}}^{s-1}} ||h^{-1}||_{\frac{1}{p-1}}^{\frac{s}{p}} ||u||_X^s. \tag{11}$$

Therefore, for any $1 \le s < p$, $X \hookrightarrow L^s(V)$ continuously and

$$\|u\|_s = \left(\int_V |u|^s d\mu\right)^{\frac{1}{s}} \le (\mu_{\min})^{\frac{1-s}{s}} ||h^{-1}||_{\frac{1}{p-1}}^{\frac{1}{p}} ||u||_X. \tag{12}$$

By (8) and (12), we can obtain that there exists a constant ξ depending on $s, p, \mu_{\min}, h_0$ and $||h^{-1}||_{\frac{1}{p-1}}$ such that for any $u \in X$,

$$||u||_s \le \xi ||u||_X.$$

Let p^* be the exponent conjugate to p. Each element $v \in L^{p^*}(V)$ defines a linear functional ϕ_v on $L^p(V)$ via

$$\phi_v(u) = \int_V uv d\mu, \quad u \in L^p(V).$$

Noting that X is reflexive, for $\{u_k\}$ bounded in X, we have that, up to a subsequence, $u_k \rightharpoonup u$ in X. on the other hand, $\{u_k\} \subset X$ is also bounded in $L^p(V)$ and we have $u_k \rightharpoonup u$ in $L^p(V)$, which tell us that,

$$\lim_{n\to\infty} \phi_v(u_n - u) = \lim_{k\to\infty} \int_V (u_k - u) v d\mu = \lim_{k\to\infty} \sum_{x\in V} \mu(x)(u_k(x) - u(x))v(x) = 0, \quad \forall v \in L^{p^*}(V). \tag{13}$$

Take any $x_0 \in V$ and let

$$v_0(x) = \begin{cases} 1 & x = x_0, \\ 0 & x \neq x_0. \end{cases}$$

Obviously it belongs to $L^{p^*}(V)$. By substituting v_0 into (13) we have

$$\lim_{k\to\infty} \mu(x_0)(u_k(x_0) - u(x_0)) = 0, \tag{14}$$

which implies that $\lim\limits_{k\to\infty} u_k(x) = u(x)$ for any $x \in V$.

We now prove $u_k \to u$ in $L^s(V)$ for all $1 \leq s \leq +\infty$. Since $\{u_k\}$ bounded in X and $u \in X$, there exists some constant C_1 such that

$$\int_V h|u_k - u|^p d\mu \leq C_1.$$

Let $x_0 \in V$ be fixed. For any $\epsilon > 0$, in view of (9), there exists some $R > 0$ such that

$$\int_{d(x,x_0)>R} h^{-\frac{1}{p-1}} d\mu < \epsilon^p.$$

Hence by the Hölder inequality,

$$\begin{aligned} \int_{d(x,x_0)>R} |u_k - u| d\mu &= \int_{d(x,x_0)>R} h^{-\frac{1}{p}} h^{\frac{1}{p}} |u_k - u| d\mu \\ &\leq \left(\int_{d(x,x_0)>R} h^{-\frac{1}{p-1}} d\mu \right)^{\frac{p-1}{p}} \left(\int_{d(x,x_0)>R} h|u_k - u|^p d\mu \right)^{\frac{1}{p}} \\ &\leq C_1^{\frac{1}{p}} \epsilon^{p-1}. \end{aligned} \tag{15}$$

Moreover, we have that up to a subsequence,

$$\lim_{k\to+\infty} \int_{d(x,x_0)\leq R} |u_k - u| d\mu = 0. \tag{16}$$

Combining (15) and (16), we conclude

$$\liminf_{k\to+\infty}\int_V |u_k-u|d\mu = 0.$$

In particular, there holds up to a subsequence, $u_k \to u$ in $L^1(V)$. Since

$$||u_k-u||_{L^\infty(V)} \le \frac{1}{\mu_{\min}}\int_V |u_k-u|d\mu,$$

there holds for any $1 < s < +\infty$,

$$\int_V |u_k-u|^s d\mu \le \frac{1}{\mu_{\min}^{s-1}}(\int_V |u_k-u|d\mu)^s.$$

Therefore, up to a subsequence, $u_k \to u$ in $L^s(V)$ for all $1 \le s \le +\infty$. □

3 Proof of Theorem 1.1

In this section, by using the Ekeland variational principle and the Mountain Pass theorem, we prove that the equation (S_p) admits at least two nontrivial different solutions.

Lemma 3.1 *([22, Mountain Pass theorem]). Let* $(Y, ||\cdot||)$ *be a Banach space and* $I \in C^1(Y, \mathbb{R})$ *be a functional satisfying the* $(PS)_c$ *condition. If there exist* $e \in Y$ *and* $r > 0$ *satisfying* $||e|| > r$ *such that*

$$b := \inf_{||u||=r} I(u) > I(0) \ge I(e),$$

then c *is a critical value of* I*, where*

$$c := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t))$$

and

$$\Gamma := \{\gamma \in C([0,1], Y) : \gamma(0) = 0, \gamma(1) = e\}.$$

Lemma 3.2 *([23, Ekeland variational principle]). Let* (Y, d) *be a complete metric space and* $I : Y \to \mathbb{R}\bigcup+\infty$ *be a lower-semicontinuous function which is bounded from below. Suppose* $\varepsilon > 0$ *and* v *are such that*

$$I(v) \le \inf_Y I + \varepsilon.$$

Then given any $\lambda > 0$*, there exists* $u_\lambda \in Y$ *such that*

$$I(u_\lambda) \le I(v), \quad d(u_\lambda, v) \le \lambda,$$

and

$$I(u_\lambda) < I(u) + \frac{\varepsilon}{\lambda} d(u_\lambda, u), \quad \forall u \neq u_\lambda.$$

Lemma 3.3 *If $u \in X$ is a weak solution of (S_p), then u is also a point-wise solution of (S_p).*

Proof If $u \in X$ is a weak solution of (S_p), by Lemma 2.2, for any test function $\phi \in C_c(V)$, we have

$$\int_V (-\Delta_p u\phi + h|u|^{p-2}u\phi) d\mu = \int_V f(x,u)\phi d\mu - \int_V g(x)\phi d\mu. \tag{17}$$

For any fixed $x_0 \in V$, taking a test function $\phi : V \to \mathbb{R}$ in (17) with

$$\phi(x) = \begin{cases} 1, & x = x_0, \\ 0, & x \neq x_0, \end{cases}$$

and $\phi \in X$, then we have

$$-\Delta_p u(x_0) + h|u(x_0)|^{p-2}u(x_0) - f(x_0, u(x_0)) - g(x_0) = 0.$$

Since x_0 is arbitrary, we conclude that u is a point-wise solution of (S_p). □

Lemma 3.4 *Let $(G_1) - (G_2)$, $(h_1) - (h_2)$, (f_1) and (f_3) hold and suppose that $g \in L^{\frac{p}{p-1}}(V)$, where $p \geq 2$. Then J is weakly lower semi-continuous.*

Proof Note that

$$J(u) = \Phi(u) - \Psi(u), \quad u \in X,$$

where

$$\Phi(u) = \frac{1}{p} \int_V (|\nabla u|^p + h(x)|u|^p) d\mu \ \text{ and } \ \Psi(u) = \int_V F(x,u) d\mu + \int_V g(x) u d\mu.$$

It is easy to see that Φ is weakly lower semi-continuous in X. Next, we prove that Ψ is weakly continuous in X. Let $u_n \rightharpoonup u$ in X as $n \to \infty$. Consider the Banach space $L^p(V) \cap L^q(V)$ endowed with the norm $||u||_{L^p(V)\cap L^q(V)} := ||u||_p + ||u||_q$. Then by Lemma 2.3, we have that $u_n \to u$ in $L^p(V) \cap L^q(V)$ and $u_n(x) \to u(x)$ for all $x \in V$. If (f_1) and (f_3) hold, then for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(x,u)| \leq \varepsilon |u|^{p-1} + C_\varepsilon |u|^{q-1}, \quad \forall u \in \mathbb{R}, \quad x \in V \tag{18}$$

and

$$|F(x,u)| \leq \frac{\varepsilon}{p}|u|^p + \frac{C_\varepsilon}{q}|u|^q, \quad \forall u \in \mathbb{R}, \quad x \in V. \tag{19}$$

Then, by (19) and Lemma 5.10 in [17], there exists a function $Q(x) \in L^1(V)$ such that $|F(x, u_n)| \leq Q(x)$ and $F(x, u_n(x)) \to F(x, u(x))$ as $n \to +\infty$ for any $x \in V$. Hence, by the Lebesgue dominated convergence theorem, we have $\lim\limits_{n\to\infty} \int_V F(x, u_n)d\mu = \int_V F(x, u)d\mu$. Furthermore, by the Hölder's inequality, we get

$$\int_V g(x)(u_n - u)d\mu \leq \|g\|_{\frac{p}{p-1}} \|u_n - u\|_p \to 0 \text{ as } n \to \infty.$$

Thus, Ψ is weakly continuous in X. Therefore J is weakly lower semi-continuous in X. □

Lemma 3.5 *Assume $(f_1) - (f_3)$ hold. Then there exist positive constants ρ, σ, δ such that $J(u) \geq \sigma$ for all functions u with $||u||_X = \rho$ and all $g \in L^{\frac{p}{p-1}}(V)$ with $\|g\|_{\frac{p}{p-1}} \leq \delta$.*

Proof By (19) and the Hölder's inequality, we have

$$\begin{aligned} J(u) &= \frac{1}{p}\int_V (|\nabla u|^p + h|u|^p)d\mu - \int_V F(x, u)d\mu - \int_V g(x)ud\mu \\ &\geq \frac{1}{p}\|u\|_X^p - \frac{\varepsilon}{p}\|u\|_p^p - \frac{C_\varepsilon}{q}\|u\|_q^q - \|g\|_{\frac{p}{p-1}}\|u\|_p \\ &\geq \frac{1}{p}\|u\|_X^p - \frac{\varepsilon}{p}\xi_1\|u\|_X^p - \frac{C_\varepsilon}{q}\xi_2\|u\|_X^q - \xi_3\|g\|_{\frac{p}{p-1}}\|u\|_X \\ &= \|u\|_X\left[\left(\frac{1}{p} - \frac{\varepsilon}{p}\xi_1\right)\|u\|_X^{p-1} - \frac{C_\varepsilon}{q}\xi_2\|u\|_X^{q-1} - \xi_3\|g\|_{\frac{p}{p-1}}\right]. \end{aligned}$$

Taking $\varepsilon = \frac{1}{2\xi_1}$ and setting

$$\eta(t) = \frac{1}{2p}t^{p-1} - \frac{C_\varepsilon}{q}\xi_2 t^{q-1}, \quad \forall t \in [0, +\infty),$$

we see that there exists $\rho > 0$ such that $\max\limits_{t\in[0,+\infty)} \eta(t) = \eta(\rho)$, since $q > p \geq 2$. Taking $\delta = \frac{\eta(\rho)}{2\xi_3}$, we obtain that $J(u) \geq \sigma := \frac{\eta(\rho)\rho}{2} > 0$ for all u in X with $\|u\|_X = \rho$ and for all $g \in L^{\frac{p}{p-1}}(V)$ with $\|g\|_{\frac{p}{p-1}} \leq \delta$. □

Lemma 3.6 *Assume $(f_1) - (f_3)$ holds. Then there exists some non-negative function $u \in X$ such that $J(tu) \to -\infty$ as $t \to +\infty$.*

Proof We obtain from (f_2) and (f_3) that there exist positive constants C_1 and C_2 such that

$$F(x, s) \geq C_1|s|^\alpha - C_2|s|^p, \quad \forall (x, s) \in V \times \mathbb{R}, \tag{20}$$

where $\alpha > p \geq 2$. Let x_0 be fixed. Take a function

$$u(x) = \begin{cases} 1 & x = x_0; \\ 0 & x \neq x_0. \end{cases}$$

Note that $F(x, 0) = 0$, then we have

$$\begin{aligned} J(tu) &= \frac{t^p}{p}\int_V (|\nabla u|^p + h|u|^p)d\mu - \int_V F(x, tu)d\mu - \int_V g(x)tud\mu \\ &= \frac{t^p}{p}\sum_{x\in V}|\nabla u|^p(x)\mu(x) + \frac{t^p}{p}\sum_{x\in V}h(x)|u(x)|^p\mu(x) \\ &\quad -\sum_{x\in V}\mu(x)F(x, tu(x)) - \sum_{x\in V}\mu(x)g(x)tu(x) \\ &= \frac{t^p}{p}\sum_{x\in V}|\nabla u|^p(x)\mu(x) + \frac{t^p}{p}h(x_0)|u(x_0)|^p\mu(x_0) \\ &\quad -\mu(x_0)F(x_0, tu(x_0)) - \mu(x_0)g(x_0)tu(x_0) \\ &= \frac{t^p}{p}\sum_{x\in V}|\nabla u|^p(x)\mu(x) + \frac{t^p}{p}h(x_0)\mu(x_0) \\ &\quad -\mu(x_0)F(x_0, t) - \mu(x_0)g(x_0)t \\ &\leq \frac{t^p}{p}\sum_{x\in V}|\nabla u|^p(x)\mu(x) + \frac{t^p}{p}h(x_0)\mu(x_0) - \mu(x_0)C_1t^\alpha \\ &\quad +\mu(x_0)C_2t^p - \mu(x_0)g(x_0)t. \end{aligned}$$

By the definition of $u(x)$, the nonzero terms of $\sum\limits_{x\in V}|\nabla u|^p(x)\mu(x)$ are finite, since $G = (V, E)$ is locally finite graph. Then $\sum\limits_{x\in V}|\nabla u|^p(x)\mu(x)$ is bounded. Therefore,

$$\begin{aligned} J(tu) &\leq \frac{t^p}{p}\sum_{x\in V}|\nabla u|^p(x)\mu(x) + \frac{t^p}{p}h(x_0)\mu(x_0) - \mu(x_0)C_1t^\alpha + \mu(x_0)C_2t^p \\ &\quad -\mu(x_0)g(x_0)t \to -\infty \end{aligned}$$

as $t \to +\infty$, since $\alpha > p \geq 2$. □

Next, we prove that J satisfies $(PS)_c$ condition. And first we need the following two lemmas.

Lemma 3.7 *For any $u, v \in X$, it holds that*

$$(\Phi'(u) - \Phi'(v), u - v) \geq (||u||_X^{p-1} - ||v||_X^{p-1})(||u||_X - ||v||_X). \tag{21}$$

Proof We follow the idea of the proof of Lemma 3.1 in [24]. By direct computations, we have

$$
\begin{aligned}
(\Phi'(u)-\Phi'(v),u-v) &= (\Phi'(u),u-v)-(\Phi'(v),u-v)\\
&= \int_V (|\nabla u|^{p-2}\Gamma(u,u-v)+h|u|^{p-2}u(u-v))d\mu\\
&\quad -\int_V (|\nabla v|^{p-2}\Gamma(v,u-v)+h|v|^{p-2}v(u-v))d\mu\\
&= \int_V (|\nabla u|^{p-2}\Gamma(u,u)-|\nabla u|^{p-2}\Gamma(u,v)\\
&\quad + h|u|^{p-2}(u^2-uv))d\mu\\
&\quad -\int_V (|\nabla v|^{p-2}\Gamma(v,u)-|\nabla v|^{p-2}\Gamma(v,v)\\
&\quad + h|v|^{p-2}(vu-v^2))d\mu\\
&= \int_V (|\nabla u|^{p}+|\nabla v|^{p}-|\nabla u|^{p-2}\Gamma(u,v)\\
&\quad -|\nabla v|^{p-2}\Gamma(v,u))d\mu\\
&\quad +\int_V h(|u|^p+|v|^p-|u|^{p-2}uv-|v|^{p-2}vu)d\mu\\
&= ||u||_X^p+||v||_X^p-\int_V (|\nabla u|^{p-2}\Gamma(u,v)+h|u|^{p-2}uv)d\mu\\
&\quad -\int_V (|\nabla v|^{p-2}\Gamma(v,u)+h|v|^{p-2}vu)d\mu.
\end{aligned}
$$

Applying Hölder's inequality,

$$
\begin{aligned}
&\int_V (|\nabla u|^{p-2}\Gamma(u,v)+h|u|^{p-2}uv)d\mu\\
&= \int_V (|\nabla u|^{p-2}\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}(u(y)-u(x))(v(y)-v(x))+h|u|^{p-2}uv)d\mu\\
&= \int_V (|\nabla u|^{p-2}\frac{1}{2\mu(x)}\sum_{y\sim x}\omega_{xy}^{\frac{1}{2}}(u(y)-u(x))\omega_{xy}^{\frac{1}{2}}(v(y)-v(x))+h|u|^{p-2}uv)d\mu\\
&\leq \int_V (|\nabla u|^{p-2}(\Gamma(u))^{\frac{1}{2}}(\Gamma(v))^{\frac{1}{2}}+h|u|^{p-2}uv)d\mu\\
&= \int_V (|\nabla u|^{p-2}|\nabla u||\nabla v|+h|u|^{p-2}uv)d\mu\\
&\leq \left(\int_V |\nabla u|^p d\mu\right)^{\frac{p-1}{p}}\left(\int_V |\nabla v|^p d\mu\right)^{\frac{1}{p}}+\left(\int_V h|u|^p d\mu\right)^{\frac{p-1}{p}}\left(\int_V h|v|^p d\mu\right)^{\frac{1}{p}}.
\end{aligned}
$$

Using the following inequality

$$(a+b)^{\beta}(c+d)^{1-\beta} \geq a^{\beta}c^{1-\beta} + b^{\beta}d^{1-\beta}, \tag{22}$$

which holds for any $\beta \in (0,1)$ and for any $a, b, c, d \geq 0$. Set $\beta = \frac{p-1}{p}$ and

$$a = \int_V |\nabla u|^p d\mu, \quad b = \int_V h|u|^p d\mu, \quad c = \int_V |\nabla v|^p d\mu, \quad d = \int_V h|v|^p d\mu, \tag{23}$$

we can get that

$$\begin{aligned}
&\int_V (|\nabla u|^{p-2}\Gamma(u,v) + h|u|^{p-2}uv)d\mu \\
&\leq \left(\int_V (|\nabla u|^p + h|u|^p)d\mu\right)^{\frac{p-1}{p}} \left(\int_V (|\nabla v|^p + h|v|^p)d\mu\right)^{\frac{1}{p}} \\
&= ||u||_X^{p-1}||v||_X.
\end{aligned} \tag{24}$$

Similarly, we can obtain

$$\int_V (|\nabla v|^{p-2}\Gamma(v,u) + h|v|^{p-2}vu)d\mu \leq ||v||_X^{p-1}||u||_X.$$

Therefore, we have

$$\begin{aligned}
(\Phi'(u) - \Phi'(v), u - v) &\geq ||u||_X^p + ||v||_X^p - ||u||_X^{p-1}||v||_X - ||v||_X^{p-1}||u||_X \\
&= (||u||_X^{p-1} - ||v||_X^{p-1})(||u||_X - ||v||_X).
\end{aligned}$$

□

Lemma 3.8 *If $u_n \rightharpoonup u$ in X and $(\Phi'(u_n), u_n - u) \to 0$, then $u_n \to u$ in X.*

Proof Since X is a reflexive Banach space, weak convergence and norm convergence imply strong convergence. Therefore we only need to show that $||u_n||_X \to ||u||_X$. Note that

$$\lim_{n\to\infty} (\Phi'(u_n) - \Phi'(u), u_n - u) = \lim_{n\to\infty} (\Phi'(u_n), u_n - u) - (\Phi'(u), u_n - u) = 0.$$

By Lemma 3.7 we have,

$$(\Phi'(u_n) - \Phi'(u), u_n - u) \geq (||u_n||_X^{p-1} - ||u||_X^{p-1})(||u_n||_X - ||u||_X).$$

Hence $||u_n||_X \to ||u||_X$ as $n \to \infty$ and the assertion follows. □

Now, we prove that J satisfies $(PS)_c$ condition.

Lemma 3.9 *Assume* $(h_1), (h_2)$ *and* $(f_1) - (f_3)$ *holds. Then* J *satisfies the* $(PS)_c$ *condition for any* $c \in \mathbb{R}$.

Proof Note that $J(u_k) \to c$ and $J'(u_k) \to 0$ as $k \to +\infty$ are equivalent to

$$\frac{1}{p}||u_k||_X^p - \int_V F(x, u_k)d\mu - \int_V g(x)u_k d\mu = c + o_k(1), \tag{25}$$

and

$$(\Phi'(u_k), v) - (\Psi'(u_k), v) = o_k(1)||v||_X, \quad \forall v \in X. \tag{26}$$

Here, $o_k(1) \to 0$ as $k \to +\infty$. Taking $v = u_k$ in (26), we have

$$||u_k||_X^p = \int_V f(x, u_k)u_k d\mu + \int_V g(x)u_k d\mu + o_k(1)||u_k||_X. \tag{27}$$

In view of (f_2), we have by combining (25) and (27) that

$$\begin{aligned}
||u_k||_X^p \leq & \frac{p}{\alpha}\left[\int_V f(x, u_k)u_k d\mu + \int_V g(x)u_k d\mu\right] \\
& + \left(p - \frac{p}{\alpha}\right)\int_V g(x)u_k d\mu + pc + o_k(1) \\
\leq & \frac{p}{\alpha}||u_k||_X^p + \left(p - \frac{p}{\alpha}\right)\|g\|_{\frac{p}{p-1}}\|u_k\|_p + o_k(1)||u_k||_X + pc + o_k(1) \\
\leq & \frac{p}{\alpha}||u_k||_X^p + \left(p - \frac{p}{\alpha}\right)\xi\|g\|_{\frac{p}{p-1}}\|u\|_X + o_k(1)||u_k||_X + pc + o_k(1).
\end{aligned}$$

Then

$$\left(1 - \frac{p}{\alpha}\right)||u_k||_X^p \leq \left(p - \frac{p}{\alpha}\right)\xi\|g\|_{\frac{p}{p-1}}\|u\|_X + o_k(1)||u_k||_X + pc + o_k(1),$$

which implies that $\{u_k\}$ is bounded in X, since $\alpha > p \geq 2$. Then Lemma 2.3 implies that up to a subsequence, there exists $u \in X$ such that $u_k \rightharpoonup u$ in X and $u_k \to u$ in $L^s(V)$, $1 \leq s \leq +\infty$.

It follows from (18) that

$$\begin{aligned}
\left|\int_V f(x, u_k)(u_k - u)d\mu\right| &\leq \int_V |f(x, u_k)||u_k - u|d\mu \\
&\leq \int_V (\varepsilon|u_k|^{p-1} + C_\varepsilon|u_k|^{q-1})|u_k - u|d\mu \\
&\leq \varepsilon\|u_k\|_p^{p-1}\|u_k - u\|_p + C_\varepsilon\|u_k\|_q^{q-1}\|u_k - u\|_q \\
&\leq \varepsilon\xi_1\|u_k\|_X^{p-1}\|u_k - u\|_p + C_\varepsilon\xi_2\|u_k\|_X^{q-1}\|u_k - u\|_q \\
&= o_k(1).
\end{aligned}$$

Note that $\int_V g(x)(u_k - u)d\mu \leq \|g\|_{\frac{p}{p-1}} \|u_k - u\|_p = o_k(1)$. Replacing φ by $u_k - u$ in (26), we have

$$\begin{aligned}(\Phi'(u_k), u_k - u) &= \int_V (|\nabla u_k|^{p-2}\Gamma(u_k, u_k - u) + h|u_k|^{p-2}u_k(u_k - u))d\mu \\ &= \int_V f(x, u_k)(u_k - u)d\mu + \int_V g(x)(u_k - u)d\mu + o_k(1)||u_k - u||_X \\ &= o_k(1),\end{aligned}$$

thus $(\Phi'(u_k), u_k - u) \to 0$ as $k \to \infty$. Then, by Lemma 3.8 we have $u_k \to u$ in X as $k \to \infty$. □

We have the following conclusion about the perturbation term g.

Lemma 3.10 *Suppose that $g \in L^{\frac{p}{p-1}}(V)$ and $g \not\equiv 0$. Then there exists a function $\varphi \in X$ such that $\int_V g(x)\varphi(x)d\mu > 0$.*

Proof By Lemma 2.1, we know that $C_c(V)$ is dense in $L^p(V)$. Since $|g|^{\frac{p}{p-1}-2}g \in L^p(V)$, there exists a sequence $\{g_n\}$ in $C_c(V)$ such that $g_n \to |g|^{\frac{p}{p-1}-2}g$ in $L^p(V)$. Hence, there exists $n_0 > 0$ such that

$$\|g_{n_0} - |g|^{\frac{p}{p-1}-2}g\|_p \leq \frac{1}{2}\|g\|^{\frac{1}{p-1}}_{\frac{p}{p-1}}.$$

Obviously, $g_{n_0} \in X$ and taking $\varphi = g_{n_0}$, we have

$$\begin{aligned}\int_V g(x)\varphi(x)d\mu &= \int_V g_{n_0}(x)g(x)d\mu \\ &\geq -\int_V \left|g_{n_0}(x) - |g(x)|^{\frac{p}{p-1}-2}g(x)\right| |g(x)|d\mu + \int_V |g(x)|^{\frac{p}{p-1}}d\mu \\ &\geq -\|g_{n_0} - |g|^{\frac{p}{p-1}-2}g\|_p\|g\|_{\frac{p}{p-1}} + \|g\|^{\frac{p}{p-1}}_{\frac{p}{p-1}} \\ &\geq -\frac{1}{2}\|g\|^{\frac{1}{p-1}}_{\frac{p}{p-1}}\|g\|_{\frac{p}{p-1}} + \|g\|^{\frac{p}{p-1}}_{\frac{p}{p-1}} \\ &= \frac{1}{2}\|g\|^{\frac{p}{p-1}}_{\frac{p}{p-1}} > 0.\end{aligned}$$

□

Proof of Theorem 1.1 The proof of this theorem is divided into two steps.

Step 1. In this step we prove that there exists a function $u_0 \in X$ such that $J'(u_0) = 0$ and $J(u_0) < 0$. In fact, by Lemma 3.10 and (20), there exists $\varphi \in X$ such that

$$J(t\varphi) \leq \frac{t^p}{p}\|\varphi\|^p_X - t^\alpha C_1\|\varphi\|^\alpha_\alpha + t^p C_2\|\varphi\|^p_p - t\int_V g(x)\varphi d\mu < 0,$$

for $t \in (0, 1)$ small enough and $\|\varphi\|_X \leq \rho$, where $\rho > 0$ is given in Lemma 3.5. Thus we get $c_0 = \inf\{J(u) : u \in \overline{B}_\rho\} < 0$, where $B_\rho = \{u \in X : \|u\|_X < \rho\}$. By the Ekeland variational principle, Lemma 3.4 and Lemma 3.5, there exists a sequence $\{u_n\} \subset B_\rho$ such that $c_0 \leq J(u_n) \leq c_0 + \frac{1}{n}$ and $J(v) \geq J(u_n) - \frac{1}{n}\|v - u_n\|_X$ for all $v \in \overline{B}_\rho$. Then a standard procedure gives that $\{u_n\}$ is a bounded (PS) sequence of J. Therefore, Lemma 3.9 imply that there exists a function $u_0 \in B_\rho$ such that $J'(u_0) = 0$ and $J(u_0) = c_0 < 0$.

Step 2. In this step we prove that there exists a function $u_1 \in E$ such that $J'(u_1) = 0$ and $J(u_1) > 0$. By Lemma 3.5, Lemma 3.6 and Lemma 3.9, J satisfies all the assumptions of the Mountain Pass theorem. Thus we obtain that $c_1 = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$ is the critical value of J, where $\Gamma := \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = e\}$. In particular, there exists some $u_1 \in X$ such that $J(u) = c_1$. By Lemma 3.5, $J(u_1) = c_1 \geq \sigma > 0$. Thus, $0 \neq u_1 \neq u_0$.

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