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Oscillation of linear and half-linear differential equations via generalized Riccati technique

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Abstract
In this paper, we analyse the oscillation of second-order half-linear differential equations whose coefficients are given by the products of (typically bounded) functions and power functions. Applying the generalized Riccati technique, we find a very general oscillation criterion for the studied equations. This criterion covers several half-linear equations with unbounded non-power parts of coefficients, but it is new also for linear equations with bounded non-power parts. This fact is appropriately documented by new corollaries.

Keywords Half-linear equation · Linear equation · Riccati equation · Oscillation · Riccati technique

Mathematics Subject Classification 34C10 · 34C15

1 Introduction

We study the oscillation of the half-linear differential equation

$$\left(R(t)\Phi\left(x'(t)\right)\right)' + S(t)\Phi(x(t)) = 0, \quad \Phi(x) = |x|^{p-1}\operatorname{sgn} x, \quad p > 1, \quad (1.1)$$

where coefficients $R > 0$, S are continuous functions. Function Φ generates the so-called one dimensional p-Laplacian. This function connects the considered half-linear equations with partial differential equations (see, e.g., [3,12]). Half-linear equations generalize linear equations which form a special case of Eq. (1.1) for $p = 2$. We point

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out that the basic difference between linear and half-linear equations is the lack of the additivity of solution spaces in the half-linear case. This discrepancy motivates the used nomenclature. Nevertheless, the Sturm separation theorem remains valid for half-linear equations (see again [3,12]). Thus, analogously as in the linear case, one can classify half-linear equations as oscillatory and non-oscillatory, i.e., if one non-trivial solution is oscillatory (which means that it has zero points tending to infinity), then every solution is oscillatory.

Now we mention two known oscillation theorems which give specific motivation for our current research. In the first theorem, we consider equations whose coefficients are constant.

Theorem 1 *Let us consider the equation*

$$\left(t^{\alpha} r^{1-p} \Phi(x'(t))\right)' + t^{\alpha-p} s \Phi(x(t)) = 0, \tag{1.2}$$

where $\alpha \neq p-1$ *and* $r > 0$ *and* $s \in \mathbb{R}$ *are constants.*

(i) *If* $p^p s > |p-\alpha-1|^p r^{1-p}$*, then Eq. (1.2) is oscillatory.*
(ii) *If* $p^p s \leq |p-\alpha-1|^p r^{1-p}$*, then Eq. (1.2) is non-oscillatory.*

Proof We refer, e.g., to [12, Theorem 1.4.4] (or directly to [15,16]). □

The strongest known oscillation criterion for equations in the form of Eq. (1.2) reads as follows.

Theorem 2 *Let a positive continuously differentiable function* f *and a continuous function* $g \geq 1$ *be such that*

$$\lim_{t\to\infty} f'(t) g(t) = 0, \quad \lim_{t\to\infty} \frac{f(t) g^2(t)}{t} = 0.$$

Let us consider the equation

$$\left(t^{\alpha} r^{1-p}(t) \Phi\left(x'(t)\right)\right)' + t^{\alpha-p} s(t) \Phi(x(t)) = 0, \tag{1.3}$$

where $\alpha \neq p-1$ *and* $r > 0$*,* s *are continuous functions satisfying*

$$\limsup_{t\to\infty} \frac{\int_t^{t+f(t)} r(\tau)\,\mathrm{d}\tau}{f(t)g(t)} < \infty, \quad \limsup_{t\to\infty} \frac{\int_t^{t+f(t)} |s(\tau)|\,\mathrm{d}\tau}{f(t)g(t)} < \infty.$$

Let

$$r_f := \liminf_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} r(\tau)\,\mathrm{d}\tau \in \mathbb{R}, \quad s_f := \liminf_{t\to\infty} \frac{1}{f(t)} \int_t^{t+f(t)} s(\tau)\,\mathrm{d}\tau \in \mathbb{R}.$$

If $p^p s_f > |p-\alpha-1|^p r_f^{1-p}$*, then Eq. (1.3) is oscillatory.*

Proof See [36]. □

The aim of this paper is to contribute to the rapidly developing oscillation theory of half-linear equations in the form of Eq. (1.2). More precisely, we want to obtain an oscillation criterion which enhances Theorem 1 substantially and which is not based on estimations of average values of coefficients, i.e., which differs from Theorem 2 significantly. Using the generalized Riccati technique, we find such a criterion (applicable also for coefficients whose average values do not exist). We point out that the method used in [36] is the modified half-linear Prüfer angle and that a non-oscillatory counterpart of Theorem 2 is proved in [48]. Note that our main result is new even for linear equations which is demonstrated in the last section of this paper.

In this paragraph, we mention references that are connected to the treated topic. Other oscillation criteria for equations in the form of Eq. (1.2) are obtained, e.g., in [14, 29]. We add also papers [24–26,35,37,47] and references cited therein for important special cases. The oscillation theory of general half-linear equations is systematically presented, e.g., in the already mentioned books [3,12]. The oscillation of special types of corresponding discrete equations is studied in [23,28,38,43] (and also in [10,53]) in the case of difference equations and in [30,34,39,51] (and also in [18,45]) in the case of dynamic equations on time scales. For possible generalizations in the discrete case, see [1,2,54,55]; for more general non-linear equations, see [4,22,46,49] (and also [41,50]); for half-linear advanced (delay) differential equations, see [5–8]; and, for applications in the theory of PDE's, see at least [11,19,40,52].

The rest of this paper is divided into three sections. The next section contains the detailed specification of the studied equations together with all considered assumptions and the description of the used generalized Riccati transformation. The main result is proved in Sect. 3. The last section is devoted to corollaries of our main result, where we demonstrate the novelty of the main result and its impact to linear equations.

To conclude this section, we mention the basic considered notation. Let $p > 1$ be given and let q be the number conjugated with p, i.e., $p + q = pq$. Since we consider all equations for very large t, we consider $t \geq a$ for simplicity, where $a > 0$ is given (and arbitrarily large).

2 Treated equations

In this paper, we study the Euler type second-order half-linear differential equation

$$\left(t^{\alpha} r^{1-p}(t) \Phi(x'(t))\right)' + t^{\alpha-p} s(t) \Phi(x(t)) = 0, \quad t > a, \tag{2.1}$$

where $\alpha < p - 1$ and $r : [a, \infty) \to (0, \infty)$ and $s : [a, \infty) \to (-\infty, \infty)$ are continuous functions such that

$$\int_a^\infty \tau^{\alpha(1-q)} r(\tau)\, \mathrm{d}\tau = \infty, \quad \int_a^\infty \tau^{\alpha-p} s(\tau)\, \mathrm{d}\tau \in \mathbb{R}. \tag{2.2}$$

We consider Eq. (2.1) with

$$\int_a^\infty \frac{p^p s(\tau) - (p-\alpha-1)^p r^{1-p}(\tau)}{\tau} \, \mathrm{d}\tau = \infty. \tag{2.3}$$

Now we mention the used generalized Riccati equation. Let x be a non-zero solution of Eq. (2.1). Using the transformation

$$\zeta(t) = -\Phi\left(\frac{t x'(t)}{r(t) x(t)}\right),$$

from Eq. (2.1), we obtain the equation

$$\zeta'(t) = \frac{1}{t}\left((p-\alpha-1)\zeta(t) + s(t) + (p-1) r(t) |\zeta(t)|^q\right), \quad t > a. \tag{2.4}$$

In fact, for a non-zero solution x of Eq. (2.1), the well-known Riccati transformation

$$w(t) = t^\alpha r(t) \Phi\left(\frac{x'(t)}{x(t)}\right)$$

leads to the Riccati half-linear equation

$$w'(t) + t^{\alpha-p} s(t) + (p-1) t^{\alpha(1-q)} r(t) |w(t)|^q = 0, \quad t > a. \tag{2.5}$$

Then, the transformation

$$\zeta(t) = -t^{p-\alpha-1} w(t) \tag{2.6}$$

gives Eq. (2.4). We add that the derivation of Eq. (2.4) is described, e.g., in [35] (see also [12, Eq. (9.2.3) on p. 439]) and that Eq. (2.4) is called the generalized Riccati equation (this designation is motivated by the consistency with used versions of the Riccati equation in the literature). At the end of this section, we highlight that the aim of this paper is to prove the oscillation of Eq. (2.1) using Eq. (2.4).

3 Main result

We begin with a known theorem containing a condition which is equivalent to the non-oscillation of Eq. (2.1).

Theorem 3 *Equation (2.1) is non-oscillatory if and only if there exist $b \geq a$ and a solution $w\colon [b,\infty) \to \mathbb{R}$ of Eq. (2.5) satisfying either*

$$w(t) = \int_t^\infty \tau^{\alpha-p} s(\tau) \, \mathrm{d}\tau + (p-1) \int_t^\infty \tau^{\alpha(1-q)} r(\tau) |w(\tau)|^q \, \mathrm{d}\tau \geq 0, \quad t \geq b, \tag{3.1}$$

or

$$w(t) = \int_t^\infty \tau^{\alpha-p} s(\tau)\,\mathrm{d}\tau + (p-1)\int_t^\infty \tau^{\alpha(1-q)} r(\tau)|w(\tau)|^q\,\mathrm{d}\tau \le 0, \quad t \ge b. \tag{3.2}$$

Proof We refer, e.g., to [12, Theorems 2.2.4 and 2.2.5], where it is necessary to use the divergence of $\int_a^\infty \tau^{\alpha(1-q)} r(\tau)\,\mathrm{d}\tau$ and the convergence of $\int_a^\infty \tau^{\alpha-p} s(\tau)\,\mathrm{d}\tau$ [see (2.2)]. □

To prove the announced result, we need the lemma below which describes the connection between the non-oscillation of solutions of Eq. (2.1) and the non-positivity of a solution of the associated generalized Riccati equation (2.4).

Lemma 1 *Let us consider Eq.* (2.1)*, where* $\alpha < p-1$ *and* $r : [a,\infty) \to (0,\infty)$ *and* $s : [a,\infty) \to (-\infty,\infty)$ *are continuous functions satisfying* (2.2)*, and let* (2.3) *be valid. If Eq.* (2.1) *is non-oscillatory, then there exists a solution* $\zeta : [b,\infty) \to (-\infty, 0]$ *of the generalized Riccati equation* (2.4).

Proof At first [see (2.2)], we recall that

$$\int_t^\infty \tau^{\alpha-p} s(\tau)\,\mathrm{d}\tau \in \mathbb{R}, \quad t \ge a.$$

We show that there exists a sequence $\{a_k\}_{k\in\mathbb{N}} \subset [a,\infty)$ satisfying

$$\lim_{k\to\infty} a_k = \infty, \quad \int_{a_k}^\infty \tau^{\alpha-p} s(\tau)\,\mathrm{d}\tau \ge 0, \quad k \in \mathbb{N}. \tag{3.3}$$

On the contrary, we will assume the existence of $a_0 \ge a$ such that

$$\int_t^\infty \tau^{\alpha-p} s(\tau)\,\mathrm{d}\tau < 0, \quad t \ge a_0. \tag{3.4}$$

From (2.3), considering the positivity od r, we obtain

$$\int_a^\infty \frac{s(\tau)}{\tau}\,\mathrm{d}\tau = \infty. \tag{3.5}$$

For any interval $[t_1, t_2] \subset [a_0, \infty)$, the second mean value theorem for definite integrals guarantees the existence of $t_3 \in (t_1, t_2]$ such that

$$\int_{t_1}^{t_2} \tau^{\alpha-p} s(\tau)\,\mathrm{d}\tau = \int_{t_1}^{t_2} \frac{s(\tau)}{\tau^{p-\alpha}}\,\mathrm{d}\tau = \frac{1}{t_1^{p-\alpha-1}} \int_{t_1}^{t_3} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau. \tag{3.6}$$

The inequality (3.4) for $t = a_0$ implies the existence of $t^1 > a_0$ with the property that

$$\int_{a_0}^{t^1} \tau^{\alpha-p} s(\tau)\, \mathrm{d}\tau < 0. \tag{3.7}$$

From (3.6) for $t_1 = a_0$ and $t_2 = t^1$ and from (3.7), it follows the existence of $t^\star \in \left(a_0, t^1\right]$ such that

$$\int_{a_0}^{t^\star} \frac{s(\tau)}{\tau}\, \mathrm{d}\tau < 0,$$

i.e., the set

$$\mathcal{T} := \left\{ t^\star > a_0;\ \int_{a_0}^{t^\star} \frac{s(\tau)}{\tau}\, \mathrm{d}\tau < 0 \right\} \tag{3.8}$$

is non-empty. Then, we denote

$$T := \sup \mathcal{T}. \tag{3.9}$$

Let us assume that $T < \infty$. In particular [see directly (3.8) and (3.9)],

$$\int_{a_0}^{T} \frac{s(\tau)}{\tau}\, \mathrm{d}\tau = 0 \tag{3.10}$$

and

$$\int_{a_0}^{t} \frac{s(\tau)}{\tau}\, \mathrm{d}\tau > 0, \quad t > T. \tag{3.11}$$

From (3.4) for $t = T$, we obtain the existence of $t^2 > T$ such that

$$\int_{T}^{t^2} \tau^{\alpha-p} s(\tau)\, \mathrm{d}\tau < 0. \tag{3.12}$$

Next, we use (3.6) for $t_1 = T$ and $t_2 = t^2$. Considering (3.12), there exists $\tilde{t} \in (T, t^2]$ for which

$$\int_T^{\tilde{t}} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau < 0. \tag{3.13}$$

From (3.10) and (3.13), we have

$$\int_{a_0}^{\tilde{t}} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau < 0. \tag{3.14}$$

Of course, (3.14) gives a contradiction [see (3.11)], i.e., $T = \infty$.

Thus, for any $c > a_0$, there exists $t(c) > c$ with the property that

$$\int_{a_0}^{t(c)} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau < 0. \tag{3.15}$$

It is seen that (3.15) contradicts (3.5). Indeed, (3.15) implies

$$\liminf_{t\to\infty} \int_a^t \frac{s(\tau)}{\tau}\,\mathrm{d}\tau \le \int_a^{a_0} \frac{s(\tau)}{\tau}\,\mathrm{d}\tau < \infty.$$

Therefore, (3.4) is not true and we have proved that there exists a sequence $\{a_k\}_{k\in\mathbb{N}} \subset [a, \infty)$ satisfying (3.3).

Now the statement of the lemma follows from Theorem 3, because [consider the positivity of r and see (3.3)]

$$\int_{a_k}^{\infty} \tau^{\alpha-p} s(\tau)\,\mathrm{d}\tau + (p-1)\int_{a_k}^{\infty} \tau^{\alpha(1-q)} r(\tau)|w(\tau)|^q\,\mathrm{d}\tau \ge 0, \quad k \in \mathbb{N}, \quad \lim_{k\to\infty} a_k = \infty,$$

i.e., (3.2) can be valid only if $w \equiv 0$ (and (3.1) gives $w(t) \ge 0$ for all $t \ge b$). Theorem 3 says that there exists a solution $w : [b, \infty) \to [0, \infty)$ of Eq. (2.5). Thus [see (2.6)], there exists a non-positive solution ζ of Eq. (2.4) on $[b, \infty)$. □

Now we can prove the announced oscillation criterion.

Theorem 4 *Let us consider Eq.* (2.1)*, where* $\alpha < p-1$ *and* $r : [a, \infty) \to (0, \infty)$ *and* $s : [a, \infty) \to (-\infty, \infty)$ *are continuous functions satisfying* (2.2)*. If* (2.3) *is valid, then Eq.* (2.1) *is oscillatory.*

Proof On the contrary, let us asumme that Eq. (2.1) is non-oscillatory. We apply Lemma 1 which says that there exists a solution $\zeta : [b, \infty) \to (-\infty, 0]$ of Eq. (2.4). We have [see directly Eq. (2.4)]

$$t\,\zeta'(t) = (p-\alpha-1)\zeta(t) + s(t) + (p-1)r(t)|\zeta(t)|^q, \quad t > b. \tag{3.16}$$

Putting

$$A(t) := \left(\frac{p-\alpha-1}{p}\right)^p r^{1-p}(t), \quad B(t) := (p-1)r(t)|\zeta(t)|^q, \quad t > b, \tag{3.17}$$

we can rewrite (3.16) into the form

$$t\,\zeta'(t) = s(t) - A(t) + B(t) + (p-\alpha-1)\zeta(t) + A(t), \quad t > b. \tag{3.18}$$

To proceed further, we recall the well-known Young inequality

$$\frac{x^p}{p} + \frac{y^q}{q} \geq xy, \quad x, y \geq 0. \tag{3.19}$$

For

$$x = (pA(t))^{1/p}, \quad y = (qB(t))^{1/q}, \quad t > b,$$

we have

$$\frac{x^p}{p} = \left(\frac{p-\alpha-1}{p}\right)^p r^{1-p}(t), \quad \frac{y^q}{q} = (p-1)r(t)|\zeta(t)|^q, \quad t > b, \tag{3.20}$$

and

$$\begin{aligned} x \cdot y &= (p-\alpha-1)\, p^{\frac{1-p}{p}} r^{\frac{1-p}{p}}(t) \cdot (q(p-1))^{\frac{1}{q}}\, r^{\frac{1}{q}}(t)|\zeta(t)| \\ &= (p-\alpha-1)\, p^{-\frac{1}{q}} r^{-\frac{1}{q}}(t) \cdot p^{\frac{1}{q}} r^{\frac{1}{q}}(t)|\zeta(t)| \\ &= (p-\alpha-1)\,|\zeta(t)| = -\,(p-\alpha-1)\,\zeta(t), \quad t > b. \end{aligned} \tag{3.21}$$

Thus, considering (3.19), (3.20), and (3.21), it holds [see also (3.17)]

$$B(t) + (p-\alpha-1)\zeta(t) + A(t) = \frac{y^q}{q} - xy + \frac{x^p}{p} \geq 0, \quad t > b. \tag{3.22}$$

From (3.18) and (3.22), we obtain

$$\zeta'(t) \geq \frac{s(t) - A(t)}{t}, \quad t > b. \tag{3.23}$$

At the same time, from (2.3), we obtain

$$\begin{aligned}\int_b^\infty \frac{s(\tau)-A(\tau)}{\tau}\,\mathrm{d}\tau &= \int_b^\infty \frac{s(\tau)-\left(\frac{p-\alpha-1}{p}\right)^p r^{1-p}(\tau)}{\tau}\,\mathrm{d}\tau \\ &= p^{-p}\left(\int_a^\infty \frac{p^p s(\tau)-(p-\alpha-1)^p\, r^{1-p}(\tau)}{\tau}\,\mathrm{d}\tau\right. \\ &\quad \left. - \int_a^b \frac{p^p s(\tau)-(p-\alpha-1)^p\, r^{1-p}(\tau)}{\tau}\,\mathrm{d}\tau\right) = \infty.\end{aligned} \tag{3.24}$$

Finally, (3.23) gives

$$\zeta(t) = \int_b^t \zeta'(\tau)\,\mathrm{d}\tau + \zeta(b) \geq \int_b^t \frac{s(\tau)-A(\tau)}{\tau}\,\mathrm{d}\tau + \zeta(b), \quad t > b,$$

and, consequently, (3.24) implies

$$\liminf_{t\to\infty} \zeta(t) \geq \int_b^\infty \frac{s(\tau)-A(\tau)}{\tau}\,\mathrm{d}\tau + \zeta(b) = \infty.$$

Especially, the considered solution ζ of Eq. (2.4) is positive in a neighbourhood of ∞. This contradiction proves the oscillation of Eq. (2.1). □

Remark 1 A very intensively studied case of the considered equations is given by the choice $\alpha = 0$. Nevertheless, Theorem 4 gives new results for $\alpha = 0$ and even for $p = 2$ (in the linear case). See Corollaries 1 and 2 below. We add that the strongest known results concerning the oscillation of the studied equations are proved in [32,35,37] for $\alpha = 0$ and in [14,29,36] for more general $\alpha \neq p-1$. We point out that any result in those papers does not cover corollaries and examples in the next section.

Remark 2 Theorem 4 covers the case when

$$p^p s(t) \geq (p-\alpha-1)^p\, r^{1-p}(t) + \varepsilon$$

for some $\varepsilon > 0$ and all large t. See Corollary 3 below [or directly (2.3)]. Note that the statement of Theorem 4 is not true if

$$p^p s(t) = (p-\alpha-1)^p\, r^{1-p}(t)$$

for all large t. It suffices to consider Theorem 1, (ii) (see directly [15,16] or [44] for generalizations). Thus, the presented oscillation criterion cannot be substantially improved. In addition, it is known (see, e.g., [9,13,17,20,21,42]) that the case

$$\int_a^\infty \frac{p^p s(\tau) - (p-\alpha-1)^p r^{1-p}(\tau)}{\tau} \, \mathrm{d}\tau \in \mathbb{R} \tag{3.25}$$

is not generally solvable, i.e., there exist continuous functions $r = r_1 > 0$, $s = s_1$ satisfying (2.2) and (3.25) for which Eq. (2.1) is oscillatory and continuous functions $r = r_2 > 0$, $s = s_2$ satisfying (2.2) and (3.25) for which Eq. (2.1) is non-oscillatory.

Remark 3 The oscillation behaviour of Eq. (2.1) in the case $\alpha = p-1$ differs from the analysed case $\alpha < p-1$ substantially. It is sufficient to consider [12, p. 43]. For details, we refer at least to our papers [27,31,33] together with references cited therein.

Remark 4 Theorem 2 was proved for any $\alpha \neq p-1$, but Theorem 4 for $\alpha < p-1$. This distinction is caused by different used methods (the generalized Riccati technique in this paper and the adapted Prüfer angle in [36]). Similarly, due the applied process, only $\alpha \leq 0$ is considered in [29].

4 Corollaries

To illustrate the impact of Theorem 4, we mention the following new corollaries and some surprisingly simple examples which are not covered by any previously known results (to the best of our knowledge). In illustrative examples, for simplicity, we consider only $a = \mathrm{e}$ and the both coefficients given by the same functions containing sin and logarithms. We remark that log denotes the natural logarithm.

At first, we formulate two corollaries in the most studied case $\alpha = 0$. In fact, our main result is so general that the power function t^α can be incorporated into coefficients (see also Example 1 below).

Corollary 1 *Let us consider the equation*

$$\left(r^{1-p}(t)\Phi(x'(t))\right)' + \frac{s(t)}{t^p}\Phi(x(t)) = 0, \quad t > a, \tag{4.1}$$

where $r : [a,\infty) \to (0,\infty)$ *and* $s : [a,\infty) \to (-\infty,\infty)$ *are continuous functions such that*

$$\int_a^\infty r(\tau)\,\mathrm{d}\tau = \infty, \quad \int_a^\infty \frac{s(\tau)}{\tau^p}\,\mathrm{d}\tau \in \mathbb{R}. \tag{4.2}$$

If

$$\int_a^\infty \frac{s(\tau) - q^{-p} r^{1-p}(\tau)}{\tau}\,\mathrm{d}\tau = \infty, \tag{4.3}$$

then Eq. (4.1) *is oscillatory.*

Proof It suffices to put $\alpha = 0$ in Theorem 4, when (2.2) reduces to (4.2) and (2.3) is equivalent with (4.3). ◻

Corollary 2 *Let us consider the equation*

$$\left(\frac{x'(t)}{r(t)}\right)' + \frac{s(t)}{t^2}\, x(t) = 0, \quad t > a, \tag{4.4}$$

where $r : [a, \infty) \to (0, \infty)$ *and* $s : [a, \infty) \to (-\infty, \infty)$ *are continuous functions such that*

$$\int_a^\infty r(\tau)\, \mathrm{d}\tau = \infty, \quad \int_a^\infty \frac{s(\tau)}{\tau^2}\, \mathrm{d}\tau \in \mathbb{R}.$$

If

$$\liminf_{t\to\infty} \left(s(t) - \frac{1}{4r(t)}\right) > 0, \tag{4.5}$$

then Eq. (4.4) *is oscillatory.*

Proof See Corollary 1 for $p = 2$, where (4.5) gives the existence of $b > a$ and $\varepsilon > 0$ such that

$$s(t) - \frac{1}{4r(t)} > \varepsilon, \quad t \geq b,$$

i.e.,

$$\begin{aligned}\int_a^\infty \frac{s(\tau) - q^{-p} r^{1-p}(\tau)}{\tau}\, \mathrm{d}\tau &= \int_a^b \frac{s(\tau) - \frac{1}{4r(\tau)}}{\tau}\, \mathrm{d}\tau + \int_b^\infty \frac{s(\tau) - \frac{1}{4r(\tau)}}{\tau}\, \mathrm{d}\tau \\ &\geq \int_a^b \frac{s(\tau) - \frac{1}{4r(\tau)}}{\tau}\, \mathrm{d}\tau + \int_b^\infty \frac{\varepsilon}{\tau}\, \mathrm{d}\tau = \infty.\end{aligned} \tag{4.6}$$

◻

Example 1 For arbitrary real numbers $\lambda_2 > \lambda_1$, let us consider the equation

$$\begin{aligned}&\left(\left(\sqrt[3]{t}\, \log t\, (2 + \sin t) + \lambda_1\right) x'(t)\right)' \\ &\quad + \frac{\log t\, (2 + \sin t) + \frac{\lambda_2}{\sqrt[3]{t}}}{4\sqrt[3]{t^5}}\, x(t) = 0, \quad t > \mathrm{e}.\end{aligned} \tag{4.7}$$

Putting

$$r(t) = \frac{1}{\sqrt[3]{t}\,\log t\left(2 + \sin t + \frac{\lambda_1}{\sqrt[3]{t}\,\log t}\right)}$$

for sufficiently large t and

$$s(t) = \frac{1}{4}\sqrt[3]{t}\,\log t\left(2 + \sin t + \frac{\lambda_2}{\sqrt[3]{t}\,\log t}\right), \quad t \geq \mathrm{e},$$

in Corollary 2, we obtain the oscillation of Eq. (4.7).

Similarly, we mention two corollaries for equations with bounded coefficients.

Corollary 3 *Let us consider Eq.* (2.1)*, where* $\alpha < p - 1$ *and* $r : [a, \infty) \to (0, \infty)$ *and* $s : [a, \infty) \to (-\infty, \infty)$ *are continuous functions such that*

$$0 < \liminf_{t\to\infty} r(t), \quad \limsup_{t\to\infty} |s(t)| < \infty. \tag{4.8}$$

If

$$\liminf_{t\to\infty}\left(p^p s(t) - (p - \alpha - 1)^p\, r^{1-p}(t)\right) > 0, \tag{4.9}$$

then Eq. (2.1) *is oscillatory.*

Proof The corollary follows from Theorem 4, where (4.8) (together with $\alpha < p - 1$) gives (2.2) and (4.9) gives (2.3) (as in the proof of Corollary 2). □

Corollary 4 *Let us consider the equation*

$$\left(\frac{t^\alpha}{r(t)}\,x'(t)\right)' + t^{\alpha-2}s(t)x(t) = 0, \quad t > a, \tag{4.10}$$

where $\alpha < 1$ *and* $r : [a, \infty) \to (0, \infty)$ *and* $s : [a, \infty) \to (-\infty, \infty)$ *are continuous functions satisfying* (4.8)*. If*

$$\liminf_{t\to\infty}\left(4s(t) - \frac{(1-\alpha)^2}{r(t)}\right) > 0, \tag{4.11}$$

then Eq. (4.10) *is oscillatory.*

Proof It suffices to consider Corollary 3 for $p = 2$. □

Example 2 Let $\lambda_2 > \lambda_1 > 1$. Considering Corollary 4 for $\alpha = 1/2$ and

$$r(t) = \frac{1}{\lambda_1 + \sin(\log t)}, \quad s(t) = \frac{\lambda_2 + \sin(\log t)}{16}, \quad t \geq \mathrm{e},$$

we can see that the equation

$$\left(\sqrt{t}\,(\lambda_1 + \sin(\log t))\,x'(t)\right)' + \frac{\lambda_2 + \sin(\log t)}{16\sqrt{t^3}}\,x(t) = 0, \quad t > \mathrm{e},$$

is oscillatory.

At the end of this paper, we remark that the presented corollaries can be easily improved. For example, one can replace (4.5) by

$$\liminf_{t\to\infty}\left(\log t\left(s(t) - \frac{1}{4r(t)}\right)\right) > 0$$

and Corollary 2 remains valid. Indeed, it suffices to consider (4.6) in the proof of Corollary 2 together with

$$\int_{\mathrm{e}}^{\infty} \frac{1}{\tau\log\tau}\,\mathrm{d}\tau = \infty.$$

Analogously, (4.9) can be replaced by

$$\liminf_{t\to\infty}\left(\log t\left(p^p s(t) - (p-\alpha-1)^p\, r^{1-p}(t)\right)\right) > 0$$

in Corollary 3 and (4.11) by

$$\liminf_{t\to\infty}\left(\log t\left(4s(t) - \frac{(1-\alpha)^2}{r(t)}\right)\right) > 0$$

in Corollary 4. We formulate only these special cases of Theorem 4, because we want to highlight the novelty of our result in these cases.

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