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Fractional powers of the Schrödinger operator on weigthed Lipschitz spaces

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Abstract
In the setting of the semigroup generated by the Schrödinger operator $L = -\Delta + V$ with the potential V satisfying an appropriate reverse Hölder condition, we consider some non-local fractional differentiation operators. We study their behaviour on suitable weighted smoothness spaces. Actually, we obtain such continuity results for positive powers of L as well as for the mixed operators $L^{\alpha/2}V^{\sigma/2}$ and $L^{-\alpha/2}V^{\sigma/2}$ with $\sigma > \alpha$, together with their adjoints.

Keywords Schrödinger operator · Weights · Regularity spaces

Mathematics Subject Classification Primary 42B20 · Secondary 35J10

1 Introduction

We will work in the frame of the harmonic analysis related to the Schrödinger differential operator in $\mathbb{R}^d$ with $d > 2$ as given in [11] by Shen,

$$Lu = -\Delta u + V\, u\, ,$$

where the potential V is a non-negative locally integrable function belonging to RH_q for some $q > d/2$. We remind that the last property means that there exists a constant C such that

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$$\left(\frac{1}{|B|}\int_B V^q\right)^{1/q} \leq C\frac{1}{|B|}\int_B V, \tag{1}$$

holds for any ball $B \subset \mathbb{R}^d$. When the left hand side is replaced by $\sup_B V$, we say that $V \in RH_\infty$. Our work relies strongly on Shen's estimates which are valid only for $d > 2$. His main tool is a comparison between the fundamental solutions of L and $-\Delta$. As it is known, the fundamental solution of the Laplacian has a different form when $d = 2$.

During the last two decades many authors have been working in this context, studying the behaviour of associate integral operators such as negative powers of L, the maximal semigroup operator, Riesz Transforms, the Littlewood-Paley function, among others. Results have been obtained for boundedness on weighted L^p spaces as well as on appropriate regularity spaces and their weighted versions.

In order to describe the latter spaces we recall an important feature of this environment. Our assumption on V allows to define a critical radius function $\rho : \mathbb{R}^d \longmapsto \mathbb{R}^+$ as

$$\rho(x) = \sup\left\{r : \frac{r^2}{|B(x,r))|}\int_{B(x,r)} V \;\leq 1\right\}.$$

Under the present assumptions on the potential it is possible to show that $0 < \rho(x) < \infty$ for all x and that the following inequalities hold

$$c_\rho^{-1}\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N_0} \;\leq\; \rho(y) \;\leq\; c_\rho\,\rho(x)\left(1+\frac{|x-y|}{\rho(x)}\right)^{\frac{N_0}{N_0+1}}, \tag{2}$$

for some constants c_ρ and N_0, independent of x and y. For a proof of these facts see [11].

It turns out that this function ρ plays a special role in defining the suitable weighted regularity spaces we are interested in. Let us point out that the first appearance of the space BMO_L, the appropriate substitute of the John-Nirenberg space BMO, can be found in [6]. The main difference with the classical case is that, besides the condition on the oscillations, averages over balls $B(x,r)$ with $r > \rho(x)$ must be also uniformly bounded. This reveals the fact that $-L$ is a perturbation of the Laplacian. Later on, appropriated variants of integral Lipschitz spaces were introduced as well as weighted versions. We remind their precise definition.

Given a weight w and some β, $0 \leq \beta < 1$, we say that a locally integrable function f belongs to $BMO_\rho^\beta(w)$ if there is a constant C such that

$$\frac{1}{w(B)}\int_B |f - f_B| \leq C|B|^{\beta/d}, \tag{3}$$

for any ball B, where f_B stands for the average with respect to Lebesgue measure, and

$$\frac{1}{w(B)}\int_B |f| \le C|B|^{\beta/d} \tag{4}$$

for any ball $B = B(x,r)$ with $r \ge \rho(x)$. Clearly condition (4) implies (3) and moreover, under a mild restriction on the weight, it is enough to know that (4) is satisfied on critical balls, that is, balls of the type $B(x, \rho(x))$. See [1] for the details.

As in the case of Laplacian, these weighted integral Lipschitz spaces can be identified with functions satisfying some point-wise smoothness under certain restriction on the weight. To be precise, we introduce doubling classes of weights adapted to this context. For $\mu \ge 1$ let us denote $D_\mu^\rho = \bigcup_{\theta\ge 0} D_\mu^{\rho,\theta}$, where $D_\mu^{\rho,\theta}$ is the class of weights w such that for some constant C,

$$w(B(x,R)) \le Cw(B(x,r))\left(\frac{R}{r}\right)^{d\mu}\left(1+\frac{R}{\rho(x)}\right)^{\theta}, \tag{5}$$

for any $x \in \mathbb{R}^d$ and $0 < r \le R$.

It turns out that when $w \in D_\mu^\rho$ for some $\mu \ge 1$ and $0 < \beta < 1$, $BMO_\rho^\beta(w)$ has a point-wise description and, in fact, it coincides with $\Lambda_\rho^\beta(w)$, defined as the class of functions f satisfying

$$|f(x)| \le CW_\beta(x, \rho(x)) \tag{6}$$

and

$$|f(x) - f(z)| \le C[W_\beta(x, |x-z|) + W_\beta(z, |x-z|)] \tag{7}$$

for all $x,\ z \in \mathbb{R}^d$ such that $|x - z| \le \rho(x)$, where

$$W_\beta(x,r) = \int_{B(x,r)} \frac{w(u)}{|u-x|^{d-\beta}}du. \tag{8}$$

For a proof of this equivalence we refer to Proposition 4 in [4]. When we deal with the unweighted case $w = 1$ we will simply denote $\Lambda_\rho^\beta(1) = \Lambda_\rho^\beta$.

The aim of this work is the study of the behaviour of non-local fractional differentiation operators acting on the above mentioned weighted regularity spaces $\Lambda_\rho^\beta(w)$ with $0 < \beta < 1$. Being differentiation operators we expect them to reduce smoothness, and such is the case for positive fractional powers of L. In fact, in Sect. 3 we are able to prove that $L^{\alpha/2}$ maps $\Lambda_\rho^\beta(w)$ into $\Lambda_\rho^{\beta-\alpha}(w)$ under suitable assumptions on w. For the case of $w = 1$, this result was obtained in [9]. However they achieve that by using a different description of Λ_ρ^β based on extensions of functions to the positive half space $\mathbb{R}^d \times (0, \infty)$ by means of the generated semigroup. Here we give a direct proof, obtaining the needed pointwise estimates, and our approach also works for a wide class of weighted spaces.

In Sect. 4 we also deal with some mixed fractional differentiation operators that combine powers of L, positives or negatives, with multiplication by positive powers of the potential V. In order to prove continuity of these operators we analyse general pointwise multiplication operators by a function ϕ, which are bounded from $\Lambda_\rho^\beta(w)$ into $\Lambda_\rho^\eta(w)$ with $0 < \eta \le \beta \le 1$. We find sufficient conditions on the function ϕ that for $w = 1$ turn to be also necessary. As a consequence we derive boundedness results for fractional differentiation operators as $L^{\alpha/2}V^{\sigma/2}$ and $L^{-\alpha/2}V^{\sigma/2}$ with $\sigma > \alpha$, under stronger assumptions on the potential.

2 Preliminaries

All along this work we deal with weights in the doubling classes D_μ^ρ defined above. Let us notice that any weight in the class A_p^ρ, appearing in [5], belongs to D_p^ρ. We recall that For a given $p > 1$, $A_p^\rho = \bigcup_{\theta \ge 0} A_p^{\rho,\theta}$, where $A_p^{\rho,\theta}$ is defined as those weights w such that

$$\left(\int_B w\right)^{1/p} \left(\int_B w^{-\frac{1}{p-1}}\right)^{1/p'} \le C|B| \left(1 + \frac{r}{\rho(x)}\right)^\theta, \tag{9}$$

for every ball $B = B(x, r)$, where $|B|$ stands for the Lebesgue measure of B. Similarly, when $p = 1$, we denote $A_1^\rho = \bigcup_{\theta \ge 0} A_1^{\rho,\theta}$, where $A_1^{\rho,\theta}$ is the class of weights w such that

$$\frac{1}{|B|} \int_B w \le C \left(1 + \frac{r}{\rho(x)}\right)^\theta \inf_B w, \tag{10}$$

for every ball $B = B(x, r)$.

These classes contain the classical Muckenhoupt families of weights but as it was shown in [5] are strictly larger. For instance if $\rho \equiv 1$, $w(x) = 1 + |x|^\gamma$ with $\gamma > d(p-1)$ belongs to A_p^ρ but it is not in A_p. Moreover, this weight belongs to D_μ^ρ for any $\mu \ge 1$ but it is not in D_μ when $\mu < 1 + \gamma/d$.

We will devote this section to develop some results that will be useful in what follows. We start giving some properties for the function W_β. The first one states that $W_\beta(x, \cdot)$ inherits some kind of doubling property from w.

Lemma 1 *Let $w \in D_\mu^{\rho,\theta}$, $x \in \mathbb{R}^d$ and $r > 0$. Then, there exist a constant $C > 0$ such that for every $\tau \ge 1$,*

$$W_\beta(x, \tau r) \le C\tau^{d\mu - d + \beta} W_\beta(x, r) \left(1 + \frac{\tau r}{\rho(x)}\right)^\theta$$

Proof Let $\tau > 1$ and $j_0 \in \mathbb{Z}$ such that $2^{j_0} \leq \tau < 2^{j_0+1}$. Then,

$$\begin{aligned}
W_\beta(x, \tau r) &= \int_{B(x,r)} \frac{w(z)}{|x-z|^{d-\beta}} dz + \sum_{j=0}^{j_0} \int_{2^j r \leq |x-z| < 2^{j+1} r} \frac{w(z)}{|x-z|^{d-\beta}} dz \\
&\leq W_\beta(x, r) + \sum_{j=0}^{j_0} (2^j r)^{\beta - d} w(B(x, 2^j r)) \\
&\leq W_\beta(x, r) + C \sum_{j=0}^{j_0} (2^j r)^{\beta - d} w(B(x, r)) 2^{jd\mu} \left(1 + \frac{2^j r}{\rho(x)}\right)^\theta \\
&\leq W_\beta(x, r) + C \frac{w(B(x, r))}{r^{d-\beta}} \left(1 + \frac{2^{j_0} r}{\rho(x)}\right)^\theta \sum_{j=0}^{j_0} (2^j)^{d\mu - d + \beta} \\
&\leq W_\beta(x, r) + C W_\beta(x, r) \left(1 + \frac{\tau r}{\rho(x)}\right)^\theta \tau^{d\mu - d + \beta}.
\end{aligned}$$

□

Next we give some estimates concerning the integral of W_β over a ball. First notice that from the definition of W_β, the inequality $W_\beta(x, R) \geq w(B(x, R)) R^{\beta - d}$ holds while the opposite is not always true. The next lemma provides a weaker version of that.

Lemma 2 *Let $w \in D^\rho_\mu$ for some $\mu \geq 1$ and $\beta > 0$. Then, for any $x \in \mathbb{R}^d$ and $R > 0$,*

$$\int_{B(x,R)} W_\beta(y, R) dy \leq C R^\beta w(B(x, R)).$$

Proof Let $x \in \mathbb{R}^d$ and $R > 0$. Applying the definition of W_β together with Lemma 1,

$$\begin{aligned}
\int_{B(x,R)} W_\beta(y, R) dy &= \int_{B(x,R)} \int_{B(y,R)} \frac{w(z)}{|y-z|^{d-\beta}} dz dy \\
&\leq \int_{B(x,2R)} w(z) \int_{B(x,R)} \frac{dy}{|y-z|^{d-\beta}} dz \\
&\leq \int_{B(x,2R)} w(z) \int_{B(z,3R)} \frac{dy}{|y-z|^{d-\beta}} dz \\
&\leq C w(B(x, R)) R^\beta.
\end{aligned}$$

□

Lemma 3 *Let $w \in D^\rho_\mu$, $x \in \mathbb{R}^d$, $R > 0$ and $\beta > 0$. If $0 < r \leq \rho(x)$,*

$$\frac{1}{|B(x, r)|} \int_{B(x,r)} W_\beta(y, \rho(y)) dy \leq C W_\beta(x, \rho(x)).$$

Proof Let $0 < r \leq \rho(x)$ and $y \in B(x,r)$. By (2), taking $c = 2c_\rho$ we have $B(y,\rho(y)) \subset B(x,c\rho(x))$. Therefore, we may write

$$\begin{aligned}\int_{B(x,r)} W_\beta(y,\rho(y))dy &= \int_{B(x,r)}\int_{B(y,\rho(y))} \frac{w(z)}{|y-z|^{d-\beta}}dzdy \\ &\leq \int_{B(x,r)}\int_{B(x,c\rho(x))} \frac{w(z)}{|y-z|^{d-\beta}}dzdy \\ &= \int_{B(x,c\rho(x))} w(z)\int_{B(x,r)} \frac{dy}{|y-z|^{d-\beta}}dz.\end{aligned} \tag{11}$$

Now, we split the integration domain $B(x,c\rho(x)) = B(x,2r) \cup B(x,c\rho(x)) \setminus B(x,2r)$ obtaining two terms. For the first one, since w is doubling

$$\begin{aligned}\int_{B(x,2r)} w(z)\int_{B(x,r)} \frac{dy}{|y-z|^{d-\beta}}dz &\leq Cr^\beta w(B(x,2r)) \\ &\leq Cr^d W_\beta(x,r) \\ &\leq Cr^d W_\beta(x,\rho(x)).\end{aligned}$$

For the remaining term, we may use that $|y-z| \geq |x-z|/2$ when $y \in B(y,r)$ and $z \notin B(x,2r)$ together with Lemma 1 to obtain

$$\begin{aligned}\int_{B(x,c\rho(x))\setminus B(x,2r)} w(z)\int_{B(x,r)} \frac{dy}{|y-z|^{d-\beta}}dz &\leq Cr^d \int_{B(x,c\rho(x))} \frac{w(z)}{|x-z|^{d-\beta}}dz \\ &\leq Cr^d W_\beta(x,\rho(x)).\end{aligned}$$

□

We finish this section with some results concerning the diffusion semigroup $\{T_t\}_{t>0}$ associated to $-L$, where

$$T_t f(x) = e^{-tL}f(x) = \int_{\mathbb{R}^d} k_t(x,y)f(y)dy.$$

For this kernel k_t, the Feynman-Kac formula assures that

$$0 \leq k_t(x,y) \leq h_t(x-y) = (4\pi t)^{-d/2}e^{-\frac{|x-y|^2}{4t}},$$

where h_t is the kernel associated to the classical heat diffusion semigroup. In fact, under our assumptions on V, it is shown in [11] that for each $N > 0$ there exists a constant C_N such that

$$k_t(x,y) \leq \frac{C_N}{t^{d/2}}e^{-\frac{|x-y|^2}{5t}}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \tag{12}$$

for $x, y \in \mathbb{R}^d$ and $t > 0$.

It is known that the classical heat kernel satisfies the following smoothness estimate: there exist positive constants c and C such that

$$|h_t(x-y) - h_t(z-y)| \le C \frac{|x-z|}{\sqrt{t}} t^{-d/2} e^{-\frac{c|x-y|^2}{t}}, \tag{13}$$

for any $y \in \mathbb{R}^d$ and $|x-z| \le \sqrt{t}$.

A similar estimate can be obtained for the kernel k_t. We state it in the following Lemma. For a proof of this result we refer to Proposition 4.11 in [8].

Lemma 4 *For any* $0 < \delta < \delta_0 = \min\{1, 2 - d/q\}$ *and* $N > 0$, *there exist constants* c *and* C_N *such that*

$$|k_t(x, y) - k_t(z, y)| \le C_N \left(\frac{|x-z|}{\sqrt{t}}\right)^{\delta} t^{-d/2} e^{-\frac{c|x-y|^2}{t}} \left(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\right)^{-N}, \tag{14}$$

for any $y \in \mathbb{R}^d$ *and* $|x-z| \le \sqrt{t}$.

Now we present the following lemmas that give size and smoothness estimates for the difference between k_t and h_t. They will be essential in what follows.

Lemma 5 *Let* $V \in RH_q$ *with* $q > d/2$. *There exists a constant* $C > 0$ *such that*

$$|k_t(x, y) - h_t(x-y)| \le C t^{-d/2} e^{-\frac{C|x-y|^2}{t}} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{2-d/q}, \tag{15}$$

for any $x, y \in \mathbb{R}^d$ *and* $t > 0$.

Lemma 6 *Let* $V \in RH_q$ *with* $q > d/2$. *For any* $0 < \delta < \delta_0 = \min\{1, 2 - d/q\}$ *there exists a constant* $C > 0$ *such that for every* x, y *and* $z \in \mathbb{R}^d$ *such that* $4|x-z| \le |x-y|$ *and* $|x-z| \le \rho(x)$,

$$|k_t(x, y) - h_t(x-y) - [k_t(z, y) - h_t(z-y)]| \le C t^{-d/2} e^{-\frac{C|x-y|^2}{t}} \left(\frac{|x-z|}{\rho(x)}\right)^{\delta}. \tag{16}$$

For a proof of these estimates we refer to Proposition 2.16 and Proposition 2.17 in [7].

3 Positive fractional powers of *L*

In this section we study the behaviour of the operator $L^{\alpha/2}$, with $0 < \alpha < 2$, acting on functions belonging to the appropriate weighted smoothness spaces we just introduced. Let us remind that this operator can be written in terms of the semigroup kernel as

$$L^{\alpha/2} f(x) = \int_0^\infty \left(e^{-tL} - I\right) f(x) \frac{dt}{t^{1+\alpha/2}}.$$

The precise result is stated in the following theorem.

Theorem 1 *Let $V \in RH_q$ for some $q > d/2$ and $0 < \alpha < \beta < \delta_0 = \min\{1, 2-d/q\}$. Then $L^{\alpha/2}$ is bounded from $\Lambda_\rho^\beta(w)$ into $\Lambda_\rho^{\beta-\alpha}(w)$ as long as $w \in D_\mu^\rho$ with $1 \le \mu < 1 + \frac{\delta_0-\beta}{d}$.*

Before proving this result, we present a few technical lemmas that summarize some of the properties we will need. Observe that estimates like (14), (15) and (16) involve exponentials functions with perhaps different exponents. For that reason the next results are worked out for a generic function of that type.

Lemma 7 *Let $\gamma, \theta \ge 0$ and $\tilde{h}_t$ a function of the form*

$$\tilde{h}_t(x) = t^{-d/2} e^{-\frac{C|x|^2}{t}}, \tag{17}$$

for some constant $C > 0$. Then there exists a constant C' such that

$$\int_{\mathbb{R}^d} \tilde{h}_t(x-y) \left(\frac{|x-y|}{\sqrt{t}}\right)^\gamma \left(1 + \frac{|x-y|}{\rho(x)}\right)^\theta dy \le C' \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^\theta,$$

for $x \in \mathbb{R}^d$ and $t > 0$.

Proof To prove this inequality we are going to split the integration domain on $B(x, \rho(x))$ and its complement. For the integral over the ball we obtain

$$\begin{aligned}
&\int_{B(x,\rho(x))} \tilde{h}_t(x-y) \left(\frac{|x-y|}{\sqrt{t}}\right)^\gamma \left(1 + \frac{|x-y|}{\rho(x)}\right)^\theta dy \\
&\quad \le C t^{-d/2} \int_{B(x,\rho(x))} e^{-\frac{C|x-y|^2}{t}} \left(\frac{|x-y|}{\sqrt{t}}\right)^\gamma dy \le C,
\end{aligned}$$

by performing a simple change of variables.

For the remaining term we have

$$\begin{aligned}
&\int_{B(x,\rho(x))^c} \tilde{h}_t(x-y) \left(\frac{|x-y|}{\sqrt{t}}\right)^\gamma \left(1 + \frac{|x-y|}{\rho(x)}\right)^\theta dy \\
&\le C t^{-d/2} \left(\frac{\sqrt{t}}{\rho(x)}\right)^\theta \int_{B(x,\rho(x))^c} e^{-\frac{C|x-y|^2}{t}} \left(\frac{|x-y|}{\sqrt{t}}\right)^{\gamma+\theta} dy \\
&\le C \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^\theta.
\end{aligned}$$

□

Lemma 8 *Let $\tilde{h}_t$ be a function as in (17), $w \in D^{\rho}_{\mu}$ for some $\mu \geq 1$ and $0 < \beta \leq 1$. Then, for any $x \in \mathbb{R}^d$, $t > 0$ and $0 \leq r < R \leq \infty$, we have*

$$\int_{r<|x-y|\leq R} \tilde{h}_t(x-y) W_\beta(y, |x-y|) dy \leq C \int_{r<|x-y|\leq 4R} \tilde{h}_t((x-y)/4) W_\beta(x, |x-y|) dy$$

Proof Suppose first that $r > 0$ and $R < \infty$ and let j_0 and $k_0 \in \mathbb{Z}$ such that

$$2^{j_0}\sqrt{t} < r \leq 2^{j_0+1}\sqrt{t},$$
$$2^{k_0}\sqrt{t} < R \leq 2^{k_0+1}\sqrt{t}.$$

Applying Lemma 2,

$$\begin{aligned}
&\int_{r\leq|x-y|\leq R} \tilde{h}_t(x-y) W_\beta(y, |x-y|) dy \\
&\leq \sum_{j=j_0}^{k_0} \int_{2^j\sqrt{t}<|x-y|\leq 2^{j+1}\sqrt{t}} \tilde{h}_t(x-y) W_\beta(y, |x-y|) dy \\
&\leq \sum_{j=j_0}^{k_0} t^{-d/2} \tilde{h}_1(2^j) \int_{2^j\sqrt{t}<|x-y|\leq 2^{j+1}\sqrt{t}} W_\beta(y, 2^{j+1}\sqrt{t}) dy \\
&\leq C \sum_{j=j_0}^{k_0} t^{-d/2} \tilde{h}_1(2^j)(2^{j+1}\sqrt{t})^d W_\beta(x, 2^{j+1}\sqrt{t}) \\
&\leq C \sum_{j=j_0}^{k_0} t^{-d/2} \int_{2^{j+1}\sqrt{t}<|x-y|\leq 2^{j+2}\sqrt{t}} \tilde{h}_t(|x-y|/4) W_\beta(x, |x-y|) dy \\
&\leq C \int_{r<|x-y|\leq 4R} \tilde{h}_t(|x-y|/4) W_\beta(x, |x-y|) dy.
\end{aligned}$$

If $r = 0$ or $R = \infty$ the proof is analogous performing the sum from $j = -\infty$ or until $j = \infty$ respectively.

□

Lemma 9 *Let $\tilde{h}$ be a function as in (17), $w \in D^{\rho,\theta}_{\mu}$ and $0 < \beta \leq 1$. Then, for any $x \in \mathbb{R}^d$, we have that*

$$\int_{\mathbb{R}^d} \tilde{h}_t(x-y) W_\beta(y, \rho(y)) dy \leq C W_\beta(x, \rho(x)) \left(1 + \frac{\sqrt{t}}{\rho(x)}\right)^{d\mu - d + \beta + \theta}.$$

Proof First we are going to split $\mathbb{R}^d$ and integrate over $B(x, \rho(x))$ and its complement. For the first integral we consider $k_0 \in \mathbb{Z}$ such that $2^{k_0}\sqrt{t} \leq \rho(x) < 2^{k_0+1}\sqrt{t}$. Then we may write,

$$\int_{B(x,\rho(x))} \tilde{h}_t(x-y)W_\beta(y,\rho(y))dy \leq \sum_{k=-\infty}^{k_0} \int_{2^k\sqrt{t}<|x-y|\leq 2^{k+1}\sqrt{t}} \tilde{h}_t(x-y)W_\beta(y,\rho(y))dy$$
$$\leq \sum_{k=-\infty}^{k_0} \frac{\tilde{h}_1(2^k)}{t^{d/2}} \int_{B(x,2^{k+1}\sqrt{t})} W_\beta(y,\rho(y))dy.$$

Therefore, applying Lemma 3 we get

$$\begin{aligned}
&\int_{B(x,\rho(x))} \tilde{h}_t(x-y)W_\beta(y,\rho(y))dy \\
&\leq C W_\beta(x,\rho(x)) \sum_{k=-\infty}^{k_0} \frac{\tilde{h}_1(2^k)}{t^{d/2}} (2^k\sqrt{t})^d \\
&\leq C W_\beta(x,\rho(x)) \sum_{k=-\infty}^{\infty} \int_{2^k\sqrt{t}<|x-y|\leq 2^{k+1}\sqrt{t}} \tilde{h}_t(x-y)dy \\
&\leq C W_\beta(x,\rho(x)).
\end{aligned}$$

To deal with the integral over the complement we use that $W_\beta(y,\cdot)$ is an increasing function and apply Lemma 8 to obtain

$$\begin{aligned}
&\int_{|x-y|\geq\rho(x)} \tilde{h}_t(x-y)W_\beta(y,\rho(y))dy \\
&\leq \int_{|x-y|\geq\rho(x)} \tilde{h}_t(x-y)W_\beta(y,|x-y|)dy \qquad (18)\\
&\leq C \int_{|x-y|\geq c\rho(x)} \tilde{h}_t((x-y)/4)W_\beta(x,|x-y|)dy.
\end{aligned}$$

Now, we apply the doubling property of W_β stated on Lemma 1 together with Lemma 7 in the following way,

$$\begin{aligned}
&\int_{|x-y|\geq\rho(x)} \tilde{h}_t(x-y)W_\beta(y,\rho(y))dy \\
&\leq C \frac{W_\beta(x,\rho(x))}{\rho(x)^{d\mu-d+\beta+\theta}} \int \tilde{h}_t((x-y)/4)|x-y|^{d\mu-d+\beta+\theta}dy \qquad (19)\\
&\leq C W_\beta(x,\rho(x)) \left(\frac{\sqrt{t}}{\rho(x)}\right)^{d\mu-d+\beta+\theta}.
\end{aligned}$$

□

Now we can give the proof of the main result of this section.

Proof of Theorem 1. Let $w \in D_\mu^\rho$ and $f \in \Lambda_\rho^\beta(w)$ for $0 < \beta < \delta_0$. We can assume, without loss of generality, that $\|f\|_{\Lambda_\rho^\beta(w)} = 1$. First, we are going to check condition (6). Let $x \in \mathbb{R}^d$, we may write

$$L^{\alpha/2} f(x) = \int_0^\infty \left(e^{-tL} - I\right) f(x) \frac{dt}{t^{1+\alpha/2}} \pm \int_0^{\rho^2(x)} \left(e^{t\Delta} - I\right) f(x) \frac{dt}{t^{1+\alpha/2}}$$
$$= I + II + II,$$

where

$$I = \int_0^{\rho^2(x)} (e^{-tL} - e^{t\Delta}) f(x) \frac{dt}{t^{1+\alpha/2}},$$
$$II = \int_0^{\rho(x)^2} (e^{t\Delta} - I) f(x) \frac{dt}{t^{1+\alpha/2}},$$
$$III = \int_{\rho(x)^2}^\infty (e^{-tL} - I) f(x) \frac{dt}{t^{1+\alpha/2}}.$$

To deal with I, we may apply Lemma 5 together with the size estimate for f to obtain

$$|I| \leq \int_0^{\rho^2(x)} \int_{\mathbb{R}^d} |k_t(x, y) - h_t(x - y)||f(y)|dy \frac{dt}{t^{1+\alpha/2}}$$
$$\leq C \int_0^{\rho^2(x)} \left(\frac{\sqrt{t}}{\rho(x)}\right)^\delta \int_{\mathbb{R}^d} \tilde{h}_t(x - y) W_\beta(y, \rho(y)) dy. \frac{dt}{t^{1+\alpha/2}}$$

Now, by Lemma 9,

$$|I| \leq C W_\beta(x, \rho(x)) \rho(x)^{-\delta} \int_0^{\rho^2(x)} t^{\frac{\delta-\alpha}{2}} \frac{dt}{t}$$
$$\leq C W_\beta(x, \rho(x)) \rho(x)^{-\alpha} \leq C W_{\beta-\alpha}(x, \rho(x)).$$

To bound II, we use first that $\int h_t(x - y) dy = 1$ and then the smoothness estimate for f, obtaining

$$|II| \leq \int_0^{\rho^2(x)} \int_{\mathbb{R}^d} h_t(x - y)|f(y) - f(x)| dy \frac{dt}{t^{1+\alpha/2}}$$
$$\leq \int_0^{\rho^2(x)} \left(\int_{\mathbb{R}^d} h_t(x - y) W_\beta(x, |x - y|) dy\right.$$
$$\left. + \int_{\mathbb{R}^d} h_t(x - y) W_\beta(y, |x - y|) dy\right) \frac{dt}{t^{1+\alpha/2}}.$$

We will bound the first term only, since the second one can be estimated following the same lines once we have applied Lemma (8). We divide the first term as follows,

$$\begin{aligned}&\int_0^{\rho^2(x)}\int_{\mathbb{R}^d} h_t(x-y)W_\beta(x,|x-y|)dy\frac{dt}{t^{1+\alpha/2}}\\&\quad\leq\int_0^{\rho^2(x)}\int_{|x-y|<\sqrt{t}} h_t(x-y)W_\beta(x,|x-y|)dy\frac{dt}{t^{1+\alpha/2}}\\&\qquad+\int_0^{\rho^2(x)}\int_{|x-y|\geq\sqrt{t}} h_t(x-y)W_\beta(x,|x-y|)dy\frac{dt}{t^{1+\alpha/2}}=II_1+II_2.\end{aligned}$$

For II_1 we have

$$\begin{aligned}II_1&\leq\int_0^{\rho^2(x)}W_\beta(x,\sqrt{t})\int_{|x-y|<\sqrt{t}}h_t(x-y)dy\frac{dt}{t^{1+\alpha/2}}\\&\leq\int_0^{\rho^2(x)}W_\beta(x,\sqrt{t})\frac{dt}{t^{1+\alpha/2}}\\&\leq\int_0^{\rho^2(x)}\int_{|x-u|<\sqrt{t}}\frac{w(u)}{|x-u|^{d-\beta}}du\frac{dt}{t^{1+\alpha/2}}\\&\leq\int_{B(x,\rho(x))}\frac{w(u)}{|x-u|^{d-\beta}}\int_{t\geq|x-u|^2}\frac{dt}{t^{1+\alpha/2}}du\\&\leq C\int_{B(x,\rho(x))}\frac{w(u)}{|x-u|^{d-(\beta-\alpha)}}du\\&=CW_{\beta-\alpha}(x,\rho(x)).\end{aligned}$$

For II_2 we apply Lemmas 1 and 7 since the integration is performed on $\sqrt{t}\leq\rho(x)$.

$$\begin{aligned}II_2&\leq C\int_0^{\rho^2(x)}\int_{|x-y|\geq\sqrt{t}}h_t(x-y)W_\beta(x,|x-y|)dy\frac{dt}{t^{1+\alpha/2}}\\&\leq C\int_0^{\rho^2(x)}W_\beta(x,\sqrt{t})\int_{|x-y|\geq\sqrt{t}}h_t(x-y)\left(\frac{|x-y|}{\sqrt{t}}\right)^{d\mu-d+\beta}\\&\quad\times\left(1+\frac{|x-y|}{\rho(x)}\right)^{\theta}dy\frac{dt}{t^{1+\alpha/2}}\\&\leq C\int_0^{\rho^2(x)}W_\beta(x,\sqrt{t})\frac{dt}{t^{1+\alpha/2}}\\&\leq CW_{\beta-\alpha}(x,\rho(x)),\end{aligned}$$

where for the last inequality we follow the same steps as for II_1.

Now we turn our attention to III. Since $\int k_t(x,y)dy$ is not necessarily one, we split in the two following terms

$$|III| \le \int_{\rho^2(x)}^{\infty} \int_{\mathbb{R}^d} |k_t(x,y)||f(y)|dy \frac{dt}{t^{1+\alpha/2}} + \int_{\rho^2(x)}^{\infty} |f(x)| \frac{dt}{t^{1+\alpha/2}} = III_1 + III_2.$$

To bound III_1 we use the size estimates for f and k_t given in (12). Also we apply Lemma 9 to obtain

$$\begin{aligned} III_1 &\le C_N \int_{\rho^2(x)}^{\infty} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{-N} \int_{\mathbb{R}^d} \tilde{h}_t(x-y) W_\beta(y,\rho(y))dy \frac{dt}{t^{1+\alpha/2}} \\ &\le C_N \int_{\rho^2(x)}^{\infty} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{-N} W_\beta(x,\rho(x)) \left(\frac{\sqrt{t}}{\rho(x)}\right)^{d\mu-d+\beta+\theta} \frac{dt}{t^{1+\alpha/2}} \\ &\le C_N \frac{W_\beta(x,\rho(x))}{\rho(x)^{d\mu-d+\beta+\theta-N}} \int_{\rho^2(x)}^{\infty} t^{\frac{d\mu-d+\beta+\theta-\alpha-N}{2}} \frac{dt}{t} \\ &\le C_N \frac{W_\beta(x,\rho(x))}{\rho(x)^{d\mu-d+\beta+\theta-N}} \int_{\rho^2(x)}^{\infty} t^{\frac{d\mu-d+\beta+\theta-\alpha-N}{2}} \frac{dt}{t} \\ &\le C\rho^{-\alpha}(x) W_\beta(x,\rho(x)) \\ &\le C W_{\beta-\alpha}(x,\rho(x)), \end{aligned}$$

choosing N large enough.

Finally, for III_2 we simply use the size estimate for f arriving to

$$\begin{aligned} III_2 &\le W_\beta(x,\rho(x)) \int_{\rho^2(x)}^{\infty} t^{-\alpha/2} \frac{dt}{t} \\ &\le C\rho^{-\alpha}(x) W_\beta(x,\rho(x)) \\ &\le C W_{\beta-\alpha}(x,\rho(x)). \end{aligned}$$

Now we check condition (7). For $x,\ z \in \mathbb{R}^d$ with $|x-z| \le \rho(x)$ we want to show that

$$|L^{\alpha/2} f(x) - L^{\alpha/2} f(z)| \le C\,[W_\beta(x,|x-z|) + W_\beta(z,|x-z|)]. \tag{20}$$

Let $x,\ z \in \mathbb{R}^d$ such that $|x-z| \le \rho(x)$, we write

$$\begin{aligned} L^{\alpha/2} f(x) - L^{\alpha/2} f(z) &= \int_0^{\infty} (e^{-tL} - I) f(x) \frac{dt}{t^{1+\alpha/2}} - \int_0^{\infty} (e^{-tL} - I) f(z) \frac{dt}{t^{1+\alpha/2}} \\ &\quad \pm \left[\int_0^{\rho(x)^2} (e^{t\Delta} - I) f(x) \frac{dt}{t^{1+\alpha/2}} - \int_0^{\rho(x)^2} (e^{t\Delta} - I) f(z) \frac{dt}{t^{1+\alpha/2}} \right] \\ &= IV + V + VI, \end{aligned}$$

where

$$
\begin{aligned}
IV &= \int_0^{\rho(x)^2} (e^{-tL} - e^{t\Delta}) f(x) \frac{dt}{t^{1+\alpha/2}} - \int_0^{\rho(x)^2} (e^{-tL} - e^{t\Delta}) f(z) \frac{dt}{t^{1+\alpha/2}} \\
&= \int_0^{\rho(x)^2} \int_{\mathbb{R}^d} [k_t(x,y) - h_t(x-y) - (k_t(z,y) - h_t(z-y))] f(y) dy \frac{dt}{t^{1+\alpha/2}}, \\
V &= \int_0^{\rho(x)^2} (e^{t\Delta} - I) f(x) \frac{dt}{t^{1+\alpha/2}} - \int_0^{\rho(x)^2} (e^{t\Delta} - I) f(z) \frac{dt}{t^{1+\alpha/2}}, \\
VI &= \int_{\rho(x)^2}^{\infty} (e^{-tL} - I) f(x) \frac{dt}{t^{1+\alpha/2}} - \int_{\rho(x)^2}^{\infty} (e^{-tL} - I) f(z) \frac{dt}{t^{1+\alpha/2}}.
\end{aligned}
$$

First, to estimate IV we decompose the integral in the following way,

$$
\begin{aligned}
IV &\le \int_0^{|x-z|^2} \int_{\mathbb{R}^d} [|k_t(x,y) - h_t(x,y)| + |k_t(z,y) - h_t(z,y)|] \, |f(y)| \frac{dt}{t^{1+\alpha/2}} \\
&\quad + \int_{|x-z|^2}^{\rho(x)^2} \int_{\mathbb{R}^d} |k_t(x,y) - h_t(x,y) - (k_t(z,y) - h_t(z,y))| \, |f(y)| \frac{dt}{t^{1+\alpha/2}} \\
&= IV_1 + IV_2.
\end{aligned}
$$

Now we decompose IV_1 in a sum of two terms. Both of them can be treated in the same way, so we do only the first one. Given β and μ satisfying the assumptions of the theorem, we can fix $\delta > 0$ such that $\beta < \delta < \delta_0$ and $\mu \le 1 + \frac{\delta-\beta}{d}$. Using again the size estimate for f, applying Lemma 5 for δ as above, Lemma 9 and then Lemma 1, we obtain

$$
\begin{aligned}
IV_1 &\le C \int_0^{|x-z|^2} \left(\frac{\sqrt{t}}{\rho(x)} \right)^{\delta} \int_{\mathbb{R}^d} \tilde{h}_t(x-y) W_\beta(y, \rho(y)) dy \frac{dt}{t^{1+\alpha/2}} \\
&\le C W_\beta(x, \rho(x)) \rho(x)^{-\delta} \int_0^{|x-z|^2} t^{\frac{\delta-\alpha}{2}-1} dt \\
&\le C |x-z|^{-\alpha} W_\beta(x, |x-z|) \left(\frac{|x-z|}{\rho(x)} \right)^{\delta - d\mu + d - \beta} \\
&\le C W_{\beta-\alpha}(x, |x-z|),
\end{aligned}
$$

since $\alpha < \delta$ and $\mu \le 1 + \frac{\delta-\beta}{d}$.

To handle IV_2, we break the integral on $\mathbb{R}^d$ into $B(x, 4|x-z|)$ and its complement, giving rise to IV_{21} and IV_{22}. For the first we proceed as for IV_1 producing two similar terms. For each one we apply Lemma 5 together with Lemma 3 to get

$$
\begin{aligned}
IV_{21} &\leq \int_{|x-z|^2}^{\rho(x)^2} \left(\frac{\sqrt{t}}{\rho(x)}\right)^{\delta} \int_{|x-y|\leq 4|x-z|} t^{-d/2} e^{-\frac{|x-y|^2}{t}} W_\beta(y,\rho(y))dy \frac{dt}{t^{1+\alpha/2}} \\
&\leq \rho(x)^{-\delta} \int_{|x-z|^2}^{\rho(x)^2} \int_{|x-y|\leq 4|x-z|} W_\beta(y,\rho(y))dy \frac{dt}{t^{1+\frac{\alpha-\delta+d}{2}}} \\
&\leq \rho(x)^{-\delta}|x-z|^d W_\beta(x,\rho(x)) \int_{|x-z|^2}^{\infty} \frac{dt}{t^{1+\frac{\alpha-\delta+d}{2}}}.
\end{aligned}
$$

Now, applying Lemma 1,

$$
IV_{21} \leq |x-z|^{-\alpha} W_\beta(x,|x-z|) \left(\frac{|x-z|}{\rho(x)}\right)^{d-\mu d+\delta-\beta} \leq W_{\beta-\alpha}(x,|x-z|)
$$

since $\mu \leq 1 + \frac{\delta-\beta}{d}$.

To estimate IV_{22} we use Lemma 6 with δ as above, together with Lemma 9 and Lemma 1 to obtain

$$
\begin{aligned}
IV_{22} &\leq \left(\frac{|x-z|}{\rho(x)}\right)^{\delta} \int_{|x-z|^2}^{\rho(x)^2} \int_{\mathbb{R}^d} \tilde{h}_t(x-y) W_\beta(y,\rho(y))dy \frac{dt}{t^{1+\alpha/2}} \\
&\leq \left(\frac{|x-z|}{\rho(x)}\right)^{\delta} W_\beta(x,\rho(x)) \int_{|x-z|^2}^{\rho(x)^2} \frac{dt}{t^{1+\alpha/2}} \\
&\leq C|x-z|^{-\alpha} W_\beta(x,|x-z|) \left(\frac{|x-z|}{\rho(x)}\right)^{\delta-d\mu+d-\beta} \\
&\leq C W_{\beta-\alpha}(x,|x-z|),
\end{aligned}
$$

if $\mu \leq 1 + \frac{\delta-\beta}{d}$.

Next, we turn our attention to V. First we write it in the following way

$$
\begin{aligned}
|V| \leq & \left| \int_0^{|x-z|^2} \int_{\mathbb{R}^d} h_t(x-y)[f(y)-f(x)]dy \right. \\
& \left. - \left[\int_{\mathbb{R}^d} h_t(z-y)[f(y)-f(z)]dy\right] \frac{dt}{t^{1+\alpha/2}} \right| \\
& + \left| \int_{|x-z|^2}^{\rho(x)^2} \int_{\mathbb{R}^d} h_t(x-y)[f(y)-f(x)]dy \right. \\
& \left. - \left[\int_{\mathbb{R}^d} h_t(z-y)[f(y)-f(z)]dy\right] \frac{dt}{t^{1+\alpha/2}} \right| \\
= & V_1 + V_2.
\end{aligned}
$$

In V_1 we may bound the difference by the sum of two analogous terms V_{11} and V_{12}. To deal with V_{11} we split the inner integral in $B(x,\sqrt{t})$ and its complement, obtaining the terms V_{111} and V_{112} respectively. In V_{111}, after applying the smoothness estimate for f, we obtain

$$V_{111} \leq C \int_0^{|x-z|^2} \int_{|x-y|\leq\sqrt{t}} h_t(x-y)[W_\beta(x,|x-y|) + W_\beta(y,|x-y|)]dy \frac{dt}{t^{1+\alpha/2}}.$$

Now we split the inner sum in two terms. For the first term

$$\begin{aligned}
&\int_0^{|x-z|^2} \int_{|x-y|\leq\sqrt{t}} h_t(x-y) W_\beta(x,|x-y|)dy \frac{dt}{t^{1+\alpha/2}} \\
&\leq \int_0^{|x-z|^2} W_\beta(x,\sqrt{t}) \int_{\mathbb{R}^d} h_t(x-y)dy \frac{dt}{t^{1+\alpha/2}} \\
&\leq C \int_0^{|x-z|^2} W_\beta(x,\sqrt{t}) \frac{dt}{t^{1+\alpha/2}} \\
&\leq C \int_0^{|x-z|^2} \int_{B(x,\sqrt{t})} \frac{w(u)}{|u-x|^{d-\beta}} du \frac{dt}{t^{1+\alpha/2}} \\
&\leq C \int_{B(x,|x-z|)} \frac{w(u)}{|u-x|^{d-\beta}} \int_{\sqrt{t}>|x-u|} \frac{dt}{t^{1+\alpha/2}} du \\
&\leq C \int_{B(x,|x-z|)} \frac{w(u)}{|u-x|^{d-(\beta-\alpha)}} du \\
&\leq C W_{\beta-\alpha}(x,|x-z|).
\end{aligned}$$

The second term can be treated in the same way once we have applied Lemma 8.

To handle V_{112} we use the smoothness estimate for f, obtaining

$$V_{112} \leq C \int_0^{|x-z|^2} \int_{|x-y|>\sqrt{t}} h_t(x-y)[W_\beta(x,|x-y|) + W_\beta(y,|x-y|)]dy \frac{dt}{t^{1+\alpha/2}}.$$

Again, we split the inner sum in two terms. For the first term we apply Lemma 1 together with Lemma 7 since $\sqrt{t} < |x-z| < \rho(x)$. In this way

$$\begin{aligned}
&\int_0^{|x-z|^2} \int_{|x-y|>\sqrt{t}} h_t(x-y) W_\beta(x,|x-y|)dy \frac{dt}{t^{1+\alpha/2}} \\
&\leq \int_0^{|x-z|^2} W_\beta(x,\sqrt{t}) \int_{\mathbb{R}^d} h_t(x-y) \left(\frac{|x-y|}{\sqrt{t}}\right)^{d\mu-d+\beta} \\
&\quad \left(1+\frac{|x-y|}{\rho(x)}\right)^\theta dy \frac{dt}{t^{1+\alpha/2}} \\
&\leq C \int_0^{|x-z|^2} W_\beta(x,\sqrt{t}) \frac{dt}{t^{1+\alpha/2}} \\
&\leq C W_{\beta-\alpha}(x,|x-z|),
\end{aligned}$$

where for the last line we use the same argument as for V_{111}. Once more, after applying Lemma 8, the second term can be treated in the same way.

Now we turn our attention to V_2. We add and subtract $h_t(z-y)[f(y)-f(x)]$ obtaining $V_2 \le V_{21} + V_{22}$, where

$$V_{21} = \int_{|x-z|^2}^{\rho^2(x)} \int_{\mathbb{R}^d} h_t(z-y)|f(x)-f(z)|dy \frac{dt}{t^{1+\alpha/2}}$$

and

$$V_{22} = \int_{|x-z|^2}^{\rho^2(x)} \int_{\mathbb{R}^d} |h_t(x-y)-h_t(z-y)||f(y)-f(x)|dy \frac{dt}{t^{1+\alpha/2}}.$$

In V_{21} we simply apply the smoothness estimate for f arriving to

$$\begin{aligned} V_{21} &\le C[W_\beta(x,|x-z|)+W_\beta(z,|x-z|)] \int_{|x-z|^2}^{\rho^2(x)} \int_{\mathbb{R}^d} h_t(z-y)dy \frac{dt}{t^{1+\alpha/2}} \\ &\le [W_\beta(x,|x-z|)+W_\beta(z,|x-z|)]|x-z|^{-\alpha} \\ &\le W_{\beta-\alpha}(x,|x-z|)+W_{\beta-\alpha}(z,|x-z|). \end{aligned}$$

To handle V_{22}, we use the smoothness of the heat kernel given in (13) together with the smoothness estimate for f to obtain,

$$\begin{aligned} V_{22} &= \int_{|x-z|^2}^{\rho^2(x)} \int_{\mathbb{R}^d} |h_t(x-y)-h_t(z-y)||f(y)-f(x)|dy \frac{dt}{t^{1+\alpha/2}} \\ &\le C \int_{|x-z|^2}^{\rho^2(x)} \frac{|x-z|}{\sqrt{t}} \int_{\mathbb{R}^d} \tilde{h}_t(x-y) W_\beta(x,|x-y|)dy \frac{dt}{t^{1+\alpha/2}} \\ &\quad + C \int_{|x-z|^2}^{\rho^2(x)} \frac{|x-z|}{\sqrt{t}} \int_{\mathbb{R}^d} \tilde{h}_t(x-y) W_\beta(y,|x-y|)dy \frac{dt}{t^{1+\alpha/2}} \\ &= V_{221} + V_{222}. \end{aligned}$$

Once more we estimate only V_{221} since V_{222} can be handled following the same lines, after an application of Lemma 9. First, we decompose the inner integral of V_{221} in $B(x,|x-z|)$ and its complement obtaining two terms. For the first one we simply use that W_β is increasing on the second variable as follows

$$\int_{|x-z|^2}^{\rho^2(x)} \left(\frac{|x-z|}{\sqrt{t}}\right) \int_{|x-y|<|x-z|} \tilde{h}_t(x-y) W_\beta(x,|x-y|) dy \frac{dt}{t^{1+\alpha/2}}$$
$$\leq C \int_{|x-z|^2}^{\rho^2(x)} \left(\frac{|x-z|}{\sqrt{t}}\right) W_\beta(x,|x-z|) \int_{|x-y|<|x-z|} \tilde{h}_t(x-y) dy \frac{dt}{t^{1+\alpha/2}}$$
$$\leq C W_\beta(x,|x-z|)|x-z| \int_{|x-z|^2}^{\infty} \frac{dt}{t^{1+\alpha/2+1/2}}$$
$$\leq C W_\beta(x,|x-z|)|x-z|^{-\alpha} \leq C W_{\beta-\alpha}(x,|x-z|).$$

For the second term we can apply Lemma 1 together with Lemma 7 and, since $\sqrt{t} < \rho(x)$, we get

$$\int_{|x-z|^2}^{\rho^2(x)} \frac{|x-z|}{\sqrt{t}} \int_{|x-y|\geq|x-z|} \tilde{h}_t(x-y) W_\beta(x,|x-y|) dy \frac{dt}{t^{1+\alpha/2}}$$
$$\leq C \int_{|x-z|^2}^{\rho^2(x)} \left(\frac{|x-z|}{\sqrt{t}}\right) \int_{\mathbb{R}^d} \tilde{h}_t(x-y) W_\beta(x,|x-z|) \left(\frac{|x-y|}{|x-z|}\right)^{d\mu-d+\beta}$$
$$\times \left(1+\frac{|x-y|}{\rho(x)}\right)^{\theta} dy \frac{dt}{t^{1+\alpha/2}}$$
$$\leq C W_\beta(x,|x-z|)|x-z|^{1-d\mu+d-\beta} \int_{|x-z|^2}^{\infty} \sqrt{t}^{d\mu-d+\beta-1} \frac{dt}{t^{1+\alpha/2}}$$
$$\leq C W_\beta(x,|x-z|)|x-z|^{-\alpha} \leq C W_{\beta-\alpha}(x,|x-z|).$$

Finally, we need to bound VI. First we write

$$VI = \int_{\rho(x)^2}^{\infty} \left[\int_{\mathbb{R}^d} k_t(x,y) f(y) dy - f(x) - \left(\int_{\mathbb{R}^d} k_t(z,y) f(y) dy - f(z)\right)\right] \frac{dt}{t^{1+\alpha/2}}$$
$$\leq \int_{\rho(x)^2}^{\infty} \left(\int_{\mathbb{R}^d} |k_t(x,y)-k_t(z,y)||f(y)| dy + |f(x)-f(z)|\right) \frac{dt}{t^{1+\alpha/2}}$$
$$= VI_1 + VI_2,$$

where

$$VI_1 = \int_{\rho(x)^2}^{\infty} \int_{\mathbb{R}^d} |k_t(x,y)-k_t(z,y)||f(y)| dy \frac{dt}{t^{1+\alpha/2}}$$

and

$$VI_2 = |f(x)-f(z)| \int_{\rho(x)^2}^{\infty} \frac{dt}{t^{1+\alpha/2}}.$$

To take care of VI_1, we apply Lemma 4 with δ as above, since $|x-z| < \rho(x) < \sqrt{t}$. That, together with the size estimate for f, Lemmas 1 and 9 give us

$$\begin{aligned}
VI_1 &= \int_{\rho(x)^2}^{\infty} \int_{\mathbb{R}^d} |k_t(x,y) - k_t(z,y)||f(y)|dy \frac{dt}{t^{1+\alpha/2}} \\
&\le C_N \int_{\rho(x)^2}^{\infty} \left(\frac{|x-z|}{\sqrt{t}}\right)^{\delta} \left(\frac{\rho(x)}{\sqrt{t}}\right)^{N} \int_{\mathbb{R}^d} \tilde{h}_t(x-y) W_\beta(y,\rho(y))dy \frac{dt}{t^{1+\alpha/2}} \\
&\le C_N W_\beta(x,\rho(x))|x-z|^{\delta} \int_{\rho(x)^2}^{\infty} \left(\frac{\rho(x)}{\sqrt{t}}\right)^{N-d\mu+d+-\beta-\theta} \frac{dt}{t^{1+\alpha/2+\delta/2}} \\
&\le C W_\beta(x,\rho(x)) \left(\frac{|x-z|}{\rho(x)}\right)^{\delta} \rho(x)^{-\alpha} \\
&\le C W_\beta(x,|x-z|) \left(\frac{|x-z|}{\rho(x)}\right)^{\delta-d\mu+d-\beta} \rho(x)^{-\alpha} \\
&\le C W_{\beta-\alpha}(x,|x-z|) \left(\frac{|x-z|}{\rho(x)}\right)^{\delta-d\mu+d-\beta+\alpha} \\
&\le C W_{\beta-\alpha}(x,|x-z|),
\end{aligned}$$

for N large enough and as long as $\mu \le 1 + \frac{\delta-(\beta-\alpha)}{d}$.

It only remains to bound VI_2. To do that, we just apply the smoothness estimate for f to obtain

$$\begin{aligned}
VI_2 &\le \big(W_\beta(x,|x-z|) + W_\beta(z,|x-z|)\big) \int_{\rho(x)^2}^{\infty} \frac{dt}{t^{1+\alpha/2}} \\
&\le \rho(x)^{-\alpha} \big(W_\beta(x,|x-z|) + W_\beta(z,|x-z|)\big) \\
&\le |x-z|^{-\alpha} \big(W_\beta(x,|x-z|) + W_\beta(z,|x-z|)\big) \\
&\le \big(W_{\beta-\alpha}(x,|x-z|) + W_{\beta-\alpha}(z,|x-z|)\big),
\end{aligned}$$

since $|x-z| \le \rho(x)$.

□

4 Other fractional differentiation operators

We will consider in this section other fractional differentiation operators related to L. These will be defined as the composition of powers of L (positive or negative) together with a multiplication by a power of the potential V. We have already results concerning the behaviour on $\Lambda_\rho^\beta(w)$ of powers of L so we need a tool to handle point-wise multiplication operators. That is the first aim of this section.

For a measurable function $\phi : \mathbb{R}^d \longmapsto \mathbb{R}$ we consider a multiplication operator $T_\phi f(x) = \phi(x) f(x)$. The next theorem gives sufficient conditions on the function ϕ to guarantee the boundedness of T_ϕ from $\Lambda_\rho^\beta(w)$ to $\Lambda_\rho^\sigma(w)$ when $0 < \sigma \le \beta$.

Theorem 2 *Let $w \in D^{\rho}_{\mu}$ for some $\mu \geq 1$ and η, β such that $0 < \eta \leq \beta \leq 1$. Suppose there exists a constant $C > 0$ such that the function ϕ satisfies:*

$$|\phi(x)| \leq C\rho^{\eta-\beta}(x), \textit{ for all } x \in \mathbb{R}^d \tag{21}$$

and

$$|\phi(x) - \phi(z)| \leq C\rho^{\eta-\beta}(x)\left(\frac{|x-z|}{\rho(x)}\right)^{\eta+d(\mu-1)}, \textit{ if } |x-z| \leq \rho(x). \tag{22}$$

Then, the operator T_ϕ is bounded from $\Lambda^{\beta}_{\rho}(w)$ into $\Lambda^{\eta}_{\rho}(w)$.

Proof Let $w \in D^{\rho}_{\mu}$ and $f \in \Lambda^{\beta}_{\rho}(w)$. First we are going to check condition (6). For $x \in \mathbb{R}^d$ we have that

$$\begin{aligned}|T_\phi(f)(x)| &= |\phi(x)||f(x)| \\ &\leq C\rho^{\eta-\beta}(x)W_\beta(x,\rho(x)) \\ &\leq CW_\eta(x,\rho(x)),\end{aligned}$$

where we have used (21) for ϕ and that $f \in \Lambda^{\beta}_{\rho}(w)$.

Now, to check (7) for $T_\phi f$, we apply both conditions asked to ϕ. Let $x, z \in \mathbb{R}^d$ with $|x-z| \leq \rho(x)$.

$$\begin{aligned}|T_\phi(f)(x) - T_\phi(f)(z)| &= |\phi(x)f(x) - \phi(z)f(z)| \\ &\leq |\phi(x)||f(x) - f(z)| + |\phi(x) - \phi(z)||f(z)|.\end{aligned} \tag{23}$$

For the first term of (23), we use again assumption (21) on ϕ to obtain,

$$\begin{aligned}|\phi(x)||f(x) - f(z)| &\leq C\rho^{\eta-\beta}(x)[W_\beta(x,|x-z|) + W_\beta(z,|x-z|)] \\ &\leq C[W_\eta(x,|x-z|) + W_\eta(z,|x-z|)]\end{aligned}$$

as above, since $f \in \Lambda^{\beta}_{\rho}(w)$.

As for the second term of (23), we apply now (22) in the following way,

$$\begin{aligned}|\phi(x) - \phi(z)||f(z)| &\leq C\left(\frac{|x-y|}{\rho(x)}\right)^{\eta+d(\mu-1)}\rho^{\eta-\beta}(x)W_\beta(z,\rho(z)) \\ &\leq C\left(\frac{|x-y|}{\rho(z)}\right)^{\eta+d(\mu-1)}\rho^{\eta-\beta}(z)W_\beta(z,\rho(z)) \\ &\leq C\left(\frac{|x-y|}{\rho(z)}\right)^{\eta+d(\mu-1)}W_\eta(z,\rho(z)) \\ &\leq CW_\eta(z,|x-z|).\end{aligned}$$

Here, we have used that $\rho(x) \simeq \rho(z)$ and the duplication property for W_η given in Lemma 1. □

Remark 1 In Theorem 2 the conditions on the function ϕ depend only on w through its doubling exponent μ. In particular, note that if $w \in A_1^\rho$ (hence in D_1^ρ), condition (22) on ϕ becomes

$$|\phi(x) - \phi(z)| \leq C \left(\frac{|x-z|}{\rho(x)} \right)^\eta \rho^{\eta-\beta}(x) \quad \text{if } |x-z| \leq \rho(x). \tag{24}$$

Remark 2 It is worth noting that the point-wise conditions for ϕ given in Theorem 2 are equivalent to the following integral conditions:

$$\frac{1}{|B(x,\rho(x))|} \int_{B(x,\rho(x))} |\phi(y)| dy \leq C \rho^{\eta-\beta}(x), \text{ for all } x \in \mathbb{R}^d \tag{25}$$

and

$$\begin{aligned} &\frac{1}{|B(x,s)|} \int_{B(x,s)} |\phi(y) - \phi_B| dy \\ &\quad \leq C \left(\frac{s}{\rho(x)} \right)^{\eta+d(\mu-1)} \rho^{\eta-\beta}(x), \text{ whenever } 0 < s \leq \rho(x). \end{aligned} \tag{26}$$

It is immediate that the point-wise conditions imply the integral ones. On the other hand, if ϕ satisfies the integral conditions above we can use Lebesgue Differentiation Theorem to obtain the point-wise ones. Given $x, y \in \mathbb{R}^d$ with $|x-y| \leq \rho(x)$ we write

$$\begin{aligned} |\phi(x) - \phi(y)| &\leq |\phi(x) - \phi_{B(x,|x-y|)}| + |\phi(y) - \phi_{B(y,|x-y|)}| \\ &\quad + |\phi_{B(x,|x-y|)} - \phi_{B(y,|x-y|)}| \end{aligned}$$

For the the first term, defining $B_i = B(x, 2^{-i}|x-y|)$,

$$\begin{aligned} |\phi(x) - \phi_{B(x,|x-y|)}| &\leq \lim_{m\to\infty} \left(|\phi(x) - \phi_{B_m}| + \sum_{i=0}^{m-1} |\phi_{B_{i+1}} - \phi_{B_i}| \right) \\ &\leq C \sum_{i=0}^{\infty} \frac{1}{|B_i|} \int_{B_i} |\phi(z) - \phi_{B_i}| dz \\ &\leq C \left(\frac{|x-y|}{\rho(x)} \right)^{\eta+d(\mu-1)} \rho^{\eta-\beta}(x) \sum_{i=0}^{\infty} 2^{-i(\eta+d(\mu-1))} \\ &\leq C \left(\frac{|x-y|}{\rho(x)} \right)^{\eta+d(\mu-1)} \rho^{\eta-\beta}(x). \end{aligned}$$

The second and third terms of the sum can be bounded in a similar way, obtaining (22).

Finally, to obtain (21) we set $B = B(x, \rho(x)/2)$

$$\begin{aligned}|\phi(x)| &\le |\phi(x) - \phi_B + \phi_B| \\ &\le \frac{1}{|B|}\int_B |\phi(x) - \phi(z)|dz + \frac{1}{|B|}\int_B |\phi(z)|dz \\ &\le C\rho(x)^{\eta-\beta},\end{aligned}$$

applying condition (25) and the estimate we just obtained.

If we consider the case $w = 1$ in Theorem 2, we are able to prove that the conditions imposed on ϕ are also necessary for the operator T_ϕ to be bounded from Λ_ρ^β into Λ_ρ^η, as we show in the next proposition.

Proposition 1 *Let ϕ a real function such that the operator T_ϕ is bounded from Λ_ρ^β into Λ_ρ^η, for some $0 < \eta \le \beta$. Then the function ϕ satisfies:*

$$|\phi(x)| \le C\rho^{\eta-\beta}(x), \text{ for all } x \in \mathbb{R}^d \tag{27}$$

$$|\phi(x) - \phi(z)| \le C\left(\frac{|x-z|}{\rho(x)}\right)^\eta \rho^{\eta-\beta}(x), \text{ if } |x-z| \le \rho(x). \tag{28}$$

Proof Consider first, for $x \in \mathbb{R}^d$ and $s \le \rho(x)$, the function

$$f_{x,s}(y) = \chi_{[0,s]}(|x-y|)(\rho(x)^\beta - s^\beta) + \chi_{[s,\rho(x)]}(|x-y|)(\rho(x)^\beta - |x-y|^\beta).$$

As it was shown in Lemma 2.5 of [10], this function belongs to Λ_ρ^β and its norm does not depend neither on x nor on s.

To check (27) we take $s = \rho(x)/2$. Since T_ϕ is bounded from Λ_ρ^β into Λ_ρ^η,

$$|\phi(x)f_{x,s}(x)| = |\phi(x)|(\rho(x)^\beta - s^\beta) \le C\rho(x)^\eta$$

Then,

$$|\phi(x)| \le C\rho(x)^{\eta-\beta}.$$

Now, to verify inequality (28), we take x and $z \in \mathbb{R}^d$ such that $|x-z| \le \rho(x)$. If $|x-z| \le \rho(x)/2$, choose $s = |x-z|$. Then

$$|\phi(x)f_{x,s}(x) - \phi(z)f_{x,s}(z)| = |\phi(x) - \phi(z)|(\rho(x)^\beta - s^\beta) \le C|x-z|^\eta.$$

Therefore, since $\rho(x)^\beta - s^\beta \simeq \rho(x)^\beta$,

$$|\phi(x) - \phi(z)| \le C|x-z|^\eta\rho(x)^{-\beta} = C\left(\frac{|x-z|}{\rho(x)}\right)^\eta \rho(x)^{\eta-\beta}.$$

On the other hand, if $\rho(x)/2 < |x - z| \leq \rho(x)$, then

$$|\phi(x) - \phi(y)| \leq \rho(x)^{\eta-\beta} + \rho(z)^{\eta-\beta} \leq C \left(\frac{|x-z|}{\rho(x)}\right)^{\eta} \rho(x)^{\eta-\beta}.$$

□

Remark 3 In view of Remarks 1, 2 and Proposition 1, we are able to extend Proposition 3.2 in [10] in two directions. On one side, for $\eta = \beta$, we prove that the same conditions on ϕ required in [10] imply not only the boundedness on $\Lambda_\rho^\beta(w)$ for $w = 1$, but also for any $w \in A_1^\rho$. On the other side, for $w = 1$, we obtain equivalence between some conditions on ϕ and the boundedness of the multiplication operator T_ϕ from Λ_ρ^β into Λ_ρ^η with $\eta \leq \beta$, extending the already known result for the case $\eta = \beta$.

With this tool at hand, we will continue this section by stating and proving boundedness results for some other fractional differentiation operators in this context. First we take a look at compositions of two fractional differentiation operators, namely, given two positives numbers σ and α such that $\alpha + \sigma < 1$ we consider $V^{\sigma/2}L^{\alpha/2}$ and its adjoint, $L^{\alpha/2}V^{\sigma/2}$. In this case we have the following result.

Theorem 3 *Let us assume that $V \in RH_d$ and that we are given $\sigma > 0$ and $\alpha > 0$ with $\alpha + \sigma < 1$.*

(i) *If there exists some $\varepsilon_0 > \alpha$ such that for each $0 < \varepsilon < \varepsilon_0$, V satisfies*

$$|V^{\sigma/2}(x) - V^{\sigma/2}(y)| \leq C\rho(x)^{-\sigma} \left(\frac{|x-y|}{\rho(x)}\right)^{\varepsilon} \text{ for } |x-y| \leq \rho(x), \tag{29}$$

then $L^{\alpha/2}V^{\sigma/2}$ maps continuously $\Lambda_\rho^\beta(w)$ into $\Lambda_\rho^{\beta-(\alpha+\sigma)}(w)$, for any $\alpha + \sigma < \beta < \min\{\varepsilon_0 + \sigma, 1\}$ and $w \in D_\mu^\rho$ with $1 \leq \mu < 1 + \frac{\min\{\varepsilon_0+\sigma,1\}-\beta}{d}$.

(ii) *If there exists some $\varepsilon_0 > 0$ such that for each $0 < \varepsilon < \varepsilon_0$, V satisfies (29), then $V^{\sigma/2}L^{\alpha/2}$ maps continuously $\Lambda_\rho^\beta(w)$ into $\Lambda_\rho^{\beta-(\alpha+\sigma)}(w)$, for any $\alpha + \sigma < \beta < \min\{\varepsilon_0 + \alpha + \sigma, 1\}$ and $w \in D_\mu^\rho$ with $1 \leq \mu < 1 + \frac{\min\{\varepsilon_0+\alpha+\sigma,1\}-\beta}{d}$.*

Proof We are going to prove (ii) in the first place. Applying Theorem 1 with $q = d$ we obtain that $L^{\alpha/2}$ maps continuously $\Lambda_\rho^\beta(w)$ into $\Lambda_\rho^{\beta-\alpha}(w)$, provided $0 < \alpha < \beta < 1$ and $1 \leq \mu < 1 + \frac{1-\beta}{d}$.

To conclude (ii), we make the following observation. For $x \in \mathbb{R}^d$ and $y \in B = B(x, \rho(x))$, condition (29) implies

$$V(x)^{\sigma/2} \leq V(y)^{\sigma/2} + |V(x)^{\sigma/2} - V(y)^{\sigma/2}| \leq V(y))^{\sigma/2} + \frac{C}{\rho^\sigma(x)}.$$

Therefore, averaging over $B = B(x, \rho(x))$ in the y-variable

$$V(x)^{\sigma/2} = \frac{1}{|B|}\int_B V(x)^{\sigma/2} dy \leq \frac{1}{|B|}\int_B V^{\sigma/2} + \frac{C}{\rho^\sigma(x)} \leq \frac{C'}{\rho^\sigma(x)},$$

where we have used Hölder's inequality and the definition of $\rho(x)$.

So, (29) together with the above inequality give the sufficient conditions to get that multiplication by $V^{\sigma/2}$ is bounded from $\Lambda_\rho^{\beta-\alpha}(w)$ into $\Lambda_\rho^{\beta-\alpha-\sigma}(w)$ as long as $\alpha+\sigma < \beta < \varepsilon_0+\alpha+\sigma$ and $1 \le \mu < 1+\frac{\varepsilon_0+\alpha+\sigma-\beta}{d}$. Therefore, under our assumptions on β and μ both operators are bounded and the conclusion follows.

Item (i) can be proved in an analogous way, noting that condition (29) allows us to apply Theorem 2 obtaining that multiplication by $V^{\sigma/2}$ is bounded from $\Lambda_\rho^{\beta}(w)$ into $\Lambda_\rho^{\beta-\sigma}(w)$. Applying again Theorem 1 we obtain the desired result. □

Next we consider the case of compositions of negative powers of L with multiplication by positive powers of the potential, that is, operators of the form $L^{-\alpha/2}V^{\sigma/2}$, with $\alpha < \sigma$. In order to do that we look at the operator as the composition of $L^{-\alpha/2}V^{\alpha/2}$ with multiplication by $V^{(\sigma-\alpha)/2}$. The behaviour on $\Lambda_\rho^{\beta}(w)$ of the first of these operators was analysed in [2]. However, the hypotheses considered there were different from those needed here. In fact, since we must assume some regularity on $V^{(\sigma-\alpha)/2}$ to get boundedness of the multiplication operator, we have a fortiori a size estimate for V, namely $V(x) \le C\rho(x)^{-2}$. For that reason we state and prove the following result. Before, let us recall that $A_\infty^\rho = \bigcup_{p\ge1} A_p^\rho$.

Proposition 2 *Let us assume that $V \in RH_{d/2}$ and that there exists a constant C such that $V(x) \le C\rho(x)^{-2}$. Let $0 < \alpha \le 2$, the operator $L^{-\alpha/2}V^{\alpha/2}$ is bounded on $\Lambda_\rho^{\beta}(w)$ as long as $0 \le \beta < \min\{\alpha, 1\}$ and $w \in A_\infty^\rho \cap D_\mu^\rho$ with $1 \le \mu < 1+\frac{\min\{\alpha,1\}-\beta}{d}$.*

Proof We are going to apply the general criterium given in [2], Corollary 4. Let $\mathcal{J}_{\alpha/2}$ be the kernel of the fractional integral $L^{-\alpha/2}$. By Lemma 4 in [3], we know that for each $N > 0$, there exists a constant C_N such that

$$\mathcal{J}_{\alpha/2}(x,y) \le \frac{C_N}{|x-y|^{d-\alpha}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N}$$

and

$$|\mathcal{J}_{\alpha/2}(x,y) - \mathcal{J}_{\alpha/2}(x+h,y)| \le C\frac{|h|}{|x-y|^{d-\alpha+1}}\left(1+\frac{|x-y|}{\rho(x)}\right)^{-N},$$

if $|h| \le |x-y|/2$.

Now, we consider $K_{\alpha/2} = \mathcal{J}_{\alpha/2}(x,y)V^{\alpha/2}(y)$ the kernel of $L^{-\alpha/2}V^{\alpha/2}$. First, we make the following observation. For $f \in L^1_{\text{loc}}$, by (2), the hypothesis on V and the size estimate for $\mathcal{J}_{\alpha/2}$ given above, we get

$$
\begin{aligned}
L^{-\alpha/2}V^{\alpha/2}f(x) &\le \int_{\mathbb{R}^d} |\mathcal{J}_{\alpha/2}(x,y)|V^{\alpha/2}(y)|f(y)|dy \\
&\le C_N \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-\alpha}} \left(1+\frac{|x-y|}{\rho(x)}\right)^{-N} \rho^{-\alpha}(y)|f(y)|dy \\
&\le C_N \rho^{-\alpha}(x) \int_{\mathbb{R}^d} \frac{1}{|x-y|^{d-\alpha}} \left(1+\frac{|x-y|}{\rho(x)}\right)^{-N+\alpha N_0} |f(y)|dy \\
&\le C_N \rho^{-\alpha}(x) \sum_{j=-\infty}^{j=\infty} \frac{(1+2^j)^{-N+\alpha N_0}}{(2^j\rho(x))^{d-\alpha}} \int_{|x-y|\le 2^j\rho(x)} |f(y)|dy \\
&\le C_N Mf(x) \sum_{j=-\infty}^{j=\infty} (1+2^j)^{-N+\alpha N_0} 2^{j\alpha} \\
&\le CMf(x),
\end{aligned}
$$

where M is the Hardy–Littlewood maximal function and N is chosen large enough. This point-wise estimate guarantee that $L^{-\alpha/2}V^{\alpha/2}$ is bounded from L^1 into $L^{1,\infty}$.

Now, we are going to derive size and smoothness estimates for $K_{\alpha/2}$ from the ones given for $\mathcal{J}_{\alpha/2}$. For each $N > 0$ there exists C_N such that

$$
\begin{aligned}
|V^{\alpha/2}(y)\mathcal{J}_{\alpha/2}(x,y)| &\le \frac{C_N}{\rho^\alpha(y)|x-y|^{d-\alpha}} \left(1+\frac{|x-y|}{\rho(x)}\right)^{-N} \\
&\le \frac{C_N}{|x-y|^d} \left(1+\frac{|x-y|}{\rho(x)}\right)^{-N+\alpha+\alpha N_0}.
\end{aligned}
\tag{30}
$$

In a similar way, taking $|h| \le |x-y|/2$,

$$
\begin{aligned}
|V^{\alpha/2}(y)\mathcal{J}_{\alpha/2}(x,y) &- V^{\alpha/2}(y)\mathcal{J}_{\alpha/2}(x+h,y)| \\
&\le V^{\alpha/2}(y)|\mathcal{J}_{\alpha/2}(x,y) - \mathcal{J}_{\alpha/2}(x+h,y)| \\
&\le \frac{C_N|h|}{\rho^\alpha(y)|x-y|^{d-\alpha+1}} \left(1+\frac{|x-y|}{\rho(x)}\right)^{-N} \\
&\le C_N \frac{|h|}{|x-y|^{d+1}} \left(1+\frac{|x-y|}{\rho(x)}\right)^{-N+\alpha+\alpha N_0}.
\end{aligned}
\tag{31}
$$

Finally, we are going to show a $T1$ condition for $L^{-\alpha/2}V^{\alpha/2}$. Consider $x_0 \in \mathbb{R}^d$ and $r > 0$ such that $r \le \rho(x)/2$. If $x, z \in B(x_0, r)$

$$
\begin{aligned}
&|L^{-\alpha/2}V^{\alpha/2}1(x) - L^{-\alpha/2}V^{\alpha/2}1(z)| \\
&\quad = \left| \int_{\mathbb{R}^d} \mathcal{J}_{\alpha/2}(x,y)V^{\alpha/2}(y)dy - \int_{\mathbb{R}^d} \mathcal{J}_{\alpha/2}(z,y)V^{\alpha/2}(y)dy \right| \\
&\quad \le \left| \int_{B_\rho} \mathcal{J}_{\alpha/2}(x,y)V^{\alpha/2}(y)dy - \int_{B_\rho} \mathcal{J}_{\alpha/2}(z,y)V^{\alpha/2}(y)dy \right|
\end{aligned}
$$

$$+\int_{B_\rho^c} \left| V^{\alpha/2}(y)\mathcal{J}_{\alpha/2}(x,y) - V^{\alpha/2}(y)\mathcal{J}_{\alpha/2}(z,y) \right| dy$$
$$= I + II,$$

where $B_\rho = B(x_0, \rho(x_0))$.

To bound II, we may simply apply the smoothness estimate obtained for the kernel to get

$$II \leq \int_{B_\rho^c} \frac{|x-z|}{|x-y|^{d+1}} dy \leq C\frac{r}{\rho(x_0)},$$

since $\rho(x) \simeq \rho(x_0)$.

The point-wise estimates for $\mathcal{J}_{\alpha/2}$ allow us to apply Proposition 3 in [2] to obtain that $L^{-\alpha/2}$ is bounded from L^p into $\Lambda_\rho^{\alpha-d/p}$, for any $p \in (d/\alpha, d/(\alpha-1)^+)$. From this property we obtain,

$$\begin{aligned} I &= |L^{-\alpha/2}(V^{\alpha/2}\chi_{B_\rho})(x) - L^{-\alpha/2}(V^{\alpha/2}\chi_{B_\rho})(z)| \\ &\leq C\|V^{\alpha/2}\chi_{B_\rho}\|_p |x-z|^{\alpha-d/p} \\ &\leq C\left(\int_{B_\rho} V^{p\alpha/2}(y)dy\right)^{1/p} r^{\alpha-d/p} \\ &\leq C\left(\frac{r}{\rho(x_0)}\right)^{\alpha-d/p}. \end{aligned}$$

Now, since we can take any $p \in (d/\alpha, d/(\alpha-1)^+)$, we obtain that

$$|L^{-\alpha/2}V^{\alpha/2}1(x) - L^{-\alpha/2}V^{\alpha/2}1(z)| \leq C\left(\frac{r}{\rho(x_0)}\right)^{\varepsilon}, \tag{32}$$

for any $\varepsilon < \min\{\alpha, 1\}$.

To conclude, the weak type $(1, 1)$ together with estimates (30), (31) and (32), allow us to apply Corollary 4 of [2] to obtain the stated result.

□

Remark 4 Notice that we get the same conclusions as in Theorem 6 in [2] under the assumption $V \in RH_q$ for all $q \geq 1$. However, as it was pointed out in [3], there are potentials that satisfy the hypothesis of Proposition 2 above but they are not in RH_q for all $q \geq 1$.

Now, as a consequence of Theorem 2 and the previous result, we can obtain the following boundedness properties for differentiation operators of the form $L^{-\alpha/2}V^{\sigma/2}$ and $V^{\sigma/2}L^{-\alpha/2}$, where $\sigma > \alpha$. In what follows, for a better understanding of the statements, we change the parameters α and σ by α and γ with $\gamma = \sigma - \alpha$.

Theorem 4 *Let us assume that* $V \in RH_{d/2}$, $0 < \alpha \leq 2$ *and* $0 < \gamma < 1$. *If there exists some* $\varepsilon_0 > 0$ *such that for each* $0 < \varepsilon < \varepsilon_0$, V *satisfies*

$$|V^{\gamma/2}(x) - V^{\gamma/2}(y)| \leq C\rho(x)^{-\gamma}\left(\frac{|x-y|}{\rho(x)}\right)^{\varepsilon}, \tag{33}$$

then $L^{-\alpha/2}V^{\alpha/2+\gamma/2}$ *maps continuously* $\Lambda_\rho^\beta(w)$ *into* $\Lambda_\rho^{\beta-\gamma}(w)$, *for any* $\gamma < \beta < \min\{1, \alpha+\gamma, \varepsilon_0+\gamma\}$ *and* $w \in w \in A_\infty^\rho \cap D_\mu^\rho$ *with* $1 \leq \mu < 1 + \frac{\min\{1,\alpha+\gamma,\varepsilon_0+\gamma\}-\beta}{d}$.

Proof Let $w \in A_\infty^\rho \cap D_\mu^\rho$ and $f \in \Lambda_\rho^\beta(w)$. First, observe that condition (33) implies that $V^{\gamma/2}(x) \leq C\rho^{-\gamma}(x)$, as it was done in the proof of Theorem 3.

This observation together with condition (33), let us apply Theorem 2 obtaining that the multiplication by $V^{\gamma/2}$ is an operator bounded from $\Lambda_\rho^\beta(w)$ into $\Lambda_\rho^{\beta-\gamma}(w)$, provided $\gamma < \beta < \min\{1, \gamma+\varepsilon_0\}$ and $1 \leq \mu < 1 + \frac{\min\{1,\gamma+\varepsilon_0\}-\beta}{d}$.

On the other hand, we apply Proposition 2 to assure that $L^{-\alpha/2}V^{\alpha/2}$ is bounded on $\Lambda_\rho^{\beta-\gamma}(w)$ as long as $\gamma < \beta < \min\{1, \gamma+\alpha\}$ and $1 \leq \mu < 1 + \frac{\min\{1,\gamma+\alpha\}-\beta}{d}$.

Since our assumptions on β and μ allow us to conclude the boundedness of both operators, the proof is complete.

□

Theorem 5 *Let* $V \in RH_{d/2}$ *and* $0 < \alpha < 2$, $0 < \gamma < 1$. *Assume further that for some* $\varepsilon > 0$, V *satisfies*

$$|V^{\alpha/2}(x) - V^{\alpha/2}(y)| \leq C\rho(x)^{-\alpha}\left(\frac{|x-y|}{\rho(x)}\right)^{\varepsilon},$$

for $|x-y| \leq \rho(x)$. *Then, for* $\gamma < \beta < \beta_0 = \min\{1, \varepsilon, \gamma + \varepsilon\gamma/\alpha\}$, $V^{\alpha/2+\gamma/2}L^{-\alpha/2}$ *maps continuously* $\Lambda_\rho^\beta(w)$ *into* $\Lambda_\rho^{\beta-\gamma}(w)$ *as long as* $w \in A_\infty^\rho \cap D_\mu^\rho$ *with* $1 \leq \mu < 1 + \frac{\beta_0-\beta}{d}$.

Proof Let $w \in A_\infty^\rho \cap D_\mu^\rho$ and $f \in \Lambda_\rho^\beta(w)$. The assumptions on V and w allow us to apply Theorem 1 in [3], to show that for $0 < \beta < \min\{1, \varepsilon\}$ and $1 \leq \mu < 1 + \frac{\min\{1,\varepsilon\}-\beta}{d}$,

$$\|V^{\alpha/2}L^{-\alpha/2}f\|_{\Lambda_\rho^\beta(w)} \leq C\|f\|_{\Lambda_\rho^\beta(w)}.$$

Now we will show that the same hypothesis on V allows us to establish the boundedness of the multiplier operator $Tg = V^{\gamma/2}g$ from $\Lambda_\rho^\beta(w)$ into $\Lambda_\rho^{\beta-\gamma}(w)$, via Theorem 2. It would be sufficient to find all possible $0 < \beta < 1$ and $\mu \geq 1$ such that, for $|x-y| \leq \rho(x)$,

$$|V^{\gamma/2}(x) - V^{\gamma/2}(y)| \leq C\rho(x)^{-\gamma}\left(\frac{|x-y|}{\rho(x)}\right)^{\beta-\gamma+d(\mu-1)}, \tag{34}$$

since, as in the previous results, (34) implies the size condition on V.

We will do that considering two cases: $\alpha > \gamma$ and $\alpha \leq \gamma$. Let $x, y \in \mathbb{R}^d$ such that $|x-y| \leq \rho(x)$. If $\alpha > \gamma$, we may apply the hypothesis on V to obtain

$$
\begin{aligned}
|V^{\gamma/2}(x) - V^{\gamma/2}(y)| &\leq C|V^{\alpha/2}(x) - V^{\alpha/2}(y)|^{\gamma/\alpha} \\
&\leq C\left(\frac{|x-y|}{\rho(x)}\right)^{\varepsilon\gamma/\alpha} \rho(x)^{-\gamma},
\end{aligned}
$$

so (34) holds provided $\gamma < \beta < \min\{1, \gamma+\varepsilon\gamma/\alpha\}$ and $1 \leq \mu < 1+\frac{\min\{1,\gamma+\varepsilon\gamma/\alpha\}-\beta}{d}$.

Next, if $\alpha \leq \gamma$, we can use the Mean Value Theorem together with the hypothesis on V to obtain, for a point ξ lying between x and y,

$$
\begin{aligned}
|V^{\gamma/2}(x) - V^{\gamma/2}(y)| &\leq C|V^{\alpha/2}(x) - V^{\alpha/2}(y)|V(\xi)^{\alpha(\gamma/\alpha-1)/2} \\
&\leq C\left(\frac{|x-y|}{\rho(x)}\right)^{\varepsilon} \rho(x)^{-\alpha}\rho(x)^{-\alpha(\gamma/\alpha-1)} \\
&\leq C\left(\frac{|x-y|}{\rho(x)}\right)^{\varepsilon} \rho(x)^{-\gamma},
\end{aligned}
$$

which implies (34) for $\gamma < \beta < \min\{1, \gamma+\varepsilon\}$ and $1 \leq \mu < 1+\frac{\min\{1,\gamma+\varepsilon\}-\beta}{d}$.

Therefore, the three conditions on β and μ are satisfied for $\gamma < \beta < \beta_0 = \min\{1, \varepsilon, \gamma+\varepsilon\gamma/\alpha\}$ and $1 \leq \mu < 1+\frac{\beta_0-\beta}{d}$. Altogether, we obtain that

$$
\|V^{\gamma/2}(L^{-\alpha/2}V^{\alpha/2}f)\|_{\Lambda_\rho^{\beta-\gamma}(w)} \leq C\|L^{-\alpha/2}V^{\alpha/2}f\|_{\Lambda_\rho^{\beta}(w)} \leq C\|f\|_{\Lambda_\rho^{\beta}(w)}.
$$

□

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