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# Topological groups with invariant linear spans

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## Abstract
Given a topological group $G$ that can be embedded as a topological subgroup into some topological vector space (over the field of reals) we say that $G$ has *invariant linear span* if all linear spans of $G$ under arbitrary embeddings into topological vector spaces are isomorphic as topological vector spaces. For an arbitrary set $A$ let $\mathbb{Z}^{(A)}$ be the direct sum of $|A|$-many copies of the discrete group of integers endowed with the Tychonoff product topology. We show that the topological group $\mathbb{Z}^{(A)}$ has invariant linear span. This answers a question from a paper of Dikranjan et al. (J Math Anal Appl 437:1257–1282, 2016) in positive. We prove that given a non-discrete sequential space $X$, the free abelian topological group $A(X)$ over $X$ is an example of a topological group that embeds into a topological vector space but does not have invariant linear span.

**Keywords** Topological group · Topological vector space · Embedding · Absolutely Cauchy summable · Topologically independent · Diophantine approximation

**Mathematics Subject Classification** 46A99 · 22A99

All vector spaces in this paper are considered over the field $\mathbb{R}$ of real numbers and all topological spaces are assumed to be Hausdorff. For an arbitrary non-empty set $A$ and a topological group $G$ with addition and neutral element $0_G$ let $G^A$ be the topological group given by the direct product $\Pi_{a\in A} G$ with coordinate-wise addition and the Tychonoff product topology. We denote $G^{(A)}$ the topological subgroup of $G^A$, with inherited topology, consisting of those elements $(g_a)_{a\in A}$ for which the set $\{a \in A\colon\ g_a \neq 0_G\}$ is finite. Given a subset $H$ of a group $G$ and a subset $M$ of a vector

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space $L$, we use the standard notation $\langle H \rangle$ to denote the subgroup of $G$ generated by $H$ and $\operatorname{span}(M)$ for the vector subspace of $L$ generated by $M$. For simplicity we write $\langle g \rangle$ rather than $\langle \{g\} \rangle$ for any $g \in G$ and, similarly, $\operatorname{span}(l)$ instead of $\operatorname{span}(\{l\})$ for any $l \in L$.

## 1 Introduction

In this note we study which topological groups enjoy the property stated in the following definition.

**Definition 1.1** Let $G$ be a topological group that can be embedded (as a topological subgroup) into some topological vector space. We say that $G$ has *invariant linear span* provided that all linear spans of $G$ under arbitrary embeddings into topological vector spaces are isomorphic as topological vector spaces.

A simple example of topological group with an invariant linear span is every topological vector space. Indeed, as follows from Theorem 2 on page 24 in Banach [1], given arbitrary topological vector spaces $L$ and $E$ and a continuous group homomorphism $h : L \to E$, the homomorphism $h$ is automatically linear. This observation further yields that if $L$ is embedded in $E$ as a topological subgroup, the same embedding is already an embedding of topological vector spaces. In particular, the linear span of $L$ in $E$ is (isomorphic to) the topological vector space $L$ again and hence the linear span of $L$ does not depend on the space $E$ in which $L$ embeds. Yet another simple example of a topological group with an invariant linear span is the discrete topological group $\mathbb{Z}$ of integers. Its linear span is obviously (isomorphic to) the topological vector space $\mathbb{R}$.

In our paper we show that for an arbitrary non-empty set $A$ the group $\mathbb{Z}^{(A)}$ has invariant linear span (which is isomorphic to $\mathbb{R}^{(A)}$). See Theorem 3.4 and Corollary 3.5. This answers the Question 10.6 of Dikranjan et al. [3] in positive and generalizes the well-known fact, EVT I.14 Théorème 2 [2], that all topological vector spaces of the same finite dimension are isomorphic (see Remark 3.6).

The proof of Theorem 3.4 consists of two steps. The first was done in Proposition 10.1 [3] by showing that given an injective linear map $l : \mathbb{R}^{(A)} \to L$, where $L$ is a topological vector space, the continuity of $l$ follows from the continuity of the restriction of $l$ to $\mathbb{Z}^{(A)}$. The second step is done in Theorem 2.4, where we basically show, that if the restriction of $l$ to $\mathbb{Z}^{(A)}$ is an embedding of topological groups, then $l$ is open. The proof of Theorem 2.4 is based on a Diophantine approximation done in Lemma 2.2 which resembles the classical Kronecker's approximation theorem.

We end the paper with Theorem 3.7, which shows that for an arbitrary non-discrete sequential space $X$ the free topological abelian group $A(X)$ does not have invariant linear span, as it canonically embeds in both the free topological vector space $V(X)$ and the free locally convex topological vector space $L(X)$, and the linear spans of $A(X)$ in the latter spaces are the non-isomorphic topological vector spaces $V(X)$ and $L(X)$. This theorem is based on non-trivial results of Tkachenko [9] and Gabriyelyan and Morris [4].

## 2 The main technical theorem

We begin the section by two auxiliary observations.

**Lemma 2.1** *For every neighbourhood $V$ of zero in a compact group $G$ and every $t \in G$ there is $m \in \mathbb{N}\backslash\{0\}$ such that*

$$mt \in V. \tag{1}$$

***Proof*** Pick a neighbourhood $V$ of zero in $G$ and $t \in G$ arbitrarily. There are two possibilities. If $\langle t \rangle$ is a discrete subgroup of $G$ then it is closed and, consequently, compact and therefore finite. Let $m$ be the order of $\langle t \rangle$ and observe that (1) holds. The second possibility is, that $\langle t \rangle$ is not discrete. Then every neighbourhood of zero (and $V$ in particular) contains infinitely many elements of $\langle t \rangle$. Since $t$ is a generator of $\langle t \rangle$ there is $m \in \mathbb{N}\backslash\{0\}$ satisfying (1). □

**Lemma 2.2** *Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}^F$, where $F$ is a finite set. For every neighborhood $O$ of zero in $\mathbb{R}^F$ there is $m \in \mathbb{N}\backslash\{0\}$ and a sequence $(z_n)_{n\in\mathbb{N}} \subset \mathbb{Z}^F$ such that*

$$mt_n - z_n \in O \tag{2}$$

*holds for infinitely many $n \in \mathbb{N}$.*

***Proof*** Fix $O$, a neighbourhood of zero in $\mathbb{R}^F$, arbitrarily, and let $q : \mathbb{R}^F \to (\mathbb{R}/\mathbb{Z})^F$ be the quotient map. Since $(\mathbb{R}/\mathbb{Z})^F$ is sequentially compact, the sequence $(q(t_n))_{n\in\mathbb{N}}$ has a convergent subsequence with a limit $t$. As $q$ is an open map, we may pick a neighbourhood $V$ of zero in $(\mathbb{R}/\mathbb{Z})^F$ such that

$$V + V \subset q(O).$$

By Lemma 2.1, there is a positive integer $m$ satisfying (1). Observe that the set $M$ defined as

$$M := \{n \in \mathbb{N} : mq(t_n) \in mt + V\}$$

is infinite, and for every $n \in M$ we have

$$q(mt_n) = mq(t_n) \in mt + V \subset V + V \subset q(O).$$

Thus for every $n \in M$ there is $z_n \in \mathbb{Z}^F$ such that (2) holds. □

In order to formulate the main technical result of this paper we need to recall three notions. Their importance to the topic of our manuscript will become clear from Proposition 3.3 and from the proof of Theorem 3.4.

We say that a subset $A$ of a topological vector space $L$ is

- *absolutely Cauchy summable* provided that for every neighbourhood $V$ of $0_L$ there exists a finite set $F \subset A$ such that

$$\operatorname{span}(A \setminus F) \subset V; \tag{3}$$

- *topologically independent* if $0_L \notin A$ and for every neighbourhood $W$ of $0_L$ there exists a neighbourhood $U$ of $0_L$ such that for every finite subset $F \subset A$ and every indexed set $\{z_a : a \in F\}$ of integers the inclusion $\sum_{a \in F} z_a a \in U$ implies that $z_a a \in W$ for all $a \in F$. We call this neighbourhood $U$ a $W$-*witness* of the topological independence of $A$;
- *semi-basic* if for all $a \in A$ we have

$$a \notin \overline{\operatorname{span}(A \setminus \{a\})}. \tag{4}$$

***Remark 2.3*** In Definition 3.1 [3] the notion of an absolutely Cauchy summable set was introduced in an arbitrary abelian topological group. In topological vector spaces it is equivalent to our definition by Proposition 9.2 from the same manuscript.

*Topologically independent sets* were introduced in Definition 4.1 [3] in an arbitrary abelian topological group. For further properties of these sets in precompact groups we refer to a paper of Spěvák [8].

We have adopted the name *semi-basic* from Kalton [6], where a semi-basic sequence in an $F$-space was introduced. In Bourbaki [2], a semi-basic set is called *topologically free*. Semi-basic sequences in Banach spaces are called *minimal* [5,7].

Now we are ready to state the main technical theorem of this note.

**Theorem 2.4** *If $A$ is a topologically independent and absolutely Cauchy summable subset of a topological vector space $L$, then $A$ is semi-basic.*

***Proof*** To prove the contrapositive, assume that there is $a \in A$ with

$$a \in \overline{\operatorname{span}(A \setminus \{a\})}, \tag{5}$$

and let $A$ be absolutely Cauchy summable. We will show that $A$ is not topologically independent.

If $a = 0_L$, then we are done. Otherwise we can find a neighbourhood $W$ of $0_L$ such that $za \notin W$ for every $z \in \mathbb{Z} \setminus \{0\}$. Pick an arbitrary neighborhood $U$ of $0_L$. Let us show that $U$ is not a $W$-witness of topological independence of $A$.

Fix a balanced neighborhood $V$ of $0_L$ with $V + V + V \subset U$. Since $A$ is absolutely Cauchy summable, there is a finite $F \subset A \setminus \{a\}$ such that $\operatorname{span}(A \setminus (F \cup \{a\})) \subset V$. In particular, for every finite $B \subset A \setminus \{a\}$, reals $(s_b)_{b \in B \setminus F}$ and each $n \in \mathbb{N}$ we have

$$\sum_{b \in B \setminus F} s_b b \in \frac{1}{n} V. \tag{6}$$

Given $n \in \mathbb{N}$ arbitrarily, by (5) we can fix a finite set $B \subset A \setminus \{a\}$ and an indexed set $(r_b^n)_{b \in B}$ of reals such that

$$a - \sum_{b \in B} r_b^n b \in \frac{1}{n} V. \tag{7}$$

For $b \in F \setminus B$ define $r_b^n = 0$, and observe that (6) and (7) yield

$$a - \sum_{b \in F} r_b^n b = \left( a - \sum_{b \in B} r_b^n b \right) + \left( \sum_{b \in B \setminus F} r_b^n b \right) \in \frac{1}{n} V + \frac{1}{n} V. \tag{8}$$

By continuity of vector space operations, there is a neighborhood $O$ of zero in $\mathbb{R}^F$ such that

$$\sum_{b \in F} s_b b \in V \text{ for all } (s_b)_{b \in F} \in O. \tag{9}$$

Define a sequence $(t_n)_{n \in \mathbb{N}}$ in $\mathbb{R}^F$ by $t_n = (r_b^n)_{b \in F}$, and let $m \in \mathbb{N} \setminus \{0\}$ and $(z_n)_{n \in \mathbb{N}} \subset \mathbb{Z}^F$ be as in the conclusion of Lemma 2.2. By this lemma, we may fix $n \in \mathbb{N}$ such that $n \geq m$ and (2) holds. For $b \in F$ let $z_b \in \mathbb{Z}$ be the $b$-th coordinate of $z_n$, and observe that by (2) and (9) we have

$$\sum_{b \in F} (m r_b^n - z_b) b \in V.$$

From this, (8), and the fact that $V$ is balanced and $n \geq m$ we get

$$\begin{aligned} ma - \sum_{b \in F} z_b b &= m \left( a - \sum_{b \in F} r_b^n b \right) + \left( \sum_{b \in F} (m r_b^n - z_b) b \right) \in \frac{m}{n} V \\ &+ \frac{m}{n} V + V \subset V + V + V \subset U. \end{aligned}$$

Since $m$ is a non-zero integer and $z_b$ is an integer for each $b \in F$ we conclude that $U$ is not a $W$-witness of the topological independence of $A$, because $ma \notin W$ by the choice of $W$. □

## 3 The invariance of the linear span of $\mathbb{Z}^{(A)}$

In this section we prove that the topological group $\mathbb{Z}^{(A)}$ has invariant linear span. In order to do so we need to recall the notion of a (linear) Kalton map introduced in Dikranjan et al. [3] which is useful to deal with embeddings of $\mathbb{Z}^{(A)}$ and $\mathbb{R}^{(A)}$ into topological vector spaces.

Given a non-empty subset $A$ of a topological vector space $L$ such that $0 \notin A$, we denote

$$\mathcal{K}_A : \mathbb{Z}^{(A)} \to L$$

the group homomorphism given by $\mathcal{K}_A\left((z_a)_{a\in A}\right) = \sum_{a\in A} z_a a$ for every $(z_a)_{a\in A} \in \mathbb{Z}^{(A)}$. Similarly,

$$\ell\mathcal{K}_A : \mathbb{R}^{(A)} \to L$$

is the linear operator between vector spaces defined by $\ell\mathcal{K}_A\left((r_a)_{a\in A}\right) = \sum_{a\in A} r_a a$ for every $(r_a)_{a\in A} \in \mathbb{R}^{(A)}$. As in Dikranjan et al. [3] we call $\mathcal{K}_A$ ($\ell\mathcal{K}_A$) the (linear) Kalton map associated with $A$. Since the sums in the definitions are finite, the mappings are well-defined and $\mathcal{K}_A(\mathbb{Z}^{(A)}) = \langle A\rangle \subset L$ and $\ell\mathcal{K}_A(\mathbb{R}^{(A)}) = \mathrm{span}(A) \subset L$. Notice that the (linear) Kalton map is injective if and only if $A$ is (linearly) independent.

**Fact 3.1** *(*Proposition 10.1 *[3]). Given a non-empty subset A of non-zero elements of a topological vector space, the following statements are equivalent:*

*(i) the linear Kalton map $\ell\mathcal{K}_A$ is continuous;*
*(ii) the Kalton map $\mathcal{K}_A$ is continuous;*
*(iii) the set A is absolutely Cauchy summable.*

**Lemma 3.2** *Let A be a non-empty subset of a topological vector space. The following conditions are equivalent:*

*(i) the linear Kalton map $\ell\mathcal{K}_A$ is an open injection onto* $\mathrm{span}(A)$*;*
*(ii) the set A is semi-basic.*

***Proof*** Observe that from both items (i) and (ii) it follows that $A$ is linearly independent. Therefore, if we assume either (i) or (ii), then for each $a \in A$ there is a unique linear projection $\pi_a^A : \mathrm{span}(A) \to \mathrm{span}(a)$ such that $\ker \pi_a^A = \mathrm{span}(A\backslash\{a\})$ and $\pi_a^A$ restricted to $\mathrm{span}(a)$ is the identity map.

To end the proof it suffices to show that items (i) and (ii) are both equivalent to the following fact for a linearly independent set $A$:

$$\pi_a^A : \mathrm{span}(A) \to \mathrm{span}(a) \text{ is continuous for every } a \in A. \tag{10}$$

The equivalence of (i) and (10) follows from Proposition 10.2 [3]. To establish the equivalence of (ii) and (10) it suffices to realize that the continuity of each $\pi_a^A$ is equivalent to the fact that each $\ker(\pi_a^A)$ is closed in $\mathrm{span}(A)$ and this happens if and only if (4) holds for all $a \in A$. □

**Proposition 3.3** *Given a subset A of a topological vector space, the following conditions are equivalent:*

*(i) The linear Kalton map $\ell\mathcal{K}_A$ is an embedding of topological vector spaces.*
*(ii) A is absolutely Cauchy summable and semi-basic.*

***Proof*** Assume (i). Then $A$ is absolutely Cauchy summable by Fact 3.1 and semi-basic by Lemma 3.2. Thus (ii) holds.

If (ii) holds, Lemma 3.2 implies, that the linear Kalton map $\ell\mathcal{K}_A$ is open and injective, while Fact 3.1 provides its continuity. This gives us (i). □

Our next theorem answers Question 10.6 [3] in positive.

**Theorem 3.4** *Given a subset $A$ of a topological vector space the following statements are equivalent:*

*(i) The Kalton map $\mathcal{K}_A$ is an embedding of topological groups;*
*(ii) The linear Kalton map $\ell\mathcal{K}_A$ is an embedding of topological vector spaces.*

***Proof*** Since $\mathcal{K}_A$ is a restriction of $\ell\mathcal{K}_A$, the implication (ii)⇒(i) follows.

Assume (i). Then $A$ is absolutely Cauchy summable by Fact 3.1. Further, $A$ is topologically independent by Proposition 4.7 (ii) [3]. Theorem 2.4 yields that $A$ is also semi-basic. To show (ii) it remains to apply Proposition 3.3. □

The next statement is a direct corollary of Theorem 3.4.

**Corollary 3.5** *For every non-empty set $A$ the topological group $\mathbb{Z}^{(A)}$ has invariant linear span (which is isomorphic to $\mathbb{R}^{(A)}$).*

***Remark 3.6*** Corollary 3.5 can be viewed as a generalization of the well-known fact that *all topological vector spaces of the same finite dimension are isomorphic* (EVT I.14 Théorème 2 [2]). Indeed, if $A$ is a finite basis of a topological vector space $V$, then $A$ is topologically independent by Proposition 4.11 [3]. It follows then by Proposition 4.8 [3] that the Kalton map $\mathcal{K}_A$ is an embedding of topological groups. That is, the hull $\langle A\rangle$ is (isomorphic to) $\mathbb{Z}^A$. Hence $V = \mathrm{span}(A)$ is (isomorphic to) $\mathbb{R}^A$.

We end this paper with a theorem which provides a rich source of examples of topological groups that embed in topological vector spaces and do not have invariant linear spans.

Given a Tychonoff space $X$ the symbols $A(X)$, $L(X)$ and $V(X)$ stand for the free abelian topological group, the free locally convex topological vector space and the free topological vector space over $X$ respectively. We refer the reader to Gabriyelyan and Morris [4] for definitions of these notions.

**Theorem 3.7** *Let $X$ be a Tychonoff space. The topological group $A(X)$ canonically embeds in the topological vector spaces $L(X)$ and $V(X)$. If $X$ is sequential and non-discrete, then $A(X)$ does not have an invariant linear span.*

***Proof*** By Theorem 3 [9] the topological group $A(X)$ embeds in $L(X)$ and the linear span of $A(X)$ is $L(X)$. On the other hand, by Proposition 5.1 [4], it also embeds in $V(X)$ and its linear span in $V(X)$ is $V(X)$. Finally, if $L(X)$ and $V(X)$ are isomorphic as topological vector spaces and $X$ is sequential, then $X$ is discrete by Corollary 4.5 [4]. □

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