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Pencils and critical loci on normal surfaces

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Abstract
We study linear pencils of curves on normal surface singularities. Using the minimal good resolution of the pencil, we describe the topological type of generic elements of the pencil and characterize the behaviour of special elements. Furthermore, we show that the critical locus associated to the pencil is linked to the special elements. This gives a decomposition of the critical locus through the minimal good resolution and as a consequence, some information on the topological type of the critical locus.

Keywords Normal surface singularity · Pencil · Generic fiber · Special fiber · Critical locus

Mathematics Subject Classification 14B05 · 14J17 · 32S15 · 32S45 · 32S55

1 Introduction

Let (Z, z) be a complex analytic normal surface, and let $\pi : (Z, z) \to (\mathbb{C}^2, 0)$ be a finite complex analytic morphism germ. We choose coordinates (u, v) in $(\mathbb{C}^2, 0)$ and denote $f := u \circ \pi$ and $g := v \circ \pi$. We consider the meromorphic function $h := f/g$ defined in a punctured neighbourhood V of z in Z. It can be seen as a map

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$h : V \to \mathbb{CP}^1$ defined by $h(x) := (f(x) : g(x))$. For $w = (w_1 : w_2) \in \mathbb{CP}^1$, the closure of $h^{-1}(w)$ defines the curve $w_2 f - w_1 g = 0$ on the surface (Z, z). The set $\Lambda := \{w_2 f - w_1 g \mid (w_1 : w_2) \in \mathbb{CP}^1\}$ is the *pencil* defined by f and g. Let denote ϕ_w the element of the pencil Λ equal to $w_2 f - w_1 g$. Its (non reduced) zero locus, denoted by Φ_w, is called the *fiber* defined by ϕ_w. Assume $(Z, z) \subset (\mathbb{C}^n, 0)$, then the topological type of ϕ_w is the homeomorphism class of the pair $(B_\varepsilon \cap Z, B_\varepsilon \cap \phi_w^{-1}(0))$ where B_ε is the ball of $\mathbb{C}^n$ centered at z of radius ε small enough and the components of $\phi_w^{-1}(0)$ are pondered by the multiplicity of the irreducible components of ϕ_w. If ϕ_w and $\phi_{w'}$ have the same topological type, we also say that Φ_w and $\Phi_{w'}$ are topologically equivalent.

Such linear families of curves have been studied independently and through different approaches for (Z, z) equal to $(\mathbb{C}^2, 0)$ in [8,11,16]. In the general case (it means (Z, z) is the germ of a normal complex analytic surface which is not smooth anymore), Lê and Bondil give in [3] a definition of general elements of the pencil which are characterized by the minimality of their Milnor number. In [2] Bondil gives an algebraic μ-constant theorem for linear families of plane curves. Other results have been obtained in the case where π is the restriction to (Z, z) of a linear projection of $(\mathbb{C}^n, 0)$ onto $(\mathbb{C}^2, 0)$ (see [1,4,18]). At last, the topology of the morphism π has been studied in [13,14]. In [13], the authors define rational quotients which are topological invariants of (π, u, v) and give different ways to compute them. In [14], F. Michel presents another proof of the topological invariance of this set of rational numbers and besides, she gives a decomposition of the critical locus of π in bunches linked to the set of invariants.

Let $\rho : (X, E) \to (Z, z)$ be a *good resolution of the singularity* (Z, z). It is a resolution of the singularity (Z, z) such that the exceptional divisor is a union of smooth projective curves with normal crossings. In particular three irreducible components of the exceptional divisor do not meet at the same point. The lifting $h \circ \rho$ is a meromorphic function defined in a suitable neighbourhood of E in X but in a finite set of points.

A *good resolution ρ of the pencil* Λ is a good resolution of the singularity (Z, z) in which $h \circ \rho$ is a morphism. A good resolution of the pencil Λ is said to be *minimal* if and only if by the contraction of any rational component of self-intersection -1 of the exceptional divisor we do not obtain a good resolution of Λ anymore. We will see in Sect. 2 that there exists a unique minimal good resolution of Λ.

Let $\rho : (Y, E) \to (Z, z)$ be the minimal good resolution of the pencil Λ.

Definition An irreducible component E_α of E is called *dicritical* if the restriction of $\widehat{h} = h \circ \rho$ to E_α is not constant. Let denote $\mathcal{D}$ the union of the dicritical components.

Definition We say that a subset Δ of E is a *special zone* if it is, either the closure of a connected component of $\overline{E \setminus \mathcal{D}}$ or a critical point of the restriction of $\widehat{h}$ to $\mathcal{D}$. In the last case, P can be either a smooth point of $\mathcal{D}$ or a singular point of $\mathcal{D}$. Let $SZ(\Lambda)$ denote the (finite) set of special zones.

Notice that, if Δ is a special zone then $\widehat{h}_{|\Delta}$ is constant.

Definition The set of *special values* of Λ is constituted of the values $\widehat{h}(\Delta)$ for $\Delta \in SZ(\Lambda)$. A fiber associated to a special value is called a *special fiber* of Λ.

The other values of $\mathbb{CP}^1$ are called *generic values* for the pencil Λ. A fiber associated to a generic value is called a *generic fiber* of Λ.

Notice that, a change of the set of generators (f, g) of the pencil Λ is reflected by a linear change of coordinates in $\mathbb{CP}^1$ (a projectivity). Therefore, neither the definition of a good resolution of the pencil nor the set of special zones depend on the pair of functions of Λ chosen. Obviously, the concrete set of special values depends on the pair of functions (f, g).

We prove the following results.

Theorem 1 *Let w, w' be generic values for the pencil Λ, then the fibers Φ_w and $\Phi_{w'}$ have the same topological type.*

Moreover, if $e \in \mathbb{CP}^1$ is a special value for the pencil Λ, then the fibers Φ_w and Φ_e do not have the same topological type.

The above Definitions and Theorem generalize some of the results contained in [11] (see e.g. Theorem 4.1) where the authors study pencils on $\mathbb{C}^2$. Moreover, we prove the following characterization of the special fibers in terms of the minimal resolution, which extends to the case of normal surfaces the second item of Theorems 1, 2, 3 in [8] (there for pencils defined on $\mathbb{C}^2$).

Theorem 2 *Let $\rho : (Y, E) \to (Z, z)$ be the minimal good resolution of the pencil Λ, $\Delta \in SZ(\Lambda)$, and let $e \in \mathbb{CP}^1$. Then, the strict transform of Φ_e by ρ intersects Δ if and only if Φ_e is special and $\widehat{h}(\Delta) = e$.*

In a second part we are interested in understanding the behaviour of the critical locus of the map π. We denote by $I_z(\ , \ )$ the local intersection multiplicity at z (see Sect. 2.1). The following result generalizes the third item of Theorems 1, 2, 3 of [8].

Theorem 3 *Let $\rho : (Y, E) \to (Z, z)$ be the minimal good resolution of Λ. For each element $\Delta \in SZ(\Lambda)$ there exists an irreducible component of the critical locus $C(\pi)$ of π such that its strict transform by ρ intersects Δ.*

Moreover, for each branch Γ of $C(\pi)$ there exists $\Delta \in SZ(\Lambda)$ such that the strict transform of Γ by ρ intersects Δ and the value $e = \widehat{h}(\Delta)$ is the unique one that satisfies $I_z(\phi_e, \Gamma) > I_z(\phi_w, \Gamma)$ for all $w \neq e$.

A consequence of these results is Theorem 4:

Theorem 4 *Let Φ_e be a fiber of Λ. Then the three following properties are equivalent:*

1. *Φ_e is a special fiber of Λ.*
2. $I_z(\phi_e, C(\pi)) > \min_{\phi \in \Lambda} I_z(\phi, C(\pi))$.
3. $\mu(\phi_e) > \min_{\phi \in \Lambda} \mu(\phi)$.

Remark Let (f, g) be a pair of linear forms in such a way that π is a generic plane projection of (Z, z) in the sense of Teissier (see [20]). Then Λ is a pencil of hyperplane sections of (Z, z) and the generic members of the pencil are exactly the generic hyperplane sections among them. So, the pencil is resolved by the normalized blow-up $\psi : (X, E) \to (Z, z)$ of the maximal ideal and the minimal good resolution of Λ is just the minimal good resolution of (Z, z) which factors through ψ. In this case, the branches of the critical locus $C(\pi)$ (i.e. the polar curve) appear as curvettas in the Nash modification $N : (X', E') \to (Z, z)$ of the surface. Thus, Theorem 3 is strongly related with the configuration of the irreducible components of E and E'. In a more general context, if g is a generic linear form with respect to f, then the localization of the branches of the polar curve is related with the relative Nash transform of f in the same way.

The organization of the paper is as follows. In Sect. 2, once we have set some preliminary results, we construct and study the minimal good resolution of Λ. In Sect. 3, we prove Theorems 1 and 2 and in Sect. 4 we prove Theorems 3 and 4. To finish, in Sect. 5, we present some examples.

2 Preliminary results and notations

Let (Z, z) be a normal surface singularity and let $\rho : (X, E) \to (Z, z)$ be a good resolution of it. We denote $\{E_\alpha, \alpha \in G(\rho)\}$ the set of irreducible components of the exceptional divisor E. For $\alpha \in G(\rho)$ and for each holomorphic function $f : (Z, z) \to (\mathbb{C}, 0)$ let $\nu_\alpha(f)$ denote the vanishing order of $\overline{f} = f \circ \rho : X \to \mathbb{C}$ along the irreducible exceptional curve E_α (ν_α is just the divisorial valuation defined by E_α). The divisor $(\overline{f})$ defined by $\overline{f} = f \circ \rho$ on X could be written as

$$(\overline{f}) = (\widetilde{f}) + \sum_{\alpha \in G(\rho)} \nu_\alpha(f) E_\alpha \ ,$$

where the local part $(\widetilde{f})$ is the strict transform of the germ $\{f = 0\}$. For each $\beta \in G(\rho)$ one has the known Mumford formula (see [15]):

$$(\overline{f}) \cdot E_\beta = (\widetilde{f}) \cdot E_\beta + \sum_\alpha \nu_\alpha(f)(E_\alpha \cdot E_\beta) = 0 \ . \tag{1}$$

(Here "$\cdot$" stands for the intersection form on the smooth surface X). Notice that the intersection matrix $(E_\alpha \cdot E_\beta)$ is negative definite and so $\{\nu_\alpha(f)\}$ is the unique solution of the linear system defined by Eq. (1).

2.1 Intersection multiplicity

Let $C \subset (Z, z)$ be an irreducible germ of curve in (Z, z) and let $f \in \mathcal{O}_{Z,z}$ be a function. Let $\varphi : (\mathbb{C}, 0) \to (C, z)$ be a parametrization (uniformization) of (C, z), then we define the intersection multiplicity of $\{f = 0\} \subset Z$ and C at $z \in C$ as $I_z(f, C) = \mathrm{ord}_\tau(f \circ \varphi(\tau))$ (τ is the parameter in $\mathbb{C}$). Notice that, for C fixed, $I_z(-, C)$ is the valuation defined by the irreducible germ C. Obviously the above definition could be extended by linearity to define the intersection multiplicity of a f with a (local) divisor $\sum_{i=1}^k n_i C_i$ as $I_z(f, \sum n_i C_i) = \sum n_i I_z(f, C_i)$.

Let $\rho : (X, E) \to (Z, z)$ be a good resolution of the normal singularity (Z, z) and let $E = \bigcup_{\alpha \in G(\rho)} E_\alpha$ be the exceptional divisor. Let $\widetilde{C} := \overline{\rho^{-1}(C \setminus \{z\})}$ be the strict transform of C by ρ. Then (see [15])

$$I_z(f, C) = (\overline{f}) \cdot \widetilde{C} = (\widetilde{f}) \cdot \widetilde{C} + \sum_{\alpha \in G(\rho)} \nu_\alpha(f)(E_\alpha \cdot \widetilde{C}) .$$

Let us take now a good resolution ρ such that $\widetilde{C}$ is smooth and transversal to E at a smooth point P and also with the condition $(\widetilde{f}) \cdot \widetilde{C} = 0$. This resolution could be obtained by a finite number of blowing ups of points starting on (say) the minimal good resolution of (Z, z). Let $\alpha(C) \in G(\rho)$ be such that $E_{\alpha(C)}$ is the (unique) component of E with $\widetilde{C} \cap E_{\alpha(C)} = P$. Then one has $I_z(f, C) = \nu_{\alpha(C)}(f) = I_P(f \circ \rho, \widetilde{C})$. Here $I_P(-, -)$ denotes the usual local intersection multiplicity of two germs at the smooth local surface (X, P). Notice that $\widetilde{C}$ is a *curvetta* at the point $P \in E_{\alpha(C)}$ (it means $\widetilde{C}$ is an irreducible smooth curve germ transverse to $E_{\alpha(C)}$ at P), $\widetilde{C}$ is the normalization of C and $\rho|_{\widetilde{C}} : \widetilde{C} \to C$ is a uniformization of C.

Let f, g be analytic functions on (Z, z) and let $\Lambda = \langle f, g \rangle = \{\phi_w = w_2 f - w_1 g \mid w = (w_1 : w_2) \in \mathbb{CP}^1\}$ be the pencil of analytic functions defined by f and g. As in the case of plane branches (see [7]), one has the following easy and useful result:

Proposition 1 *Let $C \subset (Z, z)$ be an irreducible germ of curve. Then there exists a unique $w_0 \in \mathbb{CP}^1$ such that $I_z(\phi_w, C)$ is constant for all $w \in \mathbb{CP}^1 \setminus \{w_0\}$ and $I_z(\phi_{w_0}, C) > I_z(\phi_w, C)$.*

Proof The statement is trivial taking into account that $I_z(\phi, C)$ is the order of the series $\phi \circ \varphi(\tau)$. □

2.2 Resolution of pencils

Let $\pi = (f, g) : (Z, z) \to (\mathbb{C}^2, 0)$ be a finite complex analytic morphism germ, let $\Lambda = \langle f, g \rangle = \{w_2 f - w_1 g \mid w = (w_1 : w_2) \in \mathbb{CP}^1\}$ be the pencil of analytic functions defined by f and g and let $h = (f/g) : V \to \mathbb{CP}^1$ be the meromorphic function defined by f/g in a suitable punctured neighbourhood of $z \in Z$.

A *good* resolution of (f, g) is a good resolution $\rho : (X, E) \to (Z, z)$ of (Z, z) such that the (reduced) divisor $|(fg \circ \rho)^{-1}(0)|$ has normal crossings. Starting on the minimal good resolution of (Z, z) it can be produced by a sequence of blowing-ups of points in the corresponding smooth surface (resolving the singularities of the reduced total transform of the curve $\{fg = 0\}$).

Let $\rho : (X, E) \to (Z, z)$ be a good resolution of (Z, z) and E_α an irreducible component of E. The *Hironaka quotient* of (f, g) on E_α is the following rational number:

$$q(E_\alpha) := \frac{\nu_\alpha(f)}{\nu_\alpha(g)}.$$

If $q(E_\alpha) > 1$ (resp. $q(E_\alpha) < 1$) then the component E_α belongs to the zero divisor (resp. pole divisor) of $h \circ \rho$. Note that if E_α is a dicritical component of E then $q(E_\alpha) = 1$. However, there may exist irreducible components E_α of E which are not dicritical and for which $q(E_\alpha) = 1$, namely all the components for which the restriction of $h \circ \rho$ on E_α is constant (and so is not dicritical) and it's neither zero nor infinity.

Proposition 2 *There exists a (unique) minimal good resolution of* Λ.

Proof Let $\rho' : (Y', E') \to (Z, z)$ be the minimal good resolution of (f, g). The indetermination points of $h \circ \rho'$ are the intersection points of irreducible components E_α and E_β of the total transform $|(fg \circ \rho')^{-1}(0)|$ for which one has $q(E_\alpha) > 1$ and $q(E_\beta) < 1$. Here, one of the components, E_α or E_β, is allowed to be the strict transform $\widetilde{\xi}$ of a branch ξ of $\{f = 0\}$ (in such a case we put $q(\widetilde{\xi}) > 1$) or $\{g = 0\}$ (respectively $q(\widetilde{\xi}) < 1$). Let P be such an indetermination point. By blowing-up at P one creates a divisor E_η of genus 0 and one has that $\nu_\eta(f) = \nu_\alpha(f) + \nu_\beta(f)$ and $\nu_\eta(g) = \nu_\alpha(g) + \nu_\beta(g)$. (If E_β is a branch ξ of $\{f = 0\}$ of multiplicity r, we have $\nu_\beta(f) = r$ and $\nu_\beta(g) = 0$. We use similar conventions for the case in which E_β is a branch of $\{g = 0\}$.) If $q(E_\eta) = 1$, then neither $E_\alpha \cap E_\eta$ nor $E_\beta \cap E_\eta$ is an indetermination point and moreover E_η is a dicritical divisor. Else, if $q(E_\eta) > 1$ (resp. $q(E_\eta) < 1$) then $E_\beta \cap E_\eta$ (resp. $E_\alpha \cap E_\eta$) is an indetermination point.

As $q(E_\alpha) > 1$ and $q(E_\beta) < 1$ we have $q(E_\alpha) > q(E_\eta) > q(E_\beta)$. So, by iterating the process, after a finite number of blow-ups there does not subsist indetermination points and so we have constructed a good resolution $\rho'' : (Y'', E'') \to (Z, z)$ of Λ.

Now, to obtain a minimal good resolution of Λ, we have to contract some rational components of self-intersection -1 of the exceptional divisor (see Theorem 5.9 of [9]). By the above construction the new components (specially the last one which is dicritical and with self-intersection -1) can not be contracted because in such a case we have an indetermination point. As a consequence a minimal good resolution of Λ is obtained from ρ'' by iterated contractions of the rational component of self-intersection -1 of the exceptional divisor which are not dicritical. Uniqueness follows as in the case of the usual minimal resolution (see for example [5] th. 6.2 p. 86). □

Let consider $\rho : (Y, E) \to (Z, z)$ the minimal good resolution of the pencil Λ and $\widehat{h} = h \circ \rho$. For $w \in \mathbb{CP}^1$ let $\widehat{h}^{-1}(w) = \widetilde{\Phi_w}$ be the strict transform of the fiber Φ_w. For a dicritical component D of E, we will denote by $\deg(\widehat{h}_{|D})$ the degree of the restriction of $\widehat{h}$ to D, $\widehat{h}_{|D} : D \to \mathbb{CP}^1$. Let recall that $\mathcal{D}$ denotes the union of the dicritical components of E.

Proposition 3 *Let w be a generic value for the pencil Λ, then:*

(a) *The resolution ρ is a good resolution of ϕ_w.*
(b) *$\widetilde{\Phi_w}$ intersects E only at smooth points of $\mathcal{D}$.*
(c) *If $D \subset \mathcal{D}$ is a dicritical component, then the number of intersection points of $\widetilde{\Phi_w}$ and D is equal to deg$(\widehat{h}_{|D})$.*

Moreover, the minimal good resolution of Λ is the minimal good resolution of any pair of generic elements of Λ.

Proof By definition of a generic value, $\widetilde{\Phi_w}$ meets the exceptional divisor E only at smooth points of $\mathcal{D}$. Let $D \subset \mathcal{D}$ be a dicritical component and P a point of $\widetilde{\Phi_w} \cap D$. Then, as P is not a critical point for $\widehat{h}$, $\widetilde{\Phi_w}$ is smooth and transversal to D at P. This implies also that

$$\deg\,(\widehat{h}_{|D}) = \sum_{P \in D} I_P(\widetilde{\phi_w}, D)\ .$$

So, one has $\deg(\widehat{h}_{|D}) = \#(\widetilde{\Phi_w} \cap D)$.

Now, let w' be another generic value. Notice that the strict transforms of $\widetilde{\Phi_w}$ and $\widetilde{\Phi_{w'}}$ intersect in the same number of points each dicritical divisor D, so both fibers have the same number of branches, just $\sum_{D \in \mathcal{D}} \deg(\widehat{h}_{|D})$. Moreover, $\widetilde{\Phi_w}$ and $\widetilde{\Phi_{w'}}$ do not intersect $\mathcal{D}$ at the same points because $\widehat{h}$ is a morphism. As a consequence the minimal good resolution of Λ is a good resolution of any pair of generic fibers. It leaves to show that it is the minimal one.

In the minimal good resolution of Λ all the components of the exceptional divisor that can be contracted (i.e. those with self-intersection -1) are dicritical (see the proof of Proposition 2). But contracting a dicritical component we create an indetermination point. Consequently, the minimal good resolution of Λ is the minimal good resolution of the pair $(\phi_w, \phi_{w'})$. □

2.3 Hironaka quotients

In 2.2 we have defined the Hironaka quotient of (f, g) on an irreducible component E_α of the exceptional divisor of a good resolution of (Z, z). In the same way we can define the Hironaka quotient of $(\phi_w, \phi_{w'})$ on E_α for any pair $(\phi_w, \phi_{w'})$ of elements of $\Lambda = \langle f, g \rangle$. It is the rational number

$$q^w_{w'}(E_\alpha) := \frac{\nu_\alpha(\phi_w)}{\nu_\alpha(\phi_{w'})}.$$

In this way $q(E_\alpha) = q^0_\infty(E_\alpha)$ (here $0 = (0 : 1) \in \mathbb{CP}^1$, $\infty = (1 : 0) \in \mathbb{CP}^1$) but to simplify the notations we will still write $q(E_\alpha)$ for the Hironaka quotient of (f, g).

Notice that an irreducible component E_α of E is dicritical if and only if $q^w_{w'}(E_\alpha) = 1$ for any pair (w, w') of elements of $\mathbb{CP}^1$. Indeed, if for some (w, w') we have $q^w_{w'}(E_\alpha) > 1$ (resp. $q^w_{w'}(E_\alpha) < 1$) then E_α lies in the zero locus of ϕ_w (resp. $\phi_{w'}$). So, E_α is not dicritical. Conversely, if E_α is not dicritical then there exists w such that $q^w_{w'}(E_\alpha) > 1$, for any $w' \neq w$.

As a consequence of Proposition 3 we have the following result:

Corollary 1 *The Hironaka quotient of any pair of generic elements of Λ associated to any irreducible component of the exceptional divisor of the minimal good resolution of Λ is equal to one.*

Proof Let $w, w' \in \mathbb{CP}^1$ be a pair of generic values of Λ. If $D \subset \mathcal{D}$ is a dicritical component, then $(\widetilde{\phi_w}) \cdot D = (\widetilde{\phi_{w'}}) \cdot D = \deg(\widehat{h}_{|D})$ (see Proposition 3). On the other hand, if E_β is a non-dicritical component of E then one has $(\widetilde{\phi_w})\cdot E_\beta = (\widetilde{\phi_{w'}})\cdot E_\beta = 0$. Now, the system of linear equations given by the Mumford formula (1) at the beginning of Sect. 2 is the same for ϕ_w and $\phi_{w'}$ and so the solutions $\{\nu_\alpha(\phi_w)\}$ and $\{\nu_\alpha(\phi_{w'})\}$ are also the same. Thus, $\nu_\alpha(\phi_w) = \nu_\alpha(\phi_{w'})$ and $q^w_{w'}(E_\alpha) = 1$ for any $\alpha \in G(\rho)$. □

Remark Let E_α be a non-dicritical component of the exceptional divisor of the minimal good resolution of the pencil Λ and let $C \subset (Z, z)$ be an irreducible curve such that its strict transform $\widetilde{C}$ is a curvetta at the point P of E_α. Assume that $P = \widetilde{C} \cap E_\alpha$ does not belong to the strict transform of any fiber Φ of Λ. Then, by Proposition 1, there exists a unique $e \in \mathbb{CP}^1$ such that $I_z(\phi_w, C) = \nu_\alpha(\phi_w)$ is constant for all $w \in \mathbb{CP}^1 \setminus \{e\}$ and $\nu_\alpha(\phi_e) > \nu_\alpha(\phi_w)$. Moreover, the above value $e \in \mathbb{CP}^1$ must be a special value of Λ.

Let $b : (Z_I, E_I) \to (Z, z)$ be the normalized blow-up of the ideal $I = (f, g)$. In [2,3] an element $\phi \in I$ is defined to be *general* if it is *superficial* (it means that its divisorial value is minimal for each irreducible component of E_I) and the strict transform of $\Phi = \{\phi = 0\}$ by b is smooth and transverse to the exceptional divisor at smooth points. (See definition 2.1 of [2].) Proposition 2.2 of [2] allows to characterize general elements in terms of any good resolution of Z_I, in particular one can use a good resolution $\rho : (X, E) \to (Z, z)$ of the pencil Λ. In these terms one has that $\phi \in \Lambda$ is general if for each $\alpha \in G(\rho)$

$$\nu_\alpha(\phi) = \nu_\alpha(I) = \min_{\psi \in I}\{\nu_\alpha(\psi)\} = \min_{\psi \in \Lambda}\{\nu_\alpha(\psi)\}$$

and, moreover, the strict transform of Φ by ρ is smooth and transversal to E. By using the definition of the Milnor number of a germ of curve given in [6], from Theorems 1 and 2 of [3] one has that $\phi \in \Lambda$ is general if and only if

$$\mu(\phi) = \mu(I) := \min_{\psi \in I}\{\mu(\psi)\} = \min_{\psi \in \Lambda}\{\mu(\psi)\} .$$

Using Proposition 3 and the above results about Hironaka quotients we have that Φ_w is a generic fiber if and only if ϕ_w is general. Moreover, one has also that $\mu(\phi_w) = \min_{\phi \in \Lambda}\{\mu(\phi)\}$ if and only if ϕ_w is generic, and therefore $\mu(\phi_{w_0}) > \min_{\phi \in \Lambda}\{\mu(\phi)\}$ if and only if w_0 is a special value of Λ. This is the equivalence of 1 and 3 in Theorem 4.

3 Topology of special fibers

3.1 Dual graph and topology

Assume $(Z, z) \subset (\mathbb{C}^n, 0)$ and let $M := Z \cap S^{2n-1}_\varepsilon$ where S^{2n-1}_ε is the boundary of the small ball B_ε of radius ε of $\mathbb{C}^n$ centered at z. The manifold M is called the *link* (see [15]) of the singularity (Z, z).

Let ϕ_w be an element of Λ and $K_{\phi_w} := \phi_w^{-1}(0) \cap M$. The *multilink* $\mathbf{K}_{\phi_w}$ *of* ϕ_w is the oriented link K_{ϕ_w} weighted by the multiplicities of the irreducible components of ϕ_w. The topological type of ϕ_w is given by the isotopy class of $\mathbf{K}_{\phi_w}$ (see [14] Sect. 5).

Let $\rho_w : (X, E) \to (Z, z)$ be the minimal good resolution of (Z, z) such that the divisor $(\phi_w \circ \rho_w)$ has normal crossings. From Neumann (see [17]), the topology of ϕ_w determines the minimal good resolution ρ_w, where the irreducible components of the strict transform of Φ_w by ρ_w are weighted with their multiplicity and the irreducible components of the exceptional divisor by their self-intersection and genus. Conversely, (see [13,21]) as the intersection matrix $(E_\alpha \cdot E_\beta)_{\alpha,\beta \in G(\rho_w)}$ is negative definite (so invertible) this implies that the linear system associated to the Mumford formulas (1) admits a unique solution, namely the set $\{\nu_\alpha(\phi_w), \alpha \in G(\rho_w)\}$. As a direct consequence, the divisor $(\widetilde{\phi_w})$ (see Sect. 2) is completely determined on X from the set $\{(\widetilde{\phi_w}) \cdot E_\alpha \mid \alpha \in G(\rho_w)\}$ and so from the minimal good resolution ρ_w.

Let $\rho : (X, E) \to (Z, z)$ be a good resolution of the normal surface singularity (Z, z) and let $E = \bigcup_{\alpha \in G(\rho)} E_\alpha$ be its exceptional divisor. It is useful to encode the information of the resolution ρ by means of the so called *dual graph* of ρ. The set of vertices of this graph is the set $G(\rho)$, each vertex α is weighted by $(\alpha, E_\alpha^2, g(E_\alpha))$ where E_α^2 is the self-intersection of E_α, and $g(E_\alpha)$ its genus. An intersection point between E_α and E_β is represented by an edge linking the vertices α and β.

If we take ρ as a good resolution of the local curve $C = \sum_{i=1}^{\ell} n_i C_i$ (in particular if $C = \{\varphi = 0\}$ for some function φ) one adds an arrow for each irreducible component C_i of C weighted by the multiplicity n_i. In the case in which we deal with a good resolution of a pair of functions (f, g), in the graph of $\{fg = 0\}$ one marks with different colors the arrows corresponding to branches of $\{f = 0\}$ and those of $\{g = 0\}$ (another possibility is to use different kinds of marks, say for example arrows for f and stars for g). The sharp extremities of the arrows are considered as somekind of special vertices of the graph. The notations $\mathcal{G}(\rho)$, $\mathcal{G}(\rho, \varphi)$ and $\mathcal{G}(\rho, f, g)$ will be used for the dual graph in each situation. Note that a good resolution ρ of the pencil $\Lambda = \langle f, g\rangle$ is encoded by the dual graph $\mathcal{G}(\rho, \phi_w, \phi_{w'})$ for a pair of generic functions $(\phi_w, \phi_{w'})$.

Following Neumann, one has:

Statement The fibers Φ_w and $\Phi_{w'}$ are topologically equivalent if and only if the graphs $\mathcal{G}(\rho_w, \phi_w)$ and $\mathcal{G}(\rho_{w'}, \phi_{w'})$ are the same.

Let $\rho : (X, E) \to (Z, z)$ be a good resolution of (f, g) and let E_α be an irreducible component of E. Let $\overset{\circ}{E}_\alpha$ denote the set of smooth points of E_α in the reduced total transform $|(fg \circ \rho)^{-1}(0)|$. An irreducible component E_α of E (or its corresponding vertex α in $\mathcal{G}(\rho, f, g)$) is a *nodal component* if $\chi(\overset{\circ}{E}_\alpha) < 0$, where χ is the Euler characteristic. Note that $\chi(\overset{\circ}{E}_\alpha)$ is equal to $2 - 2g(E_\alpha) - v(\alpha)$, where $v(\alpha)$ is the number of intersection points of E_α with other components of the total transform of $\{fg = 0\}$. Thus, the nodal components are all the rational ones that meet at least three other components of the total transform and all the non-rational irreducible components. We say that the irreducible component E_α (or its corresponding vertex in $\mathcal{G}(\rho, f, g)$) is an *end* when $\chi(\overset{\circ}{E}_\alpha) = 1$. Obviously E_α is an end if and only if E_α is rational and meets exactly one other component of the exceptional divisor.

The *neighbouring-set* of E_α in X is the set constituted of E_α union with the irreducible components of the exceptional divisor and of the strict transform of $\{fg = 0\}$ that intersect E_α. We denote it $\mathsf{Ng}(E_\alpha)$, thus $\mathsf{Ng}(E_\alpha) = \bigcup_{E_\beta \cap E_\alpha \neq \emptyset} E_\beta$.

A *chain* of length r, $r \geq 3$, in E is a finite set of irreducible components $\{E_{\alpha_1}, \ldots, E_{\alpha_r}\}$ such that, for $2 \leq i \leq r-1$:

$$\chi(\overset{\circ}{E}_{\alpha_i}) = 0 \text{ and } \mathsf{Ng}(E_{\alpha_i}) = E_{\alpha_{i-1}} \cup E_{\alpha_i} \cup E_{\alpha_{i+1}} .$$

Notice that $\bigcup_{i=1}^{r} E_{\alpha_i}$ is connected and the strict transform of $\{fg = 0\}$ does not intersect $E_{\alpha_2} \cup \ldots \cup E_{\alpha_{r-1}}$.

A *cycle* of length r, $r \geq 3$, in E is a chain such that $\mathsf{Ng}(E_{\alpha_r}) = E_{\alpha_{r-1}} \cup E_{\alpha_r} \cup E_{\alpha_1}$. A *cycle* of length 2 in E is a connected part of E constituted by two irreducible components $E_{\alpha_1}, E_{\alpha_2}$ such that $\chi(\overset{\circ}{E}_{\alpha_2}) = 0$ and $\mathsf{Ng}(E_{\alpha_2}) = E_{\alpha_1} \cup E_{\alpha_2}$.

Remark Above definitions and terminology could also be stated in terms of the dual graph $\mathcal{G}(\rho, f, g)$ and, in some sense, the names used are more natural there. Here we have chosen to do so in terms of the exceptional divisor for reasons of simplicity in the next proofs, however the dual graph provides an essential and synthetic guide to visualize all the elements involved in a simple way (see e.g. the examples in Sect. 5).

Nodal components are called *rupture components* in several papers as an analogy to the terminology used in the case of plane curves. This is the case in [13] which is an essential reference in Sect. 4.

Proposition 4 *Let $\rho : (X, E) \to (Z, z)$ be a good resolution of (f, g). Let E_α be an irreducible component of the exceptional divisor such that the strict transform of $\{fg = 0\}$ does not intersect E_α. Then there exists $E_\beta \subset \mathsf{Ng}(E_\alpha)$ such that $q(E_\beta) > q(E_\alpha)$ if and only if there exists $E_\gamma \subset \mathsf{Ng}(E_\alpha)$ such that $q(E_\gamma) < q(E_\alpha)$.*

Moreover, if $\{E_{\alpha_1}, \ldots, E_{\alpha_r}\}$, $r \geq 3$, is a chain, then one of the following facts is true:

- *$q(E_{\alpha_i}) < q(E_{\alpha_{i+1}})$ for $1 \leq i \leq r-1$.*
- *$q(E_{\alpha_i}) > q(E_{\alpha_{i+1}})$ for $1 \leq i \leq r-1$.*
- *$q(E_{\alpha_i})$ is constant for $1 \leq i \leq r$.*

In particular, if E_{α_r} is an end, then $q(E_{\alpha_i})$ is constant for $1 \leq i \leq r$ and if $\{E_{\alpha_1}, \ldots, E_{\alpha_r}\}$ is a cycle, then $q(E_{\alpha_i})$ is constant for $1 \leq i \leq r$.

The result previously stated is a direct generalization of Proposition 1 and Corollary 1 of [8] and the proof is similar. However, here the framework and the notations are slightly different. As Proposition 4 is also a key result in the remaining proofs of this paper, we will give an outline of the proof here.

Proof By using Eq. (1) for the functions f and g with respect to the same divisor E_α we have:

$$\begin{aligned} \sum_{E_\eta \subset \mathsf{Ng}(E_\alpha), \eta \neq \alpha} \nu_\eta(f)(E_\eta \cdot E_\alpha) &= (-E_\alpha^2)\, \nu_\alpha(f) \\ \sum_{E_\eta \subset \mathsf{Ng}(E_\alpha), \eta \neq \alpha} \nu_\eta(g)(E_\eta \cdot E_\alpha) &= (-E_\alpha^2)\, \nu_\alpha(g) . \end{aligned} \tag{2}$$

Let suppose that $q(E_\eta) \geq q(E_\alpha)$ for each $E_\eta \subset \mathsf{Ng}(E_\alpha)$. This condition is equivalent to: $(E_\eta \cdot E_\alpha)\nu_\eta(f)\nu_\alpha(g) \geq (E_\eta \cdot E_\alpha)\nu_\alpha(f)\nu_\eta(g)$. As $q(E_\beta) > q(E_\alpha)$, we obtain:

$$\nu_\alpha(g) \sum_{E_\eta \subset \mathsf{Ng}(E_\alpha), \eta \neq \alpha} (E_\eta \cdot E_\alpha)\nu_\eta(f) > \nu_\alpha(f) \sum_{E_\eta \subset \mathsf{Ng}(E_\alpha), \eta \neq \alpha} (E_\eta \cdot E_\alpha)\nu_\eta(g) .$$

However, by using Eq. (2), one can see that both sides of the above inequality are equal to $(-E_\alpha^2)\nu_\alpha(f)\nu_\alpha(g)$ and thus, we reach a contradiction. □

The other statements of the Proposition are easy consequences of this result.

3.2 Proof of Theorems 1 and 2

Let $\rho : (Y, E) \to (Z, z)$ be the minimal good resolution of the pencil Λ, $\widehat{h} = h \circ \rho$. Proposition 3, together with the above Statement, gives:

Corollary 2 *Let $w, w' \in \mathbb{CP}^1$ be generic values of Λ. Then, the fibers Φ_w and $\Phi_{w'}$ are topologically equivalent.*

Thus, in order to finish the proof of Theorem 1, it only remains to show that a special fiber Φ_e is not topologically equivalent to a generic one.

Let Δ be an element of $SZ(\Lambda)$ and $e = \widehat{h}(\Delta)$. We denote by Φ_e the fiber of Λ associated to e and by $\widetilde{\Phi_e}$ its strict transform by ρ. The remaining part of Theorems 1 and 2 are direct consequences of the three following lemmas, one for each possibility of Δ. Unless otherwise specified, w denotes a generic value of Λ.

Lemma 1 *If e is the special value of Λ associated to a connected component Δ of $\overline{E \setminus \mathcal{D}}$, then the strict transform of Φ_e by ρ intersects Δ.*

Proof Let assume that $\widetilde{\Phi_e} \cap \Delta = \emptyset$. Notice that, if we enlarge ρ (by additional blow-ups) in order to have a good resolution of Φ_e and Λ, then the connected set Δ remains unchanged. So, we can assume that ρ is also a resolution of Φ_e.

For any component $E_\alpha \subset \Delta$, we have $q_w^e(E_\alpha) > 1$. Let E_β be an irreducible component of Δ such that $q_w^e(E_\beta) \geq q_w^e(E_\alpha)$ for any $E_\alpha \subset \Delta$ and let Δ' be the maximal connected subset of E such that $E_\beta \subset \Delta'$ and $(q_w^e)_{|\Delta'}$ is constant and equal to $q_w^e(E_\beta)$. Notice that $E_\beta \subset \Delta' \subset \Delta$ because $q_w^e(E_\alpha) = 1$ for any E_α such that $E_\alpha \cap \Delta \neq \emptyset$ and $E_\alpha \not\subset \Delta$ (in fact such an E_α is a dicritical divisor). Now, let $E_\gamma \subset \Delta'$ be such that $\mathsf{Ng}(E_\gamma) \not\subset \Delta'$. Let $E_\alpha \subset \mathsf{Ng}(E_\gamma)$ be such that $E_\alpha \not\subset \Delta'$ then, one has $q_w^e(E_\beta) > q_w^e(E_\alpha) > 1$ if $E_\alpha \subset \Delta$ and $q_w^e(E_\beta) > q_w^e(E_\alpha) = 1$ otherwise. However, being $\Delta' \subset \Delta$, this contradicts Proposition 4 for the irreducible component E_γ.

As a consequence $\widetilde{\Phi_e} \cap \Delta \neq \emptyset$ and so Φ_e can not be topologically equivalent to Φ_w for a generic value w. □

Lemma 2 *If e is the special value of Λ associated to a smooth point P of the dicritical component $D \subset \mathcal{D}$ which is a critical point of $\widehat{h}$, then the strict transform $\widetilde{\Phi_e}$ of Φ_e by ρ intersects D at P. Moreover, $\widetilde{\Phi_e}$ cannot be smooth and transversal to D at P.*

Proof Blowing-up at P we create a divisor E_α. As $\widehat{h}(P) = e$, then P lies in the zero locus of $(\phi_e/\phi_w) \circ \rho$ for any value $w \neq e$ and so we have $q_w^e(E_\alpha) > 1$. Moreover, as

D is a dicritical component, then $q_w^e(D) = 1$. Now, if we assume that $P \notin \widetilde{\Phi_e}$ then one can use Proposition 4 for the new divisor E_α and reach a contradiction.

Assume that $\widetilde{\Phi_e}$ is smooth and transversal to D at the point P. In this case we can choose local coordinates $\{u, v\}$ on Y at P in such a way that $\widetilde{\Phi_e} = \{v = 0\}$ and $D = \{u = 0\}$ on a neighbourhood V of P. So, the function $\phi_e \circ \rho$ is $u^a v$ on V and, for a generic value w, $\phi_w \circ \rho$ is $u^b \eta(u, v)$ for a unit η. Note that $a = \nu_D(\phi_e) = \nu_D(\phi_w) = b$, being D dicritical, and so the expression of $\widehat{h}$ at P is $v\eta(u, v)^{-1}$. Now, the restriction of $(\phi_e/\phi_w) \circ \rho$ to D is given locally at P as the map $v \mapsto v$. Thus, the point P is not a critical (ramified) point of $\widehat{h}_{|D} : D \to \mathbb{CP}^1$.

As a consequence, $\widetilde{\Phi_e}$ is either not smooth or tangent to D at P, in particular Φ_e can not be topologically equivalent to Φ_w for a generic value w. □

Lemma 3 *If e is the special value of Λ associated to an intersection point P between two irreducible components of $\mathcal{D}$, then the strict transform of Φ_e by ρ intersects $\mathcal{D}$ at P.*

Proof Let $P = E_{\alpha_1} \cap E_{\alpha_2}$ such that E_{α_1} and E_{α_2} are dicritical components. Let us assume that $P \notin \widetilde{\Phi_e}$. Blowing-up at P we create a divisor E_α satisfying $\{E_{\alpha_1} \cup E_\alpha \cup E_{\alpha_2}\} = \mathsf{Ng}(E_\alpha)$. As $q_w^e(E_{\alpha_1}) = q_w^e(E_{\alpha_2}) = 1$ and $q_w^e(E_\alpha) > 1$, we reach a contradiction with Proposition 4.

As a consequence, Φ_e is not resolved by ρ and so could not be topologically equivalent to a generic fiber Φ_w. □

4 Behaviour of the critical locus

Let $\pi = (f, g) : (Z, z) \to (\mathbb{C}^2, 0)$ be a finite complex analytic morphism. Following Teissier [19], the critical locus of π is the analytic subspace defined by the zeroth Fitting ideal $F_0(\Omega_\pi)$ of the module Ω_π of relative differentials. The critical locus can have embedded components, however, we are only interested in the components of dimension one. We denote by $C(\pi)$ the divisorial part of the critical set with its non-reduced structure (i.e. with its multiplicity) and we refer to $C(\pi)$ as the critical locus of π. Note that, out of the singular point $z \in Z$, $C(\pi)$ is defined by the vanishing of the jacobian determinant $J(f, g)$. For a different pair of functions $f', g' \in \Lambda$, one has $J(f', g') = aJ(f, g)$ for some $a \in \mathbb{C}^*$ (a is just the determinant of the linear change between both basis). As a consequence, $C(\pi)$ depends on Λ and not on the pair of functions of Λ fixed to define the corresponding finite morphism. If we denote by Γ_k, (resp. n_k), $k = 1, \ldots, \ell$, the irreducible components (branches) of $C(\pi)$ (resp. their multiplicity) then $C(\pi)$ is the local (Weil) divisor $C(\pi) = \sum_{k=1}^{\ell} n_k \Gamma_k$.

Before proving Theorems 3 and 4, let first recall two results from [13,14].

Let $(\phi_w, \phi_{w'})$ be any pair of germs of the pencil Λ, let $\rho' : (Y', E') \to (Z, z)$ be the minimal good resolution of $(\phi_w, \phi_{w'})$, and let $\Gamma(w, w') := \bigcup_{k \in K} \Gamma_k$, $K \subset \{1, \ldots, \ell\}$ denote the curve consisting of the irreducible components of $C(\pi)$ which are not components of $\{\phi_w \phi_{w'} = 0\}$. Let $Z_r \subset E'$ be the set of points $P \in E'$ such that, for any irreducible exceptional component E_α with $P \in E_\alpha$, one has $q_{w'}^w(E_\alpha) = r$. The set Z_r is called the *r-zone* of E'. A connected component of Z_r which contains at least one nodal component is called an *r-nodal zone*. Then from [13] we have:

Theorem A *The set* $\left\{\frac{I_z(\phi_w, \Gamma_k)}{I_z(\phi_{w'}, \Gamma_k)}, k \in K\right\}$ *is equal to the set of Hironaka quotients of* $(\phi_w, \phi_{w'})$ *on the nodal components of* E'*.*

In [14] a repartition in bunches of the branches of $\Gamma(w, w')$ is given as follows:

Theorem B *The intersection of the strict transform of* $\Gamma(w, w')$ *with a connected component of* Z_r *is not empty if and only if it is an* r*-nodal zone. Moreover, if* Γ *is an irreducible component of* $\Gamma(w, w')$ *whose strict transform intersects an* r*-nodal zone, then* $\frac{I_z(\phi_w, \Gamma)}{I_z(\phi_{w'}, \Gamma)} = r$*.*

Remark Notice that a direct consequence of these theorems is the following: let P be a point that does not belong to a Z_r for any r, then the strict transform of $\Gamma(w, w')$ by ρ' does not go through P. Indeed, in this case P is the intersection point between two irreducible components of E' with distinct Hironaka quotient. Let suppose that there exists an irreducible component Γ of $\Gamma(w, w')$, such that its strict transform intersects the exceptional diviosr at P. Then, blowing-up at P til the strict transform of Γ intersects the exceptional divisor at a smooth point, we obtain, either $\frac{I_z(\phi_w, \Gamma)}{I_z(\phi_{w'}, \Gamma)}$ does not belong to the set of Hironaka quotients of $(\phi_w, \phi_{w'})$ on the nodal components of E', which contradicts Theroem A, or $\frac{I_z(\phi_w, \Gamma)}{I_z(\phi_{w'}, \Gamma)}$ is equal to a Hironaka quotient of $(\phi_w, \phi_{w'})$ but the strict transform of Γ intersects the exceptional divisor in a zone which is not an r-nodal zone. This contradicts Theorem B.

The following Lemma treats the case of irreducible components of the critical locus which are also components of a fiber.

Lemma 4 *Let* $\Phi_e = \sum_{i=1}^{t} r_i \xi_i$*, where* $\xi_1, \ldots, \xi_t$ *are the irreducible components of the fiber* Φ_e*,* $\xi_i \neq \xi_j$ *if* $i \neq j$*. Then, for* $i = 1, \ldots, t$*, one has* $r_i > 1$ *if and only if* ξ_i *is an irreducible component of* $C(\pi)$*.*

Proof Let ξ be an irreducible component of a fiber Φ_e. Let $w \in \mathbb{CP}^1$ be a generic value and let $\rho' : (Y', E') \to (Z, z)$ be the minimal good resolution of (ϕ_e, ϕ_w). Let $\widetilde{\xi}$ be the strict transform of ξ by ρ' and let P be the intersection point of $\widetilde{\xi}$ with the exceptional divisor E', $P = \widetilde{\xi} \cap E_\alpha = \widetilde{\xi} \cap E'$. We can choose a local system of coordinates (u, v) in a neighbourhood $U \subset Y'$ of $P = (0, 0)$ such that $u = 0$ is an equation of E_α, $v = 0$ is an equation of $\widetilde{\xi}$ and the equation of the total transform $\overline{\Phi_e}$ of Φ_e at P is $u^a v^k$, where $a = \nu_\alpha(\phi_e)$ and k is the multiplicity of the branch ξ in Φ_e. On the other hand, the equation of $\overline{\Phi_w}$ at P is $u^b \eta(u, v)$, with $b = \nu_\alpha(\phi_w)$ and $\eta(u, v)$ being a unit. So, the expression of $(\phi_e/\phi_w) \circ \rho$ at $P \in U$ is $u^{a-b} v^k (\eta(u, v))^{-1}$.

Let first suppose that ξ belongs to $C(\pi)$. Let Q be a point of $\widetilde{\xi} \setminus \{P\}$, say Q has local coordinates $(u_0, 0)$. The restriction of $\widehat{h}$ to a small disc $D(u_0, 0)$ centered at Q in $u = u_0$ is $v^k \eta_0(u_0, v)$ with $\eta_0(u_0, v)$ a unit and $k > 1$ because ξ lies in the ramification locus. So, as k is the multiplicity of ξ in Φ_e, ξ is non-reduced.

Conversely, if ξ is an irreducible component of a fiber Φ_e which is not reduced, the multiplicity k of ξ in Φ_e satisfies $k > 1$. Moreover the local equation of $\widehat{h}$ on any small disc $D(t,0)$ centered at any point of local coordinates $(t,0)$ in U is $v^k\eta(t,v)$ with $\eta(t,v)$ a unit. As $k>1$, each point $(t,0)$ is a ramification point and so $\widetilde{\xi}$ lies in the ramification locus. Hence, ξ is an irreducible component of $C(\pi)$. □

4.1 Proof of Theorem 3 for singular points of $\mathcal{D}$ and critical points of the restriction of $\widehat{h}$ to $\mathcal{D}$

Hereafter, let $\rho:(Y,E)\to(Z,z)$ denote the minimal good resolution of Λ and $\mathcal{D}$ the dicritical locus of E.

Proposition 5 *Let $P\in\mathcal{D}$ be such that $P\notin\overline{E\setminus\mathcal{D}}$. Then, P is a singular point of $\mathcal{D}$ or a critical point of $\widehat{h}_{|\mathcal{D}}$ if and only if there exists an irreducible component Γ of $C(\pi)$ whose strict transform intersects $\mathcal{D}$ at P. Moreover, if $\widehat{h}(P)=e$, then $I_z(\phi_e,\Gamma)>I_z(\phi_w,\Gamma)$ for any $w\in\mathbb{CP}^1$, $w\neq e$.*

Proof Let assume that there exists an irreducible component Γ of $C(\pi)$ whose strict transform intersects $\mathcal{D}$ at P and let $e=\widehat{h}(P)$. If Γ is a branch of Φ_e then, by the above Lemma, it must be a multiple component and, as a consequence, the point P is a critical point of $\widehat{h}|\mathcal{D}$.

So, let consider the case in which Γ is not a branch of Φ_e and assume that P is not a singular point of $\mathcal{D}$, i.e. P is a smooth point of $\mathcal{D}$ in the exceptional divisor E. Let D denote the irreducible component of $\mathcal{D}$ such that $P\in D$.

If the strict transform $\widetilde{\Phi}_e$ of Φ_e at P has normal crossings with $\mathcal{D}$, then there exists an irreducible branch ξ of Φ_e such that its strict transform $\widetilde{\xi}$ coincides with $(\widetilde{\Phi}_e)_P$, i.e. $\widetilde{\xi}$ is smooth, transversal to D and ξ is not a multiple branch of Φ_e by Lemma 4. By Theorem B there exists a r-nodal zone R in the minimal good resolution of (ϕ_e,ϕ_w) (here w is assumed to be a generic value) such that the strict transform of Γ intersects R and moreover, $I_z(\phi_e,\Gamma)/I_z(\phi_w,\Gamma)=r$ with $r>1$, because $P\in\widetilde{\Gamma}\cap\widetilde{\Phi_e}$. Taking into account that $\widetilde{\Phi_e}$ is smooth and transversal to the dicritical divisor D, then one has that $P=\widetilde{\Gamma}\cap E\subset D\subset R$ and so, by Theorem A,

$$\frac{I_z(\phi_e,\Gamma)}{I_z(\phi_w,\Gamma)}=q_w^e(D)=\frac{\nu_D(\phi_e)}{\nu_D(\phi_w)}\ .$$

However, this is impossible because the last quotient is equal to 1, being D dicritical. Thus, as a consequence, $(\widetilde{\Phi_e})_P$ must be singular or tangent to D. In both cases P is a critical point of $\widehat{h}|_{\mathcal{D}}$ (i.e. ϕ_e is a special function of Λ). □

Conversely, let P be a singular point of $\mathcal{D}$ or a smooth point of $\mathcal{D}$ which is a critical point of $\widehat{h}_{|\mathcal{D}}$ and let $e=\widehat{h}(P)$, then from Theorem 2, Φ_e is a special fiber of Λ. If the irreducible component of $\widetilde{\Phi}_e$ that intersects $\mathcal{D}$ at P is non-reduced, then from Lemma 4 we have finished. Thus, we assume that $\widetilde{\Phi}_e$ is reduced at P.

Before all, note that $\widetilde{\Phi}_e$ has not normal crossings with E at P: if P is a singular point of $\mathcal{D}$, then there are at least three components of the total transform intersecting at P. Otherwise, if P is smooth on $\mathcal{D}$ then $\widetilde{\Phi}_e$ is either singular or tangent to $\mathcal{D}$.

Let $w,w'\in\mathbb{CP}^1$ be generic values and let $\rho':(Y',E')\to(Z,z)$ be the minimal good resolution of $\{\phi_w\phi_{w'}\phi_e=0\}$. Note that $\rho'=\rho\circ\sigma$, where σ is a sequence of blowing-ups of points, each of them produces a new irreducible rational exceptional component. In particular, $\Delta = \sigma^{-1}(P) \subset E'$ is a connected exceptional part and must contain a nodal component $E_\alpha \subset E'$. Notice that no component of Δ is contracted in the minimal good resolution $\rho'' : (Y'', E'') \to (Z, z)$ of the pair (ϕ_e, ϕ_w); i.e. $\Delta \subset E''$. As a consequence, $E_\alpha \subset E''$ is also a nodal component of E''. Let R be the corresponding nodal zone in E'' which contains E_α. Note that, for each $E_\beta \subset R \subset \Delta$ one has $r = q_w^e(E_\beta) = \nu_\beta(\phi_e)/\nu_\beta(\phi_w) > 1$.

Now, Theorem B implies that there exists a branch Γ of $C(\pi)$ such that its strict transform by ρ'' intersects Δ and also that

$$\frac{I_z(\phi_e, \Gamma)}{I_z(\phi_w, \Gamma)} = \frac{\nu_\alpha(\phi_e)}{\nu_\alpha(\phi_w)} = r > 1.$$

Taking into account that $R \subset \Delta$ and $\sigma(\Delta) = P$, one has that the strict transform of Γ by ρ intersects E at the point P and moreover $I_z(\phi_e, \Gamma) > I_z(\phi_w, \Gamma)$. Note that the above inequality is true for any irreducible component Γ of $C(\pi)$ whose strict transform by ρ intersects $\mathcal{D}$ at P. Thus, the special fiber ϕ_e is the unique fiber with the condition $I_z(\phi_e, \Gamma) > \min_w I_z(\phi_w, \Gamma)$.

Remark Notice that, if $\{P\} \subset \mathcal{D}$ is an isolated special zone, $\widehat{h}(P) = e$ and Γ is the corresponding irreducible component of $C(\pi)$ then, for any fibers Φ_a and $\Phi_{a'}$ different from Φ_e we have $I_z(\phi_a, \Gamma) = I_z(\phi_{a'}, \Gamma)$ (see Proposition 1).

4.2 Proof of Theorem 3 for the connected components of $\overline{E \backslash \mathcal{D}}$

Let recall that $\rho : (Y, E) \to (Z, z)$ is the minimal good resolution of Λ and $\mathcal{D}$ the dicritical locus of E. Let Δ be a connected component of $\overline{E \backslash \mathcal{D}}$ such that $(h \circ \rho)(\Delta) = e$. Let w, w' be generic values of Λ and let denote by $\rho' : (Y', E') \to (Z, z)$ the minimal good resolution of $\{\phi_w \phi_{w'} \phi_e = 0\}$. Let $\tau : (Y', E') \to (Y, E)$ denote the composition of blowing-ups of points which produces Y' from (Y, E):

$$(Y', E') \xrightarrow{\tau} (Y, E) \xrightarrow{\rho} (Z, z) .$$

Lastly, let Δ' be the pull-back of Δ by τ. Note that Δ' is a connected component of $\overline{E' \backslash \mathcal{D}'}$ because the dicritical locus $\mathcal{D}'$ on E' is just the strict transform of $\mathcal{D}$ by τ. We distinguish two cases, according to the existence of a nodal component $E_\alpha \subset \Delta'$ (with respect to ϕ_w and ϕ_e).

Case 1 There exist a nodal component E_α, with $E_\alpha \subset \Delta'$.

For each component $E_\beta \subset \Delta'$ one has $q_{w'}^w(E_\beta) = 1$ and $q_w^e(E_\beta) > 1$. Let R be the nodal zone of E' such that $E_\alpha \subset R$. Then $R \subset \Delta'$ because q_w^e is constant and > 1 on R and $q_w^e(D) = 1$ for any dicritical divisor, in particular for any dicritical D such that $D \cap \Delta' \neq \emptyset$.

Now, from Theorem B, there exists a branch Γ of the critical locus $C(\pi)$ whose strict transform by ρ', denoted by $\widetilde{\Gamma}$, intersects R. As a consequence, the strict transform of Γ by ρ, $\tau(\widetilde{\Gamma})$, intersects Δ. Again Theorem B implies that $q_w^e(E_\alpha) =$

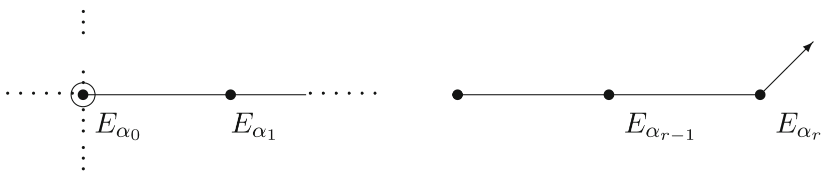

Fig. 1 Graph in Case 2

$I_z(\phi_e, \Gamma)/I_z(\phi_w, \Gamma)$ and hence, the special value e is the unique one such that $I_z(\phi_e, \Gamma) > I_z(\phi_{w'}, \Gamma)$ for any generic value w'.

Case 2 There are no nodal components in Δ'.

In this case one has that $\Delta' = E_{\alpha_1} \cup \ldots \cup E_{\alpha_r}$ and there exists a dicritical component $D \subset \mathcal{D}'$ such that $\{D = E_{\alpha_0}, E_{\alpha_1}, \ldots, E_{\alpha_r}\}$ is a chain and $\chi(\overset{\circ}{E}_{\alpha_r}) \geq 0$. Now, note that the strict transform $\widetilde{\Phi_e}$ of Φ_e by ρ' intersects Δ' (see Theorem 2). Therefore, the only way to avoid the existence of a nodal component with respect to $\phi_w\phi_e$ is if E_{α_r} is an end (i.e. it is rational and is connected only with the previous one $E_{\alpha_{r-1}}$) and $\widetilde{\Phi_e}$ intersects E_{α_r}. Moreover, $\widetilde{\Phi_e}$ with its reduced structure must be smooth and transversal to E_{α_r}. It means that the minimal good resolution of Λ is also a resolution of the reduced irreducible component ξ_e of Φ_e whose strict transform meets Δ at E_{α_r}. Indeed, otherwise in order to resolve ξ_e, one has to blow-up at $\xi_e \cap E_{\alpha_r}$ and by this process a nodal component is produced.

Lemma 5 *Let $v_0, \ldots, v_r, e_1, \ldots, e_r$ be sequences of integers such that $v_{i-1} = e_i v_i - v_{i+1}$ for $i = 1, \ldots, r-1$. Let $q_0, \ldots, q_{r-1} \in \mathbb{Z}$ be defined recursively as $q_0 = 1$, $q_1 = e_1$ and, for $i \geq 2$, $q_i = e_i q_{i-1} - q_{i-2}$. Then, for $i \geq 1$ one has* $\gcd(q_i, q_{i-1}) = 1$ *and $v_0 = q_i v_i - q_{i-1} v_{i+1}$.*

Proof Obviously $\gcd(q_0, q_1) = 1$ and from the definition of q_i, if $\gcd(q_{i-1}, q_{i-2}) = 1$ then $\gcd(q_{i-1}, q_i) = 1$. The equality $v_0 = q_i v_i - q_{i-1} v_{i+1}$ is obvious for $i = 1$ and, by induction, using the equality $v_{i-1} = e_i v_i - v_{i+1}$ in the inductive hypothesis $v_0 = q_{i-1} v_{i-1} - q_{i-2} v_i$, one has:

$$v_0 = q_{i-1} v_{i-1} - q_{i-2} v_i = q_{i-1}(e_i v_i - v_{i+1}) - q_{i-2} v_i = q_i v_i - q_{i-1} v_{i+1} \ .$$

□

Now, the proof of the case 2 is a consequence of the following:

Proposition 6 *The irreducible curve ξ_e is a branch of Φ_e with multiplicity bigger than 1. As a consequence ξ_e is also a branch of $C(\pi)$ and thus, $C(\pi)$ intersects Δ.*

Proof Recall that w is a generic element of Λ. For the sake of simplicity let denote $v_i = \nu_{\alpha_i}(\phi_w)$ and $e_i = -E_{\alpha_i}^2$ for $i = 0, \ldots, r$. Then, using the formula

$$\left((\widetilde{\phi_w}) + \sum_{\alpha \in G(\rho')} \nu_\alpha(\phi_w) E_\alpha \right) \cdot E_{\alpha_i} = 0 \tag{3}$$

for $i = 1, \dots, r$ one has that

$$\begin{aligned} v_0 &= e_1 v_1 - v_2 \\ v_1 &= e_2 v_2 - v_3 \\ &\cdots \\ v_{r-2} &= e_{r-1} v_{r-1} - v_r \\ v_{r-1} &= e_r v_r \end{aligned} \tag{4}$$

By Lemma 5 one has that $v_0 = q_r v_r$. Moreover, taking into account that $e_i = -E_{\alpha_i}^2 \geq 2$, one can easily prove that $q_r > q_{r-1} > \cdots > q_1 > q_0 = 1$.

Now, let consider the special fiber Φ_e and write $v_i' = \nu_{\alpha_i}(\phi_e)$ for $i = 0, \dots, r$. Equations (3) applied for ϕ_e (instead of ϕ_w) gives a sequence of equalities $v_{i-1}' = e_i v_i' - v_{i+1}'$, for $i = 1, \dots, r-1$ (like in (4) above with v_i' instead of v_i) together with the last one:

$$v_{r-1}' = e_r v_r' - (\widetilde{\phi_e}) \cdot E_{\alpha_r} = e_r v_r' - k \ .$$

Lemma 5 implies that $v_0' = q_r v_r' - q_{r-1} k$. As $E_{\alpha_0} = D$ is a dicritical divisor, one has that $v_0' = \nu_{\alpha_0}(\phi_e) = \nu_{\alpha_0}(\phi_w) = v_0$, i.e.

$$q_r v_r = q_r v_r' - q_{r-1} k \ .$$

By Lemma 5 again, $\gcd(q_r, q_{r-1}) = 1$ and so q_r divides k. In particular $k = (\widetilde{\phi_e}) \cdot E_\sigma > 1$ and the irreducible germ ξ_e appears repeated k times in Φ_e. □

4.3 Special fibers and critical locus

Let $C(\pi) = \sum_{i=1}^{\ell} n_i \Gamma_i$ be the decomposition of the critical locus in irreducible components. For each $i \in \{1, \dots, \ell\}$ the intersection multiplicity $I_z(\phi, \Gamma_i)$ is constant, except for the unique special value $\varepsilon(\Gamma_i)(= \varepsilon(i))$ such that $I_z(\phi_{\varepsilon(i)}, \Gamma_i) > I_z(\phi, \Gamma_i)$, for $\phi \neq \phi_{\varepsilon(i)}$. So, as in [8], one has a surjective map $\varepsilon : \mathcal{B}(C(\pi)) \to Sp(\Lambda)$ from the set of branches of the critical locus to the set of special values of Λ.

If $w \in \mathbb{CP}^1$ is a generic value one has that

$$I_z(\phi_w, C(\pi)) = \sum_{i=1}^{\ell} n_i I_z(\phi_w, \Gamma_i) = \min\{I_z(\phi, C(\pi)) \ , \ \phi \in \Lambda\}$$

and, on the other hand, for a special value $e \in \mathbb{CP}^1$ there is

$$I_z(\phi_e, C(\pi)) = \sum_{i=1}^{\ell} n_i I_z(\phi_e, \Gamma_i) > \sum_{i=1}^{\ell} n_i I_z(\phi_w, \Gamma_i) = \min\{I_z(\phi, C(\pi)) \ , \ \phi \in \Lambda\} \ .$$

As a consequence, one has the following result which gives the equivalence of items 1 and 2 in Theorem 4.

Corollary 3 Φ_e *is a special fiber of* Λ *if and only if*

$$I_z(\phi_e, C(\pi)) > \min\{I_z(\phi, C(\pi)), \phi \in \Lambda\} \ .$$

Remark As in [8], the map $\varepsilon : \mathcal{B}(C(\pi)) \to Sp(\Lambda)$, defined above, can be factorized through the set of special zones $SZ(\Lambda)$ as $\varepsilon = \widehat{h} \circ \psi$:

$$\mathcal{B}(C(\pi)) \overset{\psi}{\to} SZ(\Lambda) \overset{\widehat{h}}{\to} Sp(\Lambda) \ .$$

The map ψ associates to the branch Γ the special zone Δ such that the strict transform of Γ in the minimal good resolution intersects Δ. Obviously one can decompose the branches of $C(\pi)$ in bunches by means of ψ.

Let $\rho' : (Y', E') \to (Z, z)$ be a good resolution of all the fibers of Λ (i.e. a good resolution of the product of all the special fibers and a pair of generic ones). Let e (resp. w) be a special value (resp. a generic one). An 1-nodal zone for the pair (ϕ_e, ϕ_w) can be decomposed in some different zones when we use different special values. This fact can be used to determine a finer decomposition in bunches of the branches of the critical locus $C(\pi)$. To do that, one can use the determination of all the nodal zones in E' with respect to all the pairs (ϕ_e, ϕ_w), when e varies in the set of special values and w is a fixed generic value.

5 Examples

As seen in Sect. 3.1, to the minimal good resolution ρ of the pencil Λ, one can associate its dual graph $\mathcal{G}(\rho)$. The following examples illustrate Theorems 1, 2 and 3 in terms of the dual graph. To construct $\mathcal{G}(\rho)$, we follow the method of Laufer described in [10,12] and also [13]. It consists, first, in establishing the graph of the minimal embedded resolution of the discriminant curve, which is the image by π of the critical locus $C(\pi)$ of π. This graph is constructed as follows. To each irreducible component E'_i of the exceptional divisor of the minimal resolution of the discriminant curve, we associate a vertex weighted by $(i, (E'_i)^2, v(E'_i))$, where $(E'_i)^2$ is the self-intersection of E'_i and $v(E'_i)$ is the valuation of the discriminant function along E'_i. An intersection point between two irreducibe components of the exceptional divisor is represented by an edge linking the associated vertices. Second, from the dual graph of the discriminant curve, we deduce the graph of the minimal good resolution of (Z, z) and so the one of ρ, using in particular Propositions 2.6.1, 2.7.1 (examples 2 and 3) 3.6.1 and 3.7.1 (example 1) of [12]. As in Fig. 1 of Sect. 4.2, we use a different mark for the vertices representing dicritical divisors.

5.1 Example 1

Let (Z, z) be defined by $z^3 = h(x, y)$ with $h(x, y) = (y + x^2)(y - x^2)(y + 2x^2)(x + y^2)(x - y^2)(x + 2y^2)$ and let π be the projection on the (x, y)-plane. In this way $(u, v) = (x, y)$, $f = u \circ \pi = x$ and $g = v \circ \pi = y$.

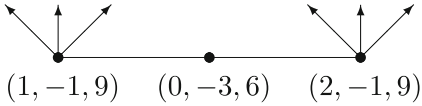

Fig. 2 Graph of the minimal resolution of the discriminant of π

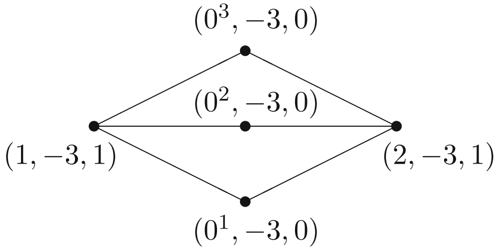

Fig. 3 The graph of the minimal good resolution of (Z, z)

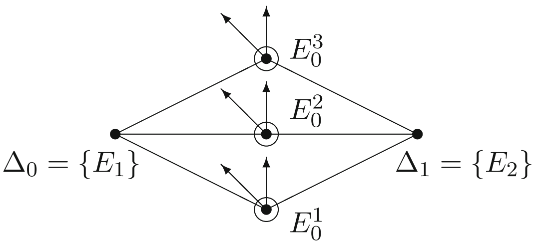

Fig. 4 The graph of the minimal good resolution of Λ

The discriminant curve of π is the curve $h(u, v) = 0$. The dual graph of its minimal embedded resolution is represented in Fig. 2.

From Proposition 3.6.1 and 3.7.1 of [12] we deduce the graph of the minimal good resolution of (Z, z) (see Fig. 3).

As the minimal embedded resolution of the discriminant curve $h(u, v) = 0$ of π is also the minimal good resolution of the product $uv(\lambda u + \mu v)h(u, v) = 0$, for $(\lambda : \mu) \in \mathbb{CP}^1 \backslash \{(1 : 0), (0 : 1)\}$, from Propositions 3.6.1 and 3.7.1 of [12] we can deduce the dual graph of the minimal good resolution of Λ (Fig. 4), the one of (f, g) and as a consequence the one of the minimal good resolution of $(\phi_w \phi_{w'} f g)^{-1}(0)$ where w and w' are generic values of Λ (Fig. 5). Notice that the minimal good resolution of Λ is also the minimal good resolution of (f, g).

The dicritical components of E are E_0^1, E_0^2, E_0^3. We have $SZ(\Lambda) = \{\Delta_1, \Delta_2\}$ with $\Delta_1 = \{E_1\}$ and $\Delta_2 = \{E_2\}$. The map $(f/g) \circ \rho$ has no critical point on $\mathcal{D}$ and $\mathcal{D}$ has no singular point neither. The special fiber associated to Δ_1 is $\{f = 0\}$ and the one associated to Δ_2 is $\{g = 0\}$. We conclude that Λ admits two special elements f and g; the special value associated to Δ_1 is $(0 : 1)$ and the one associated to Δ_2 is $(1 : 0)$. The Hironaka quotients are $q(E_1) = 2$ and $q(E_2) = 1/2$.

Moreover, using the minimal resolution of the discriminant curve (see Fig. 2), we deduce that, for each Δ_i, there exists three irreducible components of the reduced critical locus of π whose strict transform intersects Δ_i.

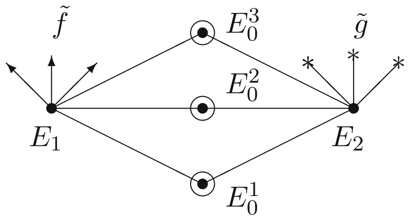

Fig. 5 Minimal good resolution of (f, g)

5.2 Example 2

Let (Z, z) be the D_6 singularity defined by the equation $z^2 = y(x^2 + y^4)$. The graph of its minimal resolution is shown in Fig. 6.

On this surface we make two examples for two different projections (pencils). Firstly, let $\pi = (f, g) : (Z, z) \to (\mathbb{C}^2, 0)$ be defined by $f(x, y, z) = u \circ \pi = x$ and $g(x, y, z) = v \circ \pi = y$. The discriminant curve of π is the curve $v(u^2 + v^4) = 0$. Notice that this projection is not a generic one because the image of the curve $\{g = 0\}$ is an irreducible component of the discriminant curve and the image of $\{f = 0\}$ is tangent to the discriminant curve.

Furthermore, using the minimal good resolution of the discriminant curve (Fig. 7) and Proposition 2.6.1 and 2.7.1 of [12], we obtain that the minimal good resolution of Λ is the one of (Z, z) and there exists a unique dicritical component E_1: the divisor with weight $(1, -2, 0)$. Thus, $\Delta_0 = \{E_0\}$ and $\Delta_1 = E_2 \cup E_3 \cup E_4 \cup E_5$ (see Fig. 8 for the notations) are two special zones. Moreover $C(\pi)$ intersects the exceptional divisor at E_0, E_4, E_5 and so $SZ(\Lambda) = \{\Delta_0, \Delta_1\}$. The Hironaka quotients corresponding to each vertex are: $q(E_0) = 1/2, q(E_1) = 1, q(E_2) = 3/2$ and $q(E_3) = q(E_4) = q(E_5) = 2$.

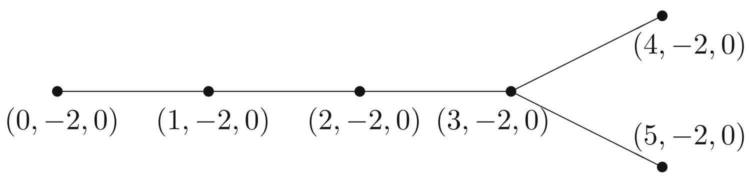

Fig. 6 The graph of the minimal good resolution of D_6

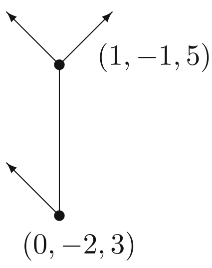

Fig. 7 Graph of the minimal resolution of the discriminant of π

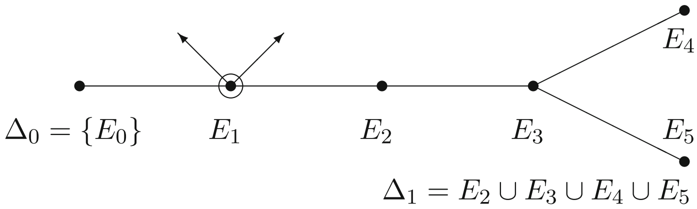

Fig. 8 The graph of the minimal good resolution of Λ

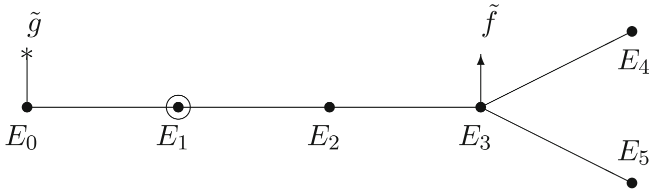

Fig. 9 The graph of the minimal good resolution of (f, g)

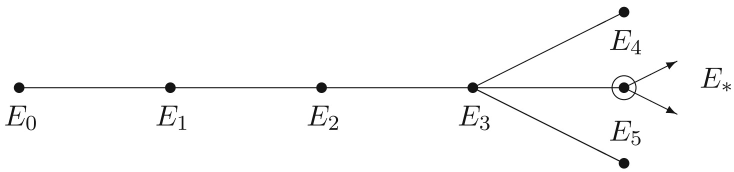

Fig. 10 The graph of the minimal good resolution of Λ

The connected component Δ_0 does not contain any nodal component and Δ_1 has a nodal component with Hironaka quotient equal to 2. The special fiber associated to Δ_1 is $\{f = 0\}$ whose strict transform meets Δ_1 at E_3, and there are two irreducible components of $C(\pi)$ intersecting Δ_1 at E_4 and E_5. The special fiber of Λ associated to Δ_0 is $\{g = 0\}$ which is also a non reduced irreducible component of the critical locus and intersects Δ_0 at E_0. The minimal good resolution of the pencil Λ is also the minimal good resolution of (f, g), so the corresponding graph of the minimal good resolution of $fg = 0$ is represented in Fig. 9.

Let us consider the morphism $\pi = (f, g) : (Z, z) \to (\mathbb{C}^2, 0)$ defined on D_6 by $f(x, y, z) = x + 2iy^2 = u$ and $g(x, y, z) = x^2 + y^3 = v$. In this case the minimal good resolution of (Z, z) coincides with the one of $\{fg = 0\}$. It is not the case of the minimal good resolution of Λ whose graph of resolution is represented in Fig. 10.

There exists a unique dicritical component represented by E_* and exactly one special zone consisting of $E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$. The Hironaka quotients of (f, g) corresponding to each vertex are: $q(E_*) = 1, q(E_0) = q(E_1) = q(E_2) = 1/2$, $q(E_3) = q(E_4) = q(E_5) = 2/3$.

In this case f is a generic element of the pencil Λ and g is the unique special element of Λ.

The graph of the minimal good resolution of $\{fg = 0\}$ is represented in Fig. 11.

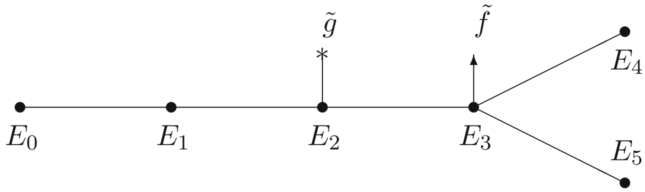

Fig. 11 The graph of the minimal good resolution of (f, g)

5.3 Example 3

With this example, issued from [13], we illustrate the case where a special zone is a singular point of the dicritical locus.

Let (Z, z) be defined by $z^2 = (x^2 + y^5)(y^2 + x^3)$ and let $\pi = (f, g) : (Z, z) \to (\mathbb{C}^2, 0)$ be the projection on the (x, y)-plane. The dual graph of the minimal embedded resolution of the discriminant curve $(u^2 + v^5)(v^2 + u^3) = 0$ of π and the coordinate axes is shown in Fig. 12. The arrows representing the strict transforms of the coordinate axes are depicted by double-arrows.

The graph of the minimal good resolution of Λ is in Fig. 13. The components E_{0^1} and E_{0^2} are dicritical. Thus, there exists two special zones Δ_0 and Δ_1 with $\Delta_0 = E_{1^1} \cup E_{1^2}$ and $\Delta_1 = E_{0^1} \cap E_{0^2} = \{P\}$ where P is the singular point of $\mathcal{D}$.

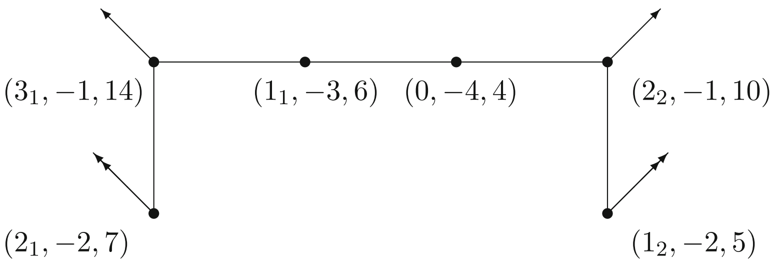

Fig. 12 Graph of the minimal resolution of the discriminant of π and the coordinates axes

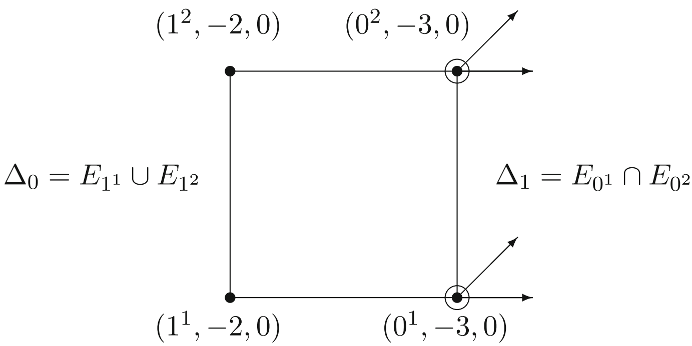

Fig. 13 The graph of the minimal good resolution of Λ

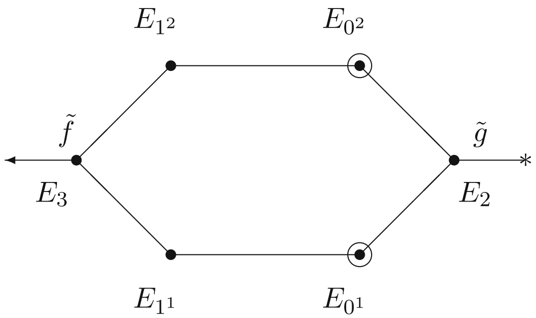

Fig. 14 The graph of the minimal good resolution of (f, g)

The special fibers associated to Δ_0 and Δ_1 are respectively $\{f = 0\}$ and $\{g = 0\}$. The graph of the minimal good resolution of (f, g) is shown Fig. 14.

The Hironaka quotients of the rational components (of self-intersection -1) E_2 and E_3 are respectively $2/3$ and $5/2$ and there exists two irreducible components of $C(\pi)$ whose strict transform intersects E_2 and two others whose strict transform intersects E_3.

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