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Pointwise Convergence along non-tangential direction for the Schrödinger equation with Complex Time

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Abstract
We study the pointwise convergence to the initial data in a cone region for the fractional Schrödinger operator $P^t_{a,\gamma}$ with complex time. By stationary phase analysis, we establish the maximal estimate for $P^t_{a,\gamma}$ in a cone region. As a consequence of the maximal estimate, the pointwise convergence holds through a standard argument. Our results extend those obtained by Cho–Lee–Vargas (J Fourier Anal Appl 18:972–994, 2012) and Shiraki (arXiv:1903.02356v1) from the real value time to the complex value time.

Keywords Pointwise convergence · Fractional Schrödinger operator · Maximal estimate

Mathematics Subject Classification 42B25 · 35Q56 · 47A63

1 Introduction

We define the Schrödinger type operator $P^t_{a,\gamma}$ as follows

$$P^t_{a,\gamma} f = e^{ig(t)(-\Delta)^{\frac{a}{2}}} f = \int_{\mathbb{R}} \hat{f}(\xi) e^{ix\xi} e^{it|\xi|^a} e^{-t^\gamma |\xi|^a} \, d\xi, \tag{1.1}$$

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where $g(t) = t + it^{\gamma}$ with $t > 0$, $\gamma > 0$ and $a \geq 1$.

For $\gamma = 1$, (1.1) coincides with the solution of the Ginzburg-Landau equation(see [3]).

For $g(t) = t$ and $a = 2$, then (1.1) is the solution to the most basic and universal form of the Schrödinger equation

$$\begin{cases} i\partial_t u - \Delta u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R} \\ u(x, 0) = f(x), & x \in \mathbb{R}. \end{cases} \tag{1.2}$$

For (1.2), Carleson [2] put forward a question about the range of exponent s for the Sobolev space $H^s(\mathbb{R})$ such that for $f \in H^s(\mathbb{R})$, there is

$$e^{it\Delta} f(x) \to f(x) \qquad \text{a. e.} \quad x \in \mathbb{R}^n,$$

as the time t tends to 0. He proved the almost everywhere convergence for the exponent $s \geq \frac{1}{4}$ in dimension one, which is sharp by the counterexamples given by Dahlberg and Kenig [5].

For the operator $P^t_{a,\gamma}$, Sjölin [11,12] together with Soria studied the pointwise convergence in $\mathbb{R}$ for the classical Schrödinger operator with complex time in the case $a = 2$, and $\gamma > 0$. Later, using Kolmogrov–Selierstov–Plessner method, Bailey [1] improved their results to the case $a > 1$.

This paper is devoted to the study of the pointwise convergence problem in $\mathbb{R}$ for the operator $P^t_{a,\gamma}$ along the non-tangential directions.

Let Θ be a compact region in $\mathbb{R}$. We define

$$\Gamma_x = \{x + t\theta : t \in (0, 1), \theta \in \Theta\}, \tag{1.3}$$

which is associated to the directions of the non-tangential convergence. And we study the pointwise convergence to the initial data in the region Γ_x for the Schrödinger type operator $P^t_{a,\gamma}$, that is

$$\lim_{\substack{(y,t)\to(x,0^+) \\ y\in\Gamma_x}} P^t_{a,\gamma} f(y) = f(x), \quad \text{for } a.e. \ x \in \mathbb{R}. \tag{1.4}$$

To do this, we first recall that the upper Minkowski dimension of Θ is defined by

$$\beta(\Theta) = \inf\{r > 0 : \limsup_{\delta\to 0} N(\Theta, \delta)\delta^r = 0\},$$

where $N(\Theta, \delta)$ is the minimal number of δ-intervals which cover Θ.

In $\mathbb{R}$, Cho–Lee–Vargas [4] considered the non-tangential convergence for the operator $e^{it\Delta}$ whose directions are determined by Θ, and they proved that non-tangential convergence holds for $s > \frac{\beta(\Theta)+1}{4}$. Shiraki [9] extended this result to the operator $e^{it(-\Delta)^{\frac{a}{2}}}$ with $a > 1$. In $\mathbb{R}^n$, by Sobolev embedding, it is easy to see that the non-tangential pointwise convergence for $e^{it\Delta}$ holds in the cone region Γ_x, which is defined by (1.3), if $s > \frac{n}{2}$, and Sögren and Sjölin [10] proved that this result is sharp. They showed that there exists a function $f \in H^{\frac{n}{2}}(\mathbb{R}^n)$ such that

$$\limsup_{\substack{(y,t)\to(x,0)\\|y-x|<\omega(t),t>0}} |e^{it\Delta} f(y)| = \infty, \quad \text{for } x \in \mathbb{R}^n,$$

where $\omega(t)$ is a strictly increasing function with $\omega(0) = 0$.

In a more general case, we consider the non-tangential pointwise convergence problem for $P^t_{a,\gamma}$ defined by (1.1) with complex time. Our result is the following.

Theorem 1.1 *Let $\Theta \subset \mathbb{R}$ be a compact set, then*

(i) *let $\gamma > 1$, if*

$$s > \begin{cases} \min\left\{\dfrac{(\beta(\Theta)+1)a}{4}(1-\frac{1}{\gamma}), \dfrac{\beta(\Theta)+1}{4}\right\}, & a > 1, \quad (1.5)\\ \min\left\{\dfrac{(\beta(\Theta)+1)}{2}(1-\frac{1}{\gamma}), \dfrac{1}{2}\right\}, & a = 1, \quad (1.6)\end{cases}$$

we have

$$\left\| \sup_{(t,\theta)\in(0,1)\times\Theta} |P^t_{a,\gamma} f(x+t\theta)| \right\|_{L^2(B(0,1))} \lesssim \|f\|_{H^s}, \quad \text{for } f \in H^s. \qquad (1.7)$$

(ii) *for $a \geq 1$, $\gamma \in (0, 1]$, and $0 < a < 1$, $\gamma \in (0, a]$, the maximal estimate holds in $L^p(\mathbb{R})$ for $1 < p \leq \infty$, that is,*

$$\left\| \sup_{(t,\theta)\in(0,1)\times\Theta} |P^t_{a,\gamma} f(x+t\theta)| \right\|_{L^p(\mathbb{R})} \lesssim \|f\|_{L^p(\mathbb{R})}, \quad \textit{for } f \in L^p(\mathbb{R}). \qquad (1.8)$$

For $p = 1$, we have

$$\left| \left\{ x \in \mathbb{R} : \sup_{(t,\theta)\in(0,1)\times\Theta} |P^t_{a,\gamma} f(x+t\theta)| > \lambda \right\} \right| < C\frac{\|f\|_{L^1}}{\lambda}, \qquad (1.9)$$

for $f \in L^1(\mathbb{R})$, $\lambda > 0$.

Remark 1.1 For $\gamma \geq \frac{a}{a-1}$ with $a > 1$, notice that the dispersion effect is stronger than the dissipation effect arising from the operator $P^t_{a,\gamma}$, then by the same argument as in Shiraki [9] one can obtain that the pointwise convergence holds for the operator $P^t_{a,\gamma}$ in the cone region for $s > \frac{\beta(\Theta)+1}{4}$, which is a better result than $s > \frac{(\beta(\Theta)+1)a}{4}(1-\frac{1}{\gamma})$. Thus, for (1.5) in Theorem 1.1 (i), we just need to discuss the case for $1 < \gamma < \frac{a}{a-1}$.

The sharpness of the results in Theorem 1.1 is remained to be solved.

The proof of Theorem 1.1 (i) is based on some oscillatory estimates, the Littlewood–Paley decomposition and the fact that the compact region Θ can be covered by a finite number of intervals (see Sect. 3). Theorem 1.1 (ii) is proved by showing that the maximal estimate for the operator $P^t_{a,\gamma}$ along non-tangential direction is bounded by the Hardy-Littlewood maximal functions.

As a direct consequence of this theorem, by standard arguments, we obtain the pointwise convergence result for the operator $P^t_{a,\gamma}$.

Corollary 1.2 *Let $\Theta \subset \mathbb{R}$ be a compact set, then*

(i) *let $\gamma > 1$ and s be as in Theorem 1.1 (i), then we have for $f \in H^s$,*

$$\lim_{\substack{(y,t)\to(x,0^+)\\ y-x\in t\Theta}} P^t_{a,\gamma} f(y) = f(x), \quad \text{for } a.e.\ x \in \mathbb{R}. \tag{1.10}$$

(ii) *for $a \geq 1$ with $\gamma \in (0,1]$, and $0 < a < 1$ with $\gamma \in (0,a]$, for $f(x) \in L^p(\mathbb{R})$ with $1 \leq p < \infty$, (1.10) holds.*

Remark 1.2 (1) For $a > 1$, $0 < t < 1$, if we set $\gamma = \infty$, then our results in Theorem 1.1 and Corollary 1.2 coincide with those of Cho–Lee–Vargas [4] and Shiraki [9].

(2) For $a > 1$, $\gamma > 0$, if we take $\Theta = \{0\}$, then $\beta(\Theta) = 0$ and the pointwise convergence results for the operator $P^t_{a,\gamma}$ coincide with the results of Sjölin–Soria [11,12] and Bailey [1].

(3) For $a = 1$, $0 < t < 1$, if we take $\gamma = \infty$ and $\Theta = \{0\}$, then the corresponding results in Theorem 1.1(i) and Corollary 1.2(i) coincide with those obtained by Rogers and Villarroya via Littlewood–Paley decomposition and the Strichartz inequality in [8], which are almost sharp up to the end point.

2 Preliminaries

In this section, we first give the proof of Theorem 1.1 (ii). Next we introduce some useful tools for latter use.

2.1 Proof of Theorem 1.1 (ii)

For the proof of the Theorem 1.1 (ii), we need the kernel estimate for the operator $P^t_{a,\gamma}$ with $0 < \gamma \leq 1$.

Lemma 2.1 *For $a > 0$ and $0 < \gamma \leq 1$, we have*

$$\left| \int_{\mathbb{R}} e^{ix\xi} e^{it|\xi|^a} e^{-t^\gamma |\xi|^a}\, d\xi \right| \lesssim \frac{t^\gamma}{(t^{\frac{\gamma}{a}} + |x|)^{a+1}} \tag{2.1}$$

where $x \in \mathbb{R}$ and $0 < t < 1$.

Proof Let

$$L(x,t)=\int e^{ix\xi}e^{it^{1-\gamma}|\xi|^a}e^{-|\xi|^a}\,\mathrm{d}\xi,$$

then

$$\text{LHS } of \ (2.1)=t^{-\frac{\gamma}{a}}|L(t^{-\frac{\gamma}{a}}x,t)|.$$

Hence, it suffices to prove that

$$|L(x,t)|\le C\min\{1,|x|^{-a-1}\}. \tag{2.2}$$

Since the finiteness of $|L(x,t)|$ is trivial, we just consider the case $|x|\gg 1$. By integration by parts, we have

$$\begin{aligned}
|L(x,t)|&=\left|\int e^{ix\xi}e^{i(t^{1-\gamma}+i)|\xi|^a}\,\mathrm{d}\xi\right|\\
&=\frac{1}{|x|}\left|\int i(t^{1-\gamma}+i)a|\xi|^{a-2}\xi e^{ix\xi}e^{i(t^{1-\gamma}+i)|\xi|^a}\,\mathrm{d}\xi\right|\\
&\lesssim\frac{1}{|x|}\int_{|\xi|<\frac{1}{|x|}}(t^{1-\gamma}+1)|\xi|^{a-1}e^{-|\xi|^a}\,\mathrm{d}\xi\\
&\quad+\frac{1}{|x|}\left|\int_{|\xi|>\frac{1}{|x|}}(t^{1-\gamma}+i)|\xi|^{a-2}\xi e^{ix\xi}e^{i(t^{1-\gamma}+i)|\xi|^a}\,\mathrm{d}\xi\right|\\
&\le C|x|^{-1-a}+|x|^{-1}\Sigma,
\end{aligned}$$

where

$$\Sigma=\left|\int_{|\xi|>\frac{1}{|x|}}(t^{1-\gamma}+i)|\xi|^{a-2}\xi e^{ix\xi}e^{i(t^{1-\gamma}+i)|\xi|^a}\,\mathrm{d}\xi\right|.$$

By integration by parts again, we can obtain

$$\begin{aligned}
\Sigma&\lesssim\frac{1}{|x|}\left|i(t^{1-\gamma}+i)|\xi|^{a-2}\xi e^{ix\xi}e^{i(t^{1-\gamma}+i)|\xi|^a}\Big|_{1/|x|}^{\infty}\right|\\
&\quad+\frac{1}{|x|}\left|\int_{|\xi|>\frac{1}{|x|}}e^{ix\xi}(i(t^{1-\gamma}+i)|\xi|^{a-2}+[i(t^{1-\gamma}+i)]^2|\xi|^{2a-2})e^{i(t^{1-\gamma}+i)|\xi|^a}\,\mathrm{d}\xi\right|\\
&=M_1+M_2.
\end{aligned}$$

Since $0 < t < 1$ and $0 < \gamma \leq 1$, then

$$t^{1-\gamma} + 1 \leq 2,$$
$$|(t^{1-\gamma} + i)^2| = |2it^{1-\gamma} + t^{2-2\gamma} - 1| \leq C.$$

Also note that $|\xi|^{2a} e^{-|\xi|^a} \leq C$. Then we have $M_1 \lesssim |x|^{-a}$, and

$$M_2 \lesssim \frac{1}{|x|} \int_{|\xi| > \frac{1}{|x|}} (|\xi|^{a-2} + |\xi|^{2a-2}) e^{-|\xi|^a} \, \mathrm{d}\xi \leq |x|^{-a}.$$

In conclusion, we have

$$|L(x,t)| \lesssim |x|^{-a-1}.$$

The proof is completed. □

Next we show that how we can prove Theorem 1.1 (ii) by Lemma 2.1.

Proof of Theorem 1.1 (ii) Since $\Theta \subset \mathbb{R}$ is a compact set, then we have

$$\sup_{\substack{0<t<1 \\ y-x\in t\Theta}} |P^t_{a,\gamma} f(y)| \leq \sup_{\substack{0<t<1 \\ |y-x|<Ct}} |P^t_{a,\gamma} f(y)|. \tag{2.3}$$

For a fixed $x \in \mathbb{R}$, set

$$\Gamma^1_x = \{(y,t) : 0 < t < 1, |y-x| < Ct\};$$
$$\Gamma^2_x = \{(y,t) : 0 < t < 1, |y-x| < Ct^{\frac{\gamma}{a}}\}.$$

Since $\gamma \in (0,1]$ with $a \geq 1$, and $\gamma \in (0,a]$ with $0 < a < 1$, we have $\Gamma^1_x \subset \Gamma^2_x$. Then by (2.3) and

$$\sup_{\Gamma^1_x} |P^t_{a,\gamma} f(y)| \leq \sup_{\Gamma^2_x} |P^t_{a,\gamma} f(y)|, \tag{2.4}$$

it is reduced to consider the maximal estimate for the operator $P^t_{a,\gamma}$ on the region Γ^2_x.

By Lemma 2.1, if $0 < \gamma \leq \min\{a,1\}$, then $\frac{\gamma}{a} \leq 1$, and for $|y-x| < Ct^{\frac{\gamma}{a}}$ with $0 < t < 1$, we have

$$\begin{aligned}
|P^t_{a,\gamma} f(y)| &\leq \int \left| \int e^{i(y-z)\xi} e^{it|\xi|^a} e^{-t^\gamma |\xi|^a} \, \mathrm{d}\xi \right| |f(z)| \, \mathrm{d}z \\
&\lesssim \int \frac{t^\gamma}{(t^{\frac{\gamma}{a}} + |y-z|)^{a+1}} |f(z)| \, \mathrm{d}z \\
&\leq \int_{|x-z|<2t^{\frac{\gamma}{a}}} \frac{t^\gamma}{(t^{\frac{\gamma}{a}} + |y-z|)^{a+1}} |f(z)| \, \mathrm{d}z
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k=1}^{\infty}\int_{2^k t^{\frac{\gamma}{a}}\le|x-z|<2^{k+1}t^{\frac{\gamma}{a}}}\frac{t^{\gamma}}{(t^{\frac{\gamma}{a}}+|y-z|)^{a+1}}|f(z)|\,\mathrm{d}z\\
&\le\int_{|x-z|<2t^{\frac{\gamma}{a}}}t^{-\frac{\gamma}{a}}|f(z)|\,\mathrm{d}z+\sum_{k=1}^{\infty}\frac{2^{k+1}}{2^{(k-1)(a+1)}}\frac{1}{2^{k+1}t^{\frac{\gamma}{a}}}\\
&\quad\times\int_{|x-z|<2^{k+1}t^{\frac{\gamma}{a}}}|f(z)|\,\mathrm{d}z\\
&\lesssim\mathcal{M}(f)(x)+4\sum_{k=1}^{\infty}2^{-(k-1)a}\mathcal{M}(f)(x)\\
&\lesssim\mathcal{M}(f)(x),
\end{aligned}
$$

where $\mathcal{M}$ is the Hardy-Littlewood maximal operator.

Then for $a>0$ and $0<\gamma\le\min\{a,1\}$, if $1<p\le\infty$, we have

$$
\left\|\sup_{\substack{0<t<1\\|y-x|<t^{\frac{\gamma}{a}}}}|P^t_{a,\gamma}f(y)|\right\|_{L^p(\mathbb{R})}\lesssim\|\mathcal{M}(f)(x)\|_{L^p(\mathbb{R})}\lesssim\|f\|_{L^p(\mathbb{R})}; \tag{2.5}
$$

if $p=1$, we have

$$
\left|\left\{x\in\mathbb{R}:\sup_{\substack{0<t<1\\|y-x|<t^{\frac{\gamma}{a}}}}|P^t_{a,\gamma}f(y)|>\lambda\right\}\right|<C\frac{\|f\|_{L^1}}{\lambda}, \tag{2.6}
$$

where $\lambda>0$.

Combining the estimates (2.3), (2.5) and (2.6), we obtain the result in Theorem 1.1 (ii) in this case.

The proof of Theorem 1.1 (ii) is finished. □

Remark 2.1 For $\gamma\in(a,1]$ with $0<a<1$, we have $\Gamma^2_x\subset\Gamma^1_x$. In this case, we cannot bound the maximal function $\sup_{\Gamma^1_x}|P^t_{a,\gamma}f(y)|$ by $\sup_{\Gamma^2_x}|P^t_{a,\gamma}f(y)|$, then it seems that the estimates (2.5) and (2.6) cannot be used to obtain the maximal estimate in Theorem 1.1 for $\gamma\in(a,1]$ with $0<a<1$.

2.2 Necessary tool

In order to prove Theorem 1.1 (i) in next section, we introduce the following two useful lemmas.

The following lemma is crucial for the oscillatory integral estimate in the proof of Theorem 1.1 (i) in Section 3.

Lemma 2.2 (Van der Corput lemma, [13]) *Suppose ϕ is real-valued and smooth in (a,b), ψ is complex-valued and smooth, and that $|\phi^{(k)}(x)| \geq 1$ for all $x \in (a,b)$. Then*

$$\Big|\int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx\Big| \leq c_k\lambda^{-\frac{1}{k}}\Big[|\psi(b)| + \int_a^b |\psi'(x)|\,dx\Big] \tag{2.7}$$

holds when

(i) $k \geq 2$ *or*
(ii) $k = 1$ *and $\phi'(x)$ is monotonic.*

The bound c_k is independent of ϕ and λ.

Next we introduce another useful lemma, which is associated to the maximal estimate for the operator $P^t_{a,\gamma}$. It is easy to see that the lemma below is a result of the Hardy–Littlewood–Sobolev inequality, which can be found in [7].

Lemma 2.3 *For $\frac{1}{2} < \alpha < 1$, we have*

$$\Big|\iint_{B(0,1)\times B(0,1)}\iint_{[0,1]\times[0,1]} f(x,t)g(\tilde{x},\tilde{t})|x-\tilde{x}|^{-\alpha}\,\mathrm{d}x\mathrm{d}\tilde{x}\mathrm{d}t\mathrm{d}\tilde{t}\Big| \lesssim \|f\|_{L^2_xL^1_t}\|g\|_{L^2_xL^1_t}. \tag{2.8}$$

Proof Let $F(x) = \|f(x,\cdot)\|_{L^1_t}$ and $G(\tilde{x}) = \|g(\tilde{x},\cdot)\|_{L^1_t}$. Then, it is easy to see that

$$\begin{aligned}&\Big|\iint_{B(0,1)\times B(0,1)}\iint_{[0,1]\times[0,1]} f(x,t)g(\tilde{x},\tilde{t})|x-\tilde{x}|^{-\alpha}\,\mathrm{d}x\mathrm{d}\tilde{x}\mathrm{d}t\mathrm{d}\tilde{t}\Big|\\ &\leq \int_{-1}^1\int_{-1}^1 F(x)G(\tilde{x})|x-\tilde{x}|^{-\alpha}\,\mathrm{d}x\mathrm{d}\tilde{x}.\end{aligned}$$

By Hölder's inequality and Hardy–Littlewood–Sobolev inequality (see [6]), we have

$$\begin{aligned}\int_{-1}^1\int_{-1}^1 F(x)G(\tilde{x})|x-\tilde{x}|^{-\alpha}\,\mathrm{d}x\mathrm{d}\tilde{x} &\leq \|F\|_{L^2_x(B(0,1))}\Big\|\int_{-1}^1 G(\tilde{x})|x-\tilde{x}|^{-\alpha}\,\mathrm{d}\tilde{x}\Big\|_{L^2_x(B(0,1))}\\ &\lesssim \|F\|_{L^2_x(B(0,1))}\|G\|_{L_x^{\frac{2}{3-2\alpha}}(B(0,1))}\\ &\lesssim \|F\|_{L^2_x(B(0,1))}\|G\|_{L^2_x(B(0,1))}\\ &\leq \|f\|_{L^2_xL^1_t}\|g\|_{L^2_xL^1_t}.\end{aligned}$$

The proof is completed. □

3 Proof of Theorem 1.1 (*i*)

Take a function $\psi(\xi) \in C_c^\infty(\mathbb{R})$ such that

$$\mathrm{supp}\ \psi(\xi) \subset \{\xi \in \mathbb{R} : \tfrac{1}{2} \leq |\xi| \leq 2\},\ \ \psi(\mathbb{R}) \subset [0,1].$$

Let $\psi_k(\xi) = \psi(\frac{\xi}{2^{k-1}})$, and use $\psi(\xi)$ to obtain the Littlewood–Paley decomposition, that is

$$\varphi_0(\xi) + \sum_{k\geq 1} \psi_k(\xi) = 1,$$

where $\varphi_0(\xi) \in C_c^\infty(\mathbb{R})$ satisfies that

$$\text{supp } \varphi_0(\xi) \subset [-1, 1], \ \varphi_0(\mathbb{R}) \subset [0, 1], \ \varphi_0(\xi) = 1 \text{ on } [-\tfrac{1}{2}, \tfrac{1}{2}].$$

We define the operator Δ_k by

$$\widehat{\Delta_0 f}(\xi) = \varphi_0(\xi)\widehat{f}(\xi),$$
$$\widehat{\Delta_k f}(\xi) = \psi_k(\xi)\widehat{f}(\xi), \ \text{ for } k \geq 1.$$

Let $M_\Theta f(x) = \sup\{|P^t_{a,\gamma} f(x + t\theta)| : t \in (0, 1), \theta \in \Theta\}$.

With the Littlewood–Paley decomposition, we have

$$\|M_\Theta f\|_{L^2(B(0,1))} \leq \|M_\Theta \Delta_0 f\|_{L^2(B(0,1))} + \sum_{k\geq 1} \|M_\Theta \Delta_k f\|_{L^2(B(0,1))}. \tag{3.1}$$

For the first term in RHS above, it is easy to see that

$$\|M_\Theta \Delta_0 f\|_{L^2(B(0,1))} \lesssim \int \varphi_0 |\hat{f}| \, d\xi \lesssim \|f\|_{L^2}. \tag{3.2}$$

Then we just need to deal with these terms $\|M_\Theta \Delta_k f\|_{L^2(B(0,1))}$, $k \geq 1$.

Let $\sigma = \frac{a}{2}(1 - \frac{1}{\gamma})$. Later we will see that this parameter is associated to the structure of the phase function $\phi(\xi)$ in (3.16) below and the corresponding oscillatory integral estimate.

Since Θ is a compact set in $\mathbb{R}$, without loss of generality, we can assume $\Theta \subset [-1, 1]$.

Let $N(\Theta, \lambda^{-\sigma})$ denote the smallest numeber of $\lambda^{-\sigma}$-intervals $\Omega_j(\lambda)$ with $|\Omega_j(\lambda)| < \lambda^{-\sigma}$ which cover Θ. Then for each $\lambda > 0$, we have

$$\Theta = \bigcup_{j=1}^{N(\Theta,\lambda^{-\sigma})} \Omega_j(\lambda).$$

For a fixed k and $x \in B(0, 1)$, we have by $l^2 \hookrightarrow l^\infty$

$$|M_\Theta \Delta_k f(x)|^2 \leq \sum_{j=1}^{N(\Theta,2^{-k\sigma})} |M_{\Omega_{k,j}} \Delta_k f(x)|^2,$$

where $\Omega_{k,j} = \Omega_j(2^k)$. Therefore we have

$$\sum_{k\geq 1} \|M_\Theta \Delta_k f\|_{L^2(B(0,1))} \leq \sum_{k\geq 1} \left(\sum_{j=1}^{N(\Theta,2^{-k\sigma})} \|M_{\Omega_{k,j}} \Delta_k f\|^2_{L^2(B(0,1))} \right)^{\frac{1}{2}}. \tag{3.3}$$

In order to estimate (3.3), we just need the following estimates.

Lemma 3.1 *Assume $a \geq 1$ and $\gamma > 1$. Let Ω be an interval with $|\Omega| < 2^{-k\sigma}$, then we have*

(1) *for $a > 1$, $\gamma \in (1, \frac{a}{a-1})$,*

$$\|M_\Omega \Delta_k f\|_{L^2(B(0,1))} \lesssim 2^{k[\frac{a}{4}(1-\frac{1}{\gamma})+\epsilon]} \|f\|_{L^2} \tag{3.4}$$

for all $f \in L^2$ and $0 < \epsilon \ll 1$.

(2) *for $a = 1$, $\gamma \in (1, \infty)$,*

$$\|M_\Omega \Delta_k f\|_{L^2(B(0,1))} \lesssim 2^{k[\frac{1}{2}(1-\frac{1}{\gamma})+\epsilon]} \|f\|_{L^2} \tag{3.5}$$

for all $f \in L^2$ and $0 < \epsilon \ll 1$.

We postpone the proof of the above lemma, and first look at that how we get our results in Theorem 1.1 (i) by Lemma 3.1.

Since $\sigma = \frac{a}{2}(1 - \frac{1}{\gamma})$, by the definition of the upper Minkowski dimension $\beta(\Theta)$, for each $\epsilon > 0$, there exists C_ϵ such that for each $k \geq 1$,

$$N(\Theta, 2^{-k\sigma}) \leq C_\epsilon 2^{k\sigma(\beta(\Theta)+\epsilon)}. \tag{3.6}$$

Let

$$\tilde{\psi}(\xi) = \begin{cases} 1, & \text{if } \frac{1}{2} < |\xi| < 2, \\ 0, & \text{if } |\xi| < \frac{1}{4} \ or \ |\xi| > 4, \end{cases}$$

and set $\tilde{\psi}_k(\xi) = \tilde{\psi}(\frac{\xi}{2^{k-1}})$ and the operator $\tilde{\Delta}_k$ such that

$$\widehat{\tilde{\Delta}_k f} = \tilde{\psi}_k(\xi)\widehat{f}(\xi), \quad \text{for } k \geq 1. \tag{3.7}$$

For $a > 1$, by the estimate (3.4) in Lemma 3.1 and (3.6), we have

$$\begin{aligned}\|M_{\Theta}\Delta_k f\|^2_{L^2(B(0,1))} &\lesssim \sum_{j=1}^{N(\Theta,2^{-k\sigma})} \|M_{\Omega_{k,j}}\Delta_k\tilde{\Delta}_k f\|^2_{L^2(B(0,1))}\\ &\lesssim \sum_{j=1}^{N(\Theta,2^{-k\sigma})} 2^{k(\frac{a}{2}(1-\frac{1}{\gamma})+\epsilon)}\|\tilde{\Delta}_k f\|^2_{L^2}\\ &\lesssim 2^{-k\epsilon}\|f\|^2_{H^{\frac{(\beta(\Theta)+1)a}{4}(1-\frac{1}{\gamma})+\epsilon}},\end{aligned} \tag{3.8}$$

then

$$\sum_{k\geq 1}\|M_{\Theta}\Delta_k f\|_{L^2(B(0,1))} \lesssim \|f\|_{H^{\frac{(\beta(\Theta)+1)a}{4}(1-\frac{1}{\gamma})+\epsilon}}. \tag{3.9}$$

For $a = 1$, by the same argument as in (3.8), we obtain

$$\sum_{k\geq 1}\|M_{\Theta}\Delta_k f\|_{L^2(B(0,1))} \lesssim \|f\|_{H^{\frac{(\beta(\Theta)+1)}{2}(1-\frac{1}{\gamma})+\epsilon}}. \tag{3.10}$$

Combining the estimates (3.1), (3.2), (3.3), (3.8) and (3.10), we obtain the results of Theorem 1.1 (i).

Now we turn to the proof of Lemma 3.1.

Proof of Lemma 3.1 Let $\lambda = 2^k$. Set

$$Tf = \chi(x,t,\theta)\int_{\mathbb{R}} e^{i(x+t\theta)\xi}e^{it|\xi|^a}e^{-t^\gamma|\xi|^a}\psi(\tfrac{\xi}{\lambda})f(\xi)\,\mathrm{d}\xi,$$

where $\chi(x,t,\theta) = \chi_{B(0,1)\times[0,1]\times\Omega}$. It suffices to show

$$\|Tf\|_{L^2_x L^\infty_t L^\infty_\theta} \lesssim \lambda^{\alpha(a,\gamma)+\epsilon}\|f\|_{L^2}, \tag{3.11}$$

where

$$\alpha(a,\gamma) = \begin{cases}\frac{a}{4}(1-\frac{1}{\gamma}), & \text{if } a>1,\ \gamma\in(1,\frac{a}{a-1});\\ \frac{1}{2}(1-\frac{1}{\gamma}), & \text{if } a=1,\ \gamma\in(1,\infty).\end{cases} \tag{3.12}$$

Indeed, with (3.11) in hand, we get

$$\|M_{\Omega}\Delta_k f\|_{L^2(B(0,1))} \lesssim \|T\hat{f}\|_{L^2_x L^\infty_t L^\infty_\theta} \lesssim \lambda^{\alpha(a,\gamma)+\epsilon}\|\hat{f}\|_{L^2} = \lambda^{\alpha(a,\gamma)+\epsilon}\|f\|_{L^2}.$$

By duality, it is reduced to prove

$$\|T^*F\|_{L^2} \lesssim \lambda^{\alpha(a,\gamma)+\epsilon}\|F\|_{L^2_x L^1_t L^1_\theta}, \tag{3.13}$$

where

$$T^*F = \psi(\tfrac{\xi}{\lambda}) \iiint e^{-i(x+t\theta)\xi} e^{-it|\xi|^a} e^{-t^\gamma|\xi|^a} F(x,t,\theta)\chi(x,t,\theta)\,\mathrm{d}x\mathrm{d}t\mathrm{d}\theta.$$

Next we turn to look at (3.13). We denote $u=(x,t,\theta)$ and $U=B(0,1)\times(0,1)\times\Omega$. By direct computation, we have

$$\begin{aligned}
&\|T^*F\|_{L^2}^2 \\
&\quad = \lambda \int \psi(\xi)^2 \iiint \iiint e^{i\lambda(x-\tilde{x}+t\theta-\tilde{t}\tilde{\theta})\xi} e^{i\lambda^a(t-\tilde{t})|\xi|^a} e^{-\lambda^a(t^\gamma+\tilde{t}^\gamma)|\xi|^a} \\
&\qquad \chi(x,t,\theta)\chi(\tilde{x},\tilde{t},\tilde{\theta})\overline{F}(x,t,\theta)F(\tilde{x},\tilde{t},\tilde{\theta})\,\mathrm{d}x\mathrm{d}t\mathrm{d}\theta\mathrm{d}\tilde{x}\mathrm{d}\tilde{t}\mathrm{d}\tilde{\theta}\mathrm{d}\xi \\
&\quad = \lambda \int_U \int_{\widetilde{U}} \chi(u)\chi(\tilde{u})\overline{F}(u)F(\tilde{u})K_\lambda(u,\tilde{u})\,\mathrm{d}u\mathrm{d}\tilde{u}, \qquad (3.14)
\end{aligned}$$

where

$$K_\lambda = \int e^{i\phi(\lambda\xi)} e^{-\lambda^a(t^\gamma+\tilde{t}^\gamma)|\xi|^a} \psi(\xi)^2\,\mathrm{d}\xi, \qquad (3.15)$$

$$\phi(\xi) = (x-\tilde{x}+t\theta-\tilde{t}\tilde{\theta})\xi + (t-\tilde{t})|\xi|^a. \qquad (3.16)$$

3.1 Proof of Lemma 3.1 in the case $a>1$

Split the integral (3.14) into three parts as follows

$$\|T^*F\|_{L^2}^2 = \lambda \sum_{m=1}^{3} \iint_{V_m} \chi(u)\chi(\tilde{u})\overline{F}(u)F(\tilde{u})K_\lambda(u,\tilde{u})\,\mathrm{d}u\mathrm{d}\tilde{u} \triangleq \sum_{m=1}^{3} E_m,$$

where

$$\begin{aligned}
V_1 &= \{(u,\tilde{u}) \in U\times\widetilde{U} : t+\tilde{t} > \lambda^{-\frac{a}{\gamma}+\frac{\delta}{\gamma}}\}, \\
V_2 &= \{(u,\tilde{u}) \in U\times\widetilde{U} : |x-\tilde{x}| < \lambda^{-[1-\frac{a}{2}(1-\frac{1}{\gamma})]+\epsilon}, t,\tilde{t} < \lambda^{-\frac{a}{\gamma}+\frac{\delta}{\gamma}}\}, \\
V_3 &= \{(u,\tilde{u}) \in U\times\widetilde{U} : |x-\tilde{x}| > \lambda^{-[1-\frac{a}{2}(1-\frac{1}{\gamma})]+\epsilon}, t,\tilde{t} < \lambda^{-\frac{a}{\gamma}+\frac{\delta}{\gamma}}\},
\end{aligned}$$

with $0<\delta\ll\epsilon$. The decomposition of the integral region is associated to the structure of the phase function $\phi(\xi)$ in (3.16) and the corresponding oscillatory integral estimate.

To obtain the estimate (3.13), we just need to prove that

$$E_m \lesssim \lambda^{\frac{a}{2}(1-\frac{1}{\gamma})+\epsilon} \|F\|_{L_x^2L_t^1L_\theta^1}^2, \quad \text{for } m=1,2,3. \qquad (3.17)$$

Step 1. Estimate for E_1. Since $t+\tilde{t} > \lambda^{-\frac{a}{\gamma}+\frac{\delta}{\gamma}}$, then for $\xi \in \operatorname{supp}\psi(\xi) \subset \{\xi : \frac{1}{2} < |\xi| < 2\}$, we have

$$e^{-\lambda^a(t^\gamma+\tilde{t}^\gamma)|\xi|^{-a}} < e^{-\lambda^a(t^\gamma+\tilde{t}^\gamma)2^{-a}} \lesssim_N (\lambda^a(t^\gamma+\tilde{t}^\gamma))^{-N} \\ \le \lambda^{-\delta N} < \lambda^{-[1-\frac{a}{2}(1-\frac{1}{\gamma})]}, \tag{3.18}$$

where we choose $N \in \mathbb{N}$ such that $\delta N > 1-\frac{a}{2}(1-\frac{1}{\gamma})$.

By (3.18), we have

$$|K_\lambda(u,\tilde{u})| \le \int |\psi(\xi)^2|\,\mathrm{d}\xi \cdot e^{-\lambda^a(t^\gamma+\tilde{t}^\gamma)2^{-a}} \lesssim \lambda^{-[1-\frac{a}{2}(1-\frac{1}{\gamma})]}.$$

Then

$$E_1 \le \lambda\lambda^{-[1-\frac{a}{2}(1-\frac{1}{\gamma})]} \iint_{V_1} \chi(u)\chi(\tilde{u})\overline{F}(u)F(\tilde{u})\,\mathrm{d}u\mathrm{d}\tilde{u} \lesssim \lambda^{\frac{a}{2}(1-\frac{1}{\gamma})}\|F\|^2_{L_x^2L_t^1L_\theta^1}. \tag{3.19}$$

Step 2. Estimate for E_2. Let $\tilde{\sigma} = 1-\frac{a}{2}(1-\frac{1}{\gamma})$. Since $|K_\lambda(u,\tilde{u})| < C$, by the definition of the set V_2 and Young's inequality, then we have

$$\begin{aligned} E_2 &\lesssim \lambda \int_{V_2} \chi(u)\chi(\tilde{u})\overline{F}(u)F(\tilde{u})\chi_{[-\lambda^{-\tilde{\sigma}+\epsilon},\lambda^{-\tilde{\sigma}+\epsilon}]}(x-\tilde{x})\,\mathrm{d}u\mathrm{d}\tilde{u} \\ &\lesssim \lambda \int \|F(x)\|_{L_t^1L_\theta^1}\|F(\tilde{x})\|_{L_{\tilde{t}}^1L_{\tilde{\theta}}^1}\chi_{[-\lambda^{-\tilde{\sigma}+\epsilon},\lambda^{-\tilde{\sigma}+\epsilon}]}(x-\tilde{x})\,\mathrm{d}x\mathrm{d}\tilde{x} \\ &\lesssim \lambda^{1-\tilde{\sigma}+\epsilon}\|F\|^2_{L_x^2L_t^1L_\theta^1} \\ &\lesssim \lambda^{\frac{a}{2}(1-\frac{1}{\gamma})+\epsilon}\|F\|^2_{L_x^2L_t^1L_\theta^1}. \end{aligned}$$

Step 3. Estimate for E_3. Since $\partial_\xi\phi = (x-\tilde{x}+t\theta-\tilde{t}\tilde{\theta})+a(t-\tilde{t})|\xi|^{a-2}\xi$, in order to bound $|x-\tilde{x}+t\theta-\tilde{t}\tilde{\theta}|$ from below and use Van der Corput lemma to estimate K_λ, we further split the region V_3 into several parts as follows

$$V_{31} = \{(u,\tilde{u}) \in V_3 : |x-\tilde{x}| < 4|t-\tilde{t}|\};$$
$$V_{32} = \{(u,\tilde{u}) \in V_3 : 4|t-\tilde{t}| < |x-\tilde{x}|\}.$$

Let

$$E_{3j} = \lambda \iint_{V_{3j}} \chi(u)\chi(\tilde{u})\overline{F}(u)F(\tilde{u})K_\lambda(u,\tilde{u})\,\mathrm{d}u\mathrm{d}\tilde{u}, \quad \text{for } j=1,2.$$

Then $E_3 = E_{31}+E_{32}$.

For the region V_{31}, by the support of $\psi(\xi)$, we have

$$|\partial_\xi^2[\phi(\lambda\xi)]| = |a(a-1)|\lambda^a|t-\tilde{t}||\xi|^{a-2} \gtrsim \lambda^a|x-\tilde{x}|,$$

then by Lemma 2.2 (i) and the inequality $t^\gamma + \tilde{t}^\gamma \gtrsim |t-\tilde{t}|^\gamma$, we can get

$$\begin{aligned}
|K_\lambda| &\lesssim (\lambda^a|x-\tilde{x}|)^{-\frac{1}{2}}\left(\int |(e^{-\lambda^a(t^\gamma+\tilde{t}^\gamma)|\xi|^a}\psi^2(\xi))'|\,\mathrm{d}\xi + \|e^{-\lambda^a(t^\gamma+\tilde{t}^\gamma)|\xi|^a}\psi^2(\xi)\|_{L^\infty}\right)\\
&\lesssim (\lambda^a|x-\tilde{x}|)^{-\frac{1}{2}} e^{-\lambda^a(t^\gamma+\tilde{t}^\gamma)2^{-a}}\\
&\lesssim (\lambda^a|x-\tilde{x}|)^{-\frac{1}{2}}\lambda^{-a\beta}|t-\tilde{t}|^{-\gamma\beta}\\
&= \lambda^{-\frac{a}{2}-a\beta}|x-\tilde{x}|^{-\frac{1}{2}-\gamma\beta}.
\end{aligned}$$

We take $\beta = \frac{1}{2\gamma} - \epsilon$, then by Lemma 2.3

$$\begin{aligned}
E_{31} &\lesssim \lambda^{1-\frac{a}{2}-a\beta}\int |F(u)|\chi(u)|F(\tilde{u})|\chi(\tilde{u})|x-\tilde{x}|^{-\frac{1}{2}-\gamma\beta}\,\mathrm{d}u\,\mathrm{d}\tilde{u}\\
&\lesssim \lambda^{\frac{a}{2}(1-\frac{1}{\gamma})+a\epsilon}\int |F(u)|\chi(u)|F(\tilde{u})|\chi(\tilde{u})|x-\tilde{x}|^{-1+\gamma\epsilon}\,\mathrm{d}u\,\mathrm{d}\tilde{u}\\
&\lesssim \lambda^{\frac{a}{2}(1-\frac{1}{\gamma})+a\epsilon}\|F\|^2_{L_x^2L_t^1L_\theta^1}.
\end{aligned}$$

For the region V_{32}, since

$$-\frac{a}{\gamma} - \sigma = -\frac{a}{\gamma} - \frac{a}{2}\Big(1-\frac{1}{\gamma}\Big) = -\frac{a}{2}\Big(1+\frac{1}{\gamma}\Big) < -1 + \frac{a}{2}\Big(1-\frac{1}{\gamma}\Big),$$

then

$$|t|\cdot|\theta-\tilde{\theta}| < \frac{1}{4}\lambda^{-\frac{a}{\gamma}+\frac{\delta}{\gamma}}\lambda^{-\sigma} < \frac{1}{4}\lambda^{-1+\frac{a}{2}(1-\frac{1}{\gamma})+\epsilon} < \frac{1}{4}|x-\tilde{x}|,$$

and

$$|t\theta - \tilde{t}\tilde{\theta}| \le |t||\theta-\tilde{\theta}| + |t-\tilde{t}||\tilde{\theta}| < \frac{1}{4}|x-\tilde{x}| + \frac{1}{4}|x-\tilde{x}| < \frac{1}{2}|x-\tilde{x}|, \quad (3.20)$$

where $0 < \delta \ll \epsilon$.

These inequalities yield

$$|x-\tilde{x}+t\theta-\tilde{t}\tilde{\theta}| \ge |x-\tilde{x}| - |t\theta-\tilde{t}\tilde{\theta}| > \frac{1}{2}|x-\tilde{x}|.$$

Through a direct computation, we have the first order derivative for the phase function for

$$\partial_\xi[\phi(\lambda\xi)] = \lambda(x-\tilde{x}+t\theta-\tilde{t}\tilde{\theta}) + a\lambda^a(t-\tilde{t})|\xi|^{a-2}\xi.$$

By stationary phase analysis, we split the integral $K_\lambda(u, \tilde{u})$ into two parts as follows

$$
\begin{aligned}
K_\lambda(u, \tilde{u}) &= \int_{W_1} e^{i\phi(\lambda\xi)} e^{-\lambda^a(t^\gamma+\tilde{t}^\gamma)|\xi|^a} \psi(\xi)^2 \, \mathrm{d}\xi \\
&\quad + \int_{W_2} e^{i\phi(\lambda\xi)} e^{-\lambda^a(t^\gamma+\tilde{t}^\gamma)|\xi|^a} \psi(\xi)^2 \, \mathrm{d}\xi \\
&= J_1 + J_2,
\end{aligned}
$$

where

$$
\begin{aligned}
W_1 &= \{\xi \in \mathbb{R} : |x - \tilde{x} + t\theta - \tilde{t}\tilde{\theta}| > 2a|t - \tilde{t}||\lambda\xi|^{a-1}\}, \\
W_2 &= \{\xi \in \mathbb{R} : |x - \tilde{x} + t\theta - \tilde{t}\tilde{\theta}| < 2a|t - \tilde{t}||\lambda\xi|^{a-1}\}.
\end{aligned}
$$

For J_1, since

$$
\begin{aligned}
|\partial_\xi[\phi(\lambda\xi)]| &\geq \lambda|x - \tilde{x} + t\theta - \tilde{t}\tilde{\theta}| - a\lambda|t - \tilde{t}||\lambda\xi|^{a-1} \\
&\geq \frac{1}{2}\lambda|x - \tilde{x} + t\theta - \tilde{t}\tilde{\theta}| \gtrsim \lambda|x - \tilde{x}| \\
&\geq \lambda^{\frac{a}{2}(1-\frac{1}{\gamma})+\epsilon} > 1
\end{aligned}
$$

and $\partial_\xi[\phi(\lambda\xi)]$ is monotonic with respect to ξ on W_2, then by Lemma 2.2 (ii), we have

$$
|J_1| \lesssim (\lambda|x - \tilde{x}|)^{-1} < (\lambda|x - \tilde{x}|)^{-[1-\frac{a}{2}(1-\frac{1}{\gamma})]}. \tag{3.21}
$$

Notice that in (3.21), in order to keep $\frac{1}{2} < 1 - \frac{a}{2}(1 - \frac{1}{\gamma}) < 1$, we need $1 < \gamma < \frac{a}{a-1}$ for $a > 1$.

For J_2, by supp $\psi(\xi) \subset \{\frac{1}{2} < |\xi| < 2\}$ and the definition of the set W_2, it is easy to see that

$$
|t - \tilde{t}| \gtrsim \lambda^{1-a}|x - \tilde{x} + t\theta - \tilde{t}\tilde{\theta}| \gtrsim \lambda^{1-a}|x - \tilde{x}|. \tag{3.22}
$$

Since

$$
|\partial_\xi^2[\phi(\lambda\xi)]| = a(a-1)\lambda^a|t - \tilde{t}||\xi|^{a-2} \gtrsim \lambda|x - \tilde{x}|
$$

and

$$
t^\gamma + \tilde{t}^\gamma \gtrsim (t + \tilde{t})^\gamma > |t - \tilde{t}|^\gamma,
$$

then by Lemma 2.2 (i) and the inequality (3.22), we can obtain

$$\begin{aligned}|J_2| &\lesssim (\lambda|x-\tilde{x}|)^{-\frac{1}{2}}(\|\psi^2(\xi)e^{-\lambda^a(t^\gamma+\tilde{t}^\gamma)|\xi|^a}\|_{L^\infty(W_2)} \\ &\quad + \int_{W_2} |(\psi^2(\xi)e^{-\lambda^a(t^\gamma+\tilde{t}^\gamma)|\xi|^a})'|\,\mathrm{d}\xi) \\ &\lesssim (\lambda|x-\tilde{x}|)^{-\frac{1}{2}}e^{-\lambda^a(t^\gamma+\tilde{t}^\gamma)|2^{-2}} \\ &\lesssim (\lambda|x-\tilde{x}|)^{-\frac{1}{2}}\lambda^{-a\beta}|t-\tilde{t}|^{-\gamma\beta} \\ &\lesssim \lambda^{-\frac{1}{2}-a\beta+(a-1)\gamma\beta}|x-\tilde{x}|^{-\frac{1}{2}-\gamma\beta},\end{aligned}$$

where $0<\beta=\frac{1}{2\gamma}-\epsilon$, so

$$|J_2| \lesssim \lambda^{-1+\frac{a}{2}(1-\frac{1}{\gamma})+[a-(a-1)\gamma]\epsilon}|x-\tilde{x}|^{-1+\gamma\epsilon}. \tag{3.23}$$

From (3.21) and (3.23), we have by Lemma 2.3

$$\begin{aligned}E_{32} &\lesssim \lambda^{\frac{a}{2}(1-\frac{1}{\gamma})+\epsilon}\int_{V_{22}} \chi(u)\chi(\tilde{u})\overline{F}(u)F(\tilde{u}) \\ &\quad (|x-\tilde{x}|^{-[1-\frac{a}{2}(1-\frac{1}{\gamma})]}+|x-\tilde{x}|^{-1+\gamma\epsilon})\,\mathrm{d}u\mathrm{d}\tilde{u} \\ &\lesssim \lambda^{\frac{a}{2}(1-\frac{1}{\gamma})+\epsilon}\|F\|^2_{L^2_xL^1_tL^1_\theta}.\end{aligned}$$

Then by the estimates of E_{31} and E_{32}, we have

$$E_3 \lesssim \lambda^{\frac{a}{2}(1-\frac{1}{\gamma})+\epsilon}\|F\|^2_{L^2_xL^1_tL^1_\theta}.$$

In conclusion, (3.17) has been proved.

3.2 Proof of Lemma 3.1 in the case $a=1$

For $a=1$, notice that $|\partial_\xi\phi(\xi)|$ does not depend on the value of ξ but its direction, which is different from the the case $a>1$, thus we consider the case $a=1$ alone.

In this case, rewrite the equalities (3.15) and (3.16) as follows

$$K_\lambda = \int e^{i\phi(\lambda\xi)}e^{-\lambda(t^\gamma+\tilde{t}^\gamma)|\xi|}\psi(\xi)^2\,\mathrm{d}\xi, \tag{3.24}$$

$$\phi(\xi) = (x-\tilde{x}+t\theta-\tilde{t}\tilde{\theta})\xi+(t-\tilde{t})|\xi|. \tag{3.25}$$

Split the integral (3.14) into three parts as follows

$$\|T^*F\|_{L^2}^2 = \lambda \sum_{m=1}^{3} \iint_{V_m} \chi(u)\chi(\tilde{u})\overline{F}(u)F(\tilde{u})K_\lambda(u,\tilde{u})\,\mathrm{d}u\mathrm{d}\tilde{u} = \sum_{m=1}^{3} E_m,$$

where

$$V_1 = \{(u,\tilde{u}) \in U \times \widetilde{U} : t + \tilde{t} > \lambda^{-\frac{1}{\gamma}+\frac{\delta}{\gamma}}\},$$
$$V_2 = \{(u,\tilde{u}) \in U \times \widetilde{U} : |x - \tilde{x}| < \lambda^{-\frac{1}{\gamma}+\epsilon}, t, \tilde{t} < \lambda^{-\frac{1}{\gamma}+\frac{\delta}{\gamma}}\},$$
$$V_3 = \{(u,\tilde{u}) \in U \times \widetilde{U} : |x - \tilde{x}| > \lambda^{-\frac{1}{\gamma}+\epsilon}, t, \tilde{t} < \lambda^{-\frac{1}{\gamma}+\frac{\delta}{\gamma}}\},$$

with $0 < \delta \ll \epsilon$. The decomposition of the integral region is based on the fact that $\partial_\xi^2\phi(\xi) \equiv 0$. We just need to prove that

$$E_m \lesssim \lambda^{1-\frac{1}{\gamma}+\epsilon}\|F\|_{L_x^2 L_t^1 L_\theta^1}^2, \quad \text{for } m = 1, 2, 3. \tag{3.26}$$

We will see that the estimate for E_1 and E_2 is similar to the corresponding terms for $a > 1$, and the only different term we need to consider is E_3, which shows the difference of the property of the phase function for the cases $a > 1$ and $a = 1$.

Step 1. Estimate for E_1. Since $t + \tilde{t} > \lambda^{-\frac{1}{\gamma}+\frac{\delta}{\gamma}}$, then for $\xi \in \operatorname{supp}\psi(\xi) \subset \{\xi : \frac{1}{2} < |\xi| < 2\}$, we have

$$\begin{aligned} e^{-\lambda(t^\gamma+\tilde{t}^\gamma)|\xi|^{-1}} &< e^{-\lambda(t^\gamma+\tilde{t}^\gamma)2^{-1}} \lesssim_N (\lambda(t^\gamma+\tilde{t}^\gamma))^{-N} \\ &\le \lambda^{-\delta N} < \lambda^{-\frac{1}{\gamma}}, \end{aligned} \tag{3.27}$$

where we choose $N \in \mathbb{N}$ such that $\delta N > \frac{1}{\gamma}$. This implies that

$$K_\lambda(u,\tilde{u}) \le \int |\psi(\xi)^2|\,\mathrm{d}\xi \cdot e^{-\lambda(t^\gamma+\tilde{t}^\gamma)2^{-1}} \lesssim \lambda^{-\frac{1}{\gamma}}.$$

Then

$$\begin{aligned} E_1 &\le \lambda\lambda^{-\frac{1}{\gamma}} \iint_{V_1} \chi(u)\chi(\tilde{u})\overline{F}(u)F(\tilde{u})\,\mathrm{d}u\mathrm{d}\tilde{u} \\ &\lesssim \lambda^{1-\frac{1}{\gamma}}\|F\|_{L_x^2 L_t^1 L_\theta^1}^2. \end{aligned} \tag{3.28}$$

Step 2. Estimate for E_2. Since $K_\lambda(u,\tilde{u}) < C$, by the definition of the set V_2 and Young's inequality, then we have

$$
\begin{aligned}
E_2 &\lesssim \lambda \int_{V_3} \chi(u)\chi(\tilde{u})\overline{F}(u)F(\tilde{u})\chi_{[-\lambda^{-\frac{1}{\gamma}+\epsilon},\lambda^{-\frac{1}{\gamma}+\epsilon}]}(x-\tilde{x})\,\mathrm{d}u\mathrm{d}\tilde{u} \\
&\lesssim \lambda \int \|F(x)\|_{L_t^1 L_\theta^1}\|F(\tilde{x})\|_{L_{\tilde{t}}^1 L_{\tilde{\theta}}^1}\chi_{[-\lambda^{-\frac{1}{\gamma}+\epsilon},\lambda^{-\frac{1}{\gamma}+\epsilon}]}(x-\tilde{x})\,\mathrm{d}x\mathrm{d}\tilde{x} \\
&\lesssim \lambda^{1-\frac{1}{\gamma}+\epsilon}\|F\|^2_{L_x^2 L_t^1 L_\theta^1}.
\end{aligned}
$$

Step 3. Estimate for E_3. By the definition of the set V_3, for $(u, \tilde{u}) \in V_3$, we have

$$
|x-\tilde{x}| \gg (|t|+|\tilde{t}|),
$$

then

$$
|x-\tilde{x}+t\theta-\tilde{t}\tilde{\theta}| \geq |x-\tilde{x}| - |t\theta-\tilde{t}\tilde{\theta}| \geq |x-\tilde{x}| - (|t|+|\tilde{t}|) \gtrsim |x-\tilde{x}|.
$$

Since

$$
\begin{aligned}
|\partial_\xi[\phi(\lambda\xi)]| &\geq \lambda|x-\tilde{x}+t\theta-\tilde{t}\tilde{\theta}| - \lambda|t-\tilde{t}| \\
&\gtrsim \lambda|x-\tilde{x}+t\theta-\tilde{t}\tilde{\theta}| \gtrsim \lambda|x-\tilde{x}| \\
&\geq \lambda^{1-\frac{1}{\gamma}+\epsilon} > 1,
\end{aligned}
$$

by Lemma 2.2 (ii), we have

$$
|K_\lambda| \lesssim (\lambda|x-\tilde{x}|)^{-1} < (\lambda^{\frac{1}{\gamma}}|x-\tilde{x}|)^{-1+\epsilon}\lambda^{-1+\frac{1}{\gamma}} = \lambda^{-1+\frac{\epsilon}{\gamma}}|x-\tilde{x}|^{-1+\epsilon}, \quad (3.29)
$$

where $\epsilon > 0$ is small enough such that $\gamma > 1+\epsilon$. By this inequality, we have

$$
\begin{aligned}
E_3 &\lesssim \lambda\lambda^{-1+\frac{\epsilon}{\gamma}} \int_{V_2} \chi(u)\chi(\tilde{u})\overline{F}(u)F(\tilde{u})|x-\tilde{x}|^{-1+\epsilon}\,\mathrm{d}u\mathrm{d}\tilde{u} \\
&\lesssim \lambda^{1-\frac{1}{\gamma}}\|F\|^2_{L_x^2 L_t^1 L_\theta^1}.
\end{aligned}
$$

In conclusion, (3.26) has been proved. □

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