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Orthogonally additive polynomials on non-commutative L^p-spaces

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Abstract

Let $\mathscr{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ. We prove that every continuous m-homogeneous polynomial P from $L^p(\mathscr{M}, \tau)$, with $0 < p < \infty$, into each topological linear space X with the property that $P(x+y) = P(x)+P(y)$ whenever x and y are mutually orthogonal positive elements of $L^p(\mathscr{M}, \tau)$ can be represented in the form $P(x) = \Phi(x^m)$ $(x \in L^p(\mathscr{M}, \tau))$ for some continuous linear map $\Phi\colon L^{p/m}(\mathscr{M}, \tau) \to X$.

Keywords Non-commutative L^p-space · Schatten classes · Orthogonally additive polynomial

Mathematics Subject Classification 46L10 · 46L52 · 47H60

1 Introduction

In [16], the author succeeded in providing a useful representation of the orthogonally additive homogeneous polynomials on the spaces $L^p([0, 1])$ and ℓ^p with $1 \le p < \infty$. In [12] (see also [6]), the authors obtained a similar representation for the space $C(K)$, for a compact Hausdorff space K. These results were generalized to Banach

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lattices [4] and Riesz spaces [9]. Further, the problem of representing the orthogonally additive homogeneous polynomials has been also considered in the context of Banach function algebras [1,19] and non-commutative Banach algebras [2,3,11]. Notably, [11] can be thought of as the natural non-commutative analogue of the representation of orthogonally additive polynomials on $C(K)$-spaces, and the purpose to this paper is to extend the results of [16] on the representation of orthogonally additive homogeneous polynomials on L^p-spaces to the non-commutative L^p-spaces.

The non-commutative L^p-spaces that we consider are those associated with a von Neumann algebra $\mathscr{M}$ equipped with a normal semifinite faithful trace τ. From now on, $S(\mathscr{M},\tau)$ stands for the linear span of the positive elements x of $\mathscr{M}$ such that $\tau\big(\operatorname{supp}(x)\big)<\infty$; here $\operatorname{supp}(x)$ stands for the support of x. Then $S(\mathscr{M},\tau)$ is a $*$-subalgebra of $\mathscr{M}$ with the property that $|x|^p\in S(\mathscr{M},\tau)$ for each $x\in S(\mathscr{M},\tau)$ and each $0<p<\infty$. For $0<p<\infty$, we define $\|\cdot\|_p\colon S(\mathscr{M},\tau)\to\mathbb{R}$ by $\|x\|_p=\tau(|x|^p)^{1/p}$ $(x\in S(\mathscr{M},\tau))$. Then $\|\cdot\|_p$ is a norm or a p-norm according to $1\le p<\infty$ or $0<p<1$, and the space $L^p(\mathscr{M},\tau)$ can be defined as the completion of $S(\mathscr{M},\tau)$ with respect to $\|\cdot\|_p$. Nevertheless, for our purposes here, it is important to realize the elements of $L^p(\mathscr{M},\tau)$ as measurable operators. Specifically, the set $L^0(\mathscr{M},\tau)$ of measurable closed densely defined operators affiliated to $\mathscr{M}$ is a topological $*$-algebra with respect to the strong sum, the strong product, the adjoint operation, and the topology of the convergence in measure. The algebra $\mathscr{M}$ is a dense $*$-subalgebra of $L^0(\mathscr{M},\tau)$, the trace τ extends to the positive cone of $L^0(\mathscr{M},\tau)$ in a natural way, and we can define

$$\|x\|_p=\tau\big(|x|^p\big)^{1/p}\quad(x\in L^0(\mathscr{M},\tau)),$$
$$L^p(\mathscr{M},\tau)=\big\{x\in L^0(\mathscr{M},\tau):\|x\|_p<\infty\big\}.$$

Also we set $L^\infty(\mathscr{M},\tau)=\mathscr{M}$ (with $\|\cdot\|_\infty:=\|\cdot\|$, the operator norm). Operators $x,y\in L^0(\mathscr{M},\tau)$ are mutually orthogonal, written $x\perp y$, if $xy^*=y^*x=0$. This condition is equivalent to requiring that x and y have mutually orthogonal left, and right, supports. Further, for $x,y\in L^p(\mathscr{M},\tau)$ with $0<p<\infty$, the condition $x\perp y$ implies that $\|x+y\|_p^p=\|x\|_p^p+\|y\|_p^p$, and conversely, if $\|x\pm y\|_p^p=\|x\|_p^p+\|y\|_p^p$ and $p\neq 2$, then $x\perp y$ (see [14, Fact 1.3]). The orthogonal additivity considered in [16] for the spaces $L^p([0,1])$ and ℓ^p can of course equally well be considered for the space $L^p(\mathscr{M},\tau)$. Let P be a map from $L^p(\mathscr{M},\tau)$ into a linear space X. Then P is:

(i) *orthogonally additive* on a subset $\mathscr{S}$ of $L^p(\mathscr{M},\tau)$ if

$$x,y\in\mathscr{S},\ x\perp y=0\ \Rightarrow\ P(x+y)=P(x)+P(y);$$

(ii) an *m-homogeneous polynomial* if there exists an m-linear map φ from $L^p(\mathscr{M},\tau)^m$ into X such that

$$P(x)=\varphi(x,\ldots,x)\quad(x\in L^p(\mathscr{M},\tau)).$$

Here and subsequently, $m\in\mathbb{N}$ is fixed with $m\ge 2$ and the superscript m stands for the m-fold Cartesian product. Such a map is unique if it is required to be symmetric.

Further, in the case where X is a topological linear space, the polynomial P is continuous if and only if the symmetric m-linear map φ associated with P is continuous.

Given a continuous linear map $\Phi\colon L^{p/m}(\mathscr{M},\tau)\to X$, where X is an arbitrary topological linear space, the map $P_\Phi\colon L^p(\mathscr{M},\tau)\to X$ defined by

$$P_\Phi(x)=\Phi(x^m)\quad(x\in L^p(\mathscr{M},\tau))$$

is a natural example of a continuous m-homogeneous polynomial which is orthogonally additive on $L^p(\mathscr{M},\tau)_{\text{sa}}$ (Theorem 4), and we will prove that every continuous m-homogeneous polynomial which is orthogonally additive on $L^p(\mathscr{M},\tau)_{\text{sa}}$ is actually of this special form (Theorem 5). Here and subsequently, the subscripts "sa" and $+$ are used to denote the self-adjoint and the positive parts of a given subset of $L^0(\mathscr{M},\tau)$, respectively.

We require a few remarks about the setting of our present work. Throughout the paper we are concerned with m-homogeneous polynomials on the space $L^p(\mathscr{M},\tau)$ with $0<p$, and thus one might wish to consider polynomials with values in the space $L^q(\mathscr{M},\tau)$, especially with $q\le p$. Further, in the case case where $p/m<1$ and the von Neumann algebra $\mathscr{M}$ has no minimal projections, there are no non-zero continuous linear functionals on $L^{p/m}(\mathscr{M},\tau)$; since one should like to have non-trivial "orthogonally additive" polynomials on $L^p(\mathscr{M},\tau)$, some weakening of the normability must be allowed to the range space (see Corollary 2). For these reasons, throughout the paper, X will be a (complex and Hausdorff) topological linear space. In the case where the von Neumann algebra $\mathscr{M}$ is commutative, the prototypical polynomials P_Φ mentioned above are easily seen to be orthogonally additive on the whole domain. In contrast, we will point out in Propositions 1 and 3 that this is not the case for the von Neumann algebra $\mathscr{B}(H)$ of all bounded operators on a Hilbert space H whenever $\dim H\ge 2$.

We assume a basic knowledge of C^*-algebras and von Neumann algebras, tracial non-commutative L^p-spaces, and polynomials on topological linear spaces. For the relevant background material concerning these topics, see [5,7,8,10,13,17,18].

2 C^*-algebras and von Neumann algebras

Our approach to the problem of representing the orthogonally additive homogeneous polynomials on the non-commutative L^p-spaces relies on the representation of those polynomials on the von Neumann algebras.

Recall that two elements x and y of a C^*-algebra $\mathscr{A}$ are mutually orthogonal if $xy^*=y^*x=0$, in which case the identity $\|x+y\|=\max\{\|x\|,\|y\|\}$ holds. The reader should be aware that we have chosen the standard definition of orthogonality in the setting of non-commutative L^p-spaces. This definition is slightly different from the one used in [11], which is the standard one in the setting of Banach algebras. In [11] the orthogonality of two elements x and y is defined by the relation $xy=yx=0$, and, further, the orthogonally additive polynomials on the self-adjoint part of a C^*-algebra are automatically orthogonally additive on the whole algebra. The important point to note here is that both the definitions of orthogonality agree on the self-adjoint part of the C^*-algebra. Thus, for a polynomial on a C^*-algebra, the property of being orthogonally additive on the self-adjoint part according to our definition is the same as being orthogonally additive according to [11]. Nevertheless, in contrast to [11], there are no non-zero orthogonally additive polynomials from the von Neumann algebra $\mathscr{B}(H)$ into any topological Banach space according to our definition (Proposition 1).

Suppose that $\mathscr{A}$ is a linear space with an involution $*$. Recall that for a linear functional $\Phi\colon \mathscr{A} \to \mathbb{C}$, the map $\Phi^*\colon \mathscr{A} \to \mathbb{C}$ defined by $\Phi^*(x) = \overline{\Phi(x^*)}$ ($x \in \mathscr{A}$) is a linear functional, and Φ is said to be *hermitian* if $\Phi^* = \Phi$. Similarly, for an m-homogeneous polynomial $P\colon \mathscr{A} \to \mathbb{C}$, the map $P^*\colon \mathscr{A} \to \mathbb{C}$ defined by $P^*(x) = \overline{P(x^*)}$ ($x \in \mathscr{A}$) is an m-homogeneous polynomial, and we call P *hermitian* if $P^* = P$.

Lemma 1 *Let X and Y be linear spaces, and let $P\colon X \to Y$ be an m-homogeneous polynomial. Suppose that P vanishes on a convex set $C \subset X$. Then P vanishes on the linear span of C.*

Proof Set $x_1, x_2, x_3, x_4 \in C$. Let $\eta\colon Y \to \mathbb{C}$ be a linear functional, and define $f\colon \mathbb{C}^4 \to \mathbb{C}$ by

$$f(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \eta\big(P(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \alpha x_4)\big) \quad (\alpha_1, \alpha_2, \alpha_3, \alpha_4 \in \mathbb{C}).$$

Then f is a complex polynomial function in four complex variables that vanishes on the set

$$\big\{(\rho_1, \rho_2, \rho_3, \rho_4) \in \mathbb{R}^4 : 0 \le \rho_1, \rho_2, \rho_3, \rho_4,\ \rho_1 + \rho_2 + \rho_3 + \rho_4 = 1\big\}.$$

This implies that f is identically equal to 0 on $\mathbb{C}^4$, and, in particular,

$$\eta\big(P(\rho_1 x_1 - \rho_2 x_2 + i\rho_3 x_3 - i\rho_4 x_4)\big) = f(\rho_1, -\rho_2, i\rho_3, -i\rho_4) = 0$$

for all $\rho_1, \rho_2, \rho_3, \rho_4 \ge 0$. Since this identity holds for each linear functional η, it may be concluded that $P(\rho_1 x_1 - \rho_2 x_2 + i\rho_3 x_3 - i\rho_4 x_4) = 0$ for all $\rho_1, \rho_2, \rho_3, \rho_4 \ge 0$. Thus P vanishes on the set

$$\big\{\rho_1 x_1 - \rho_2 x_2 + i\rho_3 x_3 - i\rho_4 x_4 : \rho_j \ge 0,\ x_j \in C\ (j = 1, 2, 3, 4)\big\},$$

which is exactly the linear span of the set C. ◻

Theorem 1 *Let $\mathscr{A}$ be a C^*-algebra, let X be a topological linear space, and let $\Phi\colon \mathscr{A} \to X$ be a continuous linear map. Then:*

(i) *the map $P_\Phi\colon \mathscr{A} \to X$ defined by $P_\Phi(x) = \Phi(x^m)$ ($x \in \mathscr{A}$) is a continuous m-homogeneous polynomial which is orthogonally additive on $\mathscr{A}_{\mathrm{sa}}$;*
(ii) *the polynomial P_Φ is uniquely specified by the map Φ.*

Suppose, further, that X is a q-normed space, $0 < q \le 1$. Then:

(iii) $2^{-1/q}\|\Phi\| \le \|P_\Phi\| \le \|\Phi\|$.

Moreover, in the case where $X = \mathbb{C}$*,*

(iv) *the functional* Φ *is hermitian if and only if the polynomial* P_Φ *is hermitian, in which case* $\|P_\Phi\| = \|\Phi\|$.

Proof (i) It is clear that the map P_Φ is continuous and that P_Φ is the m-homogeneous polynomial associated with the symmetric m-linear map $\varphi\colon \mathscr{A}^m \to X$ defined by

$$\varphi(x_1, \dots, x_m) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \Phi\left(x_{\sigma(1)} \cdots x_{\sigma(m)}\right) \quad (x_1, \dots, x_m \in \mathscr{A});$$

here and subsequently, we write $\mathfrak{S}_m$ for the symmetric group of order m.

Suppose that $x, y \in \mathscr{A}_{\text{sa}}$ are such that $x \perp y$. Then $xy = yx = 0$, and so $(x + y)^m = x^m + y^m$, which gives

$$P_\Phi(x + y) = \Phi\big((x + y)^m\big) = \Phi\big(x^m + y^m\big) = \Phi\big(x^m\big) + \Phi\big(y^m\big) = P_\Phi(x) + P_\Phi(y).$$

(ii) Assume that $\Psi\colon \mathscr{A} \to X$ is a linear map with the property that $P_\Psi = P_\Phi$. If $x \in \mathscr{A}_+$, then

$$\Phi(x) = \Phi\big((x^{1/m})^m\big) = P(x^{1/m}) = \Psi\big((x^{1/m})^m\big) = \Psi(x).$$

By linearity we also get $\Psi(x) = \Phi(x)$ for each $x \in \mathscr{A}$.

(iii) Next, assume that X is a q-normed space. For each $x \in \mathscr{A}$, we have

$$\|P_\Phi(x)\| = \|\Phi(x^m)\| \le \|\Phi\|\|x^m\| \le \|\Phi\|\|x\|^m,$$

which implies that $\|P_\Phi\| \le \|\Phi\|$. Now take $x \in \mathscr{A}$, and let $\omega \in \mathbb{C}$ with $\omega^m = -1$. Then $x = \Re x + i\Im x$, where

$$\Re x = \frac{1}{2}(x^* + x),\ \Im x = \frac{i}{2}(x^* - x) \in \mathscr{A}_{\text{sa}},$$

and, further, $\|\Re x\|, \|\Im x\| \le \|x\|$. Moreover, $\Re x = x_1 - x_2$ and $\Im x = x_3 - x_4$, where $x_1, x_2, x_3, x_4 \in \mathscr{A}_+$, $x_1 \perp x_2$, and $x_3 \perp x_4$. Since $x_1 \perp x_2$ and $x_3 \perp x_4$, it follows that $x_1^{1/m} \perp x_2^{1/m}$ and $x_3^{1/m} \perp x_4^{1/m}$. Consequently,

$$\begin{aligned} \|\Re x\| &= \max\big\{\|x_1\|, \|x_2\|\big\}, \\ \|\Im x\| &= \max\big\{\|x_3\|, \|x_4\|\big\}, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \big\|x_1^{1/m} + \omega x_2^{1/m}\big\| &= \max\big\{\big\|x_1^{1/m}\big\|, \big\|x_2^{1/m}\big\|\big\}, \\ \big\|x_3^{1/m} + \omega x_4^{1/m}\big\| &= \max\big\{\big\|x_3^{1/m}\big\|, \big\|x_4^{1/m}\big\|\big\}. \end{aligned} \tag{2}$$

Since

$$\|x_1^{1/m}\| = \|x_1\|^{1/m},\ \|x_2^{1/m}\| = \|x_2\|^{1/m},\ \|x_3^{1/m}\| = \|x_3\|^{1/m},\ \|x_4^{1/m}\| = \|x_4\|^{1/m},$$

it follows, from (1) and (2), that

$$\begin{aligned} \|x_1^{1/m} + \omega x_2^{1/m}\|^m &= \max\{\|x_1\|, \|x_2\|\} = \|\Re x\|, \\ \|x_3^{1/m} + \omega x_4^{1/m}\|^m &= \max\{\|x_3\|, \|x_4\|\} = \|\Im x\|. \end{aligned} \tag{3}$$

On the other hand, we have

$$\big(x_1^{1/m} + \omega x_2^{1/m}\big)^m = x_1 - x_2 = \Re x, \quad \big(x_3^{1/m} + \omega x_4^{1/m}\big)^m = x_3 - x_4 = \Im x,$$

and so

$$\begin{aligned} \Phi(x) &= \Phi(\Re x) + i\Phi(\Im x) = \Phi\big(\big(x_1^{1/m} + \omega x_2^{1/m}\big)^m\big) + i\Phi\big(\big(x_3^{1/m} + \omega x_4^{1/m}\big)^m\big) \\ &= P_\Phi\big(x_1^{1/m} + \omega x_2^{1/m}\big) + i P_\Phi\big(x_3^{1/m} + \omega x_4^{1/m}\big). \end{aligned}$$

Hence, by (3),

$$\begin{aligned} \|\Phi(x)\|^q &\le \big\|P_\Phi\big(x_1^{1/m} + \omega x_2^{1/m}\big)\big\|^q + \big\|P_\Phi\big(x_3^{1/m} + \omega x_4^{1/m}\big)\big\|^q \\ &\le \|P_\Phi\|^q \big\|x_1^{1/m} + \omega x_2^{1/m}\big\|^{mq} + \|P_\Phi\|^q \big\|x_3^{1/m} + \omega x_4^{1/m}\big\|^{mq} \\ &= \|P_\Phi\|^q \big(\|\Re x\|^q + \|\Im x\|^q\big) \\ &\le \|P_\Phi\|^q 2\|x\|^q. \end{aligned}$$

This clearly forces $\|\Phi\| \le 2^{1/q}\|P_\Phi\|$, as claimed.

(iv) It is straightforward to check that $P_\Phi^* = P_{\Phi^*}$. Consequently, if Φ is hermitian, then $P_\Phi^* = P_{\Phi^*} = P_\Phi$ so that P_Φ is hermitian. Conversely, if P_Φ is hermitian, then $P_{\Phi^*} = P_\Phi^* = P_\Phi$ and (ii) implies that $\Phi^* = \Phi$. Finally, assume that Φ is a hermitian functional. For the calculation of $\|P_\Phi\|$ it suffices to check that $\|\Phi\| \le \|P_\Phi\|$. For this purpose, let $\varepsilon \in \mathbb{R}^+$, and choose $x \in \mathscr{A}$ such that $\|x\| = 1$ and $\|\Phi\| - \varepsilon < |\Phi(x)|$. We take $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and $|\Phi(x)| = \alpha\Phi(x)$, so that

$$\|\Phi\| - \varepsilon < |\Phi(x)| = \Phi(\alpha x) = \overline{\Phi(\alpha x)} = \Phi\big((\alpha x)^*\big).$$

Note that $\|\Re(\alpha x)\| \le 1$ and $\|\Phi\| - \varepsilon < \Phi(\Re(\alpha x))$. Now we consider the decomposition $\Re(\alpha x) = x_1 - x_2$ with $x_1, x_2 \in \mathscr{A}_+$ and $x_1 \perp x_2$ and take $\omega \in \mathbb{C}$ with $\omega^m = -1$. As in (3), we see that $\big\|x_1^{1/m} + \omega x_2^{1/m}\big\| = \|\Re(\alpha x)\|^{1/m} \le 1$. Moreover, we have

$$P_\Phi\big(x_1^{1/m} + \omega x_2^{1/m}\big) = \Phi\big(\big(x_1^{1/m} + \omega x_2^{1/m}\big)^m\big) = \Phi(\Re(\alpha x)),$$

which gives $\|\Phi\| - \varepsilon < \|P_\Phi\|$. □

Lemma 2 *Let $\mathscr{A}$ be a C^*-algebra, let $\mathscr{R}$ be a $*$-subalgebra of $\mathscr{A}$, let X be a topological linear space, and let $\Phi\colon \mathscr{R} \to X$ be a linear map. Suppose that the polynomial $P\colon \mathscr{R} \to X$ defined by $P(x) = \Phi(x^m)$ $(x \in \mathscr{R})$ is continuous and that $\mathscr{R}$ satisfies the following conditions:*

(i) $|x| \in \mathscr{R}$ *for each* $x \in \mathscr{R}_{\mathrm{sa}}$*;*
(ii) $x^{1/m} \in \mathscr{R}$ *for each* $x \in \mathscr{R}_+$.

Then Φ is continuous.

Proof Let U be a neighbourhood of 0 in X. Let V be a balanced neighbourhood of 0 in X with $V + V + V + V \subset U$. The set $P^{-1}(V)$ is a neighbourhood of 0 in $\mathscr{R}$, which implies that there exists $r \in \mathbb{R}^+$ such that $P(x) \in V$ whenever $x \in \mathscr{R}$ and $\|x\| < r$. Take $x \in \mathscr{R}$ with $\|x\| < r^m$. Since $\mathscr{R}$ is a $*$-subalgebra of $\mathscr{A}$, we see that $\Re x, \Im x \in \mathscr{R}_{\mathrm{sa}}$. We write $\Re x = x_1 - x_2$ and $\Im x = x_3 - x_4$, as in the proof of Theorem 1, where, on account of the condition (i),

$$x_1 = \tfrac{1}{2}\big(|\Re x| + \Re x\big) \in \mathscr{R}_+, \qquad x_2 = \tfrac{1}{2}\big(|\Re x| - \Re x\big) \in \mathscr{R}_+,$$
$$x_3 = \tfrac{1}{2}\big(|\Im x| + \Im x\big) \in \mathscr{R}_+, \qquad x_4 = \tfrac{1}{2}\big(|\Im x| - \Im x\big) \in \mathscr{R}_+.$$

For each $j \in \{1, 2, 3, 4\}$, condition (ii) gives $x_j^{1/m} \in \mathscr{R}$, and, further, we have $\|x_j^{1/m}\| = \|x_j\|^{1/m} \le \|x\|^{1/m} < r$. Hence

$$\begin{aligned}\Phi(x) &= \Phi\big((x_1^{1/m})^m - (x_2^{1/m})^m + i(x_3^{1/m})^m - i(x_4^{1/m})^m\big)\\ &= \Phi\big((x_1^{1/m})^m\big) - \Phi\big((x_2^{1/m})^m\big) + i\Phi\big((x_3^{1/m})^m\big) - i\Phi\big((x_4^{1/m})^m\big)\\ &= P\big(x_1^{1/m}\big) - P\big(x_2^{1/m}\big) + iP\big(x_3^{1/m}\big) - iP\big(x_4^{1/m}\big) \in V + V + V + V \subset U,\end{aligned}$$

which establishes the continuity of Φ. □

Theorem 2 *Let $\mathscr{A}$ be a C^*-algebra, let X be a locally convex space, and let $P\colon \mathscr{A} \to X$ be a continuous m-homogeneous polynomial. Then the following conditions are equivalent:*

(i) *there exists a continuous linear map $\Phi\colon \mathscr{A} \to X$ such that $P(x) = \Phi(x^m)$ $(x \in \mathscr{A})$;*
(ii) *the polynomial P is orthogonally additive on $\mathscr{A}_{\mathrm{sa}}$;*
(iii) *the polynomial P is orthogonally additive on $\mathscr{A}_+$.*

If the conditions are satisfied, then the map Φ is unique.

Proof Theorem 1 gives (i)⇒(ii), and obviously (ii)⇒(iii). The task is now to prove that (iii)⇒(i).

Suppose that (iii) holds. For each continuous linear functional $\eta\colon X \to \mathbb{C}$, set $P_\eta = \eta \circ P$. Then P_η is a complex-valued continuous m-homogeneous polynomial. We claim that P_η is orthogonally additive on $\mathscr{A}_{\mathrm{sa}}$. Take $x, y \in \mathscr{A}_{\mathrm{sa}}$ with $x \perp y$. Then we can write $x = x_+ - x_-$ and $y = y_+ - y_-$ with $x_+, x_-, y_+, y_- \in \mathscr{A}_+$ mutually orthogonal. Define $f\colon \mathbb{C}^2 \to \mathbb{C}$ by

$$f(\alpha,\beta) = P_\eta(x_+ + \alpha x_- + y_+ + \beta y_-) - P_\eta(x_+ + \alpha x_-) - P_\eta(y_+ + \beta y_-) \quad (\alpha,\beta \in \mathbb{C}^2).$$

Then f is a complex polynomial function in two complex variables. If $\alpha,\beta \in \mathbb{R}^+$, then $x_+ + \alpha x_-, y_+ + \beta y_- \in \mathscr{A}_+$ are mutually orthogonal, and so, by hypothesis, $P(x_+ + \alpha x_- + y_+ + \beta y_-) = P(x_+ + \alpha x_-) + P(y_+ + \beta y_-)$. This shows that $f(\alpha,\beta) = 0$. Since f vanishes on $\mathbb{R}^+ \times \mathbb{R}^+$, it follows that f vanishes on $\mathbb{C}^2$, which, in particular, implies

$$P_\eta(x+y) - P_\eta(x) - P_\eta(y) = f(-1,-1) = 0.$$

Having proved that P_η is orthogonally additive on $\mathscr{A}_{\mathrm{sa}}$ we can apply [11, Theorem 2.8] to obtain a unique continuous linear functional Φ_η on $\mathscr{A}$ such that

$$\eta\big(P(x)\big) = \Phi_\eta(x^m) \quad (x \in \mathscr{A}). \tag{4}$$

Each $x \in \mathscr{A}$ can be written in the form $x_1^m + \cdots + x_k^m$ for suitable $x_1, \ldots, x_k \in \mathscr{A}$, and we define

$$\Phi(x) = \sum_{j=1}^{k} P(x_j).$$

Our next goal is to show that Φ is well-defined. Suppose that $x_1, \ldots, x_k \in \mathscr{A}$ are such that $x_1^m + \cdots + x_k^m = 0$. For each continuous linear functional η on X, (4) gives

$$\eta\left(\sum_{j=1}^{k} P(x_j)\right) = \sum_{j=1}^{k} \eta\big(P(x_j)\big) = \sum_{j=1}^{k} \Phi_\eta(x_j^m) = \Phi_\eta\left(\sum_{j=1}^{k} x_j^m\right) = 0.$$

Since X is locally convex, we conclude that $\sum_{j=1}^{k} P(x_j) = 0$.

It is a simple matter to check that Φ is linear and, by definition, $P(x) = \Phi(x^m)$ $(x \in \mathscr{A})$. The continuity of Φ then follows from Lemma 2.

The uniqueness of the map Φ follows from Theorem 1(ii). □

The assumption that the space X be locally convex can be removed by requiring that the C^*-algebra $\mathscr{A}$ be sufficiently rich in projections. The real rank zero is the most important existence of projections property in the theory of C^*-algebras. We refer the reader to [5, Section V.3.2] and [7, Section V.7] for the basic properties and examples of C^*-algebras of real rank zero. This class of C^*-algebras contains the von Neumann algebras and the C^*-algebras $\mathscr{K}(H)$ of all compact operators on any Hilbert space H. Let us remark that every C^*-algebra of real rank zero has an approximate unit of projections (but not necessarily increasing).

Theorem 3 *Let $\mathscr{A}$ be a C^*-algebra of real rank zero, let X be a topological linear space, and let $P\colon \mathscr{A} \to X$ be a continuous m-homogeneous polynomial. Suppose that $\mathscr{A}$ has an increasing approximate unit of projections. Then the following conditions are equivalent:*

(i) *there exists a continuous linear map $\Phi\colon \mathscr{A} \to X$ such that $P(x) = \Phi(x^m)$ $(x \in \mathscr{A})$;*
(ii) *the polynomial P is orthogonally additive on $\mathscr{A}_{\mathrm{sa}}$;*
(iii) *the polynomial P is orthogonally additive on $\mathscr{A}_+$.*

If the conditions are satisfied, then the map Φ is unique.

Proof Theorem 1 gives (i)⇒(ii), and it is clear that (ii)⇒(iii). We will henceforth prove that (iii)⇒(i).

We first note that such a map Φ is necessarily unique, because of Theorem 1(ii).

Suppose that (iii) holds and that $\mathscr{A}$ is unital. Let $\varphi\colon \mathscr{A}^m \to X$ be the symmetric m-linear map associated with P and define $\Phi\colon \mathscr{A} \to X$ by

$$\Phi(x) = \varphi(x, 1, \ldots, 1) \quad (x \in \mathscr{A}).$$

Let $Q\colon \mathscr{A} \to X$ be the m-homogeneous polynomial defined by

$$Q(x) = \Phi(x^m) \quad (x \in \mathscr{A}).$$

We will prove that $P = Q$. On account of Lemma 1, it suffices to show that $P(x) = Q(x)$ for each $x \in \mathscr{A}_{\mathrm{sa}}$.

First, consider the case where $x \in \mathscr{A}_{\mathrm{sa}}$ has finite spectrum, say $\{\rho_1, \ldots, \rho_k\} \subset \mathbb{R}$. This implies that x can be written in the form

$$x = \sum_{j=1}^{k} \rho_j e_j,$$

where $e_1, \ldots, e_k \in \mathscr{A}$ are mutually orthogonal projections (specifically, the projection e_j is defined by using the continuous functional calculus for x by $e_j = \chi_{\{\rho_j\}}(x)$ for each $j \in \{1, \ldots, k\}$). We also set $e_0 = 1 - (e_1 + \cdots + e_k)$, so that the projections $e_0, e_1, \ldots, e_k$ are mutually orthogonal, and $\rho_0 = 0$. We claim that if $j_1, \ldots, j_m \in \{0, \ldots, k\}$ and $j_l \neq j_{l'}$ for some $l, l' \in \{1, \ldots, m\}$, then

$$\varphi(e_{j_1}, \ldots, e_{j_m}) = 0. \tag{5}$$

Let $\Lambda_1 = \{n \in \{1, \ldots, m\} : j_n = j_l\}$ and $\Lambda_2 = \{n \in \{1, \ldots, m\} : j_n \neq j_l\}$. For each $\alpha_1, \ldots, \alpha_m \in \mathbb{R}^+$, the elements $\sum_{n\in\Lambda_1} \alpha_n e_{j_n}$ and $\sum_{n\in\Lambda_2} \alpha_n e_{j_n}$ are positive and mutually orthogonal, so that the orthogonal additivity of P on $\mathscr{A}_+$ gives

$$P\left(\sum_{n=1}^{m} \alpha_n e_{j_n}\right) = P\left(\sum_{n\in\Lambda_1} \alpha_n e_{j_n}\right) + P\left(\sum_{n\in\Lambda_2} \alpha_n e_{j_n}\right).$$

This implies that, for each linear functional $\eta\colon X \to \mathbb{C}$, the function $f\colon \mathbb{C}^m \to \mathbb{C}$ defined by

$$f(\alpha_1, \ldots, \alpha_m) = \eta\left(P\left(\sum_{n=1}^{m} \alpha_n e_{j_n}\right) - P\left(\sum_{n\in\Lambda_1} \alpha_n e_{j_n}\right) - P\left(\sum_{n\in\Lambda_2} \alpha_n e_{j_n}\right)\right),$$

for all $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$, is a complex polynomial function in m complex variables vanishing in $\left(\mathbb{R}^+\right)^m$. Therefore f vanishes on $\mathbb{C}^m$. Moreover, we observe that the coefficient of the monomial $\alpha_1 \cdots \alpha_m$ is given by $n!\eta\big(\varphi(e_{j_1}, \ldots, e_{j_m})\big)$, because both Λ_1 and Λ_2 are different from $\{1, \ldots, m\}$. We thus get

$$n!\eta\big(\varphi(e_{j_1}, \ldots, e_{j_m})\big) = 0.$$

Since this identity holds for each linear functional η, our claim follows. Property (5) now leads to

$$\begin{aligned} P(x) &= \varphi\left(\sum_{j=1}^{k} \rho_j e_j, \ldots, \sum_{j=1}^{k} \rho_j e_j\right) = \sum_{j_1,\ldots,j_m=1}^{k} \rho_{j_1} \cdots \rho_{j_m} \varphi\left(e_{j_1}, \ldots, e_{j_m}\right) \\ &= \sum_{j=1}^{k} \rho_j^m \varphi\left(e_j, \ldots, e_j\right) \end{aligned}$$

and

$$\begin{aligned} Q(x) &= \varphi\left(\left(\sum_{j=0}^{k} \rho_j e_j\right)^m, \sum_{j=0}^{k} e_j, \ldots, \sum_{j=0}^{k} e_j\right) = \varphi\left(\sum_{j=0}^{k} \rho_j^m e_j, \sum_{j=0}^{k} e_j, \ldots, \sum_{j=0}^{k} e_j\right) \\ &= \sum_{j_1,\ldots,j_m=0}^{k} \rho_{j_1}^m \varphi\left(e_{j_1}, \ldots, e_{j_m}\right) = \sum_{j=1}^{k} \rho_j^m \varphi\left(e_j, \ldots, e_j\right). \end{aligned}$$

We thus get $P(x) = Q(x)$.

Now suppose that $x \in \mathscr{A}_{\mathrm{sa}}$ is an arbitrary element. Since $\mathscr{A}$ has real rank zero, it follows that there exists a sequence (x_n) in $\mathscr{A}_{\mathrm{sa}}$ such that each x_n has finite spectrum and $\lim x_n = x$. On account of the above case, we have $P(x_n) = Q(x_n)$ $(n \in \mathbb{N})$, and the continuity of both P and Q now yields $P(x) = \lim P(x_n) = \lim Q(x_n) = Q(x)$, as required.

We are now in a position to prove the non-unital case. By hypothesis, there exists an increasing approximate unit of projections $(e_\lambda)_{\lambda\in\Lambda}$. For each $\lambda \in \Lambda$, set $\mathscr{A}_\lambda = e_\lambda \mathscr{A} e_\lambda$. Then $\mathscr{A}_\lambda$ is a unital C^*-algebra (with identity e_λ) and has real rank zero (because $\mathscr{A}_\lambda$ is a hereditary C^*-subalgebra of $\mathscr{A}$). From what has previously been proved, it follows that there exists a unique continuous linear map $\Phi_\lambda\colon \mathscr{A}_\lambda \to X$ such that

$$P(x) = \Phi_\lambda(x^m) \quad (x \in \mathscr{A}_\lambda). \tag{6}$$

Define

$$\mathscr{R} = \bigcup_{\lambda \in \Lambda} \mathscr{A}_\lambda$$

and, for each $x \in \mathscr{R}$, set

$$\Phi(x) = \Phi_\lambda(x),$$

where $\lambda \in \Lambda$ is such that $x \in \mathscr{A}_\lambda$. We will show that Φ is well-defined. Suppose $\lambda, \mu \in \Lambda$ are such that $x \in \mathscr{A}_\lambda \cap \mathscr{A}_\mu$. Then there exists $\nu \in \Lambda$ with $\lambda, \mu \leq \nu$. Since the net $(e_\lambda)_{\lambda \in \Lambda}$ is increasing, we see that $e_\lambda, e_\mu \leq e_\nu$ and therefore $\mathscr{A}_\lambda, \mathscr{A}_\mu \subset \mathscr{A}_\nu$. The uniqueness of the representation of P on both $\mathscr{A}_\lambda$ and $\mathscr{A}_\mu$ implies that $\Phi_\nu \mid_{\mathscr{A}_\lambda} = \Phi_\lambda$ and $\Phi_\nu \mid_{\mathscr{A}_\mu} = \Phi_\mu$, which implies that $\Phi_\lambda(x) = \Phi_\nu(x) = \Phi_\mu(x)$. We now show that $\mathscr{R}$ is a $*$-subalgebra of $\mathscr{A}$ and that Φ is linear. Take $x, y \in \mathscr{R}$ and $\alpha, \beta \in \mathbb{C}$. We take $\lambda, \mu \in \Lambda$ such that $x \in \mathscr{A}_\lambda$ and $y \in \mathscr{A}_\mu$. Then $x^* \in \mathscr{A}_\lambda \subset \mathscr{R}$. Now set $\nu \in \Lambda$ with $\lambda, \mu \leq \nu$. Hence $x, y \in \mathscr{A}_\nu$, so that $\alpha x + \beta y, xy \in \mathscr{A}_\nu \subset \mathscr{R}$, which shows that $\mathscr{R}$ is a subalgebra of $\mathscr{A}$. Further, we have

$$\Phi(\alpha x + \beta y) = \Phi_\nu(\alpha x + \beta y) = \alpha \Phi_\nu(x) + \beta \Phi_\nu(y) = \alpha \Phi(x) + \beta \Phi(y),$$

which shows that Φ is linear.

From (6) we deduce that $P(x) = \Phi(x^m)$ for each $x \in \mathscr{R}$.

Our next goal is to show that $\mathscr{R}$ satisfies the conditions of Lemma 2. If $x \in \mathscr{R}_{\mathrm{sa}}$ ($x \in \mathscr{R}_+$), then there exists $\lambda \in \Lambda$ such that $x \in (\mathscr{A}_\lambda)_{\mathrm{sa}}$ ($x \in (\mathscr{A}_\lambda)_+$, respectively) and therefore $|x| \in \mathscr{A}_\lambda \subset \mathscr{R}$ ($x^{1/m} \in \mathscr{A}_\lambda \subset \mathscr{R}$, respectively). Since the polynomial $P \mid_{\mathscr{R}}$ is continuous, Lemma 2 shows that the map Φ is continuous.

Since $(e_\lambda)_{\lambda \in \Lambda}$ is an approximate unit, it follows that $\mathscr{R}$ is dense in $\mathscr{A}$, and hence that the map Φ extends uniquely to a continuous linear map from $\mathscr{A}$ into the completion of X. By abuse of notation we continue to write Φ for this extension. Since both P and Φ are continuous, it may be concluded that $P(x) = \Phi(x^m)$ for each $x \in \mathscr{A}$. We next prove that the image of Φ is actually contained in X. Of course, it suffices to show that Φ takes $\mathscr{A}_+$ into X. If $x \in \mathscr{A}_+$, then

$$\Phi(x) = \Phi\left(\left(x^{1/m}\right)^m\right) = P\left(x^{1/m}\right) \in X,$$

as required. □

Since every von Neumann algebra is unital and has real rank zero, Theorem 3 applies in this setting and gives the following.

Corollary 1 *Let $\mathscr{M}$ be a von Neumann algebra, let X be a topological linear space, and let $P: \mathscr{M} \to X$ be a continuous m-homogeneous polynomial. Then the following conditions are equivalent:*

(i) *there exists a continuous linear map $\Phi: \mathscr{M} \to X$ such that $P(x) = \Phi(x^m)$ ($x \in \mathscr{M}$);*

(ii) *the polynomial P is orthogonally additive on $\mathscr{M}_{\mathrm{sa}}$;*
(iii) *the polynomial P is orthogonally additive on $\mathscr{M}_{+}$.*

If the conditions are satisfied, then the map Φ is unique.

Proposition 1 *Let H be a Hilbert space with* $\dim H \geq 2$*, let X be a topological linear space, and let $P \colon \mathscr{B}(H) \to X$ be a continuous m-homogeneous polynomial. Suppose that P is orthogonally additive in $\mathscr{B}(H)$. Then $P = 0$.*

Proof For each unitary $v \in \mathscr{B}(H)$, the map $P_v \colon \mathscr{B}(H) \to X$ defined by

$$P_v(x) = P(vx) \quad (x \in \mathscr{B}(H))$$

is easily seen to be a continuous m-homogeneous polynomial that is orthogonally additive on $\mathscr{B}(H)$. In particular, P_v is orthogonally additive on $\mathscr{B}(H)_{\mathrm{sa}}$, and Corollary 1 then gives a unique continuous linear map $\Phi_v \colon \mathscr{B}(H) \to X$ such that

$$P(vx) = \Phi_v(x^m) \quad (x \in \mathscr{B}(H)).$$

We claim that, if $e, e' \in \mathscr{B}(H)$ are equivalent projections with $e \perp e'$, then $P(e) = P(e') = 0$. Let $u \in \mathscr{B}(H)$ be a partial isometry such that $u^*u = e$ and $uu^* = e'$. Then

$$\left\|u^2\right\|^4 = \left\|(u^2)^*u^2\right\|^2 = \left\|\left((u^2)^*u^2\right)^2\right\| = \left\|u^*ee'eu\right\| = 0,$$

which gives $u^2 = 0$. From this we see that $u \perp u^*$, and therefore

$$P(vu + vu^*) = P_v(u + u^*) = P_v(u) + P_v(u^*) = \Phi_v\left(u^m\right) + \Phi_v\left((u^*)^m\right) = 0. \quad (7)$$

We now take $\omega \in \mathbb{C}$ with $\omega^m = -1$, and define

$$v = 1 + u + u^* - e - e',$$
$$v_\omega = 1 + \omega u + u^* - e - e'.$$

It is immediately seen that both v and v_ω are unitary, and so applying (7) (and using the orthogonal additivity of P and that $e \perp e'$), we see that

$$0 = P(vu + vu^*) = P(e + e') = P(e) + P(e'),$$
$$0 = P(v_\omega u + v_\omega u^*) = P(e + \omega e') = P(e) + P(\omega e') = P(e) - P(e').$$

By comparing both identities, we conclude that $P(e) = P(e') = 0$, as claimed.

Our next objective is to prove that $P(e) = 0$ for each projection $e \in \mathscr{B}(H)$. Suppose that $e \in \mathscr{B}(H)$ is a rank-one projection. Since $\dim H \geq 2$, it follows that there exists an equivalent projection e' such that $e' \perp e$. Then it follows from the above claim that $P(e) = 0$. Let $e \in \mathscr{B}(H)$ be a finite projection. Then there exist mutually orthogonal projections $e_1, \ldots, e_n$ such that $e_1 + \cdots + e_n = e$. Using the preceding observation and the orthogonal additivity of P we get $P(e) = P(e_1) + \cdots + P(e_n) = 0$. We now assume that $e \in \mathscr{B}(H)$ is an infinite projection. Then there exist mutually orthogonal, equivalent projections e_1 and e_2 such that $e_1 + e_2 = e$. By the claim, we have $P(e) = P(e_1) + P(e_2) = 0$.

We finally proceed to show that $P = 0$. By Lemma 1, it suffices to show that $P(x) = 0$ for each $x \in \mathscr{B}(H)_+$. Suppose that $x \in \mathscr{B}(H)_+$ can be written in the form $x = \sum_{j=1}^{k} \rho_j e_j$, where $e_1, \ldots, e_k \in \mathscr{B}(H)$ are mutually orthogonal projections and $\rho_1, \ldots, \rho_k \in \mathbb{R}^+$. Then we have $P(x) = \sum_{j=1}^{k} \rho_j{}^m P(e_j) = 0$. Now let $x \in \mathscr{B}(H)_+$ be an arbitrary element. From the spectral decomposition we deduce that there exists a sequence (x_n) in $\mathscr{B}(H)_+$ such that each x_n is a positive linear combination of mutually orthogonal projections and $\lim x_n = x$. On account of the preceding observation, $P(x_n) = 0$ $(n \in \mathbb{N})$, and the continuity of P implies that $P(x) = \lim P(x_n) = 0$, as required. □

3 Non-commutative L^p-spaces

Before giving the next results we make the following preliminary remarks.

A fundamental fact for us is the behaviour of the product of $L^0(\mathscr{M}, \tau)$ when restricted to the L^p-spaces. Specifically, if $0 < p, q, r \leq \infty$ are such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, then the Hölder inequality states that

$$x \in L^p(\mathscr{M}, \tau),\ y \in L^q(\mathscr{M}, \tau) \Rightarrow xy \in L^r(\mathscr{M}, \tau) \text{ and } \|xy\|_r \leq \|x\|_p \|y\|_q. \tag{8}$$

Suppose that $x, y \in L^p(\mathscr{M}, \tau)_+$, $0 < p < \infty$, are mutually orthogonal and that $\omega \in \mathbb{C}$ with $|\omega| = 1$. Then it is immediately seen that $|x + \omega y| = x + y$, and it follows, by considering the spectral resolutions of x, y, and $x + y$, that $(x + y)^p = x^p + y^p$. Hence

$$\|x + \omega y\|_p^p = \|x\|_p^p + \|y\|_p^p. \tag{9}$$

Each $x \in L^p(\mathscr{M}, \tau)$ can be written in the form

$$\begin{aligned} x = x_1 - x_2 + i(x_3 - x_4),\ \text{with } & x_1, x_2, x_3, x_4 \in L^p(\mathscr{M}, \tau)_+, \\ & x_1 \perp x_2,\ x_3 \perp x_4, \\ & \|x_1\|_p^p + \|x_2\|_p^p = \|x_1 - x_2\|_p^p \leq \|x\|_p^p, \\ & \|x_3\|_p^p + \|x_4\|_p^p = \|x_3 - x_4\|_p^p \leq \|x\|_p^p. \end{aligned} \tag{10}$$

Indeed, first we write $x = \Re x + i \Im x$, where

$$\Re x = \frac{1}{2}(x^* + x),\ \Im x = \frac{i}{2}(x^* - x) \in L^p(\mathscr{M}, \tau)_{\mathrm{sa}},$$

and, since $\|x^*\|_p = \|x\|_p$, it follows that $\|\Re x\|_p, \|\Im x\|_p \leq \|x\|_p$. Further, we take the positive operators

$$x_1 = \tfrac{1}{2}\left(|\Re x| + \Re x\right),\ x_2 = \tfrac{1}{2}\left(|\Re x| - \Re x\right),\ x_3 = \tfrac{1}{2}\left(|\Im x| + \Im x\right),\ x_4 = \tfrac{1}{2}\left(|\Im x| - \Im x\right).$$

Then $x_1, x_2, x_3, x_4 \in L^p(\mathscr{M}, \tau)$, $\Re x = x_1 - x_2$ with $x_1 \perp x_2$, so that (9) gives

$$\|\Re x\|_p^p = \|x_1\|_p^p + \|x_2\|_p^p,$$

and $\Im x = x_3 - x_4$ with $x_3 \perp x_4$, so that (9) gives

$$\|\Im x\|_p^p = \|x_3\|_p^p + \|x_4\|_p^p.$$

Theorem 4 *Let $\mathscr{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ, let X be a topological linear space, and let $\Phi\colon L^{p/m}(\mathscr{M}, \tau) \to X$ be a continuous linear map with $0 < p < \infty$. Then:*

(i) *the map $P_\Phi\colon L^p(\mathscr{M}, \tau) \to X$ defined by $P_\Phi(x) = \Phi(x^m)$ $(x \in L^p(\mathscr{M}, \tau))$ is a continuous m-homogeneous polynomial which is orthogonally additive on $L^p(\mathscr{M}, \tau)_{\mathrm{sa}}$;*
(ii) *the polynomial P_Φ is uniquely specified by the map Φ.*

Suppose, further, that X is a q-normed space, $0 < q \le 1$. Then:

(iii) $2^{-1/q}\|\Phi\| \le \|P_\Phi\| \le \|\Phi\|$.

Moreover, in the case where $X = \mathbb{C}$,

(iv) *the functional Φ is hermitian if and only if the polynomial P_Φ is hermitian, in which case $\|P_\Phi\| = \|\Phi\|$.*

Proof The proof of this result is similar to that establishing Theorem 1.
(i) It follows immediately from (8) that, for each $x_1, \ldots, x_m \in L^p(\mathscr{M}, \tau)$,

$$x_1 \cdots x_m \in L^{p/m}(\mathscr{M}, \tau) \text{ and } \|x_1 \cdots x_m\|_{p/m} \le \|x_1\|_p \cdots \|x_m\|_p. \tag{11}$$

On the one hand, this clearly implies that the map P_Φ is well-defined, on the other hand, the map $x \mapsto x^m$ from $L^p(\mathscr{M}, \tau)$ into $L^{p/m}(\mathscr{M}, \tau)$ is continuous, and so P_Φ is continuous. Further, P_Φ is the m-homogeneous polynomial associated with the symmetric m-linear map $\varphi\colon L^p(\mathscr{M}, \tau)^m \to X$ defined by

$$\varphi(x_1, \ldots, x_m) = \frac{1}{m!} \sum_{\sigma \in \mathfrak{S}_m} \Phi\left(x_{\sigma(1)} \cdots x_{\sigma(m)}\right) \quad (x_1, \ldots, x_m \in L^p(\mathscr{M}, \tau)).$$

Suppose that $x, y \in L^p(\mathscr{M}, \tau)_{\mathrm{sa}}$ are such that $x \perp y$. Then $xy = yx = 0$, and so $(x + y)^m = x^m + y^m$, which gives

$$P_\Phi(x + y) = \Phi\big((x + y)^m\big) = \Phi\big(x^m + y^m\big) = \Phi\big(x^m\big) + \Phi\big(y^m\big) = P_\Phi(x) + P_\Phi(y).$$

(ii) Suppose that $\Psi\colon L^{p/m}(\mathscr{M}, \tau) \to X$ is a linear map such that $P_\Psi = P_\Phi$. For each $x \in L^{p/m}(\mathscr{M}, \tau)_+$, we have $x^{1/m} \in L^p(\mathscr{M}, \tau)$ and

$$\Phi(x) = \Phi\big(\big(x^{1/m}\big)^m\big) = P\big(x^{1/m}\big) = \Psi\big(\big(x^{1/m}\big)^m\big) = \Psi(x).$$

By linearity we obtain $\Phi = \Psi$.

(iii) Next, assume that X is a q-normed space. For each $x \in L^p(\mathscr{M}, \tau)$, by (11), we have

$$\|P_\Phi(x)\| = \|\Phi(x^m)\| \le \|\Phi\|\|x^m\|_{p/m} \le \|\Phi\|\|x\|_p^m,$$

which clearly implies that $\|P_\Phi\| \le \|\Phi\|$. Now take $x \in L^{p/m}(\mathscr{M}, \tau)$, and take $\omega \in \mathbb{C}$ with $\omega^m = -1$. Write

$$x = \Re x + i\Im x = x_1 - x_2 + i(x_3 - x_4)$$

as in (10) (with p/m instead of p). Since $x_1 \perp x_2$ and $x_3 \perp x_4$, it follows that $x_1^{1/m} \perp x_2^{1/m}$ and $x_3^{1/m} \perp x_4^{1/m}$, so that (9) gives

$$\begin{aligned} \|\Re x\|_{p/m}^{p/m} &= \|x_1\|_{p/m}^{p/m} + \|x_2\|_{p/m}^{p/m}, \\ \|\Im x\|_{p/m}^{p/m} &= \|x_3\|_{p/m}^{p/m} + \|x_4\|_{p/m}^{p/m}, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \big\|x_1^{1/m} + \omega x_2^{1/m}\big\|_p^p &= \big\|x_1^{1/m}\big\|_p^p + \big\|x_2^{1/m}\big\|_p^p, \\ \big\|x_3^{1/m} + \omega x_4^{1/m}\big\|_p^p &= \big\|x_3^{1/m}\big\|_p^p + \big\|x_4^{1/m}\big\|_p^p. \end{aligned} \tag{13}$$

Further, we have $x_1^{1/m}, x_2^{1/m}, x_3^{1/m}, x_4^{1/m} \in L^p(\mathscr{M}, \tau)$ and

$$\big\|x_1^{1/m}\big\|_p = \|x_1\|_{p/m}^{1/m},\ \big\|x_2^{1/m}\big\|_p = \|x_2\|_{p/m}^{1/m},\ \big\|x_3^{1/m}\big\|_p = \|x_3\|_{p/m}^{1/m},\ \big\|x_4^{1/m}\big\|_p = \|x_4\|_{p/m}^{1/m},$$

so that (12) and (13) give

$$\begin{aligned} \big\|x_1^{1/m} + \omega x_2^{1/m}\big\|_p^p &= \|\Re x\|_{p/m}^{p/m}, \\ \big\|x_3^{1/m} + \omega x_4^{1/m}\big\|_p^p &= \|\Im x\|_{p/m}^{p/m}. \end{aligned} \tag{14}$$

On the other hand, we have

$$\Big(x_1^{1/m} + \omega x_2^{1/m}\Big)^m = x_1 - x_2 = \Re x, \quad \Big(x_3^{1/m} + \omega x_4^{1/m}\Big)^m = x_3 - x_4 = \Im x,$$

whence

$$\begin{aligned} \Phi(x) &= \Phi(\Re x) + i\Phi(\Im x) = \Phi\big(\big(x_1^{1/m} + \omega x_2^{1/m}\big)^m\big) + i\Phi\big(\big(x_3^{1/m} + \omega x_4^{1/m}\big)^m\big) \\ &= P_\Phi\big(x_1^{1/m} + \omega x_2^{1/m}\big) + iP_\Phi\big(x_3^{1/m} + \omega x_4^{1/m}\big). \end{aligned}$$

Hence, by (14),

$$\begin{aligned}\|\Phi(x)\|^q &\le \left\|P_\Phi\left(x_1^{1/m}+\omega x_2^{1/m}\right)\right\|^q+\left\|P_\Phi\left(x_3^{1/m}+\omega x_4^{1/m}\right)\right\|^q\\ &\le \|P_\Phi\|^q\left\|x_1^{1/m}+\omega x_2^{1/m}\right\|_p^{mq}+\|P_\Phi\|^q\left\|x_3^{1/m}+\omega x_4^{1/m}\right\|_p^{mq}\\ &= \|P_\Phi\|^q\left(\|\Re x\|^q+\|\Im x\|^q\right)\\ &\le \|P_\Phi\|^q 2\|x\|^q.\end{aligned}$$

This clearly forces $\|\Phi\| \le 2^{1/q}\|P_\Phi\|$, as claimed.

(iv) It is straightforward to check that $P_\Phi^* = P_{\Phi^*}$. From this deduce that Φ is hermitian if and only if P_Φ is hermitian as in the proof of Theorem 1(iv). Suppose that Φ is a hermitian functional. By direct calculation, we see that P_Φ is hermitian, and it remains to prove that $\|P_\Phi\| = \|\Phi\|$. We only need to show that $\|\Phi\| \le \|P_\Phi\|$. To this end, let $\varepsilon \in \mathbb{R}^+$, and choose $x \in L^{p/m}(\mathscr{M},\tau)$ such that $\|x\|_{p/m} = 1$ and $\|\Phi\| - \varepsilon < |\Phi(x)|$. We take $\alpha \in \mathbb{C}$ with $|\alpha| = 1$ and $|\Phi(x)| = \alpha\Phi(x)$, so that

$$\|\Phi\| - \varepsilon < |\Phi(x)| = \Phi(\alpha x) = \overline{\Phi(\alpha x)} = \Phi\big((\alpha x)^*\big).$$

We see that $\Re(\alpha x) \in L^{p/m}(\mathscr{M},\tau)_{\mathrm{sa}}$, $\|\Re(\alpha x)\|_{p/m} \le 1$, and $\|\Phi\| - \varepsilon < \Phi(\Re(\alpha x))$. Now we consider the decomposition $\Re(\alpha x) = x_1 - x_2$ as in (10) (with p/m instead of p), and take $\omega \in \mathbb{C}$ with $\omega^m = -1$. As in (14), we see that $\left\|x_1^{1/m} + \omega x_2^{1/m}\right\| = \|\Re(\alpha x)\|^{1/m} \le 1$. Moreover, we have

$$P_\Phi\left(x_1^{1/m} + \omega x_2^{1/m}\right) = \Phi\left(\left(x_1^{1/m} + \omega x_2^{1/m}\right)^m\right) = \Phi(\Re(\alpha x)),$$

and so $\|\Phi\| - \varepsilon < \|P_\Phi\|$. □

Theorem 5 *Let $\mathscr{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ, let X be a topological linear space, and let $P\colon L^p(\mathscr{M},\tau) \to X$ be a continuous m-homogeneous polynomial with $0 < p < \infty$. Then the following conditions are equivalent:*

(i) *there exists a continuous linear map $\Phi\colon L^{p/m}(\mathscr{M},\tau) \to X$ such that $P(x) = \Phi(x^m)$ $(x \in L^p(\mathscr{M},\tau))$;*
(ii) *the polynomial P is orthogonally additive on $L^p(\mathscr{M},\tau)_{\mathrm{sa}}$;*
(iii) *the polynomial P is orthogonally additive on $S(\mathscr{M},\tau)_+$.*

If the conditions are satisfied, then the map Φ is unique.

Proof Theorem 4 shows that (i)⇒(ii), and it is obvious that (ii)⇒(iii). We proceed to prove that (iii)⇒(i).

Suppose that (iii) holds. Let $e \in \mathscr{M}$ be a projection such that $\tau(e) < \infty$, and consider the von Neumann algebra $\mathscr{M}_e = e\mathscr{M}e$. We claim that $\mathscr{M}_e \subset S(\mathscr{M},\tau)$ and that there exists a unique continuous linear map $\Phi_e\colon \mathscr{M}_e \to X$ such that

$$P(x) = \Phi_e(x^m) \quad (x \in \mathscr{M}_e). \tag{15}$$

Set $x \in \mathscr{M}_e$, and write $x = (x_1 - x_2) + i(x_3 - x_4)$ with $x_1, x_2, x_3, x_4 \in \mathscr{M}_{e+}$. Then $\operatorname{supp}(x_j) \le e$ and therefore $\tau\big(\operatorname{supp}(x_j)\big) \le \tau(e) < \infty$ ($j \in \{1, 2, 3, 4\}$). This shows that $x_j \in S(\mathscr{M}, \tau)$ ($j \in \{1, 2, 3, 4\}$), whence $x \in S(\mathscr{M}, \tau)$. Our next goal is to show that the restriction $P\mid_{\mathscr{M}_e}$ is continuous (with respect to the norm that $\mathscr{M}_e$ inherits as a closed subspace of $\mathscr{M}$). Let $x \in \mathscr{M}_e$, and let $U \subset X$ be a neighbourhood of $P(x)$. Since P is continuous, the set $P^{-1}(U)$ is a neighbourhood of x in $L^p(\mathscr{M}, \tau)$, which implies that there exists $r \in \mathbb{R}^+$ such that $P(y) \in U$ whenever $y \in L^p(\mathscr{M}, \tau)$ and $\|y - x\|_p < r$. If $y \in \mathscr{M}_e$ is such that $\|y - x\| < r/\|e\|_p$, then, from (8), we obtain

$$\|y - x\|_p = \|e(y - x)\|_p \le \|e\|_p \|y - x\| < r$$

and therefore $P(y) \in U$. Hence $P\mid_{\mathscr{M}_e}$ is continuous. Since, by hypothesis, the polynomial $P\mid_{\mathscr{M}_e}$ is orthogonally additive on $\mathscr{M}_{e+}$, Corollary 1 states that there exists a unique continuous linear map $\Phi_e\colon \mathscr{M}_e \to X$ such that (15) holds.

For each $x \in S(\mathscr{M}, \tau)$, define

$$\Phi(x) = \Phi_e(x),$$

where $e \in \mathscr{M}$ is any projection such that

$$ex = xe = x \quad \text{and} \quad \tau(e) < \infty. \tag{16}$$

We will show that Φ is well-defined. For this purpose we first check that, if $x \in S(\mathscr{M}, \tau)$, then there exists a projection e such that (16) holds. Indeed, we write $x = \sum_{j=1}^{k} \alpha_j x_j$ with $\alpha_1, \dots, \alpha_k \in \mathbb{C}$ and $x_1, \dots, x_k \in S(\mathscr{M}, \tau)_+$, and define $e = \operatorname{supp}(x_1) \vee \cdots \vee \operatorname{supp}(x_k)$. Then $ex = xe = x$ and $\tau(e) \le \sum_{j=1}^{k} \tau\big(\operatorname{supp}(x_j)\big) < \infty$, as required. Suppose that $x \in S(\mathscr{M}, \tau)$ and that $e_1, e_2 \in \mathscr{M}$ are projections satisfying (16). Then the projection $e = e_1 \vee e_2$ satisfies (16) and $\mathscr{M}_{e_1}, \mathscr{M}_{e_2} \subset \mathscr{M}_e$. The uniqueness of the representation (15) on both $\mathscr{M}_{e_1}$ and $\mathscr{M}_{e_2}$ gives $\Phi_e\mid_{\mathscr{M}_{e_1}} = \Phi_{e_1}$ and $\Phi_e\mid_{\mathscr{M}_{e_2}} = \Phi_{e_2}$, which implies that $\Phi_{e_1}(x) = \Phi_e(x) = \Phi_{e_2}(x)$.

We now show that Φ is linear. Take $x_1, x_2 \in S(\mathscr{M}, \tau)$ and $\alpha, \beta \in \mathbb{C}$. Let $e_1, e_2 \in \mathscr{M}$ be projections such that $e_j x_j = x_j e_j = x_j$ and $\tau(e_j) < \infty$ ($j \in \{1, 2\}$). Then the projection $e = e_1 \vee e_2$ satisfies

$$ex_j = x_j e = x_j \quad (j \in \{1, 2\}),$$
$$e(\alpha x_1 + \beta x_2) = (\alpha x_1 + \beta x_2)e = \alpha x_1 + \beta x_2,$$

and

$$\tau(e) \le \tau(e_1) + \tau(e_2) < \infty.$$

Thus

$$\Phi(x_j) = \Phi_e(x_j) \quad (j \in \{1, 2\})$$

and

$$\Phi(\alpha x_1 + \beta x_2) = \Phi_e(\alpha x_1 + \beta x_2) = \alpha\Phi_e(x_1) + \beta\Phi_e(x_2) = \alpha\Phi(x_1) + \beta\Phi(x_2).$$

We see from the definition of Φ that

$$P(x) = \Phi(x^m) \quad (x \in S(\mathscr{M}, \tau)). \tag{17}$$

Our next concern will be the continuity of Φ with respect to the norm $\|\cdot\|_{p/m}$. Let U be a neighbourhood of 0 in X. Let V be a balanced neighbourhood of 0 in X with $V + V + V + V \subset U$. The set $P^{-1}(V)$ is a neighbourhood of 0 in $L^p(\mathscr{M}, \tau)$, which implies that there exists $r \in \mathbb{R}^+$ such that $P(x) \in V$ whenever $x \in L^p(\mathscr{M}, \tau)$ and $\|x\|_p < r$. Take $x \in S(\mathscr{M}, \tau)$ with $\|x\|_{p/m} < r^m$, and write $x = (x_1 - x_2) + i(x_3 - x_4)$ as in (10) (with p/m instead of p). Then it is immediate to check that actually $x_1, x_2, x_3, x_4 \in S(\mathscr{M}, \tau)_+$ and, further, $\|x_j\|_{p/m} \leq \|x\|_{p/m}$ $(j \in \{1, 2, 3, 4\})$. For each $j \in \{1, 2, 3, 4\}$, we have

$$\begin{aligned}\left\|x_j^{1/m}\right\|_p &= \tau\left(x_j^{p/m}\right)^{1/p} = \left(\tau\left(x_j^{p/m}\right)^{m/p}\right)^{1/m} = \|x_j\|_{p/m}{}^{1/m}\\ &\leq \|x\|_{p/m}^{1/m} < r,\end{aligned}$$

whence

$$\begin{aligned}\Phi(x) &= \Phi\left(\left(x_1^{1/m}\right)^m - \left(x_2^{1/m}\right)^m + i\left(x_3^{1/m}\right)^m - i\left(x_4^{1/m}\right)^m\right)\\ &= \Phi\left(\left(x_1^{1/m}\right)^m\right) - \Phi\left(\left(x_2^{1/m}\right)^m\right) + i\Phi\left(\left(x_3^{1/m}\right)^m\right) - i\Phi\left(\left(x_4^{1/m}\right)^m\right)\\ &= P\left(x_1^{1/m}\right) - P\left(x_2^{1/m}\right)\\ &\quad + iP\left(x_3^{1/m}\right) - iP\left(x_4^{1/m}\right) \in V + V + V + V \subset U,\end{aligned}$$

which establishes the continuity of Φ. Since $S(\mathscr{M}, \tau)$ is dense in $L^{p/m}(\mathscr{M}, \tau)$, the map Φ extends uniquely to a continuous linear map from $L^{p/m}(\mathscr{M}, \tau)$ into the completion of X. By abuse of notation we continue to write Φ for this extension. Since both P and Φ are continuous, (17) gives $P(x) = \Phi(x^m)$ for each $x \in L^p(\mathscr{M})$. The task is now to show that the image of Φ is actually contained in X. Of course, it suffices to show that Φ takes $L^{p/m}(\mathscr{M}, \tau)_+$ into X. Let $x \in L^{p/m}(\mathscr{M}, \tau)_+$. Then $x^{1/m} \in L^p(\mathscr{M}, \tau)_+$ and

$$\Phi(x) = \Phi\left(\left(x^{1/m}\right)^m\right) = P\left(x^{1/m}\right) \in X,$$

as required.

The uniqueness of the map Φ is given by Theorem 4(ii). □

Let us note that the space of all continuous m-homogeneous polynomials from $L^p(\mathscr{M}, \tau)$ into any topological linear space X which are orthogonally additive on $S(\mathscr{M}, \tau)_+$ is sufficiently rich in the case where $p/m \geq 1$, because of the existence of continuous linear functionals on $L^{p/m}(\mathscr{M},\tau)$. However, some restriction on the space X must be imposed when we consider the case $p/m < 1$ and the von Neumann algebra $\mathscr{M}$ has no minimal projections, because in this case the dual of $L^{p/m}(\mathscr{M},\tau)$ is trivial ([15]). In fact, there are no non-zero continuous linear maps from $L^{p}(\mathscr{M},\tau)$ into any q-normed space X with $q > p$. We think that this property is probably well-known, but we have not been able to find any reference, so that we next present a proof of this result for completeness.

Proposition 2 *Let $\mathscr{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ and with no minimal projections, let X be a q-normed space, $0 < q \le 1$, and let $\Phi\colon L^p(\mathscr{M},\tau) \to X$ be a continuous linear map with $0 < p < q$. Then $\Phi = 0$.*

Proof The proof will be divided in a number of steps.

Our first step is to show that for each projection $e_0 \in \mathscr{M}$ with $\tau(e_0) < \infty$ and each $0 \le \rho \le \tau(e_0)$, there exists a projection $e \in \mathscr{M}$ such that $e \le e_0$ and $\tau(e) = \rho$. Set

$$\mathscr{P}_1 = \big\{e \in \mathscr{M} : e \text{ is a projection}, \ e \le e_0,\ \tau(e) \ge \rho\big\}.$$

Note that $e_0 \in \mathscr{P}_1$, so that $\mathscr{P}_1$ is non-empty. Let $\mathscr{C}$ be a chain in $\mathscr{P}_1$, and let $e' = \wedge_{e\in\mathscr{C}} e$. Then e' is a projection and $e' \le e_0$. For each $e \in \mathscr{C}$, since $\tau(e_0) < \infty$, it follows that $\tau(e_0) - \tau(e) = \tau(e_0 - e)$. From the normality of τ we now deduce that

$$\begin{aligned}\tau(e_0) - \inf_{e\in\mathscr{C}} \tau(e) &= \sup_{e\in\mathscr{C}}\big(\tau(e_0) - \tau(e)\big) = \sup_{e\in\mathscr{C}} \tau(e_0 - e)\\ &= \tau\left(\vee_{e\in\mathscr{C}}(e_0 - e)\right) = \tau(e_0 - e').\end{aligned}$$

Hence $\tau(e') = \inf_{e\in\mathscr{C}} \tau(e) \ge \rho$, which shows that e' is a lower bound of $\mathscr{C}$, and so, by Zorn's lemma, $\mathscr{P}_1$ has a minimal element, say e_1. We now consider the set

$$\mathscr{P}_2 = \big\{e \in \mathscr{M} : e \text{ is a projection}, \ e \le e_1,\ \tau(e) \le \rho\big\}.$$

Note that $0 \in \mathscr{P}_2$, so that $\mathscr{P}_2$ is non-empty. Let $\mathscr{C}$ be a chain in $\mathscr{P}_2$, and let $e' = \vee_{e\in\mathscr{C}} e$. Then $e' \le e_1$, and the normality of τ yields

$$\tau(e') = \sup_{e\in\mathscr{C}} \tau(e) \le \rho.$$

This implies that e' is an upper bound of $\mathscr{C}$, and so, by Zorn's lemma, $\mathscr{P}_2$ has a maximal element, say e_2. Assume towards a contradiction that $e_1 \ne e_2$. Since, by hypothesis, $\mathscr{M}$ has no minimal projections, it follows that there exists a non-zero projection $e < e_1 - e_2$. Since $e \perp e_2$, we see that $e_2 + e$ is a projection. Further, we have $e_2 < e_2 + e < e_1$. The maximality of e_2 implies that $\tau(e_2 + e) > \rho$, which implies that $e_2 + e \in \mathscr{P}_1$, contradicting the minimality of e_1. Thus $e_1 = e_2$, and this clearly implies that $\tau(e_1) = \tau(e_2) = \rho$.

Our next goal is to show that $\Phi(e_0) = 0$ for each projection e_0 with $\tau(e_0) < \infty$. From the previous step, it follows that there exists a projection $e \le e_0$ with $\tau(e) = \frac{1}{2}\tau(e_0)$. Set $e' = e_0 - e$. Then $\tau(e') = \frac{1}{2}\tau(e_0)$. Further,

$$\|\Phi(e_0)\|^q = \|\Phi(e) + \Phi(e')\|^q \le \|\Phi(e)\|^q + \|\Phi(e')\|^q,$$

and therefore either $\|\Phi(e)\|^q \ge \frac{1}{2}\|\Phi(e_0)\|^q$ or $\|\Phi(e')\|^q \ge \frac{1}{2}\|\Phi(e_0)\|^q$. We define e_1 to be any of the projections e, e' for which the inequality holds. We thus get $e_1 \le e_0$, $\tau(e_1) = \frac{1}{2}\tau(e_0)$, and $\|\Phi(e_1)\| \ge 2^{-1/q}\|\Phi(e_0)\|$. By repeating the process, we get a decreasing sequence of projections (e_n) such that

$$\tau(e_n) = 2^{-n}\tau(e_0) \quad \text{and} \quad \|\Phi(e_n)\| \ge 2^{-n/q}\|\Phi(e_0)\| \quad (n \in \mathbb{N}).$$

Then

$$\left\|2^{n/q}e_n\right\|_p = 2^{n/q}\tau(e_n)^{1/p} = 2^{n(1/q-1/p)}\tau(e_0)^{1/p},$$

which converges to zero, because $p < q$. Since Φ is continuous and $\|\Phi(e_0)\| \le \left\|\Phi\big(2^{n/q}e_n\big)\right\|_p$ $(n \in \mathbb{N})$, it may be concluded that $\Phi(e_0) = 0$, as claimed.

Our next concern is to show that Φ vanishes on $S(\mathscr{M}, \tau)$. Of course, it suffices to show that Φ vanishes on $S(\mathscr{M}, \tau)_+$. Take $x \in S(\mathscr{M}, \tau)_+$, and let $e = \operatorname{supp}(x)$, so that $\tau(e) < \infty$. The spectral decomposition implies that there exists a sequence (x_n) in $\mathscr{M}_+$ such that $\lim x_n = x$ with respect to the operator norm and each x_n is of the form $x_n = \sum_{j=1}^k \rho_j e_j$, where $\rho_1, \dots, \rho_k \in \mathbb{R}^+$ and $e_1, \dots, e_k \in \mathscr{M}$ are mutually orthogonal projections with $e_j e = e e_j = e_j$ $(j \in \{1, \dots, k\})$. From the previous step, we conclude that $\Phi(x_n) = 0$ $(n \in \mathbb{N})$. Further, from (8) we deduce that

$$\|x - x_n\|_p = \|e(x - x_n)\|_p \le \|e\|_p\|x - x_n\| \to 0,$$

and the continuity of Φ implies that $\Phi(x) = 0$, as required.

Finally, since $S(\mathscr{M}, \tau)$ is dense in $L^p(\mathscr{M}, \tau)$ and Φ is continuous, it may be concluded that $\Phi = 0$. □

Corollary 2 *Let $\mathscr{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ and with no minimal projections, let X be a q-normed space, $0 < q \le 1$, and let $P\colon L^p(\mathscr{M}, \tau) \to X$ be a continuous m-homogeneous polynomial with $0 < p/m < q$. Suppose that P is orthogonally additive on $S(\mathscr{M}, \tau)_+$. Then $P = 0$.*

Proof This is a straightforward consequence of Theorem 5 and Proposition 2. □

We now turn our attention to the complex-valued polynomials. In this setting the representation given in Theorem 5 has a particularly significant integral form, because of the well-known representation of the dual of the L^p-spaces. The trace gives rise to a distinguished contractive positive linear functional on $L^1(\mathscr{M}, \tau)$, still denoted by τ. By (8), if $\frac{1}{p} + \frac{1}{q} = 1$, for each $\zeta \in L^q(\mathscr{M}, \tau)$, the formula

$$\Phi_\zeta(x) = \tau(\zeta x) \quad (x \in L^p(\mathscr{M}, \tau)) \tag{18}$$

defines a continuous linear functional on $L^p(\mathscr{M}, \tau)$. Further, in the case where $1 \le p < \infty$, the map $\zeta \mapsto \Phi_\zeta$ is an isometric isomorphism from $L^q(\mathscr{M}, \tau)$ onto the dual space of $L^p(\mathscr{M}, \tau)$. It is immediate to see that $\Phi_\zeta^* = \Phi_{\zeta^*}$, so that Φ_ζ is hermitian if and only if ζ is self-adjoint.

Corollary 3 *Let $\mathscr{M}$ be a von Neumann algebra with a normal semifinite faithful trace τ, and let $P\colon L^p(\mathscr{M}, \tau) \to \mathbb{C}$ be a continuous m-homogeneous polynomial with $m \le p < \infty$. Then the following conditions are equivalent:*

(i) *there exists $\zeta \in L^r(\mathscr{M}, \tau)$ such that $P(x) = \tau(\zeta x^m)$ $(x \in L^p(\mathscr{M}, \tau))$, where $r = p/(p-m)$ (with the convention that $p/0 = \infty$);*
(ii) *the polynomial P is orthogonally additive on $L^p(\mathscr{M}, \tau)_{\mathrm{sa}}$;*
(iii) *the polynomial P is orthogonally additive on $S(\mathscr{M}, \tau)_+$.*

If the conditions are satisfied, then ζ is unique and $\|P\| \le \|\zeta\|_r \le 2\|P\|$; moreover, if P is hermitian, then ζ is self-adjoint and $\|\zeta\|_r = \|P\|$.

Proof This follows from Theorems 4 and 5. □

Let H be a Hilbert space. We denote by Tr the usual trace on the von Neumann algebra $\mathscr{B}(H)$. Then $L^p(\mathscr{B}(H), \mathrm{Tr})$, with $0 < p < \infty$, is the Schatten class $S^p(H)$. In the case where $0 < p < q$, we have $S^p(H) \subset S^q(H) \subset \mathscr{K}(H)$ and $\|x\| \le \|x\|_q \le \|x\|_p$ $(x \in S^p(H))$. It is clear that $S(\mathscr{B}(H), \mathrm{Tr}) = \mathscr{F}(H)$, the two-sided ideal of $\mathscr{B}(H)$ consisting of the finite-rank operators. Thus, the following result is an immediate consequence of Corollary 3.

Corollary 4 *Let H be a Hilbert space, and let $P\colon S^p(H) \to \mathbb{C}$ be a continuous m-homogeneous polynomial with $m < p < \infty$. Then the following conditions are equivalent:*

(i) *there exists $\zeta \in S^r(H)$ such that $P(x) = \mathrm{Tr}(\zeta x^m)$ $(x \in S^p(H))$, where $r = p/(p-m)$;*
(ii) *the polynomial P is orthogonally additive on $S^p(H)_{\mathrm{sa}}$;*
(iii) *the polynomial P is orthogonally additive on $\mathscr{F}(H)_+$.*

If the conditions are satisfied, then ζ is unique and $\|P\| \le \|\zeta\|_r \le 2\|P\|$; moreover, if P is hermitian, then ζ is self-adjoint and $\|\zeta\|_r = \|P\|$.

Corollary 5 *Let H be a Hilbert space, and let $P\colon \mathscr{K}(H) \to \mathbb{C}$ be a continuous m-homogeneous polynomial. Then the following conditions are equivalent:*

(i) *there exists $\zeta \in S^1(H)$ such that $P(x) = \mathrm{Tr}(\zeta x^m)$ $(x \in \mathscr{K}(H))$;*
(ii) *the polynomial P is orthogonally additive on $\mathscr{K}(H)_{\mathrm{sa}}$;*
(iii) *the polynomial P is orthogonally additive on $\mathscr{F}(H)_+$.*

If the conditions are satisfied, then ζ is unique and $\|P\| \le \|\zeta\|_1 \le 2\|P\|$; moreover, if P is hermitian, then ζ is self-adjoint and $\|\zeta\|_1 = \|P\|$.

Proof In order to prove the equivalence of the conditions we are reduced to prove that (iii)$\Rightarrow$(i). Suppose that (iii) holds. Let $x, y \in \mathscr{K}(H)_+$ such that $x \perp y$. From the spectral decomposition of both x and y we deduce that there exist sequences (x_n) and (y_n) in $\mathscr{F}(H)_+$ such that $\lim x_n = x$, $\lim y_n = y$, and $x_m \perp y_n$ $(m, n \in \mathbb{N})$. Then

$$P(x+y) = \lim P(x_n + y_n) = \lim\bigl(P(x_n) + P(y_n)\bigr) = P(x) + P(y).$$

This shows that P is orthogonally additive on $\mathscr{K}(H)_+$. Since the C^*-algebra $\mathscr{K}(H)$ has real rank zero and the net consisting of all finite-rank projections is an increasing approximate unit, Theorem 3 applies and gives a continuous linear functional Φ on $\mathscr{K}(H)$ such that $P(x) = \Phi(x^m)$ $(x \in \mathscr{K}(H))$. It is well-known that the map $\zeta \mapsto \Phi_\zeta$, as defined in (18), gives an isometric isomorphism from $S^1(H)$ onto the dual of $\mathscr{K}(H)$, so that there exists $\zeta \in S^1(H)$ such that $\Phi(x) = \mathrm{Tr}(\zeta x)$ $(x \in \mathscr{K}(H))$ and $\|\zeta\|_1 = \|\Phi\|$. Thus we obtain (i). The additional properties of the result follow from Theorem 1. □

Corollary 6 *Let H be a Hilbert space, and let $P\colon S^p(H) \to \mathbb{C}$ be a continuous m-homogeneous polynomial with $0 < p \le m$. Then the following conditions are equivalent:*

(i) *there exists $\zeta \in \mathscr{B}(H)$ such that $P(x) = \mathrm{Tr}(\zeta x^m)$ $(x \in S^p(H))$;*
(ii) *the polynomial P is orthogonally additive on $S^p(H)_{\mathrm{sa}}$;*
(iii) *the polynomial P is orthogonally additive on $\mathscr{F}(H)_+$.*

If the conditions are satisfied, then ζ is unique and $\|P\| \le \|\zeta\| \le 2\|P\|$; moreover, if P is hermitian, then ζ is self-adjoint and $\|\zeta\| = \|P\|$.

Proof By Theorems 4 and 5, it suffices to show that the map $\zeta \mapsto \Phi_\zeta$, as defined in (18), gives isometric isomorphism from $\mathscr{B}(H)$ onto the dual of $S^{p/m}(H)$. This is probably well-known, but we are not aware of any reference. Consequently, it may be helpful to include a proof of this fact. If $\zeta \in \mathscr{B}(H)$ and $x \in S^{p/m}(H)$, then, by (8), $\zeta x \in S^{p/m}(H)$, so that $\zeta x \in S^1(H)$ and

$$\big\| \mathrm{Tr}(\zeta x) \big\| \le \|\zeta x\|_1 \le \|\zeta\| \|x\|_1 \le \|\zeta\| \|x\|_{p/m},$$

which shows that Φ_ζ is a continuous linear functional on $S^{p/m}(H)$ with $\|\Phi_\zeta\| \le \|\zeta\|$. Conversely, assume that Φ is a continuous linear functional on $S^{p/m}(H)$. For each $\xi, \eta \in H$, let $\xi \otimes \eta \in \mathscr{F}(H)$ defined by

$$\big(\xi \otimes \eta\big)(\psi) = \langle \psi | \eta \rangle \xi \quad (\psi \in H),$$

and define $\varphi\colon H \times H \to \mathbb{C}$ by

$$\varphi(\xi, \eta) = \Phi(\xi \otimes \eta) \quad (\xi, \eta \in H).$$

It is easily checked that φ is a continuous sesquilinear functional with $\|\varphi\| \le \|\Phi\|$. Therefore there exists $\zeta \in \mathscr{B}(H)$ such that $\langle \zeta(\xi) | \eta \rangle = \varphi(\xi, \eta)$ for all $\xi, \eta \in H$ and $\|\zeta\| \le \|\Phi\|$. The former condition implies that

$$\Phi_\zeta(\xi \otimes \eta) = \mathrm{Tr}(\zeta \xi \otimes \eta) = \langle \zeta(\xi) | \eta \rangle = \varphi(\xi, \eta) = \Phi(\xi \otimes \eta)$$

for all $\xi, \eta \in H$, which gives $\Phi_\zeta(x) = \Phi(x)$ for each $x \in \mathscr{F}(H)$. Since $\mathscr{F}(H)$ is dense in $S^{p/m}(H)$, it follows that $\Phi_\zeta = \Phi$. Further, we have $\|\zeta\| \le \|\Phi\| = \|\Phi_\zeta\| \le \|\zeta\|$. Finally, it is immediate to see that $\Phi_\zeta^* = \Phi_{\zeta^*}$, so that Φ_ζ is hermitian if and only if ζ is self-adjoint. □

Proposition 3 *Let H be a Hilbert space with* $\dim H \geq 2$*, let X be a topological linear space, and let $P\colon S^p(H) \to X$ be a continuous m-homogeneous polynomial with* $0 < p < \infty$*. Suppose that P is orthogonally additive on $S^p(H)$. Then $P = 0$.*

Proof Since $\mathscr{F}(H)$ is dense in $S^p(H)$ and P is continuous, it suffices to prove that P vanishes on $\mathscr{F}(H)$. On account of Lemma 1, we are also reduced to prove that P vanishes on $\mathscr{F}(H)_{\mathrm{sa}}$. We continue to use the notation $\xi \otimes \eta$ which was introduced in the proof of Corollary 6.

Let $x \in \mathscr{F}(H)_{\mathrm{sa}}$. Then $x = \sum_{j=1}^{k} \alpha_j \xi_j \otimes \xi_j$, where $k \geq 2$, $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$, and $\{\xi_1, \ldots, \xi_k\}$ is an orthonormal subset of H. It is clear that the subalgebra $\mathscr{M}$ of $\mathscr{B}(H)$ generated by $\left\{\xi_i \otimes \xi_j : i, j \in \{1, \ldots, k\}\right\}$ is contained in $\mathscr{F}(H)$ and it is $*$-isomorphic to the von Neumann algebra $\mathscr{B}(K)$, where K is the linear span of the set $\{\xi_1, \ldots, \xi_k\}$. By Proposition 1, $P\mid_{\mathscr{M}} = 0$, and therefore $P(x) = 0$. We thus get $P\mid_{\mathscr{F}(H)_{\mathrm{sa}}} = 0$, as required. □

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