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Lebesgue inequalities for Chebyshev thresholding greedy algorithms

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Abstract

We establish estimates for the Lebesgue parameters of the Chebyshev weak thresholding greedy algorithm in the case of general bases in Banach spaces. These generalize and slightly improve earlier results in Dilworth et al. (Rev Mat Complut 28(2):393–409, 2015), and are complemented with examples showing the optimality of the bounds. Our results also clarify certain bounds recently announced in Shao and Ye (J Inequal Appl 2018(1):102, 2018), and answer some questions left open in that paper.

Keywords Thresholding Chebyshev greedy algorithm · Thresholding greedy algorithm · Quasi-greedy basis · Semi-greedy bases

Mathematics Subject Classification 41A65 · 41A46 · 46B15

1 Introduction

Let $\mathbb{X}$ be a Banach space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, let $\mathbb{X}^*$ be its dual space, and consider a system $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^{\infty} \subset \mathbb{X} \times \mathbb{X}^*$ with the following properties:

(a) $0 < \inf_n\{\|\mathbf{e}_n\|, \|\mathbf{e}_n^*\|\} \leq \sup_n\{\|\mathbf{e}_n\|, \|\mathbf{e}_n^*\|\} < \infty$

(b) $\mathbf{e}_n^*(\mathbf{e}_m) = \delta_{n,m}$, for all $n, m \geq 1$

(c) $\mathbb{X} = \overline{\text{span}\,\{\mathbf{e}_n : n \in \mathbb{N}\}}$

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(d) $\mathbb{X}^* = \overline{\text{span}\,\{\mathbf{e}_n^* : n \in \mathbb{N}\}}^{w^*}$.

Under these conditions $\mathscr{B} = \{\mathbf{e}_n\}_{n=1}^{\infty}$ is called a *seminormalized Markushevich basis for* $\mathbb{X}$ (or M-basis for short), with *dual system* $\{\mathbf{e}_n^*\}_{n=1}^{\infty}$. Sometimes we shall consider the following special cases

(e) $\mathscr{B}$ is a *Schauder basis* if $K_b := \sup_N \|S_N\| < \infty$, where $S_N x := \sum_{n=1}^{N} \mathbf{e}_n^*(x)\mathbf{e}_n$ is the N-th partial sum operator
(f) $\mathscr{B}$ is a *Cesàro basis* if $\sup_N \|F_N\| < \infty$, where $F_N := \frac{1}{N}\sum_{n=1}^{N} S_n$ is the N-th (C,1)-Cesàro operator. In this case we use the constant

$$\beta = \max\left\{\sup_N \|F_N\|,\ \sup_N \|I - F_N\|\right\}. \tag{1.1}$$

For the latter terminology, see e.g. [21, Def. III.11.1]. With every $x \in \mathbb{X}$, we shall associate the formal series $x \sim \sum_{n=1}^{\infty} \mathbf{e}_n^*(x)\mathbf{e}_n$, where a)-c) imply that $\lim_n \mathbf{e}_n^*(x) = 0$. As usual, we denote $\text{supp} x = \{n \in \mathbb{N} : \mathbf{e}_n^*(x) \neq 0\}$.

We recall standard notions about (weak) greedy algorithms; see e.g. the texts [23,25] for details and historical background. Fix $t \in (0, 1]$. We say that A is a *t-greedy set for x of order m*, denoted $A \in G(x, m, t)$, if $|A| = m$ and

$$\min_{n \in A} |\mathbf{e}_n^*(x)| \geq t \cdot \max_{n \notin A} |\mathbf{e}_n^*(x)|. \tag{1.2}$$

A *t-greedy operator of order m* is any mapping $\mathscr{G}_m^t : \mathbb{X} \to \mathbb{X}$ which at each $x \in \mathbb{X}$ takes the form

$$\mathscr{G}_m^t(x) = \sum_{n \in A} \mathbf{e}_n^*(x)\mathbf{e}_n, \quad \text{for some set} \quad A = A(x, \mathscr{G}_m^t) \in G(x, m, t).$$

We write $\mathbb{G}_m^t$ for the set of all t-greedy operators of order m. The approximation scheme which assigns a sequence $\{\mathscr{G}_m^t(x)\}_{m=1}^{\infty}$ to each vector $x \in \mathbb{X}$ is called a *Weak Thresholding Greedy Algorithm* (WTGA), see [16,24]. When $t = 1$ one just says Thresholding Greedy Algorithm (TGA), and drops the super-index t, that is $\mathscr{G}_m^1 = \mathscr{G}_m$, etc.

It is standard to quantify the efficiency of these algorithms, among all possible m-term approximations, in terms of *Lebesgue-type inequalities*. That is, for each $m = 1, 2, \ldots$, we look for the smallest constant $\mathbf{L}_m^t$ such that

$$\|x - \mathscr{G}_m^t(x)\| \leq \mathbf{L}_m^t \sigma_m(x), \quad \forall\, x \in \mathbb{X}, \quad \forall\, \mathscr{G}_m^t \in \mathbb{G}_m^t, \tag{1.3}$$

where

$$\sigma_m(x) := \inf\left\{\left\|x - \sum_{n \in B} b_n \mathbf{e}_n\right\| \ :\ b_n \in \mathbb{K}, \quad |B| \leq m\right\}.$$

We call the number $\mathbf{L}_m^t$ the *Lebesgue parameter* associated with the WTGA, and we just write $\mathbf{L}_m$ when $t = 1$. We refer to [25, Chapter 3] for a survey on such inequalities, and to [1,5,6,10,12,26] for recent results. It is known that $\mathbf{L}_m^t = O(1)$ holds for a fixed t if and only if it holds for all $t \in (0, 1]$, and if and only if $\mathscr{B}$ is unconditional and democratic; see [15] and [23, Thm. 1.39]. In this special case $\mathscr{B}$ is called a *greedy basis*.

In this paper we shall be interested in *Chebyshev thresholding greedy algorithms*. These were introduced by Dilworth et al. [8, §3], as an enhancement of the TGA. Here, we use the weak version considered in [10]. Namely, for fixed $t \in (0, 1]$ we say that $\mathfrak{CG}_m^t : \mathbb{X} \to \mathbb{X}$ is a *Chebyshev t-greedy operator* of order m if for every $x \in \mathbb{X}$ there is a set $A = A(x, \mathfrak{CG}_m^t) \in G(x, m, t)$ such that $\operatorname{supp}\mathfrak{CG}_m^t(x) \subset A$ and moreover

$$\|x - \mathfrak{CG}_m^t(x)\| = \min\left\{\left\|x - \sum_{n\in A} a_n \mathbf{e}_n\right\| \ : \ a_n \in \mathbb{K}\right\}.$$

Finally, we define the *weak Chebyshevian Lebesgue parameter* $\mathbf{L}_m^{\mathrm{ch},t}$ as the smallest constant such that

$$\|x - \mathfrak{CG}_m^t(x)\| \le \mathbf{L}_m^{\mathrm{ch},t}\sigma_m(x), \quad \forall\, x \in \mathbb{X}, \quad \forall\, \mathfrak{CG}_m^t \in \mathbb{G}_m^{\mathrm{ch},t},$$

where $\mathbb{G}_m^{\mathrm{ch},t}$ is the collection of all Chebyshev t-greedy operators of order m. As before, when $t = 1$ we shall omit the index t, that is $\mathbf{L}_m^{\mathrm{ch}} := \mathbf{L}_m^{\mathrm{ch},1}$.

When $\mathbf{L}_m^{\mathrm{ch}} = O(1)$ the system $\mathscr{B}$ is called semi-greedy; see [8]. We remark that the first author recently established that a Schauder basis $\mathscr{B}$ is semi-greedy if and only if is quasi-greedy and democratic; see [3].

In this paper we shall be interested in quantitative bounds of $\mathbf{L}_m^{\mathrm{ch},t}$ in terms of the quasi-greedy and democracy parameters of a general M-basis $\mathscr{B}$. Earlier bounds were obtained by Dilworth et al. [10] when $\mathscr{B}$ is a quasi-greedy basis, and very recently, some improvements were also announced by Shao and Ye [19, Theorem 3.5]. Unfortunately, various arguments in the last paper seem not to be correct, so one of our goals here is to give precise statements and proofs for the results in [19], and also settle some of the questions which are left open there.

To state our results, we recall the definitions of the involved parameters. Given a finite set $A \subset \mathbb{N}$, we shall use the following standard notation for the indicator sums:

$$\mathbf{1}_A = \sum_{n\in A} \mathbf{e}_n \quad \text{and} \quad \mathbf{1}_{\varepsilon A} = \sum_{n\in A} \varepsilon_n \mathbf{e}_n, \quad \varepsilon \in \Upsilon$$

where Υ is the set of all $\varepsilon = \{\varepsilon_n\}_n \subset \mathbb{K}$ with $|\varepsilon_n| = 1$. Similarly, we write

$$P_A(x) = \sum_{n\in A} \mathbf{e}_n^*(x)\mathbf{e}_n.$$

The relevant parameters for this paper are the following:

- Conditionality parameters:

$$k_m := \sup_{|A|\le m} \|P_A\| \quad \text{and} \quad k_m^c = \sup_{|A|\le m} \|I - P_A\|.$$

- Quasi-greedy parameters:

$$g_m := \sup_{\mathscr{G}_k \in \mathbb{G}_k,\, k \le m} \|\mathscr{G}_k\| \quad \text{and} \quad g_m^c := \sup_{\mathscr{G}_k \in \mathbb{G}_k,\, k \le m} \|I - \mathscr{G}_k\|.$$

  Below we shall also use the variant

$$\tilde{g}_m := \sup_{\substack{\mathscr{G}' < \mathscr{G} \\ \mathscr{G} \in \mathbb{G}_k,\, k \le m}} \|\mathscr{G} - \mathscr{G}'\|,$$

  where $\mathscr{G}' < \mathscr{G}$ means that $A(x, \mathscr{G}') \subset A(x, \mathscr{G})$ for all x; see [5].
- Super-democracy parameters:

$$\tilde{\mu}_m = \sup_{\substack{|A|=|B|\le m \\ |\varepsilon|=|\eta|=1}} \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \quad \text{and} \quad \tilde{\mu}_m^d = \sup_{\substack{|A|=|B|\le m,\ A\cap B=\emptyset \\ |\varepsilon|=|\eta|=1}} \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}.$$

- Quasi-greedy parameters for constant coefficients (see [5, (3.11)])

$$\gamma_m = \sup_{\substack{|\varepsilon|=1 \\ B\subset A,\ |A|\le m}} \frac{\|\mathbf{1}_{\varepsilon B}\|}{\|\mathbf{1}_{\varepsilon A}\|}.$$

Note that $\gamma_m \le g_m \le \tilde{g}_m \le 2g_m$, but in general γ_m may be much smaller than g_m; see e.g. [5, §5.5]. Likewise, in §5 below we show that $\tilde{\mu}_m^d$ may be much smaller than $\tilde{\mu}_m$, except for Schauder bases, where both quantities turn out to be equivalent; see Theorem 5.2.

Our first result is a general upper bound, which improves and extends [19, Theorem 2.4].

Theorem 1.1 *Let $\mathscr{B}$ be an M-basis in $\mathbb{X}$, and let $\mathfrak{K} = \sup_{n,j} \|\mathbf{e}_n^*\|\|\mathbf{e}_j\|$. Then,*

$$\mathbf{L}_m^{\mathrm{ch},t} \le 1 + \left(1 + \tfrac{1}{t}\right) \mathfrak{K} m, \quad \forall\, m \in \mathbb{N}, \quad t \in (0, 1]. \tag{1.4}$$

Moreover, there exists a pair $(\mathbb{X}, \mathscr{B})$ where the equality is attained for all m and t.

The second result is a slight generalization of [10, Theorem 4.1], and gives a correct version of [19, Theorem 3.5].

Theorem 1.2 *Let $\mathscr{B}$ be an M-basis in $\mathbb{X}$. Then, for all $m \ge 1$ and $t \in (0, 1]$,*

$$\mathbf{L}_m^{\mathrm{ch},t} \le g_{2m}^c + \frac{2}{t} \min\left\{ \tilde{g}_m \tilde{\mu}_m\,,\ \gamma_{2m} \tilde{g}_{2m} \tilde{\mu}_m^d \right\}. \tag{1.5}$$

Our next result concerns lower bounds for $\mathbf{L}_m^{\mathrm{ch},t}$, for which we need to introduce weaker versions of the democracy parameters with an additional separation condition. For two finite sets $A, B \subset \mathbb{N}$ and $c \ge 1$, the notation $A > cB$ will stand for $\min A > c \max B$.

- Given an integer $c \geq 2$, we define

$$\vartheta_{m,c} := \sup \left\{ \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \;:\; |\varepsilon| = |\eta| = 1,\ |A| = |B| \leq m \ \text{ with } A > cB \text{ or } B > cA \right\}. \quad (1.6)$$

Theorem 1.3 *If $\mathscr{B}$ is a Cesàro basis in $\mathbb{X}$ with constant β, then for every $c \geq 2$*

$$\mathbf{L}_m^{\mathrm{ch},t} \geq \frac{1}{t\beta^2} \frac{c-1}{c+1} \vartheta_{m,c}, \quad \forall\, m \in \mathbb{N},\ t \in (0,1].$$

We shall also establish, in Theorem 3.10 below, a similar lower bound valid for more general M-bases (not necessarily of Cesàro type), in terms of a new parameter θ_m which is invariant under rearrangements of $\mathscr{B}$.

Remark 1.4 One may compare the bounds for $\mathbf{L}_m^{\mathrm{ch}}$ above with those for $\mathbf{L}_m$ given in [5]

$$(1)\ \mathbf{L}_m \leq 1 + 3\mathfrak{K}m, \qquad (2)\ \mathbf{L}_m \leq k_{2m}^c + \tilde{g}_m \tilde{\mu}_m, \qquad \text{and} \qquad (3)\ \mathbf{L}_m \geq \tilde{\mu}_m^d,$$

which illustrate a slightly better behavior of the Chebishev TGA. Observe that one also has the trivial inequalities

$$\mathbf{L}_m^{\mathrm{ch},t} \leq \mathbf{L}_m^t \leq k_m^c\, \mathbf{L}_m^{\mathrm{ch},t}.$$

Indeed, $\mathbf{L}_m^{\mathrm{ch},t} \leq \mathbf{L}_m^t$ is direct by definition, while $\mathbf{L}_m^t \leq k_m^c \mathbf{L}_m^{\mathrm{ch},t}$ can be proved as follows: take $x \in \mathbb{X}$ and let $A = \operatorname{supp}\mathscr{G}_m^t(x)$. Pick a Chebyshev greedy operator $\mathfrak{CG}_m^t$ such that $\operatorname{supp}\mathfrak{CG}_m^t(x) = A$. Then

$$\|x - \mathscr{G}_m^t(x)\| = \|(I - P_A)x\| = \|(I - P_A)(x - \mathfrak{CG}_m^t(x))\| \leq k_m^c \|x - \mathfrak{CG}_m^t(x)\|,$$

so $\mathbf{L}_m^t \leq k_m^c \mathbf{L}_m^{\mathrm{ch},t}$. Hence, when $\mathscr{B}$ is unconditional then $\mathbf{L}_m^t \approx \mathbf{L}_m^{\mathrm{ch},t}$. However for all conditional quasi-greedy and democratic bases we have $\mathbf{L}_m^{\mathrm{ch}} = O(1)$, but $\mathbf{L}_m \to \infty$.

The paper is organized as follows. Section 2 is devoted to preliminary lemmas. In Sect. 3 we prove Theorems 1.1, 1.2 and 1.3, and also establish the more general lower bound in Theorem 3.10, giving various situations in which it applies. Section 4 is devoted to examples illustrating the optimality of the results; in particular, an optimal bound of $\mathbf{L}_m^{\mathrm{ch}}$ for the trigonometric system in $L^1(\mathbb{T})$, settling a question left open in [19]. In Sect. 5 we investigate the equivalence between $\tilde{\mu}_m^d$ and $\tilde{\mu}_m$ and show Theorem 5.2. Finally, in Sect. 6 we study the convergence of $\mathfrak{CG}_m(x)$ and $\mathscr{G}_m(x)$ to x, pointing out the role of a *strong* M-basis assumption for such results.

2 Preliminary results

We recall some basic concepts and results that will be used later in the paper; see [5,8]. For each $\alpha > 0$ we define the α-truncation of a scalar $y \in \mathbb{K}$ as

$$T_\alpha(y) = \alpha \operatorname{sign} y \ \text{ if } |y| \geq \alpha, \quad \text{and} \quad T_\alpha(y) = y \ \text{ if } |y| \leq \alpha.$$

We extend T_α to an operator in $\mathbb{X}$ by formally assigning $T_\alpha(x) \sim \sum_{n=1}^{\infty} T_\alpha(\mathbf{e}_n^*(x))\mathbf{e}_n$, that is

$$T_\alpha(x) := \alpha \mathbf{1}_{\varepsilon\Lambda_\alpha(x)} + (I - P_{\Lambda_\alpha(x)})(x),$$

where $\Lambda_\alpha(x) = \{n : |\mathbf{e}_n^*(x)| > \alpha\}$ and $\varepsilon = \{\operatorname{sign}(\mathbf{e}_n^*(x))\}$. Of course, this operator is well defined since $\Lambda_\alpha(x)$ is a finite set. In [5] we can find the following result:

Lemma 2.1 [5, Lemma 2.5] *For all $\alpha > 0$ and $x \in \mathbb{X}$, we have*

$$\|T_\alpha(x)\| \leq g^c_{|\Lambda_\alpha(x)|}\|x\|.$$

We also need a well known property from [8,9], formulated as follows.

Lemma 2.2 [5, Lemma 2.3] *If $x \in \mathbb{X}$ and $\varepsilon = \{\operatorname{sign}(\mathbf{e}_n^*(x))\}$, then*

$$\min_{n \in G} |\mathbf{e}_n^*(x)| \|\mathbf{1}_{\varepsilon G}\| \leq \tilde{g}_{|G|}\|x\|, \quad \forall G \in G(x, m, 1). \tag{2.1}$$

The following version of (2.1), valid even if G is not greedy, improves [10, Lemma 2.2].

Lemma 2.3 *Let $x \in \mathbb{X}$ and $\varepsilon = \{\operatorname{sign}(\mathbf{e}_n^*(x))\}$. For every set finite $A \subset \mathbb{N}$, if $\alpha = \min_{n\in A} |\mathbf{e}_n^*(x)|$, then*

$$\alpha\|\mathbf{1}_{\varepsilon A}\| \ \leq \ \gamma_{|A \cup \Lambda_\alpha(x)|}\, \tilde{g}_{|A\cup\Lambda_\alpha(x)|}\|x\|, \tag{2.2}$$

where $\Lambda_\alpha(x) = \{n : |\mathbf{e}_n^*(x)| > \alpha\}$.

Proof. Call $G = A \cup \Lambda_\alpha(x)$, and notice that it is a greedy set for x. Then,

$$\alpha\|\mathbf{1}_{\varepsilon A}\| \ \leq \ \alpha\, \gamma_{|G|}\|\mathbf{1}_{\varepsilon G}\| \ \leq \ \gamma_{|G|}\, \tilde{g}_{|G|}\, \|x\|,$$

using (2.1) in the last step. □

Remark 2.4 The following is a variant of (2.2) with a different constant

$$\min_{n\in A} |\mathbf{e}_n^*(x)| \ \|\mathbf{1}_{\varepsilon A}\| \ \leq \ k_{|A|}\, \|x\|. \tag{2.3}$$

A similar proof as the one in Lemma 2.3 can be seen in [4, Proposition 2.5].

Finally, we need the following elementary result, which follows directly from the convexity of the norm; see e.g [25, p. 108] (or [5, Lemma 2.7] if $\mathbb{K} = \mathbb{C}$).

Lemma 2.5 *For all finite sets $A \subset \mathbb{N}$ and scalars $a_n \in \mathbb{K}$ it holds*

$$\left\| \sum_{n\in A} a_n \mathbf{e}_n \right\| \leq \max_{n\in A} |a_n| \sup_{|\varepsilon|=1} \left\| \mathbf{1}_{\varepsilon A} \right\|.$$

3 Proof of the main results

3.1 Proof of Theorem 1.1

Let $x \in \mathbb{X}$ and $\mathfrak{CG}_m^t \in \mathbb{G}_m^{\mathrm{ch},t}$ be a fixed Chebyshev t-greedy operator. Let $A = A(x, \mathfrak{CG}_m^t) \in G(x, m, t)$. Pick any $z = \sum_{n\in B} b_n \mathbf{e}_n$ such that $|B| = m$. By definition of the Chebyshev operators,

$$\|x - \mathfrak{CG}_m^t(x)\| \leq \|x - P_{A\cap B}(x)\| \leq \|P_{B\setminus A}(x)\| + \|x - P_B(x)\|.$$

On the one hand, using (1.2),

$$\|P_{B\setminus A}(x)\| \leq \sup_n \|\mathbf{e}_n\| \sum_{j\in B\setminus A} |\mathbf{e}_j^*(x)| \leq \frac{1}{t} \sup_n \|\mathbf{e}_n\| \sum_{j\in A\setminus B} |\mathbf{e}_j^*(x - z)| \leq \frac{1}{t} \mathfrak{K}m \|x - z\|.$$

On the other hand, using the inequality (3.9) of [5],

$$\|x - P_B(x)\| = \|(I - P_B)(x - z)\| \leq k_m^c \|x - z\| \leq (1 + \mathfrak{K}m)\|x - z\|.$$

Hence, $\mathbf{L}_m^{\mathrm{ch},t} \leq 1 + \left(1 + \frac{1}{t}\right) \mathfrak{K}m$. Finally, the fact that the equality in (1.4) can be attained is witnessed by Examples 4.1 and 4.2 below.

3.2 Proof of Theorem 1.2

The scheme of the proof follows the lines in [8, Theorem 3.2] and [10, Theorem 4.1], with some additional simplifications introduced in [5].

Given $x \in \mathbb{X}$ and $\mathfrak{CG}_m^t \in \mathbb{G}_m^{\mathrm{ch},t}$, let $A = A(x, \mathfrak{CG}_m^t) \in G(x, m, t)$. Pick any $z = \sum_{n\in B} b_n \mathbf{e}_n$ such that $|B| = m$. By definition of the Chebyshev operators,

$$\|x - \mathfrak{CG}_m^t(x)\| \leq \|x - p\|, \quad \text{for any } p = \sum_{n\in A} a_n \mathbf{e}_n. \tag{3.1}$$

We make the selection of p suggested in [8]. Namely, if $\alpha = \max_{n\notin A} |\mathbf{e}_n^*(x)|$, we let

$$p = P_A(x) - P_A\big(T_\alpha(x - z)\big).$$

It is easily verified that

$$\begin{aligned} x - p &= (I - P_A)\big(x - T_\alpha(x-z)\big) + T_\alpha(x-z) \\ &= P_{B\setminus A}\big(x - T_\alpha(x-z)\big) + T_\alpha(x-z). \end{aligned} \tag{3.2}$$

Since $\Lambda_\alpha(x-z) = \{n \ : \ |\mathbf{e}_n^*(x-z)| > \alpha\} \subset A \cup B$, then Lemma 2.1 gives

$$\big\| T_\alpha(x-z) \big\| \le g_{2m}^c \|x-z\|. \tag{3.3}$$

Next we treat the first term in (3.2). Observe that $\max_{n\in B\setminus A} |\mathbf{e}_n^*(x - T_\alpha(x-z))| \le 2\alpha$, so Lemma 2.5 gives

$$\begin{aligned} \big\| P_{B\setminus A}\big(x - T_\alpha(x-z)\big)\big\| &\le 2\alpha \sup_{|\varepsilon|=1} \big\| \mathbf{1}_{\varepsilon(B\setminus A)} \big\| \\ &\le \frac{2}{t} \min_{n\in A\setminus B} |\mathbf{e}_n^*(x-z)| \sup_{|\varepsilon|=1} \big\| \mathbf{1}_{\varepsilon(B\setminus A)} \big\| = (*). \end{aligned} \tag{3.4}$$

At this point we have two possible approaches. Let $\eta_n = \operatorname{sign}[e_n^*(x-z)]$. In the first approach we pick a greedy set $\Gamma \in G(x-z, |A\setminus B|, 1)$, and control (3.4) by

$$(*) \ \le \ \frac{2}{t} \min_{n\in\Gamma} |\mathbf{e}_n^*(x-z)| \ \tilde{\mu}_m \ \big\| \mathbf{1}_{\eta\Gamma} \big\| \ \le \ \frac{2}{t} \tilde{\mu}_m \, \tilde{g}_m \|x-z\|, \tag{3.5}$$

using Lemma 2.2 in the last step. In the second approach, we argue as follows

$$(*) \ \le \ \frac{2}{t} \min_{n\in A\setminus B} |\mathbf{e}_n^*(x-z)| \ \tilde{\mu}_m^d \ \big\| \mathbf{1}_{\eta(A\setminus B)} \big\| \ \le \ \frac{2}{t} \gamma_{2m} \, \tilde{g}_{2m} \, \tilde{\mu}_m^d \, \|x-z\|, \tag{3.6}$$

using in the last step Lemma 2.3 and the fact that, if $\delta = \min_{A\setminus B} |\mathbf{e}_n^*(x-z)|$, then the set $(A\setminus B) \cup \{n \ : \ |\mathbf{e}_n^*(x-z)| > \delta\} \subset A \cup B$ and hence has cardinality $\le 2m$.

We can now combine the estimates displayed in (3.1)–(3.6) and obtain

$$\|x - \mathfrak{CG}_m^t(x)\| \le \Big[g_{2m}^c + \frac{2}{t} \min\Big\{ \tilde{g}_m \tilde{\mu}_m \, , \ \gamma_{2m} \tilde{g}_{2m} \tilde{\mu}_m^d \Big\} \Big] \|x-z\|,$$

which after taking the infimum over all z establishes Theorem 1.2. □

Remark 3.1 In [19, Theorem 3.5] a stronger inequality is stated (for $t = 1$), namely

$$\mathbf{L}_m^{\mathrm{ch}} \le g_{2m}^c + 2\tilde{g}_m \tilde{\mu}_m^d. \tag{3.7}$$

The proof, however, seems to contain a gap, and a missing factor k_m^c should also appear in the last summand. Nevertheless, it is still fair to ask whether the inequality (3.7) asserted in [19] may be true with a different proof.

Remark 3.2 Using Remark 2.4 in place of Lemma 2.3 in (3.6) above leads to an alternative and slightly simpler estimate

$$\mathbf{L}_m^{\mathrm{ch},t} \;\leq\; g_{2m}^c \;+\; \frac{2}{t}\, k_m \tilde{\mu}_m^d\,. \tag{3.8}$$

However, this would not be as efficient as (1.5) when $\mathscr{B}$ is quasi-greedy and conditional.

Remark 3.3 When $\mathscr{B}$ is quasi-greedy with constant $\mathbf{q} = \sup_m g_m < \infty$, then Theorem 1.2 implies the following

$$\mathbf{L}_m^{\mathrm{ch},t} \leq \mathbf{q} + 4t^{-1}\, \mathbf{q}^2\, \tilde{\mu}_m^d.$$

This is a slight improvement with respect to [10, Theorem 4.1].

3.3 Proof of Theorem 1.3

Recall that $S_N = \sum_{n=1}^N \mathbf{e}_n^*(\cdot)\mathbf{e}_n$ and

$$F_N(x) = \frac{1}{N}\sum_{n=1}^N S_n(x) = \sum_{n=1}^N \Big(1 - \frac{n-1}{N}\Big)\, \mathbf{e}_n^*(x)\mathbf{e}_n.$$

For $M > N$ we define the operators (of de la Vallée-Poussin type)

$$\begin{aligned} V_{N,M}(x) &= \frac{M}{M-N}\, F_M(x) - \frac{N}{M-N} F_N(x) \\ &= \sum_{n=1}^N \mathbf{e}_n^*(x)\mathbf{e}_n \;+\; \sum_{n=N+1}^M \Big(1 - \frac{n-N-1}{M-N}\Big)\, \mathbf{e}_n^*(x)\mathbf{e}_n. \end{aligned} \tag{3.9}$$

In particular, observe that, for β as in (1.1) we have

$$\max\big\{\|V_{N,M}\|, \|I - V_{N,M}\|\big\} \;\leq\; \frac{M+N}{M-N}\,\beta. \tag{3.10}$$

We next prove that, if $c \geq 2$, then for all $A, B \subset \mathbb{N}$ such that $B > cA$ with $|A| = |B| \leq m$ it holds

$$\mathbf{L}_m^{\mathrm{ch},t} \geq \frac{1}{t\beta}\, \frac{c-1}{c+1}\, \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}, \quad \forall\ |\varepsilon| = |\eta| = 1. \tag{3.11}$$

Pick any set $C > B$ such that $|B \cup C| = m$, and let

$$x = \mathbf{1}_{\varepsilon A} + t\mathbf{1}_{\eta B} + t\mathbf{1}_C.$$

Then $B \cup C \in G(x, m, t)$, and hence there is a Chebyshev t-greedy operator so that

$$x - \mathfrak{CG}_m^t(x) = \mathbf{1}_{\varepsilon A} + \sum_{n \in B \cup C} a_n \mathbf{e}_n,$$

for some scalars $a_n \in \mathbb{K}$. Clearly,

$$\|x - \mathfrak{CG}_m^t(x)\| \leq \mathbf{L}_m^{\mathrm{ch},t} \sigma_m(x) \ \leq \ \mathbf{L}_m^{\mathrm{ch},t} \ \|t\mathbf{1}_{\eta B}\|,$$

using $z = \mathbf{1}_{\varepsilon A} + t\mathbf{1}_C$ an m-term approximant. On the other hand, let $N = \max A$. Since $\min B \cup C > cN$, then (3.9) yields

$$V_{N,cN}(x - \mathfrak{CG}_m^t(x)) = \mathbf{1}_{\varepsilon A}.$$

Therefore, (3.10) implies that

$$\|x - \mathfrak{CG}_m^t(x)\| \geq \frac{\|V_{N,cN}(x - \mathfrak{CG}_m^t(x))\|}{\|V_{N,cN}\|} \geq \frac{c-1}{(c+1)\beta} \|\mathbf{1}_{\varepsilon A}\|.$$

We have therefore proved (3.11).

We next show that when $|A| = |B| \leq m$ satisfy $A > cB$ then

$$\mathbf{L}_m^{\mathrm{ch},t} \geq \frac{1}{t\beta^2} \frac{c-1}{c+1} \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}, \quad \forall\, |\varepsilon| = |\eta| = 1. \tag{3.12}$$

This together with (3.11) is enough to establish Theorem 1.3. We shall actually show a slightly stronger result:

Lemma 3.4 *Let $|A| = |B| \leq m$ and let $y \in \mathbb{X}$ be such that $|y|_\infty := \sup_n |\mathbf{e}_n^*(y)| \leq 1$ and $A > c(B \cup \operatorname{supp} y)$. Then*

$$\mathbf{L}_m^{\mathrm{ch},t} \geq \frac{1}{t\beta^2} \frac{c-1}{c+1} \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B} + y\|}, \quad \forall\, |\varepsilon| = |\eta| = 1. \tag{3.13}$$

Observe that the case $y = 0$ in (3.13) yields (3.12). We now show (3.13). Pick a large integer $\lambda > 1$ and a set $C > \lambda A$ such that $|B \cup C| = m$. Let

$$x = \mathbf{1}_{\varepsilon A} + ty + t\mathbf{1}_{\eta B} + t\mathbf{1}_C.$$

As before, $B \cup C \in G(x, m, t)$, and hence for some Chebyshev t-greedy operator we have

$$x - \mathfrak{CG}_m^t(x) = \mathbf{1}_{\varepsilon A} + ty + \sum_{n \in B \cup C} a_n \mathbf{e}_n,$$

for suitable scalars $a_n \in \mathbb{K}$. Choosing $\mathbf{1}_{\varepsilon A} + t\mathbf{1}_C$ as m-term approximant of x we see that

$$\|x - \mathfrak{CG}_m^t(x)\| \le \mathbf{L}_m^{\mathrm{ch},t} \sigma_m(x) \ \le \ \mathbf{L}_m^{\mathrm{ch},t}\, t\, \|\mathbf{1}_{\eta B} + y\|.$$

On the other hand, calling $N = \max(B \cup \operatorname{supp} y)$ and $L = \max A$ we have

$$(I - V_{N,cN}) \circ V_{L,\lambda L}\big(x - \mathfrak{CG}_m^t(x)\big) = \mathbf{1}_{\varepsilon A}$$

Thus,

$$\|x - \mathfrak{CG}_m^t(x)\| \ge \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|I - V_{N,cN}\|\,\|V_{L,\lambda L}\|} \ge \frac{c-1}{(c+1)\beta}\,\frac{\lambda-1}{(\lambda+1)\beta}\,\|\mathbf{1}_{\varepsilon A}\|.$$

Therefore we obtain

$$\mathbf{L}_m^{\mathrm{ch},t} \ge \frac{1}{t\beta^2}\,\frac{c-1}{c+1}\,\frac{\lambda-1}{\lambda+1}\,\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B} + y\|}$$

which letting $\lambda \to \infty$ yields (3.13). This completes the proof of Lemma 3.4, and hence of Theorem 1.3.

Remark 3.5 When $\mathscr{B}$ is a Schauder basis, a similar proof gives the following lower bound, which is also obtained in [19, Theorem 2.2]

$$\mathbf{L}_m^{\mathrm{ch},t} \ge \frac{1}{(K_b+1)t} \sup\left\{\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \ : \ |A| = |B| = m,\ A > B \text{ or } B > A,\ |\varepsilon| = |\eta| = 1\right\}.$$

The statement for Cesàro bases, however, will be needed for the applications in §4.3.

3.4 Lower bounds for general M-bases

Observe that

$$\vartheta_{m,c} = \sup_{|A| \le m} \vartheta_c(A), \quad \text{where} \quad \vartheta_c(A) = \sup_{\substack{B \,:\, |B|=|A| \\ B >_c A \\ \varepsilon,\eta \in \Upsilon}} \max\left\{\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}, \frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|}\right\}.$$

We consider a new parameter

$$\vartheta_m = \sup_{|A| \le m} \inf_{c \ge 1} \vartheta_c(A). \tag{3.14}$$

We remark that, unlike $\vartheta_{m,c}$, the parameter ϑ_m depends on $\{\mathbf{e}_n\}_{n=1}^\infty$ but not on the reorderings of the system. We shall give a lower bound for $\mathbf{L}_m^{\mathrm{ch},t}$ in terms of ϑ_m in a less restrictive situation than the Cesàro basis assumption on $\{\mathbf{e}_n\}_{n=1}^\infty$.

Given $\rho \geq 1$, we say that $\{\mathbf{e}_n\}_{n=1}^\infty$ is ρ-admissible if the following holds: *for each finite set* $A \subset \mathbb{N}$, *there exists* $n_0 = n_0(A) > \max A$ *such that, for all sets* B *with* $\min B \geq n_0$ *and* $|B| \leq |A|$,

$$\left\| \sum_{n \in A} \alpha_n \mathbf{e}_n \right\| \leq \rho \left\| \sum_{n \in A \cup B} \alpha_n \mathbf{e}_n \right\|, \quad \forall\, \alpha_n \in \mathbb{K}. \tag{3.15}$$

Observe that (3.15) implies that

$$\left\| \sum_{n \in B} \alpha_n \mathbf{e}_n \right\| \leq (\rho + 1) \left\| \sum_{n \in A \cup B} \alpha_n \mathbf{e}_n \right\|, \quad \forall\, \alpha_n \in \mathbb{K}. \tag{3.16}$$

This condition is clearly satisfied by all Schauder and Cesàro bases (with $\rho = K_b$ or $\rho > \beta$), but we shall see below that it also holds in more general situations.

Proposition 3.6 *Let* $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ *be an M-basis such that* $\{\mathbf{e}_n\}_{n=1}^\infty$ *is* ρ*-admissible. Then*

$$\mathbf{L}_m^{\mathrm{ch},t} \geq \frac{\vartheta_m}{(\rho+1)t}, \quad \forall\, m \in \mathbb{N}, \quad t \in (0, 1]. \tag{3.17}$$

Proof. Fix $A \subset \mathbb{N}$ such that $|A| \leq m$. Choose C disjoint with A such that $|A \cup C| = m$. Let $n_0 = n_0(A \cup C)$ be as in the above definition, so that n_0 is larger than $\max A \cup C$. Pick any B with $\min B \geq n_0$ and $|B| = |A|$, and any $\varepsilon, \eta \in \Upsilon$. Let $x = t\mathbf{1}_{\varepsilon A} + t\mathbf{1}_C + \mathbf{1}_{\eta B}$. Then $A \cup C \in G(x, m, t)$, and there is a Chebyshev t-greedy operator with $\mathfrak{CG}_m^t(x)$ supported in $A \cup C$. Thus,

$$\|x - \mathfrak{CG}_m^t(x)\| \leq \mathbf{L}_m^{\mathrm{ch},t}\, \sigma_m(x) \leq \mathbf{L}_m^{\mathrm{ch},t}\, \|x - (\mathbf{1}_{\eta B} + t\mathbf{1}_C)\| = \mathbf{L}_m^{\mathrm{ch},t}\, t\, \|\mathbf{1}_{\varepsilon A}\|.$$

On the other hand, using the property in (3.16) one obtains

$$\|x - \mathfrak{CG}_m^t(x)\| \geq \frac{\|\mathbf{1}_{\eta B}\|}{\rho + 1}.$$

Thus,

$$\mathbf{L}_m^{\mathrm{ch},t} \geq \frac{1}{(\rho+1)t} \frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|}.$$

We now assume additionally that $\min B \geq n_0 + m$, and pick $D \subset [n_0, n_0 + m - 1]$ such that $|B| + |D| = m$. Let $y = \mathbf{1}_{\varepsilon A} + t\mathbf{1}_{\eta B} + t\mathbf{1}_D$. Then $B \cup D \in G(y, m, t)$ and a similar reasoning gives

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\rho} \leq \|y - \mathfrak{CG}_m^t(y)\| \leq \mathbf{L}_m^{\mathrm{ch},t}\, \sigma_m(y) \leq \mathbf{L}_m^{\mathrm{ch},t}\, t\, \|\mathbf{1}_{\eta B}\|.$$

Thus,

$$\mathbf{L}_m^{\mathrm{ch},t} \geq \frac{1}{(\rho+1)t} \max\left\{\frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|}, \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}\right\},$$

and taking the supremum over all $|B| = |A|$ with $B \geq (n_0 + m)A$ and all $\varepsilon, \eta \in \Upsilon$, we see that

$$\mathbf{L}_m^{\mathrm{ch},t} \geq \frac{\vartheta_{n_0+m}(A)}{(\rho+1)t} \geq \frac{\inf_{c\geq 1} \vartheta_c(A)}{(\rho+1)t}.$$

Finally, a supremum over all $|A| \leq m$ leads to (3.17). □

We now give some general conditions in $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ and $\mathbb{X}$ under which ρ-admissibility holds. We recall a few standard definitions; see e.g. [13]. We use the notation $[\mathbf{e}_n]_{n\in A} = \overline{\mathrm{span}}\{\mathbf{e}_n\}_{n\in A}$, for $A \subset \mathbb{N}$. A sequence $\{\mathbf{e}_n\}_{n=1}^\infty$ is *weakly null* if

$$\lim_{n\to\infty} x^*(\mathbf{e}_n) = 0, \quad \forall\, x^* \in \mathbb{X}^*.$$

Given a subset $Y \subset \mathbb{X}^*$, we shall say that $\{\mathbf{e}_n\}_{n=1}^\infty$ is Y*-null* if

$$\lim_{n\to\infty} y(\mathbf{e}_n) = 0, \quad \forall\, y \in Y.$$

Given $\kappa \in (0, 1]$, we say that a set $Y \subset \mathbb{X}^*$ is κ-norming whenever

$$\sup_{x^*\in Y, \|x^*\|\leq 1} |x^*(x)| \geq \kappa\, \|x\|, \quad \forall\, x \in \mathbb{X}.$$

We finally introduce a new abstract definition.

Definition 3.7 We say that a biorthogonal system $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty \subset \mathbb{X} \times \mathbb{X}^*$ satisfies the *property* $\mathscr{P}(\kappa)$, for some $0 < \kappa \leq 1$, if the sequence $\{\|\mathbf{e}_n^*\|\, \mathbf{e}_n\}_{n=1}^\infty \subset \mathbb{X}$ is Y-null, for some subset $Y \subset \mathbb{X}^*$ which is κ-norming.

We remark that in every separable Banach space $\mathbb{X}$ there exists an M-basis $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ with the property $\mathscr{P}(1)$; see e.g. [21, Theorem III.8.5].[1] Other examples are given in Remark 3.9 below.

Proposition 3.8 *Let* $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ *be a biorthogonal system in* $\mathbb{X} \times \mathbb{X}^*$ *with the property* $\mathscr{P}(\kappa)$. *Then* $\{\mathbf{e}_n\}_{n=1}^\infty$ *is* ρ*-admissible for every* $\rho > 1/\kappa$.

Proof. Let $Y \subset \mathbb{X}^*$ be the κ-norming set from Definition 3.7. Consider a finite set $A \subset \mathbb{N}$ with say $|A| = m$ and denote

$$E := [\mathbf{e}_n]_{n\in A}.$$

1 The M-basis constructed in [21] satisfies that $Y = [\mathbf{e}_n^*]_{n\in\mathbb{N}}$ is 1-norming and $\sup_{n\in\mathbb{N}} \|\mathbf{e}_n\|\, \|\mathbf{e}_n^*\| < \infty$. But the latter easily implies that $\{\|\mathbf{e}_n^*\|\, \mathbf{e}_n\}_{n\geq 1}$ is Y-null.

Given $\varepsilon > 0$, one can find a finite set $S \subset Y \cap \{x^* \in \mathbb{X}^* \ : \ \|x^*\| = 1\}$ so that

$$\max_{x^* \in S} |x^*(e)| \geq (1-\varepsilon)\kappa\|e\|, \quad \forall\, e \in E. \tag{3.18}$$

Indeed, it suffices to verify the above inequality for e of norm 1. Pick an $\varepsilon\kappa/2$-net $(z_k)_{k=1}^N$ in the unit sphere of E. For any k find a norm one $z_k^* \in Y$ so that $|z_k^*(z_k)| > (1-\varepsilon/2)\kappa$. We claim that $S = \{z_k^* : 1 \leq k \leq N\}$ has the desired properties. To see this, pick a norm one $e \in E$, and find k with $\|e - z_k\| \leq \varepsilon\kappa/2$. Then

$$\max_{x^* \in S} |x^*(e)| \geq |z_k^*(e)| \geq |z_k^*(z_k)| - \|e - z_k\| \geq (1-\varepsilon/2)\kappa - \varepsilon\kappa/2 = (1-\varepsilon)\kappa.$$

Next, since the sequence $\{\|\mathbf{e}_n^*\|\, \mathbf{e}_n\}$ is Y-null, for each $\delta > 0$ we can find an integer $n_0 > \max A$ so that

$$\max_{x^* \in S} |x^*(\mathbf{e}_n)|\, \|\mathbf{e}_n^*\| \leq \frac{\delta\kappa}{m}, \quad \forall\, n \geq n_0.$$

Pick any B of cardinality m with $\min B \geq n_0$, and let

$$G := [\mathbf{e}_n]_{n \in B}.$$

For $f = \sum_{n \in B} \mathbf{e}_n^*(f)\mathbf{e}_n \in G$, we have

$$\max_{x^* \in S} |x^*(f)| \leq \max_{x^* \in S} \sum_{n \in B} |x^*(\mathbf{e}_n)|\, \|\mathbf{e}_n^*\|\, \|f\| \leq \delta\kappa\|f\|. \tag{3.19}$$

We claim that

$$\|e + f\| \geq \frac{(1-\varepsilon-\delta)\kappa}{1+\delta\kappa}\|e\|, \quad \text{for any } \ e \in E,\ f \in G. \tag{3.20}$$

To show this, we fix $\gamma > 0$ (to be chosen later), and assume first that $\|f\| \geq (1+\gamma)\|e\|$. Then,

$$\|e + f\| \geq \|f\| - \|e\| \geq \gamma\|e\|.$$

Next assume that $\|f\| < (1+\gamma)\|e\|$, then using (3.18) and (3.19) we obtain that

$$\|e + f\| \geq \max_{x^* \in S} |x^*(e+f)| \geq (1-\varepsilon)\kappa\|e\| - \delta\kappa\|f\| > (1-\varepsilon-\delta(1+\gamma))\kappa\|e\|.$$

We now choose γ so that $\gamma = (1-\varepsilon-\delta(1+\gamma))\kappa$, that is,

$$\gamma = \frac{(1-\varepsilon-\delta)\kappa}{1+\delta\kappa},$$

which shows the claim in (3.20). Now, given $\rho > 1/\kappa$, we may pick $\delta = \varepsilon$ sufficiently small so that the above number $\gamma > 1/\rho$. Then, (3.20) becomes

$$\|e + f\| \geq \frac{1}{\rho}\,\|e\|, \quad \text{for any } \ e \in [e_n]_{n\in A}, \ f \in [e_n]_{n\in B},$$

for all B with $\min B \geq n_0$ and $|B| = |A| = m$. Thus, $\{\mathbf{e}_n\}_{n=1}^{\infty}$ is ρ-admissible. □

Remark 3.9 We give some more examples where property $\mathscr{P}(\kappa)$ holds.

(1) If the sequence $\{\|\mathbf{e}_n^*\|\,\mathbf{e}_n\}_{n=1}^{\infty}$ is weakly null then $\mathscr{P}(1)$ holds (since $Y = \mathbb{X}^*$ is always 1-norming).
(2) If $\{\mathbf{e}_n\}_{n=1}^{\infty}$ is a Schauder basis then $\mathscr{P}(\kappa)$ holds with $\kappa = 1/K_b$; see [20, Theorems I.3.1 and I.12.2].
(3) Let $\mathbb{X} = C(K)$, where K is a compact Hausdorff set, and let μ be a Radon probability measure in K with $\mathrm{supp}\mu = K$. Let $\{\mathbf{e}_n\}_{n=1}^{\infty}$ be a complete system in $\mathbb{X}$ which is orthonormal with respect to μ and uniformly bounded, that is,

$$\int_K \mathbf{e}_n \overline{\mathbf{e}_m}\, d\mu = \delta_{n,m} \quad \text{and} \quad \sup_n \|\mathbf{e}_n\|_\infty < \infty.$$

Then $\{\mathbf{e}_n\}_{n=1}^{\infty}$ has the property $\mathscr{P}(1)$ in $\mathbb{X} = C(K)$. Indeed, the sequence $\{\mathbf{e}_n\}_{n=1}^{\infty}$ is $L_1(\mu)$-null in $\mathbb{X}$, while $Y = L_1(\mu)$ is 1-norming in $\mathbb{X}$ (since the natural embedding of $C(K)$ into $L_\infty(\mu)$ is isometric). Specific examples are the trigonometric system in $C[0, 1]$ (in the real or complex case), as well as certain polygonal versions of the Walsh system [7,17,27], or any reorderings of them (which may cease to be Cesàro bases).
(4) As a dual of the previous, if $\mathbb{X} = L^1(\mu)$ then every system $\{\mathbf{e}_n\}_{n=1}^{\infty}$ as in (3) is weakly null, and hence case (1) applies.
(5) If $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^{\infty}$ is an M-basis such that

$$\boldsymbol{\varphi}(m) := \sup_{|A|\leq m} \Big\| \sum_{n\in A} \mathbf{e}_n \Big\| = \mathbf{o}(m), \quad \text{as} \quad m \to \infty,$$

then $\{\mathbf{e}_n\}_{n=1}^{\infty}$ is weakly null (and in particular, $\mathscr{P}(1)$ holds). Indeed, first note that also $\tilde{\boldsymbol{\varphi}}(m) = \sup\{\|\mathbf{1}_{\eta A}\| : |A| \leq m, \ |\eta| = 1\} = \mathbf{o}(m)$. Assume that the system is not weakly null. Then there exist a norm one $x^* \in \mathbb{X}^*$ and $\varepsilon_0 > 0$ so that the set $A = \{n \in \mathbb{N} : |x^*(\mathbf{e}_n)| \geq \varepsilon_0\}$ is infinite. For every $m \geq 1$, pick a set $F \subset A$ with $|F| = m$ and let $\eta_n = \mathrm{sign}[x^*(\mathbf{e}_n)]$; then

$$\tilde{\boldsymbol{\varphi}}(m) \geq \|\mathbf{1}_{\overline{\eta} F}\| \geq \left| x^* \left(\sum_{n\in F} \overline{\eta_n} \mathbf{e}_n \right) \right| = \sum_{n\in F} |x^*(\mathbf{e}_n)| \geq m\varepsilon_0,$$

contradicting our assumption.

Finally, as a consequence of Propositions 3.6 and 3.8 one obtains

Theorem 3.10 *Let $\{\mathbf{e}_n, \mathbf{e}_n^*\}_{n=1}^\infty$ be a seminormalized M-basis with the property $\mathscr{P}(\kappa)$. Then, if ϑ_m is as in (3.14), we have*

$$\mathbf{L}_m^{\mathrm{ch},t} \geq \frac{\kappa\, \vartheta_m}{(\kappa+1)t}, \quad \forall\, m \in \mathbb{N}, \quad t \in (0,1]. \tag{3.21}$$

4 Examples

The first two examples are variants of those in [5, §5.1] and [6, §8.1].

4.1 Example 4.1: the summing basis

Let $\mathbb{X}$ be the closure of the set of all finite sequences $\mathbf{a} = (a_n)_n \in c_{00}$ with the norm

$$\|\mathbf{a}\| = \sup_m \left| \sum_{n=1}^m a_n \right|.$$

The canonical system $\mathscr{B} = \{\mathbf{e}_n\}_{n=1}^\infty$ is a Schauder basis in $\mathbb{X}$ with $K_b = 1$ and $\|\mathbf{e}_n\| = 1$ for all n. Also, $\|\mathbf{e}_1^*\| = 1$, $\|\mathbf{e}_n^*\| = 2$ if $n \geq 2$, so $\mathfrak{K} = 2$ in Theorem 1.1; see [5, §5.1]. We now show that, for this example of $(\mathbb{X}, \mathscr{B})$, the bound of Theorem 1.1 is sharp. As in [5, §5.1], we consider the element:

$$x = \Bigg(\underbrace{\frac{1}{2}, \frac{1}{t}, \frac{1}{2}}, \ldots, \underbrace{\frac{1}{2}, \frac{1}{t}, \frac{1}{2}}; \frac{1}{2}; \underbrace{-1, 1}, \ldots, \underbrace{-1, 1}, 0, \ldots, \Bigg),$$

where we have m blocks of $\left(\frac{1}{2}, \frac{1}{t}, \frac{1}{2}\right)$ and m blocks of $(-1, 1)$. Picking $A = \{n : x_n = -1\}$ as a t-greedy set of x, we see that

$$\begin{aligned} \|x - \mathfrak{CG}_m^t(x)\| &= \min_{a_i, i=1,\ldots,m} \left\| \left(\frac{1}{2}, \frac{1}{t}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{t}, \frac{1}{2}; \frac{1}{2}; a_1, 1, a_2, 1, \ldots, a_m, 1, 0, \ldots, \right) \right\| \\ &\geq \left\| \left(\frac{1}{2}, \frac{1}{t}, \frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{t}, \frac{1}{2}; \frac{1}{2}; 0, \ldots, \right) \right\| = m + \frac{m}{t} + \frac{1}{2}. \end{aligned}$$

On the other hand,

$$\begin{aligned} \sigma_m(x) &\leq \left\| x - \frac{t+1}{t}(0, 1, 0, \ldots, 0, 1, 0; 0, \ldots) \right\| \\ &= \left\| \left(\frac{1}{2}, -1, \frac{1}{2}, \ldots, \frac{1}{2}, -1, \frac{1}{2}; \frac{1}{2}; -1, 1, \ldots, -1, 1, 0 \ldots, \right) \right\| = \frac{1}{2}. \end{aligned}$$

Hence, $\mathbf{L}_m^{\mathrm{ch},t} \geq 1 + 2(1 + \frac{1}{t})m$ and we conclude that $\mathbf{L}_m^{\mathrm{ch},t} = 1 + 2(1 + \frac{1}{t})m$ by Theorem 1.1. As a consequence, observe that in this case $\mathfrak{CG}_m^t(x) = 0$.

Remark 4.1 The above example strengthens [19, Theorem 2.4], where the authors are only able to show that $1 + 4m \leq \mathbf{L}_m^{\text{ch}} \leq 1 + 6m$.

4.2 Example 4.2: the difference basis

Let $\{\mathbf{e}_n\}_{n=1}^{\infty}$ be the canonical basis in $\ell^1(\mathbb{N})$ and define the elements

$$y_1 = \mathbf{e}_1, \;\; y_n = \mathbf{e}_n - \mathbf{e}_{n-1}, \;\; n = 2, 3, \ldots$$

The new system $\mathscr{B} = \{y_n\}_{n=1}^{\infty}$ is called the difference basis of ℓ^1. We recall some basic properties used in [6, §8.1]. If $(b_n)_n \in c_{00}$ then

$$\left\| \sum_{n=1}^{\infty} b_n y_n \right\| = \sum_{n=1}^{\infty} |b_n - b_{n+1}|.$$

Also, $\mathscr{B}$ is a monotone basis with $\|y_1\| = 1$, $\|y_n\| = 2$ if $n \geq 2$, and $\|y_n^*\| = 1$ for all $n \geq 1$ (in fact, the dual system corresponds to the summing basis). So, $\mathfrak{K} = 2$ and Theorem 1.1 gives $\mathbf{L}_m^{\text{ch},t} \leq 1 + 2(1 + \frac{1}{t})m$ for all $t \in (0, 1]$. To show the equality we consider the vector $x = \sum_n b_n y_n$ with coefficients (b_n) given by

$$\left(1, \underbrace{1, 1, -\tfrac{1}{t}, 1}, \ldots, \underbrace{1, 1, -\tfrac{1}{t}, 1}, 0, \ldots, \right),$$

where the block $\left(1, 1, \frac{-1}{t}, 1\right)$ is repeated m times. If we take $\Gamma = \{2, 6, \ldots, 4m-2\}$ as a t-greedy set for x of cardinality m, then

$$\begin{aligned}
\|x - \mathfrak{CG}_m^t(x)\| &= \inf_{(a_j)_{j=1}^m} \left\| x - \sum_{j=1}^m a_j y_{4j-2} \right\| \\
&= \inf_{(a_j)_{j=1}^m} \left\| \left(1, 1 - a_1, 1, \frac{-1}{t}, 1, \ldots, 1 - a_m, 1, \frac{-1}{t}, 1, 0, \ldots, \right) \right\| \\
&= \inf_{(a_j)_{j=1}^m} 2\sum_{j=1}^m |a_j| + 2m\left(1 + \frac{1}{t}\right) + 1 = 2m\left(1 + \frac{1}{t}\right) + 1.
\end{aligned}$$

Hence, in this case we also have $\mathfrak{CG}_m^t(x) = 0$. On the other hand

$$\sigma_m(x) \leq \left\| x + \left(1 + \tfrac{1}{t}\right) \sum_{j=1}^m y_{4j} \right\| = \|(1, 1, 1, 1, 1, \ldots, 1, 1, 1, 1, 0, \ldots)\| = 1.$$

This shows that $\mathbf{L}_m^{\text{ch},t} = 1 + 2(1 + \frac{1}{t})m$.

4.3 Example 4.3: the trigonometric system in $L^p(\mathbb{T})$

Consider $\mathscr{B} = \{e^{inx}\}_{n\in\mathbb{Z}}$ in $L^p(\mathbb{T})$ for $1 \le p < \infty$, and in $C(\mathbb{T})$ if $p = \infty$. In [22], Temlyakov showed that

$$c_p m^{|\frac{1}{p}-\frac{1}{2}|} \le \mathbf{L}_m \le 1 + 3m^{\left|\frac{1}{p}-\frac{1}{2}\right|},$$

for some $c_p > 0$ and all $1 \le p \le \infty$. Adapting his argument, Shao and Ye have recently established, in [19, Theorem 2.1], that for $1 < p \le \infty$ it also holds

$$\mathbf{L}_m^{\mathrm{ch}} \approx m^{|\frac{1}{p}-\frac{1}{2}|}. \tag{4.1}$$

The case $p = 1$ is left as an open question, and only the estimate $\frac{\sqrt{m}}{\ln(m)} \lesssim \mathbf{L}_m^{\mathrm{ch}} \lesssim \sqrt{m}$ is given; see [19, (2.24)]. Moreover, the proof of the case $p = \infty$ seems to contain some gaps and may not be complete.

Here, we shall give a short proof ensuring the validity of (4.1) in the full range $1 \le p \le \infty$, with a reasoning similar to [5, §5.4]. More precisely, we shall prove the following.

Proposition 4.2 *Let $1 \le p \le \infty$. Then there exists $c_p > 0$ such that*

$$\mathbf{L}_m^{\mathrm{ch},t} \ge c_p\, t^{-1} m^{|\frac{1}{p}-\frac{1}{2}|}, \quad \forall\, m \in \mathbb{N}, \quad t \in (0,1]. \tag{4.2}$$

We remark that in the cases $p = 1$ and $p = \infty$ the trigonometric system is not a Schauder basis, but it is a Cesàro basis.[2] So we may use the lower bounds in Theorem 1.3, namely

$$\mathbf{L}_m^{\mathrm{ch},t} \ge c'_p\, t^{-1} \sup_{\substack{|A|=|B|\le m \\ A>2B \ \text{ or } \ B>2A}} \ \sup_{|\varepsilon|=|\eta|=1} \frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|}. \tag{4.3}$$

- Case $1 < p \le 2$. Assume that $m = 2\ell + 1$ or $2\ell + 2$ (that is, $\ell = \lfloor \frac{m-1}{2} \rfloor$). We choose $B = \{-\ell, \dots, \ell\}$, so that $\mathbf{1}_B = D_\ell$ is the ℓth Dirichlet kernel, and hence

$$\|\mathbf{1}_B\|_p = \|D_\ell\|_{L^p(\mathbb{T})} \approx m^{1-\frac{1}{p}}.$$

Next we take a lacunary set $A = \{2^j : j_0 \le j \le j_0 + 2\ell\}$, so that

$$\|\mathbf{1}_A\|_p \approx \sqrt{m}, \tag{4.4}$$

and where j_0 is chosen such that $2^{j_0} \ge m$, and hence $A > 2B$. Then, (4.3) implies

$$\mathbf{L}_m^{\mathrm{ch},t} \ge c_p\, t^{-1} \frac{m^{1/2}}{m^{1-\frac{1}{p}}} = c_p\, t^{-1} m^{\left|\frac{1}{p}-\frac{1}{2}\right|}.$$

[2] We equip $\mathscr{B}$ with its natural ordering $\{1, e^{ix}, e^{-ix}, e^{2ix}, e^{-2ix}, \dots\}$.

- Case $2 \leq p < \infty$. The same proof works in this case, just reversing the roles of A and B.
- Case $p = \infty$. We replace the lacunary set by a Rudin-Shapiro polynomial of the form

$$R(x) = e^{iNx} \sum_{n=0}^{2^L-1} \varepsilon_n e^{inx}, \quad \text{with } \varepsilon_n \in \{\pm 1\},$$

where L is such that $2^L \leq m < 2^{L+1}$; see e.g. [14, p. 33]. Then, $R = \mathbf{1}_{\varepsilon B}$ with $B = N + \{0, 1, \ldots, 2^L - 1\}$ and

$$\|\mathbf{1}_{\varepsilon B}\|_\infty = \|R\|_{L^\infty(\mathbb{T})} \approx \sqrt{m}.$$

If we pick $N \geq 2 \cdot 2^L$, then $B > 2A$ with $A = \{\pm 1, \ldots, \pm (2^L - 1)\}$. Finally,

$$\|\mathbf{1}_A\|_\infty = \|D_{2^L-1} - 1\|_{L^\infty(\mathbb{T})} \approx m.$$

So, (4.3) implies the desired bound.
- Case $p = 1$. We use the lower bound in Lemma 3.4, namely

$$\mathbf{L}_m^{\mathrm{ch},t} \geq c_1' \, t^{-1} \, \frac{\|\mathbf{1}_A\|}{\|\mathbf{1}_B + y\|}, \tag{4.5}$$

for all $|A| = |B| \leq m$ and all y such that $A > 2(B \cup \operatorname{supp} y)$ and $\sup_n |\mathbf{e}_n^*(y)| \leq 1$. As before, let $m = 2\ell + 1$ or $2\ell + 2$, and choose the same sets A and B as in the case $1 < p \leq 2$. Next choose y so that the vector

$$V_\ell = \mathbf{1}_B + y$$

is a de la Vallée-Poussin kernel as in [14, p. 15]. Then, the Fourier coeffients $\mathbf{e}_n^*(y)$ have modulus ≤ 1 and are supported in $\{n \ : \ \ell < |n| \leq 2\ell + 1\}$, so the condition $A > 2(B \cup \operatorname{supp} y)$ holds if $2^{j_0} \geq 2m + 1$. Finally,

$$\|\mathbf{1}_B + y\|_1 = \|V_\ell\|_{L^1(\mathbb{T})} \leq 3,$$

so the bound $\mathbf{L}_m^{\mathrm{ch},t} \gtrsim t^{-1}\sqrt{m}$ follows from (4.5).

Remark 4.3 Using the trivial upper bound $\mathbf{L}_m^{\mathrm{ch},t} \leq \mathbf{L}_m^t \lesssim t^{-1} m^{|\frac{1}{p}-\frac{1}{2}|}$, we conclude that $\mathbf{L}_m^{\mathrm{ch},t} \approx t^{-1} m^{|\frac{1}{p}-\frac{1}{2}|}$ for all $1 \leq p \leq \infty$.

5 Comparison between $\tilde{\mu}_m$ and $\tilde{\mu}_m^d$

In this section we compare the democracy constants $\tilde{\mu}_m$ and $\tilde{\mu}_m^d$ defined in §1 above. Let us first note that

$$\tilde{\mu}_m^d \leq \tilde{\mu}_m \leq (\tilde{\mu}_m^d)^2 \tag{5.1}$$

and

$$\tilde{\mu}_m^d \leq \tilde{\mu}_m \leq (1+2\kappa)\gamma_m \tilde{\mu}_m^d, \tag{5.2}$$

where $\kappa = 1$ or 2 depending if $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Indeed, the left inequality in (5.1) is immediate by definition, and the right one follows from

$$\frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|} = \frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{C}\|}\frac{\|\mathbf{1}_{C}\|}{\|\mathbf{1}_{\varepsilon A}\|} \leq (\tilde{\mu}_m^d)^2,$$

for any $|A| = |B| \leq m$ and any C disjoint with $A \cup B$ with $|C| = |A| = |B|$. Concerning the right inequality in (5.2), we use that if $|A| = |B| \leq m$ then

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \leq \frac{\|\mathbf{1}_{\varepsilon(A\setminus B)}\| + \|\mathbf{1}_{\varepsilon(A\cap B)}\|}{\|\mathbf{1}_{\eta B}\|} \leq \gamma_m \frac{\|\mathbf{1}_{\varepsilon(A\setminus B)}\|}{\|\mathbf{1}_{\eta(B\setminus A)}\|} + \frac{\|\mathbf{1}_{\varepsilon(A\cap B)}\|}{\|\mathbf{1}_{\eta B}\|} \leq \gamma_m\, \tilde{\mu}_m^d + 2\kappa\gamma_m,$$

using in the last step [5, Lemma 3.3]. From (5.2) we see that $\tilde{\mu}_m \approx \tilde{\mu}_m^d$ when $\mathscr{B}$ is quasi-greedy for constant coefficients.

In the next subsection we shall show that $\tilde{\mu}_m \approx \tilde{\mu}_m^d$ for all Schauder bases, a result which seems new in the literature.

5.1 Equivalence for Schauder bases

We begin with a simple observation.

Lemma 5.1

$$\tilde{\mu}_m^d = \sup\left\{\frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|} \;:\; |B| \leq |A| \leq m, \quad A \cap B = \emptyset, \quad |\varepsilon| = |\eta| = 1\right\}. \tag{5.3}$$

Proof. Let $|\varepsilon| = |\eta| = 1$ and $|B| \leq |A| \leq m$ with $A \cap B = \emptyset$. We must show that $\|\mathbf{1}_{\eta B}\|/\|\mathbf{1}_{\varepsilon A}\| \leq \tilde{\mu}_m^d$. Pick any set C disjoint with $A \cup B$ such that $|B| + |C| = |A|$. We now use the elementary inequality

$$\|x\| = \left\|\frac{x+y}{2} + \frac{x-y}{2}\right\| \leq \max\{\|x+y\|, \|x-y\|\}, \tag{5.4}$$

with $x = \mathbf{1}_{\eta B}$ and $y = \mathbf{1}_C$. Let $\eta' \in \Upsilon$ be such that $\eta'|_B = \eta|_B$ and $\eta'|_C = \pm 1$, according to the sign that reaches the maximum in (5.4). Then $\|\mathbf{1}_{\eta B}\| \leq \|\mathbf{1}_{\eta'(B\cup C)}\| \leq \tilde{\mu}_m^d \|\mathbf{1}_{\varepsilon A}\|$, and the result follows. □

Theorem 5.2 *If K_b is the basis constant and $\varkappa = \sup_n \|\mathbf{e}_n^*\|\|\mathbf{e}_n\|$, then*

$$\tilde{\mu}_m \leq 2(K_b+1)\tilde{\mu}_m^d + \varkappa K_b. \tag{5.5}$$

Proof. Let $|A| = |B| \leq m$, and $|\varepsilon| = |\eta| = 1$. Then

$$\frac{\|\mathbf{1}_{\eta B}\|}{\|\mathbf{1}_{\varepsilon A}\|} \leq \frac{\|\mathbf{1}_{\eta(B\setminus A)}\|}{\|\mathbf{1}_{\varepsilon A}\|} + \frac{\|\mathbf{1}_{\eta(B\cap A)}\|}{\|\mathbf{1}_{\varepsilon A}\|} = I + II.$$

Lemma 5.1 implies $I \leq \tilde{\mu}_m^d$. We now bound II. Pick an integer n_0 such that $A_1 = \{n \in A \;:\; n \leq n_0\}$ and $A_2 = A \backslash A_1$ satisfy

$$|A_1| = |A_2| \quad \text{(if } |A| \text{ is even), or} \quad |A_1| = \frac{|A|-1}{2} = |A_2| - 1 \quad \text{(if } |A| \text{ is odd).}$$

Then

$$\begin{aligned} II &\leq \frac{\|\mathbf{1}_{\eta(B\cap A_1)}\|}{\|\mathbf{1}_{\varepsilon A}\|} + \frac{\|\mathbf{1}_{\eta(B\cap A_2)}\|}{\|\mathbf{1}_{\varepsilon A}\|} \\ &\leq (K_b+1)\frac{\|\mathbf{1}_{\eta(B\cap A_1)}\|}{\|\mathbf{1}_{\varepsilon A_2}\|} + K_b\frac{\|\mathbf{1}_{\eta(B\cap A_2)}\|}{\|\mathbf{1}_{\varepsilon A_1}\|} \;=\; II_1 + II_2, \end{aligned}$$

using in the second line the basis constant bound for the denominator. Since $|B \cap A_1| \leq |A_1| \leq |A_2|$, we see that

$$II_1 \leq (K_b+1)\tilde{\mu}_m^d.$$

On the other hand, picking any number $n_1 \in B \cap A_2$, and using $\|\mathbf{e}_{n_1}^*\|\|\mathbf{1}_{\varepsilon A}\| \geq |\mathbf{e}_{n_1}^*(\mathbf{1}_{\varepsilon A})| = 1$, we see that

$$II_2 \leq K_b\frac{\|\mathbf{1}_{\eta(B\cap A_2\backslash\{n_1\})}\|}{\|\mathbf{1}_{\varepsilon A_1}\|} + K_b\|\mathbf{e}_{n_1}\|\|\mathbf{e}_{n_1}^*\| \leq K_b\tilde{\mu}_m^d + \varkappa K_b,$$

the last bound due to $|B \cap A_2\backslash\{n_1\}| \leq |A_2| - 1 \leq |A_1|$ and Lemma 5.1. Putting together the previous bounds easily leads to (5.5). □

Remark 5.3 A similar argument shows the equivalence of the standard (unsigned) democracy parameters

$$\mu_m = \sup_{|A|=|B|\leq m}\frac{\|\mathbf{1}_B\|}{\|\mathbf{1}_A\|} \quad \text{and} \quad \mu_m^d = \sup_{\substack{|A|=|B|\leq m \\ A\cap B=\emptyset}}\frac{\|\mathbf{1}_B\|}{\|\mathbf{1}_A\|}. \tag{5.6}$$

Indeed, in this case, the analog of (5.3) takes the weaker form

$$\mu_m^d \leq \sup_{\substack{|B|\leq|A|\leq m \\ A\cap B=\emptyset}}\frac{\|\mathbf{1}_B\|}{\|\mathbf{1}_A\|} \leq K_b\mu_m^d. \tag{5.7}$$

Then, (5.7) and the same proof we gave for Theorem 5.2 (with $\eta = \varepsilon \equiv 1$) leads to

$$\mu_m \leq 2(K_b+1)K_b\,\mu_m^d + \varkappa\,K_b. \tag{5.8}$$

5.2 An example where $\tilde{\mu}_m$ grows faster than $\tilde{\mu}_m^d$

The following example also seems to be new in the literature. As in (5.6), we denote by μ_m, μ_m^d the democracy parameters corresponding to constant signs.

Theorem 5.4 *There exists a Banach space $\mathbb{X}$ with an M-basis $\mathscr{B}$ such that*

$$\limsup_{m\to\infty} \frac{\tilde{\mu}_m}{[\tilde{\mu}_m^d]^{2-\varepsilon}} = \limsup_{m\to\infty} \frac{\mu_m}{[\mu_m^d]^{2-\varepsilon}} = \infty, \quad \forall\, \varepsilon > 0.$$

Proof. Let $N_0 = 1$, and define recursively $N_k = 2^{2^{N_{k-1}}}$, and $N_k' = N_1 + \cdots + N_{k-1}$. Consider the blocks of integers

$$S_k = \{N_k' + 1, \ldots, N_k' + N_k\},$$

and denote the tail blocks by $T_k = \cup_{j\geq k+1} S_j$. Finally, let

$$\mathfrak{N}_k = \left\{ (\sigma_j)_{j\in S_k} \;:\; \sigma_j \in \{\pm 1\} \quad \text{and} \quad \sum_{j\in S_k} \sigma_j = 0 \right\}.$$

We define a real Banach space $\mathbb{X}$ as the closure of c_{00} with the norm

$$\|x\| \;=\; \max\left\{ \|x\|_\infty,\; \sup_{k\geq 1} \alpha_k \sup_{\sigma\in\mathfrak{N}_k} \big|\langle \mathbf{1}_{\sigma S_k}, x\rangle\big|,\; \sup_{k\geq 1} \beta_k \sup_{\substack{S\subset T_k\\ |S|=N_k}} \sum_{j\in S} |x_j| \right\},$$

where the weights α_k and β_k are chosen as follows:

$$\alpha_k = 2^{-N_{k-1}} = \frac{1}{\log_2 N_k} \quad \text{and} \quad \beta_k = \frac{1}{\sqrt{N_k}}.$$

Observe that

$$N_k' = N_1 + \cdots + N_{k-1} \leq 2N_{k-1} = 2\log_2\log_2 N_k \quad \text{and} \quad \frac{\alpha_k}{\beta_k} = \frac{\sqrt{N_k}}{\log_2 N_k}.$$

□

Claim 1 $\tilde{\mu}_{N_k} \geq \mu_{N_k} \geq \dfrac{N_k/2}{(\log_2 N_k)\sqrt{\log_2\log_2 N_k}}$, for all $k \geq 1$.

Proof. Pick any $A \subset S_k \cup S_{k+1}$ such that $|A| = N_k$ and $|A\cap S_k| = |A\cap S_{k+1}| = N_k/2$. Then

$$\|\mathbf{1}_A\| \geq \alpha_k N_k/2 = \frac{N_k/2}{\log_2 N_k}.$$

Next, pick $B = S_k$, so that $|B| = |A| = N_k$ and

$$\|\mathbf{1}_B\| = \max\left\{\ 1,\ \alpha_k \cdot 0,\ \sup_{n\le k-1} \beta_n N_n\ \right\} = \beta_{k-1}N_{k-1} = \sqrt{N_{k-1}} = \sqrt{\log_2 \log_2 N_k}.$$

Then $\mu_{N_k} \ge \|\mathbf{1}_A\|/\|\mathbf{1}_B\| \ge \frac{N_k/2}{(\log_2 N_k)\sqrt{\log_2 \log_2 N_k}}$. □

Claim 2 $\mu^d_{N_k} \le \tilde{\mu}^d_{N_k} \le \sqrt{N_k}$, for all $k \ge 2$.

Proof. Let A, B be any pair of disjoint sets with $|A| = |B| \le N_k$, and let $|\varepsilon| = |\eta| = 1$. If $|A| = |B| \le \sqrt{N_k}$, then the trivial bounds $\|\mathbf{1}_{\varepsilon A}\| \le |A|$ and $\|\mathbf{1}_{\eta B}\| \ge 1$ give

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \le \sqrt{N_k}.$$

So, it remains to consider the cases $\sqrt{N_k} < |A| = |B| \le N_k$. We split A into three parts

$$A_0 = A \cap S_k, \quad A_+ = A \cap T_k, \quad A_- = A \cap [S_1 \cup \ldots \cup S_{k-1}].$$

Then, we have the following upper bound

$$\begin{aligned}\|\mathbf{1}_{\varepsilon A}\| &\le \max\left\{1,\ \sup_{n<k} \alpha_n |A_-|,\ \alpha_k|A_0|,\ \sup_{n>k}\alpha_n N_k,\ \sup_{n<k}\beta_n N_n,\ \sup_{n\ge k}\beta_n|A|\ \right\} \\ &\le \max\left\{\ N_k',\ \alpha_k|A_0|,\ \beta_k|A|\ \right\},\end{aligned}$$

due to the elementary inequalities

- $\sup_{n<k} \alpha_n |A_-| \le |A_-| \le N_k'$
- $\sup_{n>k} \alpha_n N_k = \alpha_{k+1} N_k = N_k 2^{-N_k} \le 1$
- $\sup_{n<k} \beta_n N_n = \sqrt{N_{k-1}} \le N_{k-1} \le N_k'$
- $\sup_{n\ge k} \beta_n |A| = \beta_k |A|$.

Moreover, since $\beta_k|A| \le \min\{\beta_k N_k = \sqrt{N_k},\ \alpha_k|A|\ \}$, we derive

$$\|\mathbf{1}_{\varepsilon A}\| \le \max\{\sqrt{N_k}, \alpha_k|A_0|\} \quad \text{and} \quad \|\mathbf{1}_{\varepsilon A}\| \le \max\{N_k', \alpha_k|A|\}. \tag{5.9}$$

We now give a lower bound for $\|\mathbf{1}_{\eta B}\|$. The key estimate will rely on the following □

Lemma 5.5 *Let* $B_0 = B \cap S_k$ *and* $B_0^c = S_k \backslash B_0$. *Then*

$$\sup_{\sigma\in\mathfrak{N}_k} \left|\langle \mathbf{1}_{\sigma S_k}, \mathbf{1}_{\eta B_0}\rangle\right| \ge \min\{|B_0|, |B_0^c|\}. \tag{5.10}$$

Proof. If $|B_0| \leq N_k/2$, then we may select any $\sigma \in \mathfrak{N}_k$ such that $\sigma|_{B_0} = \eta$ (which is possible since $|B_0^c| \geq |B_0|$), which gives

$$|\langle \mathbf{1}_{\sigma S_k}, \mathbf{1}_{\eta B_0} \rangle| = |B_0| = \min\{|B_0|, |B_0^c|\}.$$

Assume now that $|B_0| > N_k/2$. Pick any $S \subset B_0$ with $|S| = |B_0^c| = N_k - |B_0|$. Choose $\nu \in \{-1, 1\}^{B_0^c}$ so that $\sum_{i \in S} \eta_i + \sum_{i \in B_0^c} \nu_i = 0$. Choose $\tau \in \{-1, 1\}^{B_0 \setminus S}$ so that $\sum_{i \in B_0 \setminus S} \tau_i = 0$. Replacing τ by $-\tau$, if necessary, we may assume that $\sum_{i \in B_0 \setminus S} \tau_i \eta_i \geq 0$. Finally, define $\sigma \in \mathfrak{N}_k$ by setting

$$\sigma|_S = \eta|_S, \quad \sigma|_{B_0^c} = \nu|_{B_0^c}, \quad \sigma|_{B_0 \setminus S} = \tau|_{B_0 \setminus S}.$$

Then,

$$|\langle \mathbf{1}_{\sigma S_k}, \mathbf{1}_{\eta B_0} \rangle| = \sum_{i \in S} \eta_i^2 + \sum_{i \in B_0 \setminus S} \tau_i \eta_i \geq |S| = |B_0^c| = \min\{|B_0|, |B_0^c|\}. \quad \square$$

From the lemma and the definition of the norm we see that

$$\|\mathbf{1}_{\eta B}\| \geq \max\Big\{ 1, \ \alpha_k \min\{|B_0|, |B_0^c|\}, \ \beta_k |B_+| \Big\}. \tag{5.11}$$

We shall finally combine the estimates in (5.9) and (5.11) to establish Claim 2. We distinguish two cases

Case 1: $\min\{|B_0|, |B_0^c|\} = |B_0^c|$. Then, since $A_0 \subset B_0^c$, we see that

$$\alpha_k |A_0| \leq \alpha_k |B_0^c| \leq \|\mathbf{1}_{\eta B}\|,$$

and therefore the first estimate in (5.9) gives

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \leq \frac{\max\{\sqrt{N_k}, \|\mathbf{1}_{\eta B}\|\}}{\|\mathbf{1}_{\eta B}\|} \leq \sqrt{N_k}.$$

Case 2: $\min\{|B_0|, |B_0^c|\} = |B_0|$. Then, (5.11) reduces to

$$\|\mathbf{1}_{\eta B}\| \geq \max\Big\{ \alpha_k |B_0|, \ \beta_k |B_+| \Big\} \geq \beta_k \frac{|B_0| + |B_+|}{2} = \beta_k \frac{|B| - |B_-|}{2} \geq \beta_k |B|/4,$$

since $|B_-| \leq N_k' \leq \sqrt{N_k}/2 \leq |B|/2$, if $k \geq 2$. Also, the second bound in (5.9) reads

$$\|\mathbf{1}_{\varepsilon A}\| \leq \alpha_k |A|,$$

since $N_k' \leq \sqrt{N_k}/\log_2 N_k = \alpha_k \sqrt{N_k} \leq \alpha_k |A|$, if $k \geq 2$. Thus

$$\frac{\|\mathbf{1}_{\varepsilon A}\|}{\|\mathbf{1}_{\eta B}\|} \leq \frac{\alpha_k |A|}{\beta_k |B|/4} = \frac{4\alpha_k}{\beta_k} = \frac{4\sqrt{N_k}}{\log_2 N_k} \leq \sqrt{N_k}.$$

This establishes Claim 2.

From Claims 1 and 2 we now deduce that

$$\frac{\mu_{N_k}}{[\tilde{\mu}^d_{N_k}]^{2-\varepsilon}} \geq \frac{N_k^{\varepsilon/2}/2}{(\log_2 N_k)\sqrt{\log_2 \log_2 N_k}} \to \infty,$$

and therefore

$$\limsup_{N\to\infty} \frac{\mu_N}{[\mu^d_N]^{2-\varepsilon}} = \limsup_{N\to\infty} \frac{\tilde{\mu}_N}{[\tilde{\mu}^d_N]^{2-\varepsilon}} = \infty. \qquad \square$$

6 Norm convergence of $\mathfrak{CG}^t_m x$ and $\mathscr{G}^t_m x$

In this section we search for conditions on $\mathscr{B} = \{\mathbf{e}_n\}_{n=1}^\infty$ under which it holds

$$\|x - \mathfrak{CG}_m(x)\| \to 0, \quad \forall\, x \in \mathbb{X}. \tag{6.1}$$

In [19, Theorem 1.1] this convergence is asserted for every basis in $\mathbb{X}$. Here we investigate whether (6.1) may be true for a general M-basis, as defined in §1.

The solution to this question requires the notion of *strong M-basis*; see [21, Def 8.4]. We say that $\mathscr{B}$ is a strong M-basis if additionally to the conditions (a)–(d) in §1 it also holds

$$\overline{\operatorname{span}\{\mathbf{e}_n\}_{n\in A}} = \Big\{x \in \mathbb{X} \;:\; \operatorname{supp} x \subset A\Big\}, \quad \forall\, A \subset \mathbb{N}. \tag{6.2}$$

Clearly, all Schauder or Cesàro bases (in some ordering) are strong M-bases; see e.g. [18] for further examples. However, there exist M-bases which are not strong M-bases, see e.g. [21, p. 244], or [11][3] for seminormalized examples in Hilbert spaces.

Lemma 6.1 *If $\mathscr{B}$ is an M-basis which is not a strong M-basis, then there exists an $x_0 \in \mathbb{X}$ such that, for all Chebyshev greedy operators $\mathfrak{CG}_m$,*

$$\liminf_{m\to\infty} \|x_0 - \mathfrak{CG}_m(x_0)\| > 0. \tag{6.3}$$

Proof. If $\mathscr{B}$ is not a strong M-basis there exists some set $A \subset \mathbb{N}$ (necessarily infinite) and some $x_0 \in \mathbb{X}$ with $\operatorname{supp} x_0 \subset A$ such that

$$\delta = \operatorname{dist}(x_0, [\mathbf{e}_n]_A) > 0.$$

Since $\operatorname{supp}\mathfrak{CG}_m x_0$ is always a subset of A, this implies (6.3). $\square$

[3] We thank V. Kadets for kindly providing this reference.

Remark 6.2 The above reasoning also implies that $\liminf_m \|x_0 - \mathscr{G}_m x_0\| > 0$, for all greedy operators $\mathscr{G}_m$. In particular, if there exists a not strong M-basis with the quasi-greedy condition

$$C_q := \sup_{\substack{\mathscr{G}_m \in \mathbb{G}_m \\ m \in \mathbb{N}}} \|\mathscr{G}_m\| < \infty, \tag{6.4}$$

it will not occur that $\mathscr{G}_m x$ converges to x for all $x \in \mathbb{X}$. This observation suggests that in the characterization of quasi-greedy biorthogonal systems given in [28, Theorem 1] one may need to assume that $\mathscr{B}$ is a *strong M-basis*, or else clarify if this property could be a consequence of (6.4).[4]

Here we show that under the strong M-basis assumption, the conclusions of [19, Theorem 1.1] (and also of "3 ⇒ 1" in [28, Theorem 1]) hold.

Proposition 6.3 *If $\mathscr{B}$ is a strong M-basis then, for all Chebyshev t-greedy operators $\mathfrak{CG}^t_m$,*

$$\lim_{m \to \infty} \|x - \mathfrak{CG}^t_m(x)\| = 0, \quad \forall\, x \in \mathbb{X}. \tag{6.5}$$

If additionally $C_q < \infty$, then for all t-greedy operators $\mathscr{G}^t_m$,

$$\lim_{m \to \infty} \|x - \mathscr{G}^t_m(x)\| = 0, \quad \forall\, x \in \mathbb{X}. \tag{6.6}$$

Proof. Given $x \in \mathbb{X}$ and $\varepsilon > 0$, by (6.2) there exists $z = \sum_{n \in B} b_n \mathbf{e}_n$ such that $\|x - z\| < \varepsilon$, for some finite set $B \subset \operatorname{supp} x$. Let $\alpha = \min_{n \in B} |\mathbf{e}^*_n(x)|$ and

$$\bar{\Lambda}_\alpha = \{n \;:\; |\mathbf{e}^*_n(x)| \ge \alpha\}.$$

Since $\alpha > 0$, this is a finite greedy set for x which contains B. Moreover, we claim that

$$\bar{\Lambda}_\alpha \subset \operatorname{supp}\mathfrak{CG}^t_m(x) =: A, \quad \forall\, m > |\bar{\Lambda}_{t\alpha}|. \tag{6.7}$$

Indeed, if this was not the case there would exist $n_0 \in \bar{\Lambda}_\alpha \backslash A$, and since A is a t-greedy set for x, then $\min_{n \in A} |\mathbf{e}^*_n(x)| \ge t|\mathbf{e}^*_{n_0}(x)| \ge t\alpha$. So, $A \subset \bar{\Lambda}_{t\alpha}$, which is a contradiction since $m = |A| > |\bar{\Lambda}_{t\alpha}|$. Therefore, (6.7) holds and hence

$$\|x - \mathfrak{CG}^t_m(x)\| \le \|x - \sum_{n \in B} b_n \mathbf{e}_n\| < \varepsilon, \quad \forall\, m > |\bar{\Lambda}_{t\alpha}|.$$

This establishes (6.5).

We now prove (6.6). As above, let $z = \sum_{n \in B} b_n \mathbf{e}_n$ with $B \subset \operatorname{supp} x$ and $\|x - z\| < \varepsilon$. Performing if necessary a small perturbation in the b_n's, we may assume that $b_n \ne \mathbf{e}^*_n(x)$ for all $n \in B$. Let now

$$\alpha_1 = \min_{n \in B} |\mathbf{e}^*_n(x)|, \quad \alpha_2 = \min_{n \in B} |\mathbf{e}^*_n(x - z)|, \quad \text{and} \quad \alpha = \min\{\alpha_1, \alpha_2\} > 0.$$

[4] After this manuscript was completed, this question has been considered and settled in [2, Corollary 3.2]. There it is shown that a complete seminormalized biorthogonal system with the property (6.4) is necessarily a strong M-basis.

Consider the sets

$$\bar{\Lambda}_{t\alpha} = \{n \ : \ |\mathbf{e}_n^*(x)| \geq t\alpha\} = \{n \ : \ |\mathbf{e}_n^*(x-z)| \geq t\alpha\},$$

which for all $t \in (0, 1]$ are greedy sets for *both* x and $x - z$, and contain B. We claim that,

$$\text{if } m > |\bar{\Lambda}_{t\alpha}| \text{ and } A := \operatorname{supp}\mathscr{G}_m^t(x), \quad \text{then} \quad \bar{\Lambda}_\alpha \subset A \quad \text{and} \quad A \in G(x - z, m, t). \tag{6.8}$$

The assertion $\bar{\Lambda}_\alpha \subset A$ is proved exactly as in (6.7). Next, we must show that

$$\text{if } n \in A \quad \text{then} \quad |\mathbf{e}_n^*(x-z)| \ \geq \ t \max_{k \notin A} |\mathbf{e}_k^*(x-z)| \ = \ t \max_{k \notin A} |\mathbf{e}_k^*(x)|.$$

This is clear if $n \in A \backslash B$ since $\mathbf{e}_n^*(x - z) = \mathbf{e}_n^*(x)$, and $A \in G(x, m, t)$. On the other hand, if $n \in B$, then $|\mathbf{e}_n^*(x - z)| \geq \alpha_2 \geq \alpha \geq \max_{k \in A^c} |\mathbf{e}_k^*(x)|$, the last inequality due to $\bar{\Lambda}_\alpha \subset A$. Thus (6.8) holds true, and therefore

$$\mathscr{G}_m^t(x) - z = \sum_{n \in A} \mathbf{e}_n^*(x-z)\mathbf{e}_n \ = \ \bar{\mathscr{G}}_m^t(x-z),$$

for some $\bar{\mathscr{G}}_m^t \in \mathbb{G}_m^t$. Thus,

$$\|\mathscr{G}_m^t(x) - x\| \ = \ \|(I - \bar{\mathscr{G}}_m^t)(x-z)\| \ \leq \ (1 + \|\bar{\mathscr{G}}_m^t\|)\,\varepsilon,$$

and the result follows from $\sup_m \|\bar{\mathscr{G}}_m^t\| \leq (1 + 4C_q/t)C_q$, by [10, Lemma 2.1]. □

References

1. Albiac, F., Ansorena, J.L.: Characterization of 1-almost greedy bases. Rev. Mat. Complut. **30**(1), 13–24 (2017)
2. Albiac, F., Ansorena, J.L., Berná, P., Wojtaszczyk, P.: Greedy approximation for biorthogonal systems in quasi-Banach spaces (preprint) (2019). arXiv:1903.11651
3. Berná, P.M.: Equivalence between almost-greedy and semi-greedy bases. J. Math. Anal. Appl. **417**, 218–225 (2019)
4. Berná, P.M., Blasco, Ó.: Characterization of greedy bases in Banach spaces. J. Approx. Theory **215**, 28–39 (2017)
5. Berná, P.M., Blasco, Ó., Garrigós, G.: Lebesgue inequalities for the greedy algorithm in general bases. Rev. Mat. Complut. **30**, 369–392 (2017)
6. Berná, P.M., Blasco, Ó., Garrigós, G., Hernández, E., Oikhberg, T.: Embeddings and Lebesgue-type inequalities for the greedy algorithm in Banach spaces. Constr. Approx. **48**(3), 415–451 (2018)
7. Ciesielski, Z.: A bounded orthonormal system of polygonals. Stud. Math. **31**, 339–346 (1968)
8. Dilworth, S.J., Kalton, N.J., Kutzarova, D.: On the existence of almost greedy bases in Banach spaces. Stud. Math. **159**(1), 67–101 (2003)
9. Dilworth, S.J., Kalton, N.J., Kutzarova, D., Temlyakov, V.N.: The thresholding greedy algorithm, greedy bases, and duality. Constr. Approx. **19**, 575–597 (2003)
10. Dilworth, S.J., Kutzarova, D., Oikhberg, T.: Lebesgue constants for the weak greedy algorithm. Rev. Mat. Complut. **28**(2), 393–409 (2015)
11. Dovbysh, L.N., Nikolskii, N.K., Sudakov, V.N.: How good can a nonhereditary family be? J. Sov. Math. **34**(6), 2050–2060 (1986)

12. Garrigós, G., Hernández, E., Oikhberg, T.: Lebesgue-type inequalities for quasi-greedy bases. Constr. Approx. **38**(3), 447–470 (2013)
13. Hajek, P., Montesinos Santalucía, V., Vanderwerff, J., Zizler, V.: Biorthogonal Systems in Banach Spaces. Springer, Berlin (2008)
14. Katznelson, Y.: An Introduction to Harmonic Analysis, 2nd edn. Dover Publ Inc., New York (1976)
15. Konyagin, S.V., Temlyakov, V.N.: A remark on greedy approximation in Banach spaces. East. J. Approx. **5**, 365–379 (1999)
16. Konyagin, S.V., Temlyakov, V.N.: Greedy approximation with regard to bases and general minimal systems. Serdica Math. J. **28**, 305–328 (2002)
17. Ropela, S.: Properties of bounded orthogonal spline bases. In: Approximation Theory (Papers, VIth Semester, Stefan Banach International Mathematics Center, Warsaw, 1975), vol. 4, Banach Center Publ., Warsaw, pp. 197–205 (1979)
18. Ruckle, W.H.: On the classification of biorthogonal sequences. Can. J. Math. **26**, 721–733 (1974)
19. Shao, C., Ye, P.: Lebesgue constants for Chebyshev thresholding greedy algorithms. J. Inequal. Appl. **2018**(1), 102 (2018)
20. Singer, I.: Bases in Banach Spaces I. Springer, Berlin (1970)
21. Singer, I.: Bases in Banach Spaces II. Springer, Berlin (1981)
22. Temlyakov, V.N.: Greedy algorithm and n-term trigonometric approximation. Const. Approx. **14**, 569–587 (1998)
23. Temlyakov, V.N.: Greedy Approximation. Cambridge University Press, Cambridge (2011)
24. Temlyakov, V.N.: The best m-term approximation and greedy algorithms. Adv. Comput. **8**, 249–265 (1998)
25. Temlyakov, V.N.: Sparse approximation with bases. In: Tikhonov, S. (ed.) Advanced Courses in Mathematics. Springer, Berlin (2015)
26. Temlyakov, V.N., Yang, M., Ye, P.: Lebesgue-type inequalities for greedy approximation with respect to quasi-greedy bases. East J. Approx. **17**, 127–138 (2011)
27. Weisz, F.: On the Fejér means of bounded Ciesielski systems. Stud. Math. **146**(3), 227–243 (2001)
28. Wojtaszczyk, P.: Greedy algorithm for general biorthogonal systems. J. Approx. Theory **107**, 293–314 (2000)

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