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Gleason parts for algebras of holomorphic functions in infinite dimensions

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Abstract

For a complex Banach space X with open unit ball B_X, consider the Banach algebras $\mathcal{H}^\infty(B_X)$ of bounded scalar-valued holomorphic functions and the subalgebra $\mathcal{A}_u(B_X)$ of uniformly continuous functions on B_X. Denoting either algebra by $\mathcal{A}$, we study the Gleason parts of the set of scalar-valued homomorphisms $\mathcal{M}(\mathcal{A})$ on $\mathcal{A}$. Following remarks on the general situation, we focus on the case $X = c_0$, giving a complete characterization of the Gleason parts of $\mathcal{M}(\mathcal{A}_u(B_{c_0}))$ and, among other things, showing that every fiber in $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ over a point in B_{ℓ_∞} contains 2^c discs lying in different Gleason parts.

Keywords Gleason parts · Spectrum · Algebras of holomorphic functions · Bounded analytic functions

Mathematics Subject Classification 46J15 · 30H50 · 46E50 · 30H05

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Introduction

Let X be a complex Banach space with open unit ball B_X and unit sphere S_X. Using standard notation, $\mathcal{A}_u(B_X)$ denotes the Banach algebra of holomorphic (complex-analytic) functions $f : B_X \to \mathbb{C}$ that are uniformly continuous on B_X. This algebra is clearly a subalgebra of $\mathcal{H}^\infty(B_X)$, the Banach algebra of all bounded holomorphic mappings on B_X both endowed with the supremum norm $\|f\| = \sup\{|f(x)| \mid \|x\| < 1\}$. Also each function in $\mathcal{A}_u(B_X)$ extends continuously to $\overline{B}_X$. Then, the *maximal ideal space* (the *spectrum* for short) of $\mathcal{A}_u(B_X)$, that is the set of all nonzero $\mathbb{C}-$valued homomorphisms $\mathcal{M}(\mathcal{A}_u(B_X))$ on $\mathcal{A}_u(B_X)$, contains the point evaluations δ_x for all $x \in X$, $\|x\| \leq 1$. Our primary interest here will be in the structure of the set of such homomorphisms, and our specific focus will be on the Gleason parts of $\mathcal{M}(\mathcal{A}_u(B_X))$ and $\mathcal{M}(\mathcal{H}^\infty(B_X))$ when $X = c_0$. Classically, in the case of Banach algebras of holomorphic functions on a finite dimensional space, the study of Gleason parts was motivated by the search for analytic structure in the spectrum. That remains true in our case, in which the holomorphic functions have as their domain the (infinite dimensional) ball of X. However, in infinite dimensions the situation is more complicated and more interesting. For instance, in this case, we will exhibit non-trivial examples of Gleason parts intersecting more than one fiber; this phenomenon holds in the finite dimensional case in only simple, uninteresting cases. Unlike the situation when dim $X < \infty$, it is well-known (see, e.g., [3]) that $\mathcal{M}(\mathcal{A}_u(B_X))$ usually contains much more than mere evaluations at points of $\overline{B}_X$. As we will see, the study of Gleason parts of $\mathcal{M}(\mathcal{A}_u(B_X))$ in the case of an infinite dimensional X is considerably more difficult than in the easy, finite dimensional situation. Now, when the algebra considered is $\mathcal{H}^\infty(\mathbb{D})$ the seminal paper of Hoffman [17] evidences the complicated nature of the Gleason parts for its spectrum (see also [16,19,22]). So, it is not surprising that our results when $\mathbb{D}$ is replaced by B_X are incomplete. However, as we will see, much information about Gleason parts for both the $\mathcal{A}_u$ and $\mathcal{H}^\infty$ cases can be obtained when $X = c_0$.

As just mentioned, we will concentrate on the case $X = c_0$, which is the natural extension of the polydisc $\mathbb{D}^n$. After a review in Sect. 1 of necessary background and some general results, the description of Gleason parts for $\mathcal{M}(\mathcal{A}_u(B_{c_0}))$ will constitute Sect. 2. Finally, in Sect. 3 we will discuss what we have learned about Gleason parts for $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$.

For general theory of holomorphic functions we refer the reader to the monograph of Dineen [12] and for further information on uniform algebras and Gleason parts we suggest the books of Bear [5], Gamelin [14], Garnett [15] and Stout [21].

1 Background and general results

In this section, we will discuss some simple results concerning Gleason parts for $\mathcal{M}(\mathcal{A})$ where $\mathcal{A}$ is an algebra of holomorphic functions defined on the open unit ball of a general Banach space X. Namely, $\mathcal{A}$ will denote either $\mathcal{A}_u(B_X)$ or $\mathcal{H}^\infty(B_X)$. For a Banach space X, as usual X^* and X^{**} denote the dual and the bidual spaces, respectively. We begin with very short reviews of:

(i) Gleason parts ([5,14]) and
(ii) the particular Banach algebras of holomorphic functions that we are interested in.

(i) Let $\mathcal{A}$ be a uniform algebra and let $\mathcal{M}(\mathcal{A})$ denote the compact set of non-trivial homomorphisms $\varphi\colon \mathcal{A} \to \mathbb{C}$ endowed with the $w(\mathcal{A}^*, \mathcal{A})$ topology. For $\varphi, \psi \in \mathcal{M}(\mathcal{A})$, we set the *pseudo-hyperbolic distance*

$$\rho(\varphi, \psi) := \sup\{|\varphi(f)| \mid f \in \mathcal{A}, \|f\| \leq 1, \psi(f) = 0\}.$$

Recall that when $\mathcal{A} = \mathcal{A}(\mathbb{D})$ or $\mathcal{A} = \mathcal{H}^\infty(\mathbb{D})$, the pseudo-hyperbolic metric for λ and μ in the unit disc $\mathbb{D}$ is given by

$$\rho(\delta_\lambda, \delta_\mu) = \left|\frac{\lambda - \mu}{1 - \bar{\lambda}\mu}\right|.$$

Also, the formula given above remains true if $\mathcal{A} = \mathcal{A}(\mathbb{D})$ for $\lambda, \mu \in \overline{\mathbb{D}}$, if $|\lambda| = 1$ and $\lambda \neq \mu$. Clearly, in this case, $\rho(\delta_\lambda, \delta_\mu) = 1$.

The following very useful relation is well known (see, for instance, [5, Theorem 2.8]):

$$\|\varphi - \psi\| = \frac{2 - 2\sqrt{1 - \rho(\varphi, \psi)^2}}{\rho(\varphi, \psi)}. \tag{1.1}$$

Noting that it is always the case that $\|\varphi - \psi\|$ ($\equiv \sup_{\|f\|\leq 1} |\varphi(f) - \psi(f)|$) ≤ 2, the main point here being that $\|\varphi - \psi\| < 2$ if and only if $\rho(\varphi, \psi) < 1$. From this (with some work), it follows that by defining $\varphi \sim \psi$ to mean that $\rho(\varphi, \psi) < 1$ leads to a partition of $\mathcal{M}(\mathcal{A})$ into equivalence classes, called *Gleason parts*. Specifically, for each $\varphi \in \mathcal{M}(\mathcal{A})$, the Gleason part containing φ is the set

$$\mathcal{GP}(\varphi) := \{\psi \mid \rho(\varphi, \psi) < 1\}.$$

We remark that it was perhaps König [18] who coined the phrase *Gleason metric* for the metric $\|\varphi - \psi\|$.

(ii) We first recall [10] that any $f \in \mathcal{H}^\infty(B_X)$ can be extended in a canonical way to $\tilde{f} \in \mathcal{H}^\infty(B_{X^{**}})$. Moreover, the extension $f \rightsquigarrow \tilde{f}$ is a homomorphism of Banach algebras. A standard argument shows that the canonical extension takes functions in $\mathcal{A}_u(B_X)$ to functions in $\mathcal{A}_u(B_{X^{**}})$. Consequently, each point $z_0 \in B_{X^{**}}$ (resp. $\overline{B}_{X^{**}}$) gives rise to an element $\tilde{\delta}_{z_0} \in \mathcal{M}(\mathcal{H}^\infty(B_X))$ (resp. $\mathcal{M}(\mathcal{A}_u(B_X))$). Here, for a given function f, $\tilde{\delta}_{z_0}(f) = \tilde{f}(z_0)$. Note that for $f \in \mathcal{A}_u(B_X)$ and $z_0 \in X^{**}$, with $\|z_0\| = 1$, we are allowed to compute $\tilde{f}(z_0)$ and we will use this fact without further mention. Also, in order to avoid unwieldy notation we will omit the tilde over the δ, simply writing $\delta_{z_0}(f)$. We recall that either for $\mathcal{A} = \mathcal{A}_u(B_X)$ or $\mathcal{A} = \mathcal{H}^\infty(B_X)$ there is a mapping $\pi : \mathcal{M}(\mathcal{A}) \to \overline{B}_{X^{**}}$ given by $\pi(\varphi) := \varphi|_{X^*}$. Note that this makes sense since $X^* \subset \mathcal{A}$. It is not difficult to see that π is surjective [3]. As usual, for any

$z \in \overline{B}_{X^{**}}$, the *fiber over* z, will be denoted by

$$\mathcal{M}_z := \{\varphi \in \mathcal{M}(\mathcal{A}) \mid \pi(\varphi) = z\}.$$

As we will see, knowledge of the fiber structure is useful in the study of Gleason parts, in the context of the Banach algebras $\mathcal{A}_u(B_X)$ and $\mathcal{H}^\infty(B_X)$. The first instance of this occurs in part (b) of Proposition 1.1 below.

Proposition 1.1 *Let X be a Banach space and $\mathcal{M} = \mathcal{M}(\mathcal{A})$ be as above.*

(a) *The set $\{\delta_z : z \in B_{X^{**}}\}$ is contained in $\mathcal{GP}(\delta_0)$. In fact, $\rho(\delta_0, \delta_z) = \|z\|$ for each $z \in B_{X^{**}}$.*
(b) *Let $z \in S_{X^{**}}$ and $w \in B_{X^{**}}$. Then, for any $\varphi \in \mathcal{M}_z$ and $\psi \in \mathcal{M}_w$, $\rho(\varphi, \psi) = 1$. That is, φ and ψ lie in different Gleason parts.*

Proof (a) Fix $z \in B_{X^{**}}, z \neq 0$, and $f \in \mathcal{A}$, such that $\|f\| \leq 1$ and $f(0) = \delta_0(f) = 0$. By an application of the Schwarz lemma to $\tilde{f} \in \mathcal{A}(X^{**})$, we see that $|\delta_z(f)| = |\tilde{f}(z)| \leq \|z\|$. Therefore $\rho(\delta_0, \delta_z) \leq \|z\| < 1$, or in other words δ_z is in the same Gleason part as δ_0. In addition, if we apply the definition of ρ to a sequence $(x_n^*) \subset \overline{B}_{X^*} \subset \mathcal{A}$ such that $|z(x_n^*)| \to \|z\|$, we get that $\rho(\delta_0, \delta_z) \geq \|z\|$.
(b) As in part (a) and using that $\varphi \in \mathcal{M}_z$, we may choose a sequence (x_n^*) of norm one functionals on X such that $\varphi(x_n^*) = z(x_n^*) \to \|z\| = 1$. Observe that $|\psi(x_n^*)| = |w(x_n^*)| \leq \|w\| < 1$. For each $n, m \in \mathbb{N}$, the function $g_{n,m} : B_X \to \mathbb{C}$ defined as

$$g_{n,m}(\cdot) = \frac{(x_n^*(\cdot))^m - w(x_n^*)^m}{\|(x_n^*)^m - w(x_n^*)^m\|}$$

is in $\mathcal{A} = \mathcal{A}_u(B_X)$ or $\mathcal{H}^\infty(B_X)$. Evidently, $\|g_{n,m}\| = 1$ and $\psi(g_{n,m}) = 0$. In addition,

$$|\varphi(g_{n,m})| \geq \frac{|z(x_n^*)|^m - \|w\|^m}{1 + \|w\|^m},$$

which approaches 1 with n and m. Then, $\rho(\psi, \varphi) = 1$ and ψ and φ are in different parts. □

In the classical situation of $\mathcal{M}(\mathcal{H}^\infty(\mathbb{D}))$, the Gleason part containing the evaluation at the origin, δ_0, consists of the set $\{\delta_z \mid z \in \mathbb{D}\}$. This known fact is made evident in view of Proposition 1.1 and the fact that fibers over points in $\mathbb{D}$ are singletons. In the case of an infinite dimensional space X, it can happen that fibers (over interior points) are bigger than single evaluations and also the Gleason part of δ_0 could properly contain $B_{X^{**}}$. The following, which uses part (a) of Proposition 1.1, gives a glimpse at this situation.

Proposition 1.2 *Let X be a Banach space. Fix r, $0 < r < 1$ and consider $B_{X^{**}}(0, r) \approx \{\delta_z \mid z \in X^{**}, \|z\| < r\} \subset \mathcal{M}(\mathcal{A})$. Then the closure of $B_{X^{**}}(0, r)$ in $\mathcal{M}(\mathcal{A})$ is contained in $\mathcal{GP}(\delta_0)$.*

Proof Fix $\varphi \in \mathcal{M}(\mathcal{A})$, φ in the closure of $B_{X^{**}}(0,r)$, and choose any $f \in \mathcal{A}$, $f(0) = 0$, $\|f\| = 1$. By definition, for fixed $\varepsilon > 0$ such that $r + \varepsilon < 1$ there is $z \in B_{X^{**}}(0,r)$ such that $|\varphi(f) - \delta_z(f)| < \varepsilon$. Then,

$$|\varphi(f) - \delta_0(f)| \leq \varepsilon + |\delta_0(f) - \delta_z(f)| \leq \varepsilon + \rho(\delta_0, \delta_z) < \varepsilon + r.$$

Thus, $\rho(\varphi, \delta_0) < 1$, which concludes the proof. □

In many common situations, there are norm-continuous polynomials P acting on the Banach space X whose restriction to B_X is not weakly continuous. To give one very easy example, the $2-$homogeneous polynomial $P\colon \ell_2 \to \mathbb{C}$, $P(x) = \sum_n x_n^2$ is such that $1 = P(\frac{\sqrt{2}}{2}[e_1 + e_n]) \neq 1/2 = P(\frac{\sqrt{2}}{2}e_1)$. In these cases, the following corollary shows that the exact composition of $\mathcal{GP}(\delta_0)$ is somewhat more complicated.

Corollary 1.3 *Let X be a Banach space which admits a (norm) continuous polynomial that is not weakly continuous when restricted to the unit ball. Then $B_{X^{**}} \subsetneq \mathcal{GP}(\delta_0)$.*

Proof Combining [6, Corollary 2] and [6, Proposition 3] if X admits a polynomial which is not weakly continuous when restricted to the unit ball, then there is a homogeneous polynomial P on X whose canonical extension $\tilde{P}$ to X^{**} is not weak-star continuous at 0 when restricted to any ball $B_{X^{**}}(0,r)$, $0 < r < 1$. Fix any r and choose a net $(z_\alpha) \subset B_{X^{**}}(0,r)$ that is weak-star convergent to 0 and $\tilde{P}(z_\alpha) \nrightarrow 0$. Choosing a subnet if necessary, we may assume that $\tilde{P}(z_\alpha) \to b \neq 0$. Applying Proposition 1.2, if $\varphi \in \mathcal{M}(\mathcal{A})$ is a limit point of $\{\delta_{z_\alpha}\}$, then $\varphi \in \mathcal{GP}(\delta_0)$. Note that $\delta_0(P) = 0 \neq b = \varphi(P)$, so that $\delta_0 \neq \varphi$. Finally, $\varphi \in \mathcal{M}_0$, since $\pi(\varphi) = \varphi|_{X^*}$, which shows that $\varphi \in \mathcal{GP}(\delta_0) \backslash B_{X^{**}}$. □

Remark 1.4 Note that, under the hypothesis of the above result, by Proposition 1.1, each homomorphism $\varphi \in \mathcal{GP}(\delta_0) \backslash B_{X^{**}}$ should be in some fiber over points in $B_{X^{**}}$.

In the rest of this section, we will focus on the calculation of the pseudo-hyperbolic distance in some special, albeit important, situations. Here, we will have to distinguish between the cases $\mathcal{A} = \mathcal{A}_u(B_X)$ and $\mathcal{A} = \mathcal{H}^\infty(B_X)$.

Proposition 1.5 *Let X be a Banach space and $\mathcal{A} = \mathcal{A}_u(B_X)$ or $\mathcal{A} = \mathcal{H}^\infty(B_X)$. Suppose that there exists an automorphism $\Phi\colon B_X \to B_X$ and in addition for the case of $\mathcal{A}_u(B_X)$, assume Φ is uniformly continuous. Then, given $x \in B_X$ such that $\Phi(x) = 0$, for any $y \in B_X$ we have*

$$\rho(\delta_x, \delta_y) = \|\Phi(y)\|.$$

Proof We only prove the case $\mathcal{A} = \mathcal{A}_u(B_X)$. Let $f \in \mathcal{A}_u(B_X)$, $\|f\| \leq 1$, such that $\delta_x(f) = f(x) = 0$. As $f \circ \Phi^{-1}$ is in $\mathcal{H}^\infty(B_X)$, we can apply the Schwarz lemma to obtain

$$|\delta_y(f)| = |f(y)| = |f \circ \Phi^{-1}(\Phi(y))| \leq \|\Phi(y)\|.$$

Thus, from the definition of ρ, we see that $\rho(\delta_x, \delta_y) \leq \|\Phi(y)\|$.

For the reverse inequality, choose a norm one functional $x^* \in X^*$ such that $x^*(\Phi(y)) = \|\Phi(y)\|$, and set $f = x^* \circ \Phi$. Since $f \in \mathcal{A}_u(B_X)$ has norm at most 1 and satisfies $f(x) = 0$, we get that

$$\rho(\delta_x, \delta_y) \geq |\delta_y(f)| = \|\Phi(y)\|.$$

□

Note that the proof of Proposition 1.5 shows that $\rho(\delta_x, \delta_y)$ is independent of the particular choice of the automorphism Φ.

For subsequent embedding results, for a Banach space X and $\mathcal{A} = \mathcal{A}_u(B_X)$ or $\mathcal{A} = \mathcal{H}^\infty(B_X)$ we will use the Gleason metric on $\mathcal{M}(\mathcal{A})$. As we have already noted in (i) at the beginning of this section, this metric is the restriction of the usual distance given by the norm on $\mathcal{A}^*$. When we refer to the Gleason metric for elements of $B_{X^{**}}$, the open unit ball $B_{X^{**}}$ will be regarded as a subset of $\mathcal{M}(\mathcal{A})$. As we will see in the next proposition, under certain conditions, the automorphism Φ of Proposition 1.5 induces an isometry (for the Gleason metric) in the spectrum that sends some fibers onto different fibers. This type of isometry allows us to transfer information relative to Gleason parts intersecting one fiber to other fibers. Recall that a *finite type polynomial* on X is a function in the algebra generated by X^*. Also, a Banach space X is said to be *symmetrically regular* if every continuous linear mapping $T: X \to X^*$ which is symmetric (i. e. $T(x_1)(x_2) = T(x_2)(x_1)$ for all $x_1, x_2 \in X$) turns out to be weakly compact.

Proposition 1.6 *Let X be a Banach space and $\mathcal{A} = \mathcal{A}_u(B_X)$ or $\mathcal{A} = \mathcal{H}^\infty(B_X)$. Suppose that there exists an automorphism $\Phi: B_X \to B_X$ and in addition for the case of $\mathcal{A}_u(B_X)$, assume Φ and Φ^{-1} are uniformly continuous.*

(i) *The mapping Φ induces a composition operator $C_\Phi: \mathcal{A} \to \mathcal{A}$, $C_\Phi(f) = f \circ \Phi$ such that $\Lambda_\Phi := C_\Phi^t|_{\mathcal{M}(\mathcal{A})}: \mathcal{M}(\mathcal{A}) \to \mathcal{M}(\mathcal{A})$, the restriction of its transpose to $\mathcal{M}(\mathcal{A})$, is an onto isometry for the Gleason metric with inverse $\Lambda_\Phi^{-1} = \Lambda_{\Phi^{-1}}$.*

(ii) *If for every $x^* \in X^*$, $x^* \circ \Phi$ and $x^* \circ \Phi^{-1}$ are uniform limits of finite type polynomials then for any $x \in \overline{B}_X$, $\Lambda_\Phi(\mathcal{M}_x) = \mathcal{M}_{\Phi(x)}$. If in addition X is symmetrically regular, then, for any $z \in \overline{B}_{X^{**}}$, $\Lambda_\Phi(\mathcal{M}_z) = \mathcal{M}_{\widetilde{\Phi}(z)}$.*

Proof To prove (i), just notice that for $f \in \mathcal{A}$ and $\varphi \in \mathcal{M}(\mathcal{A})$,

$$\Lambda_{\Phi^{-1}}(\Lambda_\Phi(\varphi))(f) = \Lambda_\Phi(\varphi)\big(f \circ \Phi^{-1}\big) = \varphi(f).$$

Through this equality it is easily seen that $\|\Lambda_\Phi(\varphi) - \Lambda_\Phi(\psi)\| = \|\varphi - \psi\|$, for all $\varphi, \psi \in \mathcal{M}(\mathcal{A})$.

It is enough to prove (ii) in the case X is symmetrically regular. Fix $z \in \overline{B}_{X^{**}}$ and take $\varphi \in \mathcal{M}_z$. Given $x_1^*, \ldots, x_n^*$ in X^* as φ is multiplicative, we have that

$$\varphi(x_1^* \ldots x_n^*) = \varphi(x_1^*) \ldots \varphi(x_n^*) = z(x_1^*) \ldots z(x_n^*).$$

Thus, since any polynomial Q of finite type is a linear combination of elements as above, we have

$$\varphi(Q) = \widetilde{Q}(z).$$

By hypothesis, for any $x^* \in X^*$ there exists a sequence (Q_k) of polynomials of finite type that converges uniformly to $x^* \circ \Phi$ on B_X. Hence, the sequence $(\widetilde{Q}_k)$ converges to $\widetilde{x}^* \circ \widetilde{\Phi}$ uniformly on $B_{X^{**}}$ and $\widetilde{\Phi}$ admits a unique extension to $\overline{B}_{X^{**}}$ through weak-star continuity. Thus,

$$\Lambda_\Phi(\varphi)(x^*) = \varphi(x^* \circ \Phi) = \lim_k \varphi(Q_k) = \lim_k \widetilde{Q}_k(z) = (\widetilde{\Phi}(z))(x^*).$$

Consequently, $\Lambda_\Phi(\mathcal{M}_z) \subset \mathcal{M}_{\widetilde{\Phi}(z)}$. Now, the reverse inclusion follows from (i) because, since X is symmetrically regular and arguing as in the proof of [8, Corollary 2.2], we know that $\widetilde{\Phi^{-1}} \circ \widetilde{\Phi} = Id$. Therefore, $\Lambda_\Phi(\mathcal{M}_z) = \mathcal{M}_{\widetilde{\Phi}(z)}$. □

To conclude this section, we give three examples of these results.

Example 1.7 Let $X = c_0$ and fix a point $x = (x_n) \in B_{c_0}$. Define the mapping $\Phi_x : B_{c_0} \to B_{c_0}$ as follows:

$$\Phi_x(y) = (\eta_{x_1}(y_1), \eta_{x_2}(y_2), \dots),$$

where $\eta_\alpha(\lambda) = \frac{\alpha-\lambda}{1-\overline{\alpha}\lambda}$, $\alpha, \lambda \in \mathbb{D}$. In this case Φ_x is a uniformly continuous automorphism ($\Phi_x^{-1} = \Phi_x$) with $\Phi_x(x) = 0$ and so, for any $y \in B_{c_0}$,

$$\rho(\delta_x, \delta_y) = \|\Phi_x(y)\| = \sup_{n\geq 1} \left| \frac{x_n - y_n}{1 - \overline{x_n} y_n} \right| = \sup_{n\geq 1} \rho(\delta_{x_n}, \delta_{y_n}).$$

Also, Λ_{Φ_x} is an onto isometry for the Gleason metric in $\mathcal{M}(\mathcal{A})$ both for $\mathcal{A} = \mathcal{A}_u(B_{c_0})$ or $\mathcal{A} = \mathcal{H}^\infty(B_{c_0})$. Moreover, $\Lambda_{\Phi_x}(\mathcal{M}_z) = \mathcal{M}_{\widetilde{\Phi}_x(z)}$ for any $z \in B_{\ell_\infty}$.

In the next section, we will discuss the more complicated, more interesting extension of the previous example to $z \in \overline{B}_{\ell_\infty}$; see Theorem 2.4.

Example 1.8 ([2, Lemma 4.4]) Let $X = \ell_2$ and fix a point $x \in B_{\ell_2}$. Define the mapping $\beta_x : B_{\ell_2} \to B_{\ell_2}$ as follows:

$$\beta_x(y) = \frac{1}{1+\sqrt{1-\|x\|^2}} \left\langle \frac{x-y}{1-\langle y, x\rangle}, x \right\rangle x + \sqrt{1-\|x\|^2} \frac{x-y}{1-\langle y, x\rangle}$$

($y \in B_{\ell_2}$). From [20, Proposition 1, p.132], we know that β_x is an automorphism from B_{ℓ_2} onto itself, with inverse map $\beta_x^{-1} = \beta_x$ and $\beta_x(x) = 0$.

Also, by expanding $1/[1 - \langle y, x\rangle]$ as a geometric series $\sum \langle y, x\rangle^n$ and noting that the series converges uniformly on $\overline{B}_{\ell_2}$, we see that $\beta_x(y) = g(y)x + h(y)y$, where the functions g and h are in $\mathcal{A}_u(B_{\ell_2})$. Thus, β_x is uniformly continuous. Applying

Proposition 1.5, we see that for all $x, y \in B_{\ell_2}$, $\rho(\delta_x, \delta_y) = \|\beta_x(y)\|$. Also, by Proposition 1.6, Λ_{β_x} is an onto isometry for the Gleason metric in $\mathcal{M}(\mathcal{A})$, both for $\mathcal{A} = \mathcal{A}_u(B_{\ell_2})$ or $\mathcal{A} = \mathcal{H}^\infty(B_{\ell_2})$. Moreover, as Proposition 1.6 (ii) holds (see [2, Lemma 4.3]) $\Lambda_{\beta_x}(\mathcal{M}_y) = \mathcal{M}_{\beta_x(y)}$ for all $y \in \overline{B}_{\ell_2}$.

Example 1.9 Let H be an infinite dimensional Hilbert space and let $X = \mathcal{L}(H)$ be the Banach space of all bounded linear operators from H into itself. Fix $R \in B_{\mathcal{L}(H)}$ and denote by R^* its adjoint operator. Define the mapping Φ_R on $B_{\mathcal{L}(H)}$ as follows:

$$\Phi_R(T) = (I - RR^*)^{\frac{1}{2}}(T - R)(I - R^*T)^{-1}(I - R^*R)^{\frac{1}{2}},$$

$(T \in B_{\mathcal{L}(H)})$. Note that $\Phi_R \colon B_{\mathcal{L}(H)} \to B_{\mathcal{L}(H)}$ is an automorphism with inverse map Φ_{-R} and $\Phi_R(R) = 0$. As in the example above, it can be seen that Φ_R is uniformly continuous. Then, by Proposition 1.5, for $R, S \in B_{\mathcal{L}(H)}$ we obtain $\rho(\delta_R, \delta_S) = \|\Phi_R(S)\|$. Again, by Proposition 1.6, Λ_{Φ_R} is an onto isometry for the Gleason metric in $\mathcal{M}(\mathcal{A})$, both for $\mathcal{A} = \mathcal{A}_u(B_{\mathcal{L}(H)})$ or $\mathcal{A} = \mathcal{H}^\infty(B_{\mathcal{L}(H)})$.

2 Gleason parts for $\mathcal{M}(\mathcal{A}_u(B_{c_0}))$

Compared to other infinite dimensional Banach spaces, what is unusual about $X = c_0$ is that, in relative terms, there are very few continuous polynomials $P \colon c_0 \to \mathbb{C}$. All such polynomials are norm limits of finite linear combinations of elements of $c_0^* = \ell_1$. As a consequence, there are very few holomorphic functions on c_0 [12]. In particular, every $f \in \mathcal{A}_u(B_{c_0})$ is a uniform limit of such polynomials. Thus, since any homomorphism is automatically continuous, its action on $\mathcal{A}_u(B_{c_0})$ is completely determined by its action on c_0^*. In other words, $\mathcal{M}(\mathcal{A}_u(B_{c_0}))$ is precisely $\{\delta_z \mid z \in \overline{B}_{\ell_\infty}\}$. Note that if c_0 were replaced by ℓ_p, this approximation result would be false, and in fact $\mathcal{M}(\mathcal{A}_u(B_{\ell_p}))$ is considerably larger and more complicated than $\overline{B}_{\ell_p} \approx \{\delta_z \mid z \in \overline{B}_{\ell_p}\}$ (see, e.g., [13]).

Our aim here will be to get a reasonably complete description of the Gleason parts of $\mathcal{M}(\mathcal{A}_u(B_{c_0}))$. As just mentioned, our work is greatly helped by the fact that we know exactly what $\mathcal{M}(\mathcal{A}_u(B_{c_0}))$ is, namely that it can be associated with $\overline{B}_{\ell_\infty}$. A special role is played by homomorphisms δ_z where z belongs to the distinguished boundary $\mathbb{T}^{\mathbb{N}}$, the set of all elements $z = (z_n)$ such that $|z_n| = 1$ for all n. Also, notice that compared with the finite dimensional situation, there is a new and interesting "wrinkle" here in that there are unit vectors $z = (z_n) \in \overline{B}_{\ell_\infty}$ all of whose coordinates have absolute value smaller than 1. We begin with a straightforward lemma.

Lemma 2.1 *For any $\varnothing \neq \mathbb{N}_0 \subset \mathbb{N}$, let $\Gamma \colon \ell_\infty \to \ell_\infty(\mathbb{N}_0)$ be the projection mapping taking $z = (z_j)_{j\in\mathbb{N}} \mapsto \Gamma(z) = (z_j)_{j\in\mathbb{N}_0}$. Then for all $z, w \in \overline{B}_{\ell_\infty}$, the following inequality holds:*

$$\|\delta_{\Gamma(z)} - \delta_{\Gamma(w)}\| \leq \|\delta_z - \delta_w\|.$$

Proof Clearly, Γ is a linear operator having norm 1, and $\Gamma(c_0) = c_0(\mathbb{N}_0)$. Thus each $f \in \mathcal{A}_u(B_{c_0(\mathbb{N}_0)})$ generates a function $g \in \mathcal{A}_u(B_{c_0})$ given by $g = f \circ \Gamma|_{c_0}$ having the same norm as f. An easy verification shows that the extension of g to $\mathcal{A}_u(B_{\ell_\infty})$ is given by $\tilde{g} = \tilde{f} \circ \Gamma$. Therefore for all $z, w \in \ell_\infty$, $\|z\|, \|w\| \leq 1$,

$$\begin{aligned}\|\delta_{\Gamma(z)} - \delta_{\Gamma(w)}\| &= \sup\{|\tilde{f}(\Gamma(z)) - \tilde{f}(\Gamma(w))| \mid f \in \mathcal{A}_u(B_{c_0(\mathbb{N}_0)}),\ \|f\| \leq 1\} \\ &\leq \sup\{|\tilde{g}(z) - \tilde{g}(w)| \mid g \in \mathcal{A}_u(B_{c_0}),\ \|g\| \leq 1\} = \|\delta_z - \delta_w\|.\end{aligned}$$

□

Another way to restate Lemma 2.1 is as follows: if $\delta_z \in \mathcal{GP}(\delta_w)$, then $\delta_{\Gamma(z)} \in \mathcal{GP}(\delta_{\Gamma(w)})$. Since $\mathbb{N}_0$ is allowed to be finite, say of cardinal k, if δ_z and δ_w are in the same Gleason part, then their projections onto finite coordinates (viewed as being in $\mathbb{D}^k$) are also in the same Gleason part. Our next result examines the situation: Suppose that $z, w \in \overline{B}_{\ell_\infty}$ are such that δ_z and δ_w are in the same Gleason part. What can we say about the coordinates where these points differ and where these points are identical?

Lemma 2.2 *For $z, w \in \overline{B}_{\ell_\infty}$, let $\mathbb{N}_0 = \{n \in \mathbb{N} \mid z_n \neq w_n\}$ and $\Gamma\colon \ell_\infty \to \ell_\infty(\mathbb{N}_0)$ be the projection as in Lemma 2.1. Then*

$$\|\delta_z - \delta_w\| = \|\delta_{\Gamma(z)} - \delta_{\Gamma(w)}\|.$$

Proof Fix $z \in \overline{B}_{\ell_\infty}$ and define $\Theta_z\colon \ell_\infty(\mathbb{N}_0) \to \ell_\infty$ by:

$$(\Theta_z(u))_n = \begin{cases} u_n & \text{if } n \in \mathbb{N}_0, \\ z_n & \text{if } n \notin \mathbb{N}_0. \end{cases}$$

Given $g \in \mathcal{A}_u(B_{c_0})$, $\|g\| \leq 1$, let $f = \tilde{g} \circ \Theta_z|_{c_0(\mathbb{N}_0)}$. Note that f is well-defined since whenever $u \in \overline{B}_{\ell_\infty(\mathbb{N}_0)}$ then $\Theta_z(u) \in \overline{B}_{\ell_\infty}$. It is easy to check that $f \in \mathcal{A}_u(B_{c_0(\mathbb{N}_0)})$, $\|f\| \leq 1$, and that $\tilde{f} = \tilde{g} \circ \Theta_z \in \mathcal{A}_u(B_{\ell_\infty(\mathbb{N}_0)})$. From the definition of $\mathbb{N}_0$, we see that

$$\begin{aligned}\|\delta_z - \delta_w\| &= \sup\{|\tilde{g}(z) - \tilde{g}(w)| \mid g \in \mathcal{A}_u(B_{c_0}), \|g\| \leq 1\} \\ &= \sup\{|\tilde{g}(\Theta_z \circ \Gamma(z)) - \tilde{g}(\Theta_z \circ \Gamma(w))| \mid g \in \mathcal{A}_u(B_{c_0}), \|g\| \leq 1\} \\ &\leq \sup\{|\tilde{f}(\Gamma(z)) - \tilde{f}(\Gamma(w))| \mid f \in \mathcal{A}_u(B_{c_0(\mathbb{N}_0)}), \|f\| \leq 1\} \\ &= \|\delta_{\Gamma(z)} - \delta_{\Gamma(w)}\|,\end{aligned}$$

and this, with the previous lemma, completes the proof. □

One consequence of this result is that if $z \in \overline{B}_{\ell_\infty}$ with $|z_n| < 1$, for some n, then any $w \in \overline{B}_{\ell_\infty}$ such that $w_j = z_j$, for all $j \neq n$, and $|w_n| < 1$, satisfies that δ_z and δ_w are in the same Gleason part. In particular, the only Gleason parts that are singleton points are the evaluations at points in the distinguished boundary $\mathbb{T}^{\mathbb{N}}$ of $\overline{B}_{\ell_\infty}$, i.e. the points in the Shilov boundary of $\mathcal{M}(\mathcal{A}_u(B_{c_0}))$.

Lemma 2.3 *For each $n \in \mathbb{N}$, let $\Gamma_n : \ell_\infty \to \ell_\infty(\{1, 2, \ldots, n\})$ be the natural projection. If z and w are both in $\overline{B}_{\ell_\infty}$, then*

$$\|\delta_z - \delta_w\| = \lim_{n\to\infty} \|\delta_{\Gamma_n(z)} - \delta_{\Gamma_n(w)}\| = \sup_{n\in\mathbb{N}} \|\delta_{\Gamma_n(z)} - \delta_{\Gamma_n(w)}\|.$$

Proof First, Lemma 2.1 implies that the sequence $(\|\delta_{\Gamma_n(z)} - \delta_{\Gamma_n(w)}\|)$ is increasing and bounded by $\|\delta_z - \delta_w\|$. Note also that for each $u \in \overline{B}_{\ell_\infty}$, $\Gamma_n(u) \overset{w(\ell_\infty,\ell_1)}{\longrightarrow} u$, and if f is in $\mathcal{A}_u(B_{c_0})$, it follows that $\tilde{f} \in \mathcal{A}_u(B_{\ell_\infty})$ is weak-star continuous. Consequently, $\tilde{f}(\Gamma_n(u)) \to \tilde{f}(u)$ as $n \to \infty$. Therefore, for any $\varepsilon > 0$ take $f \in \mathcal{A}_u(B_{c_0})$, $\|f\| \le 1$, such that $|\tilde{f}(z) - \tilde{f}(w)| > \|\delta_z - \delta_w\| - \frac{\varepsilon}{2}$. Then, we can find $n_0 \in \mathbb{N}$ such that both of the following hold:

$$|\tilde{f}(\Gamma_{n_0}(z)) - \tilde{f}(z)| < \frac{\varepsilon}{4} \quad \text{and} \quad |\tilde{f}(\Gamma_{n_0}(w)) - \tilde{f}(w)| < \frac{\varepsilon}{4}.$$

Hence, we see that

$$|\tilde{f}(z) - \tilde{f}(w)| \le \frac{\varepsilon}{4} + |\tilde{f}(\Gamma_{n_0}(z)) - \tilde{f}(\Gamma_{n_0}(w))| + \frac{\varepsilon}{4} \le \|\delta_{\Gamma_{n_0}(z)} - \delta_{\Gamma_{n_0}(w)}\| + \frac{\varepsilon}{2}.$$

From this, we obtain that $\|\delta_z - \delta_w\| \le \|\delta_{\Gamma_{n_0}(z)} - \delta_{\Gamma_{n_0}(w)}\| + \varepsilon$, and the lemma follows. □

For the subsequent description of the Gleason parts for $\mathcal{M}(\mathcal{A}_u(B_{c_0}))$ we introduce the following notation. For each $\lambda \in \mathbb{D}$ and $0 < r < 1$, we denote the *pseudo-hyperbolic r-disc* centered at λ by

$$\mathcal{D}_r(\lambda) = \left\{\mu \in \mathbb{D} \mid \rho(\delta_\lambda, \delta_\mu) = \left|\frac{\lambda - \mu}{1 - \bar{\lambda}\mu}\right| < r\right\}.$$

Theorem 2.4 *Let $z = (z_n)$ and $w = (w_n)$ be vectors in $\overline{B}_{\ell_\infty}$. Then*

$$\|\delta_z - \delta_w\| = \sup_{n\in\mathbb{N}} \|\delta_{z_n} - \delta_{w_n}\|. \tag{2.1}$$

Moreover, if $\mathbb{N}_0 = \{n \in \mathbb{N} \mid z_n \ne w_n\}$ then

$$\rho(\delta_z, \delta_w) = \sup_{n\in\mathbb{N}} \rho(\delta_{z_n}, \delta_{w_n}) = \sup_{n\in\mathbb{N}_0} \left|\frac{z_n - w_n}{1 - \overline{z_n} w_n}\right|. \tag{2.2}$$

Hence, given $z = (z_n) \in \overline{B}_{\ell_\infty}$ we have

$$\mathcal{GP}(\delta_z) = \bigcup_{0<r<1} \{\delta_w \mid w_n = z_n \text{ if } |z_n| = 1 \text{ and } w_n \in \mathcal{D}_r(z_n) \text{ if } |z_n| < 1\}.$$

Proof By Lemma 2.3, it is enough to see that $\|\delta_{\Gamma_n(z)} - \delta_{\Gamma_n(w)}\| = \sup_{1\leq k\leq n} \|\delta_{z_k} - \delta_{w_k}\|$ for all n, where $\Gamma_n : \ell_\infty \to \ell_\infty(\{1, 2, \ldots, n\})$ is the natural projection. By Lemma 2.2, we may also assume that $z_k \neq w_k$ for $k = 1, \ldots, n$.

First, suppose that there exists k, $1 \leq k \leq n$, such that $|z_k| = 1$ or $|w_k| = 1$. Then, $\|\delta_{z_k} - \delta_{w_k}\| = 2$ and Lemma 2.1 gives the equality. Now, assume that $|z_k|, |w_k| < 1$ for all $1 \leq k \leq n$. Note that (1.1) describes $\|\delta_{\Gamma_n(z)} - \delta_{\Gamma_n(w)}\|$ in terms of $\rho(\delta_{\Gamma_n(z)}, \delta_{\Gamma_n(w)})$ by an increasing function. Using Example 1.7 we see that $\rho(\delta_{\Gamma_n(z)}, \delta_{\Gamma_n(w)}) = \sup_{1\leq k\leq n} \rho(\delta_{z_k}, \delta_{w_k})$ and both equalities (2.1) and (2.2) follow from this.

Now, from $\rho(\delta_z, \delta_w) = \sup_{n\in\mathbb{N}} \rho(\delta_{z_n}, \delta_{w_n})$, we have

$$\mathcal{GP}(\delta_z) = \bigcup_{0<r<1} \{\delta_w \mid \rho(\delta_{z_n}, \delta_{w_n}) < r, \text{ for all } n\}.$$

The conclusion trivially holds. □

Notice that if the algebra is $\mathcal{H}^\infty(B_{c_0})$ and the vectors z, w belong to the open unit ball B_{ℓ_∞}, equation (2.1) coincides with equation (6.1) of [9, Theorem 6.6]. The next example illustrates how Theorem 2.4 can be used.

Example 2.5 Consider the following points in the sphere of ℓ_∞ : $z = (1 - \frac{1}{n})_n$, $w = (1 - \frac{1}{n^2})_n$, and $u = (1 - \frac{1}{2n})_n$. Then δ_z and δ_w are in different Gleason parts, while δ_z and δ_u are in the same part.

To see this, observe that

$$\rho(\delta_z, \delta_w) = \sup_{n\in\mathbb{N}} \rho(\delta_{z_n}, \delta_{w_n}) = \sup_{n\in\mathbb{N}} \left|\frac{z_n - w_n}{1 - \overline{z_n} w_n}\right| = \sup_{n\in\mathbb{N}} \left|\frac{n - n^2}{n^2 + n - 1}\right| = 1,$$

which shows the first assertion. Similarly,

$$\rho(\delta_z, \delta_u) = \sup_{n\in\mathbb{N}} \rho(\delta_{z_n}, \delta_{u_n}) = \sup_{n\in\mathbb{N}} \left|\frac{z_n - u_n}{1 - \overline{z_n} u_n}\right| = \sup_{n\in\mathbb{N}} \frac{n}{3n - 1} = \frac{1}{2}.$$

Thus, δ_z and δ_u belong to the same Gleason part.

In order to give a more descriptive insight of the size of the Gleason parts, let us introduce some notation. Given $z = (z_n) \in \overline{B}_{\ell_\infty}$, let $\mathbb{N}_1$ be the (possibly empty) set $\mathbb{N}_1 = \{n \in \mathbb{N} \mid |z_n| = 1\}$. Now, $\mathbb{N} \setminus \mathbb{N}_1$ can be split into two disjoint sets $\mathbb{N}_2 \cup \mathbb{N}_3$ such that

$$\sup_{n\in\mathbb{N}_2} |z_n| < 1 \qquad \text{and} \qquad \sup_{n\in\mathbb{N}_3} |z_n| = 1.$$

Note that $\mathbb{N}_2$ and $\mathbb{N}_3$ could be empty and that they are not uniquely determined. For instance, if $\mathbb{N}_3$ is infinite and $\mathbb{N}_2$ is finite, we may redefine $\mathbb{N}_3$ as the union of $\mathbb{N}_3$ and $\mathbb{N}_2$ and redefine $\mathbb{N}_2$ to be empty. Also, $\mathbb{N}_3$ cannot be finite.

In this way we write $\mathbb{N}$ as a disjoint union satisfying the above conditions: $\mathbb{N} = \mathbb{N}_1 \cup \mathbb{N}_2 \cup \mathbb{N}_3$ and, therefore, the Gleason part containing δ_z satisfies:

$$\mathcal{GP}(\delta_z) = \left\{ \delta_w \mid w_n = z_n \text{ if } n \in \mathbb{N}_1, \sup_{n\in\mathbb{N}_2} |w_n| < 1 \text{ and } \sup_{n\in\mathbb{N}_3} \left| \frac{z_n - w_n}{1 - \overline{z_n} w_n} \right| < 1 \right\}.$$

Now, taking into account all the possibilities for the sets $\mathbb{N}_1$, $\mathbb{N}_2$ and $\mathbb{N}_3$ we obtain a more specific description of the different Gleason parts.

Corollary 2.6 *Given $z \in \overline{B}_{\ell_\infty}$ and $\mathbb{N}_1$, $\mathbb{N}_2$, $\mathbb{N}_3$ defined as above, the Gleason part $\mathcal{GP}(\delta_z)$ satisfies one of the following:*

(i) *If $\mathbb{N} = \mathbb{N}_2$ then $z \in B_{\ell_\infty}$ and $\mathcal{GP}(\delta_z) = \mathcal{GP}(\delta_0) = \{\delta_w \mid w \in B_{\ell_\infty}\}$. This produces the identification $\mathcal{GP}(\delta_z) \approx B_{\ell_\infty}$.*
(ii) *If $\mathbb{N} = \mathbb{N}_1$ then $z = (z_n) \in \mathbb{T}^{\mathbb{N}}$. So, $\mathcal{GP}(\delta_z) = \{\delta_z\}$.*
(iii) *If $\mathbb{N}_3 = \varnothing$ and $\mathbb{N}_1, \mathbb{N}_2 \neq \varnothing$ then $\mathcal{GP}(\delta_z) = \{\delta_w \mid w_n = z_n$ if $n \in \mathbb{N}_1$ and $\sup_{n\in\mathbb{N}_2} |w_n| < 1\}$. So,*
  - *if $\#(\mathbb{N}_2) = k$ then $\mathcal{GP}(\delta_z) \approx \mathbb{D}^k$,*
  - *if $\mathbb{N}_2$ is infinite then, $\mathcal{GP}(\delta_z) \approx B_{\ell_\infty}$.*

  Both identifications are isometries with respect to the Gleason metric.
(iv) *If $\mathbb{N}_3$ is infinite and $\mathbb{N}_2 = \varnothing$, then $\mathcal{GP}(\delta_z)$ contains $\mathbb{D}^k$ for every $k \in \mathbb{N}$ and this inclusion is an isometry for the Gleason metric. There is also a continuous injection of B_{ℓ_∞} into $\mathcal{GP}(\delta_z)$.*
(v) *If both $\mathbb{N}_2$ and $\mathbb{N}_3$ are infinite, then $\mathcal{GP}(\delta_z)$ contains an isometric copy of B_{ℓ_∞}, for the Gleason metric.*

Proof The results concerning isometries follow from Lemma 2.3 and Theorem 2.4. We only have to show the continuous injection of B_{ℓ_∞} in item (iv). If we write $\mathbb{N}_3 = \{n_k\}_k$, for each k there exists $r_k > 0$ such that whenever $|z_{n_k} - w_{n_k}| < r_k$ we have $w_{n_k} \in \mathbb{D}$ and

$$\left| \frac{z_{n_k} - w_{n_k}}{1 - \overline{z_{n_k}} w_{n_k}} \right| < \frac{1}{2}.$$

Then, denoting $C_{n_k} = r_k\mathbb{D}$ and $C_n = \{0\}$ for $n \notin \mathbb{N}_3$ we obtain that if $w \in z + \prod_{n=1}^{\infty} C_n$ then $\delta_w \in \mathcal{GP}(\delta_z)$. Since it is clear how to inject B_{ℓ_∞} onto the set $z + \prod_{n=1}^{\infty} C_n$, we derive the injection of B_{ℓ_∞} into $\mathcal{GP}(\delta_z)$. □

3 Gleason parts for $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$

Some of our knowledge about the Gleason parts of $\mathcal{M}(\mathcal{A}_u(B_X))$ passes to $\mathcal{M}(\mathcal{H}^\infty(B_X))$ if we consider the restriction mapping $\Upsilon_u \colon \mathcal{M}(\mathcal{H}^\infty(B_X)) \to \mathcal{M}(\mathcal{A}_u(B_X))$. With obvious notation, it is clear that for any $\varphi, \psi \in \mathcal{M}(\mathcal{H}^\infty(B_X))$,

$$\rho(\varphi, \psi) \geq \rho_u(\Upsilon_u(\varphi), \Upsilon_u(\psi)).$$

Therefore, if $\mathcal{GP}_{\mathcal{A}_u}(\Upsilon_u(\varphi)) \neq \mathcal{GP}_{\mathcal{A}_u}(\Upsilon_u(\psi))$ we also have $\mathcal{GP}_{\mathcal{H}^\infty}(\varphi) \neq \mathcal{GP}_{\mathcal{H}^\infty}(\psi)$.

Remark 3.1 Let $X = c_0$ and consider $z, w \in S_{\ell_\infty}$ such that $\mathcal{GP}_{\mathcal{A}_u}(\delta_z) \neq \mathcal{GP}_{\mathcal{A}_u}(\delta_w)$. Then, for any $\varphi \in \mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$ and $\psi \in \mathcal{M}_w(\mathcal{H}^\infty(B_{c_0}))$, as $\Upsilon_u(\varphi) = \delta_z$ and $\Upsilon_u(\psi) = \delta_w$, we also have $\mathcal{GP}_{\mathcal{H}^\infty}(\varphi) \neq \mathcal{GP}_{\mathcal{H}^\infty}(\psi)$. In particular, if $z \in \overline{B}_{\ell_\infty}$ belongs to the distinguished boundary $\mathbb{T}^{\mathbb{N}}$, every $\varphi \in \mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$ satisfies $\mathcal{GP}_{\mathcal{H}^\infty}(\varphi) \subset \mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$. That is, the Gleason part of φ is contained in the fiber over z.

The following is somehow a counterpart to the above remark.

Proposition 3.2 *Let $z, w \in S_{\ell_\infty}$ be such that $\mathcal{GP}_{\mathcal{A}_u}(\delta_z) = \mathcal{GP}_{\mathcal{A}_u}(\delta_w)$. Then there exist $\varphi \in \mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$ and $\psi \in \mathcal{M}_w(\mathcal{H}^\infty(B_{c_0}))$ satisfying $\mathcal{GP}_{\mathcal{H}^\infty}(\varphi) = \mathcal{GP}_{\mathcal{H}^\infty}(\psi)$.*

Proof Fix real numbers (r_k), with $|r_k| < 1$ and $r_k \nearrow 1$. Consider the sequences in B_{ℓ_∞}:

$$x^k = r_k z \to z \quad \text{and} \quad y^k = r_k w \to w.$$

Now, as $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ is weak-star compact, both (δ_{x^k}) and (δ_{y^k}) admit weak-star convergent subnets $(\delta_{x^{k(\alpha)}})_\alpha$, $(\delta_{y^{k(\alpha)}})_\alpha$ in $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$. Say

$$\delta_{x^{k(\alpha)}} \longrightarrow \varphi; \qquad \delta_{y^{k(\alpha)}} \longrightarrow \psi.$$

It is clear that $\varphi \in \mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$ and $\psi \in \mathcal{M}_w(\mathcal{H}^\infty(B_{c_0}))$. Now, as $\mathcal{GP}_{\mathcal{A}_u}(\delta_z) = \mathcal{GP}_{\mathcal{A}_u}(\delta_w)$, by Theorem 2.4 we have

$$C = \sup_n \|\delta_{z_n} - \delta_{w_n}\|_{\mathcal{M}(\mathcal{A}_u(\mathbb{D}))} = \|\delta_z - \delta_w\|_{\mathcal{M}(\mathcal{A}_u(B_{c_0}))} < 2.$$

Then, given $f \in \mathcal{H}^\infty(B_{c_0})$, $\|f\| \leq 1$, we can find α_0 so that for any $\alpha \geq \alpha_0$,

$$\left|\delta_{x^{k(\alpha)}}(f) - \varphi(f)\right| < \frac{2-C}{4} \quad \text{and} \quad \left|\delta_{y^{k(\alpha)}}(f) - \psi(f)\right| < \frac{2-C}{4}.$$

Therefore,

$$\begin{aligned} |\varphi(f) - \psi(f)| &\leq \frac{2-C}{2} + \left|\delta_{x^{k(\alpha)}}(f) - \delta_{y^{k(\alpha)}}(f)\right| \\ &\leq \frac{2-C}{2} + \left\|\delta_{x^{k(\alpha)}} - \delta_{y^{k(\alpha)}}\right\|_{\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))} \\ &= \frac{2-C}{2} + \sup_n \left\|\delta_{x_n^{k(\alpha)}} - \delta_{y_n^{k(\alpha)}}\right\|, \end{aligned}$$

where the last equality, which is a version of the statement of Theorem 2.4 for the spectrum $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$, appears in the proof of [9, Theorem 6.5]. Now, using the pseudo-hyperbolic distance for the unit disc $\mathbb{D}$ and the Schwarz–Pick theorem applied to the function $f(z) = r_{k(\alpha)}z$, for each fixed n such that $z_n \neq w_n$ we have

$$\rho(\delta_{x_n^{k(\alpha)}}, \delta_{y_n^{k(\alpha)}}) = \left| \frac{x_n^{k(\alpha)} - y_n^{k(\alpha)}}{1 - \overline{x_n^{k(\alpha)}}\, y_n^{k(\alpha)}} \right| = \left| \frac{r_{k(\alpha)}(z_n - w_n)}{1 - r_{k(\alpha)}^2 \overline{z_n} w_n} \right|$$
$$\leq \left| \frac{z_n - w_n}{1 - \overline{z_n} w_n} \right| \leq \rho_u(\delta_z, \delta_w).$$

Then, by (1.1) $\left\| \delta_{x^{k(\alpha)}} - \delta_{y^{k(\alpha)}} \right\|_{\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))} \leq \|\delta_z - \delta_w\|_{\mathcal{M}(\mathcal{A}_u(B_{c_0}))} = C$.

Finally, $|\varphi(f) - \psi(f)| \leq \frac{2-C}{2} + C = \frac{2+C}{2}$, for any $f \in \mathcal{H}^\infty(B_{c_0})$ with $\|f\| \leq 1$. Therefore, $\|\varphi - \psi\|_{\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))} \leq \frac{2+C}{2} < 2$ and the proof is complete. □

We next prove a kind of extension of the previous proposition. In [4, Lemma 2.9] it is shown that for $w \in \overline{B}_{\ell_\infty}$ and $b \in \mathbb{D}$ the fibers over w and (b, w) are homeomorphic. To recall the homeomorphism let us consider $\Lambda_b \colon B_{c_0} \to B_{c_0}$ given by $\Lambda_b(z) = (b, z)$ and let us denote by $S \colon B_{c_0} \to B_{c_0}$, the shift mapping $S(z) = (z_2, z_3, \ldots)$. Now, the homomorphism between the fibers is given by

$$\begin{aligned} R_b \colon \mathcal{M}_w &\to \mathcal{M}_{(b,w)} \\ \varphi &\mapsto (f \in \mathcal{H}^\infty(B_{c_0}) \mapsto \varphi(f \circ \Lambda_b)). \end{aligned}$$

Since both Λ_b and S map the unit ball into the unit ball and $S \circ \Lambda_b = Id$ it is easy to see that R_b is an isometry for the Gleason metric. Therefore, the fiber over w and the fiber over (b, w) (for any $w \in \overline{B}_{\ell_\infty}$) intersect the same "number" of Gleason parts.

From Remark 3.1 we know that if $z \in \mathbb{T}^{\mathbb{N}}$, then every $\varphi \in \mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$ satisfies that the Gleason part of φ is contained in the fiber over z. The next proposition will show us not only that this does not hold for the fibers over points outside $\mathbb{T}^{\mathbb{N}}$, but also that any Gleason part outside $\mathbb{T}^{\mathbb{N}}$ must have elements from different fibers (in fact, at least from a *disc* of fibers).

Proposition 3.3 *Given $b \in \mathbb{D}$, there exists $r_b > 0$ such that if $|c - b| < r_b$ then, for all $\varphi \in \mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$, $R_b(\varphi)$ and $R_c(\varphi)$ are in the same Gleason part.*

Proof By the Cauchy integral formula, $\overline{B}_{\mathcal{H}^\infty(\mathbb{D})}$ is an equicontinuous set of functions. Therefore, there exists $r_b > 0$ such that, if $|c - b| < r_b$ then $c \in \mathbb{D}$ and $|g(b) - g(c)| < 1$, for all $g \in B_{\mathcal{H}^\infty(\mathbb{D})}$.

Hence, for $f \in \mathcal{H}^\infty(B_{c_0})$ with $\|f\| \leq 1$ we have

$$|f(b, z) - f(c, z)| < 1, \qquad \text{if } |c - b| < r_b,\ z \in B_{c_0}.$$

Therefore, for every $\varphi \in \mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$,

$$\begin{aligned}\|R_b(\varphi) - R_c(\varphi)\| &= \sup_{\|f\|\le 1} |R_b(\varphi)(f) - R_c(\varphi)(f)| \\ &= \sup_{\|f\|\le 1} |\varphi(f\circ\Lambda_b - f\circ\Lambda_c)| \\ &\le \sup_{\|f\|\le 1} \|f\circ\Lambda_b - f\circ\Lambda_c\| \\ &= \sup_{\|f\|\le 1}\sup_{z\in B_{c_0}} |f(b,z) - f(c,z)| \le 1.\end{aligned}$$

□

It is clear that the previous result is also valid between the fibers over w and over $(w_1, b, w_2, \dots)$ or $(w_1, w_2, b, w_3, \dots)$ and so on. That means that the Gleason part of any morphism in the fiber over a point outside $\mathbb{T}^{\mathbb{N}}$, must have elements from other fibers. In particular, there cannot be singleton Gleason parts outside the fibers over the points in $\mathbb{T}^{\mathbb{N}}$.

Thus far, the above results show that in $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ there are Gleason parts intersecting different fibers (Propositions 3.2 and 3.3) and there are Gleason parts completely contained in a fiber (Remark 3.1). These results do not provide information on the size of the Gleason parts. In order to understand this feature a usual tool is the following result whose statement covers several versions appearing for instance in [15, Lemma 1.1, p. 393], [17, Lemma 2.1] and [21, p. 162].

Proposition 3.4 *Let X, Y be Banach spaces and $\Omega_X \subset X$, $\Omega_Y \subset Y$ be open convex subsets. Let $\mathcal{A}$ be a uniform algebra of analytic functions defined on Ω_X. Suppose that $\Phi\colon \Omega_Y \to \mathcal{M}(\mathcal{A})$ is an analytic inclusion. Then $\Phi(\Omega_Y)$ is contained in only one Gleason part.*

Remark 3.5 Using [4] and [9] it was recently proved (independently) in [7] and in [11] that for each $z \in \overline{B}_{\ell_\infty}$ the fiber over z contains an analytic copy of B_{ℓ_∞}. Moreover, this injection is a Gleason isometry. Even by the previous proposition or simply using the Gleason isometry it follows that each of these copies of B_{ℓ_∞} should be in a single Gleason part. Hence, for every $z \in \overline{B}_{\ell_\infty}$, there is a *thick* intersection of the fiber over z with a Gleason part.

Recall that given a compact set K and a uniform algebra $\mathcal{A}$ contained in $C(K)$ a point $x \in K$ is called a *strong boundary point* for $\mathcal{A}$ if for every neighborhood V of x there exists $f \in \mathcal{A}$ such that $\|f\| = f(x) = 1$ and $|f(y)| < 1$ if $y \in K \setminus V$. We see in the next result that in the fiber over each $z \in \mathbb{T}^{\mathbb{N}}$ there is a strong boundary point. Since the Gleason part of a strong boundary point is just a singleton set, by the above remark, we derive that the fiber over any $z \in \mathbb{T}^{\mathbb{N}}$ intersects a thick Gleason part and also a singleton Gleason part.

Proposition 3.6 *If $\mathcal{S}$ is the set of strong boundary points of $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ then $\pi(\mathcal{S}) = \mathbb{T}^{\mathbb{N}}$.*

Proof Denoting by $\mathcal{SB}$ the Shilov boundary of $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$, we have that $\mathcal{S} \subset \mathcal{SB}$ (see, e.g., [21, Corollary 7.24]) and thus $\pi(\mathcal{S}) \subset \pi(\mathcal{SB})$. Therefore, in order to prove $\pi(\mathcal{S}) = \mathbb{T}^{\mathbb{N}}$ it is enough to see $\pi(\mathcal{SB}) \subset \mathbb{T}^{\mathbb{N}}$ and $\mathbb{T}^{\mathbb{N}} \subset \pi(\mathcal{S})$.

To prove the first inclusion, for each $n \in \mathbb{N}$, let us consider the map $j_n : \overline{B}_{\ell_\infty} \to \overline{\mathbb{D}}$ given by $j_n(z) = z_n$. Then, $P_n = j_n \circ \pi$ is a weak-star continuous mapping from $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ into $\overline{\mathbb{D}}$.

Given $a \in \overline{B}_{\ell_\infty} \setminus \mathbb{T}^{\mathbb{N}}$, we want to show that $a \notin \pi(\mathcal{SB})$. Since $a \notin \mathbb{T}^{\mathbb{N}}$, there is n such that $|a_n| < 1$. The set $C_n = \overline{\mathbb{D}} \setminus \mathbb{D}(a_n, \frac{1-|a_n|}{2})$ is a closed subset of $\mathbb{C}$, so $P_n^{-1}(C_n)$ is weak-star closed in $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$. Also, since C_n contains spheres of radius r, with r approaching to 1, for each $f \in \mathcal{H}^\infty(B_{c_0})$ we should have

$$\sup_{z \in B_{c_0}} |f(z)| = \sup_{\varphi \in P_n^{-1}(C_n)} |\varphi(f)|.$$

Hence, $P_n^{-1}(C_n)$ is a boundary, which implies that $\mathcal{SB} \subset P_n^{-1}(C_n)$. Thus, $\pi(\mathcal{SB}) \subset \pi(P_n^{-1}(C_n))$. Since $a \notin \pi(P_n^{-1}(C_n))$, we obtain that $a \notin \pi(\mathcal{SB})$.

For the second inclusion, let $a = (a_n) \in \mathbb{T}^{\mathbb{N}}$ be given by $a_n = e^{i\theta_n}$, for all n. As $(\frac{e^{-i\theta_n}}{2^n}) \in \ell_1$ its associated function

$$x^*(x) = \sum_{n=1}^{\infty} \frac{e^{-i\theta_n}}{2^n} x_n$$

belongs to c_0^*. Hence $f(x) = 1 + x^*(x)$ is holomorphic on c_0, bounded and uniformly continuous when restricted to $\overline{B}_{\ell_\infty}$. Observe that

$$|\tilde{f}(a)| = 2; \qquad \text{while} \qquad |\tilde{f}(z)| < 2, \quad \text{for all } z \in \overline{B}_{\ell_\infty},\ z \neq a.$$

Associating f with its Gelfand transform $\widehat{f}$ and noting that $\widehat{f}$ attains its norm at a strong boundary point [21, Theorem 7.21], there is $\varphi \in \mathcal{S}$ such that $|\widehat{f}(\varphi)| = |\varphi(f)| = 2$. Finally

$$\varphi(f) = \varphi(1) + \varphi(x^*) = 1 + x^*(\pi(\varphi)) = \tilde{f}(\pi(\varphi)).$$

Therefore, $\pi(\varphi) = a$, and so $a \in \pi(\mathcal{S})$. □

Up to now our study about the relationships between fibers and Gleason parts gives information about in which fibers there are singleton Gleason parts, which fibers intersect *thick* Gleason parts and which Gleason parts contain elements of different fibers. To complete this picture we now wonder about how many Gleason parts intersect a particular fiber. Should it always be more than one?

With respect to this question note that we have already seen that in the fiber over any $z \in \mathbb{T}^{\mathbb{N}}$ there is a singleton Gleason part and also a copy of B_{ℓ_∞}. So, at least two Gleason parts are inside each of these fibers. By translations through mappings R_b (as in Proposition 3.3 and the subsequent comment) we also obtain that there are at least two Gleason parts intersecting the fiber over z for each $z \in S_{\ell_\infty}$ with all but finitely many coordinates of modulus 1.

The following results show that the fiber over any $z \in B_{\ell_\infty}$ intersects 2^c Gleason parts. First, relying on the proof of [9, Theorem 5.1] (see also [9, Corollary 5.2]) we obtain the desired result for the fiber over 0. For our purposes, we use the construction and notation given in [9].

Theorem 3.7 *Let X be an infinite dimensional Banach space. Then there is an embedding $\Psi\colon (\beta(\mathbb{N}) \setminus \mathbb{N}) \times \mathbb{D} \to \mathcal{M}_0$ that is analytic on each slice $\{\theta\} \times \mathbb{D}$ and satisfies:*

(a) $\Psi(\theta, \lambda) \notin \mathcal{GP}(\delta_0)$ *for each* (θ, λ).
(b) $\mathcal{GP}(\Psi(\theta, \lambda)) \cap \mathcal{GP}(\Psi(\tilde{\theta}, \tilde{\lambda})) = \varnothing$ *for each* $\theta, \tilde{\theta} \in \beta(\mathbb{N}) \setminus \mathbb{N}$ *with* $\theta \neq \tilde{\theta}$ *and any* $\lambda, \tilde{\lambda} \in \mathbb{D}$.

Proof The existence of the analytic embedding $\Psi\colon (\beta(\mathbb{N}) \setminus \mathbb{N}) \times \mathbb{D} \to \mathcal{M}_0$ is given in [9, Theorem 5.1]. Below, we summarize the main ingredients used in its construction.

- There exists a sequence $(z_k) \subset B_{X^{**}}$ such that $\|z_k\| < \|z_{k+1}\|$ and $\|z_k\|$ is convergent to 1.
- The sequence of norms $(\|z_k\|)$ increases so rapidly that there exists an increasing sequence (r_k), such that $0 < r_k < \|z_k\|$ and $\sum(1 - r_k)$ is finite.
- For a fixed sequence (a_k) so that $0 < a_k < 1$ and $(a_k) \in \ell_1$, there exists $(L_k) \subset X^*$ such that $\|L_k\| < 1$ and

  $\cdot\ L_k(z_k) = r_k,\quad$ for all k,
  $\cdot\ L_j(z_k) = 0,\quad 1 < k < j$,
  $\cdot\ |L_j(z_k)| < a_j,\quad$ for all $k > j$.

- There exists $0 < r < 1$ such that for all k, if $w_k\colon \mathbb{D} \to X$ is defined as $w_k(\lambda) = \left(\frac{r_k-\lambda}{1-r_k\lambda}\right)\frac{z_k}{r_k}$, then $\|w_k(\lambda)\| < 1$ for all $|\lambda| < r$.
- The Blaschke product $G\colon B_{X^{**}} \to \mathbb{C}$, given by $G(z) = \prod_{j=1}^{\infty} \frac{r_j - L_j(z)}{1-r_j L_j(z)}$ belongs to $\mathcal{H}^\infty(B_{X^{**}})$ and $|G(z)| < 1$ if $\|z\| < 1$.
- For $|\lambda| < r/2$ and each k there exists a unique $\xi_k(\lambda)$ such that $|\xi_k(\lambda)| < r$ and $G(w_k(\xi_k(\lambda))) = \lambda$ for all $|\lambda| < r/2$.
- For every k the function $z_k(\lambda)\colon = w_k(\xi_k(\lambda))$ for $|\lambda| < r/2$ is a multiple of z_k, depends analytically on λ and satisfies $\|z_k(\lambda)\| < 1$ if $|\lambda| < r/2$ with $z_k(0) = z_k$.

Note that replacing $\mathbb{D}$ by $D = \{\lambda \in \mathbb{C} \mid |\lambda| < r/2\}$, it is enough to show the result for $\beta(\mathbb{N}) \setminus \mathbb{N} \times D$. The function $\Psi\colon \mathbb{N} \times D \to \mathcal{M}$ defined by $\Psi(k, \lambda) = \delta_{z_k(\lambda)}$ extends to a map $\Psi\colon \beta(\mathbb{N}) \times D \to \mathcal{M}$ which is continuous on $\beta(\mathbb{N})$ for each fixed λ. Moreover, by [9, Theorem 5.1], we know that $\Psi(\beta(\mathbb{N}) \setminus \mathbb{N} \times D)$ lies in the fiber over 0, $\mathcal{M}_0$.

Now, let us prove that (a) holds. As Ψ is analytic on each slice, to show that $\Psi(\theta, \lambda) \notin \mathcal{GP}(\delta_0)$ for each (θ, λ) it is enough to see that $\Psi(\theta, 0) \notin \mathcal{GP}(\delta_0)$, for any θ. Given $N \in \mathbb{N}$, consider $f_N \in \mathcal{H}^\infty(B_{X^{**}})$ defined by

$$f_N(z) := \prod_{j>N}^{\infty} \frac{r_j - L_j(z)}{1 - r_j L_j(z)}.$$

Note that the restriction of f_N to B_X (which we still denote by f_N) belongs to $\mathcal{H}^\infty(B_X)$ and the canonical extension to $B_{X^{**}}$ of this restriction coincides with the original function.

Then, $\delta_0(f_N) = \prod_{j>N} r_j \to 1$ as $N \to \infty$. On the other hand, as $\Psi(k, 0) = \delta_{z_k}$, for $k > N$,

$$\Psi(k, 0)(f_N) = \prod_{j>N}^{\infty} \frac{r_j - L_j(z_k)}{1 - r_j\, L_j(z_k)} = 0.$$

Now, take $\theta \in \beta(\mathbb{N}) \setminus \mathbb{N}$. Then, there is a net $(j(\alpha)) \subset \mathbb{N}$, such that $\theta = \lim_\alpha j(\alpha)$. Thus,

$$\Psi(\theta, 0)(f_N) = \lim_\alpha \Psi(j(\alpha), 0)(f_N) = 0.$$

Therefore,

$$\rho(\delta_0, \Psi(\theta, 0)) \geq \sup_N\{|\delta_0(f_N)|\} = \sup_N\{\prod_{j>N} r_j\} = 1,$$

which shows that $\Psi(\theta, 0) \notin \mathcal{GP}(\delta_0)$.

To prove (b) let us see that if $\theta \neq \tilde{\theta}$ then $\mathcal{GP}(\Psi(\theta, D)) \cap \mathcal{GP}(\Psi(\tilde{\theta}, D)) = \varnothing$. Indeed, for $\theta \neq \tilde{\theta}$ there exists an infinite set $J \subset \mathbb{N}$ such that $\mathbb{N} \setminus J$ is also infinite and $\theta \in \overline{\{j : j \in J\}}$, $\tilde{\theta} \in \overline{\{j : j \in \mathbb{N} \setminus J\}}$.

Here, for $N \in \mathbb{N}$ consider $f_{(J,N)} \in \mathcal{H}^\infty(B_{X^{**}})$ given by

$$f_{(J,N)}(z) := \prod_{\substack{j\in J\\ j>N}} \frac{r_j - L_j(z)}{1 - r_j\, L_j(z)}.$$

Then, $\|f_{(J,N)}\| \leq 1$ and $f_{(J,N)}(z_k) = 0$ for all $k \in J, k > N$. Hence, as before, we obtain that $\Psi(\theta, 0)(f_{(J,N)}) = 0$.

On the other hand, $\tilde{\theta} = \lim_{\tilde{\alpha}} k(\tilde{\alpha})$. For these indexes $k(\tilde{\alpha}) \notin J$ with $k(\tilde{\alpha}) > N$, the corresponding factor does not appear in $f_{(J,N)}$ and

$$\Psi(k(\tilde{\alpha}), 0)(f_{(J,N)}) = \prod_{\substack{j\in J\\ N<j<k(\tilde{\alpha})}} \frac{r_j - L_j(z_{k(\tilde{\alpha})})}{1 - r_j\, L_j(z_{k(\tilde{\alpha})})} \cdot \prod_{\substack{j\in J\\ j>k(\tilde{\alpha})}} r_j.$$

Notice that $\left|\frac{r_j - L_j(z_{k(\tilde{\alpha})})}{1 - r_j L_j(z_{k(\tilde{\alpha})})}\right| > \frac{r_j - a_j}{1 + r_j a_j}$, for $k(\tilde{\alpha}) > j$. By the inequality $1 - \frac{r_j - a_j}{1 + r_j\, a_j} < (1 - r_j) + 2a_j$, the series $\sum_{j\geq 1}(1 - \frac{r_j - a_j}{1 + r_j\, a_j})$ converges, implying that the infinite product $\prod_{j\geq 1} \frac{r_j - a_j}{1 + r_j\, a_j}$ is convergent as well as the infinite product over $\{j \in J\}$.

Now, given $0 < \varepsilon < 1$ we can find $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$,

$$\prod_{\substack{j\in J\\ j>k}} r_j > 1 - \varepsilon \qquad \text{and} \qquad \prod_{\substack{j\in J\\ j>k}} \frac{r_j - a_j}{1 + r_j\, a_j} > 1 - \varepsilon.$$

Then, for $N > k_0$ and $\tilde{\alpha}$ such that $k(\tilde{\alpha}) > k_0$, we have

$$\prod_{\substack{j\in J\\ N<j<k(\tilde{\alpha})}} \left|\frac{r_j - L_j(z_{k(\tilde{\alpha})})}{1 - r_j L_j(z_{k(\tilde{\alpha})})}\right| > \prod_{\substack{j\in J\\ N<j<k(\tilde{\alpha})}} \frac{r_j - a_j}{1 + r_j a_j} > \prod_{\substack{j\in J\\ j>N}} \frac{r_j - a_j}{1 + r_j a_j} > 1 - \varepsilon.$$

Hence,

$$|\Psi(k(\tilde{\alpha}), 0)(f_{(J,N)})| > (1 - \varepsilon)^2,$$

and $|\Psi(\tilde{\theta}, 0)(f_{(J,N)})| \geq (1 - \varepsilon)^2$. Finally, for any $0 < \varepsilon < 1$

$$\rho(\Psi(\theta, 0)), \Psi(\tilde{\theta}, 0)) \geq \sup_N\{|\Psi(\tilde{\theta}, 0)(f_{(J,N)})|\} \geq (1 - \varepsilon)^2,$$

and the result follows. □

Next, we will see that there is a bijective biholomorphic mapping from B_{ℓ_∞} into B_{ℓ_∞} which is an isometry for the Gleason metric and transfers each fiber over an interior point to a different fiber. We use this fact to extend the conclusions in Theorem 3.7 to the fiber $\mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$ for any $z \in B_{\ell_\infty}$.

Lemma 3.8 *Let $\alpha \in \mathbb{D}$ and let $\eta_\alpha : \mathbb{D} \to \mathbb{D}$ be the Moebius transformation,*

$$\eta_\alpha(\lambda) = \frac{\alpha - \lambda}{1 - \overline{\alpha}\lambda}.$$

Given $|\alpha| \leq s < 1$, for any $\lambda \in \mathbb{D}$ with $|\lambda| \leq s$ the following inequality holds:

$$|\eta_\alpha(\lambda)| \leq \frac{2s}{1 + s^2}.$$

Proof Notice that

$$1 - \left|\frac{\alpha - \lambda}{1 - \overline{\alpha}\lambda}\right|^2 = \frac{|1 - \overline{\alpha}\lambda|^2 - |\alpha - \lambda|^2}{|1 - \overline{\alpha}\lambda|^2} = \frac{(1 - |\lambda|^2)(1 - |\alpha|^2)}{|1 - \overline{\alpha}\lambda|^2}.$$

Hence, the result follows for any $|\lambda| \leq s$ since

$$1 - \left|\frac{\alpha - \lambda}{1 - \overline{\alpha}\lambda}\right|^2 \geq \left(\frac{1 - s^2}{1 + s^2}\right)^2 \text{ and } \sqrt{1 - \left(\frac{1 - s^2}{1 + s^2}\right)^2} = \frac{2s}{1 + s^2}.$$

□

Proposition 3.9 *Fix $a = (a_n) \in B_{\ell_\infty}$. The mapping $\Phi_a : B_{\ell_\infty} \to B_{\ell_\infty}$, defined by*

$$\Phi_a(z) = (\eta_{a_n}(z_n))$$

is bijective and biholomorphic. Moreover, for any $x^* \in \ell_1$, the function $x^* \circ \Phi_a$ is uniformly continuous.

Proof First, let us check that $\Phi_a(B_{\ell_\infty}) \subset B_{\ell_\infty}$. Fix $z = (z_n) \in B_{\ell_\infty}$ and take $s = \max\{\|a\|, \|z\|\} < 1$. Using Lemma 3.8 we obtain

$$\|\Phi_a(z)\| = \sup_n |\eta_{a_n}(z_n)| \leq \frac{2s}{1+s^2} < 1.$$

To check that Φ_a is holomorphic, by Dunford's theorem it is enough to check that Φ_a is weak-star holomorphic, i.e. that $x^* \circ \Phi_a \in \mathcal{H}(B_{\ell_\infty})$ for every $x^* = (b_n) \in \ell_1$. Notice that $x^* \circ \Phi_a(z) = \sum_{n=1}^{\infty} b_n \eta_{a_n}(z_n)$, and

$$|b_n \eta_{a_n}(z_n)| \leq |b_n|,$$

for every $z \in B_{\ell_\infty}$ and every n. By the Weierstrass M-test, the series $\sum_{n=1}^{\infty} b_n \eta_{a_n}(z_n)$ converges absolutely and uniformly on $\overline{B}_{\ell_\infty}$ and as each $z \mapsto \eta_{a_n}(z_n)$ belongs to $\mathcal{A}_u(B_{\ell_\infty})$ we have actually proved that $x^* \circ \Phi_a \in \mathcal{A}_u(B_{\ell_\infty})$, for every $x^* \in \ell_1$. Thus $\Phi_a \in \mathcal{H}(B_{\ell_\infty}, B_{\ell_\infty})$.

Finally as $\Phi_a \circ \Phi_a(z) = z$ for every $z \in B_{\ell_\infty}$, we obtain that Φ_a has inverse $\Phi_a^{-1} = \Phi_a$ and Φ_a is biholomorphic. □

Remark 3.10 Observe that if we consider $a \in B_{c_0}$ and we restrict Φ_a to $z \in B_{c_0}$, then we obtain the biholomorphic mapping of Example 1.7.

Given $a \in B_{\ell_\infty}$ the restriction of Φ_a to B_{c_0} will be denoted by $\Phi_a\big|_{c_0}$.

Theorem 3.11 *Given $a \in B_{\ell_\infty}$, the mapping $C_{\Phi_a} : \mathcal{H}^\infty(B_{c_0}) \to \mathcal{H}^\infty(B_{c_0})$ defined by*

$$C_{\Phi_a}(f) = \tilde{f} \circ \Phi_a\big|_{c_0},$$

where $\tilde{f} : B_{\ell_\infty} \to \mathbb{C}$ is the canonical extension of each $f \in \mathcal{H}^\infty(B_{c_0})$, is an isometric isomorphism of Banach algebras.

Moreover, $\Lambda_{\Phi_a} := C^t_{\Phi_a}|_{\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))} : \mathcal{M}(\mathcal{H}^\infty(B_{c_0})) \to \mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$, the restriction of its transpose to $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$, is a surjective isometry for the Gleason metric with inverse $\Lambda_{\Phi_a}^{-1} = \Lambda_{\Phi_a}$ that satisfies

$$\Lambda_{\Phi_a}(\mathcal{M}_z) = \mathcal{M}_{\Phi_a(z)},$$

for every $z \in B_{\ell_\infty}$.

Proof Clearly C_{Φ_a} is well-defined, $\|C_{\Phi_a}\| \leq 1$ and it is an algebra homomorphism. Next we claim that given $f \in \mathcal{H}^\infty(B_{c_0})$,

$$\widetilde{\tilde{f} \circ \Phi_a\big|_{c_0}} = \tilde{f} \circ \Phi_a. \tag{3.1}$$

Let us observe that $\ell_\infty = C(\beta\mathbb{N})$ is a symmetrically regular space. Moreover, by Lemma 3.8, if $0 < s < 1$, then $m = \sup_{\|z\|\le s} \|\Phi_a(z)\| < 1$. With this in mind, by the method of proof of [8, Corollary 2.2], we have

$$\widetilde{\tilde{f} \circ \Phi_a\big|_{c_0}} = \tilde{\tilde{f}} \circ \widetilde{\Phi_a\big|_{c_0}} = \tilde{f} \circ \widetilde{\Phi_a\big|_{c_0}}.$$

By Proposition 3.9, $\Phi_a\big|_{c_0}$ is $w(c_0, \ell_1)$-uniformly continuous on B_{c_0}. Hence it has a unique extension to B_{ℓ_∞} that is $w(\ell_\infty, \ell_1)$-uniformly continuous on B_{ℓ_∞} and it coincides with its canonical extension $\widetilde{\Phi_a\big|_{c_0}}$. On the other hand, also by Proposition 3.9, Φ_a is $w(\ell_\infty, \ell_1)$-uniformly continuous on B_{ℓ_∞} and it is obviously an extension of $\Phi_a\big|_{c_0}$ to B_{ℓ_∞}. Thus, $\widetilde{\Phi_a\big|_{c_0}}(z) = \Phi_a(z)$, for all $z \in B_{\ell_\infty}$.

From this equality we derive that $C_{\Phi_a} \circ C_{\Phi_a}(f) = f$ for every $f \in \mathcal{H}^\infty(B_{c_0})$. Indeed,

$$C_{\Phi_a}\big(C_{\Phi_a}(f)\big)(z) = \left(\widetilde{\tilde{f} \circ \Phi_a\big|_{c_0}} \circ \Phi_a\big|_{c_0}\right)(z) = \tilde{f} \circ \widetilde{\Phi_a\big|_{c_0}} \circ \Phi_a(z) = \tilde{f}(z) = f(z),$$

for every $z \in B_{c_0}$. As a consequence C_{Φ_a} is an isomorphism of algebras. Also we have $\|f\| \le \|C_{\Phi_a}\|\|C_{\Phi_a}(f)\| \le \|C_{\Phi_a}(f)\|$ for every f, and therefore C_{Φ_a} is an isometry.

Hence its transpose $C^t_{\Phi_a}$ when restricted to $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$ is well-defined and its range is again in $\mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$. Moreover, $\Lambda_{\Phi_a} \circ \Lambda_{\Phi_a}(\varphi) = \varphi$ for every $\varphi \in \mathcal{M}(\mathcal{H}^\infty(B_{c_0}))$. Finally, for each $x^* \in \ell_1$, the function $\widetilde{x^* \circ \Phi_a\big|_{c_0}}$ belongs to $\mathcal{A}_u(B_{c_0})$ (as we have already observed) and so it is a uniform limit of finite type polynomials. Hence, as in the proof of Proposition 1.6, we obtain that $\Lambda_{\Phi_a}(\mathcal{M}_z) = \mathcal{M}_{\Phi_a(z)}$, for every $z \in B_{\ell_\infty}$. □

Combining this last theorem with Theorem 3.7 we obtain that for each $z \in B_{\ell_\infty}$, the fiber $\mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$ contains 2^c *discs* lying in different Gleason parts.

Corollary 3.12 *Let $z \in B_{\ell_\infty}$. Then, there is an embedding of $\Psi\colon (\beta(\mathbb{N}) \setminus \mathbb{N}) \times \mathbb{D} \to \mathcal{M}_z(\mathcal{H}^\infty(B_{c_0}))$ that is analytic on each slice $\{\theta\} \times \mathbb{D}$ and satisfies:*

(a) $\Psi(\theta, \lambda) \notin \mathcal{GP}(\delta_z)$ *for each* (θ, λ).
(b) $\mathcal{GP}(\Psi(\theta, \lambda)) \cap \mathcal{GP}(\Psi(\tilde{\theta}, \tilde{\lambda})) = \varnothing$ *for each* $\theta, \tilde{\theta} \in \beta(\mathbb{N}) \setminus \mathbb{N}$ *with* $\theta \ne \tilde{\theta}$ *and any* $\lambda, \tilde{\lambda} \in \mathbb{D}$.

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