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The rational sectional category of certain universal fibrations

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Abstract
We prove that the (reduced) rational sectional category of the universal fibration with fibre X, for X any space that satisfies a well-known conjecture of Halperin, equals one.

Keywords Sectional category · Rational homotopy theory · Universal fibration · Halperin conjecture · Lusternik–Schnirelmann category

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1 Main result

We begin with a concise résumé of the ingredients, then a statement, of our main result. Notations used and assertions made here are described in greater detail below. *Sectional category* (secat) is a numerical invariant of a fibration that extends the notion of *LS category* (cat) of a space to fibrations. We normalize these invariants: $\mathsf{secat}(p) = 0$ when the fibration $p\colon E \to B$ has a section. Sectional category plays a role in several interesting applications (e.g. see [1, §9.3] and [5]).

Fibrations with fibre a fixed space X are classified by a universal fibration with fibre X [2,17,21]. For fibrations of simply connected spaces, this universal fibration may be identified, up to homotopy, as the map on Dold–Lashof classifying spaces

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$u_X : B\mathrm{aut}_1^*(X) \to B\mathrm{aut}_1(X)$ that is induced by an inclusion $\mathrm{aut}_1^*(X) \hookrightarrow \mathrm{aut}_1(X)$ of connected monoids of self-equivalences [3,11].

A fibration $p: E \to B$ with B and E simply connected admits a rationalization $p_{\mathbb{Q}}: E_{\mathbb{Q}} \to B_{\mathbb{Q}}$ that, homotopically, represents a simplification of p. Then the *rational sectional category* (secat_0) of fibration p is defined by setting $\mathsf{secat}_0(p) := \mathsf{secat}(p_{\mathbb{Q}})$.

If the fibration p has fibre X we have general inequalities $\mathsf{secat}_0(p) \leq \mathsf{secat}_0(u_X) \leq \mathsf{cat}_0(B\mathrm{aut}_1(X))$, with $\mathsf{cat}_0(B\mathrm{aut}_1(X))$ the *rational LS category* of the classifying space, which is often infinite (see [7,8]). So it is natural to ask whether $\mathsf{secat}_0(u_X)$ may be finite.

Our main result is the following:

Theorem 1.1 *Let X be any (rationally non-trivial) F_0-space that satisfies Conjecture 1 below. Then $\mathsf{secat}_0(u_X) = 1$, where $u_X : B\mathrm{aut}_1^*(X) \to B\mathrm{aut}_1(X)$ denotes the universal fibration with fibre X.*

By an F_0-space we mean a simply connected, finite complex X with $H^*(X;\mathbb{Q})$ and $\pi_*(X) \otimes \mathbb{Q}$ both finite-dimensional and $H^{odd}(X;\mathbb{Q}) = 0$. Examples of F_0-spaces for which Conjecture 1 is satisfied include even-dimensional spheres, complex projective spaces, homogeneous spaces G/H with rank G = rank H, and finite products of any of these spaces. It follows from Theorem 1.1 that we have $\mathsf{secat}_0(p) \leq 1$ for any fibration $p: E \to B$ of simply connected spaces with fibre X to which the Theorem applies.

2 Introduction

We continue with a fuller description of the ingredients just indicated. We assume all spaces are of the homotopy type of a CW complex and simply connected of finite rational type. That is, simply connected with each rational homology group (equivalently each rational homotopy group) a finite-dimensional rational vector space. This hypothesis appears in some of the basic results from rational homotopy theory about minimal models of spaces and fibrations. For a general space X, the classifying space $B\mathrm{aut}_1(X)$ need not be of finite rational type. However, if X is a simply connected, finite complex then the universal fibration is a fibre sequence $X \to B\mathrm{aut}_1^*(X) \to B\mathrm{aut}_1(X)$ of spaces of the homotopy type of a CW complex that are simply connected and of finite rational type. We justify this assertion carefully, as follows.

For a general space X, we have $B\mathrm{aut}_1(X)$ simply connected: $\mathrm{aut}_1(X)$ is connected and we have $\pi_1\big(B\mathrm{aut}_1(X)\big) \cong \pi_0\big(\Omega B\mathrm{aut}_1(X)\big) \cong \pi_0\big(\mathrm{aut}_1(X)\big) = \{e\}$. If X is simply connected, then it follows from the long exact sequence in homotopy of the universal fibration that so too is $B\mathrm{aut}_1^*(X)$. For X a simply connected, finite complex, a result of [4] implies that $B\mathrm{aut}_1(X)$ is of the homotopy type of a CW complex of finite type. It follows that $B\mathrm{aut}_1^*(X)$ is also of the homotopy type of a CW complex (see [19]). With $B\mathrm{aut}_1(X)$ simply connected and of finite type, it is also of finite rational type, and it follows from the Serre spectral in rational homology of the universal fibration that $B\mathrm{aut}_1^*(X)$ is also of finite rational type. All this applies in particular when X is an F_0-space, since we assume that such an X is a simply connected, finite complex.

For $n \geq 1$, set $\mathsf{secat}(p) = n - 1$ if n is the minimal number of open sets U_i in a cover of B such that p admits a section over each U_i. Set $\mathsf{cat}(B) = n - 1$ if n is the minimal number of open sets U_i in a cover of B such that each inclusion $U_i \to B$ is nulhomotopic. The inequalities $\mathsf{secat}(p) \leq \mathsf{cat}(B)$ and $\mathsf{secat}(f^*(p)) \leq \mathsf{secat}(p)$, for $f^*(p)$ the pull-back of p by a map $f\colon B' \to B$, are both proved directly (see, e.g. Proposition 9.14 and Exercise 9.3 of [1], which reference also contains many facts concerning secat and cat). Generally speaking, both cat and secat are delicate invariants, and difficult to compute. It is quite surprising, therefore, that we are able to obtain a global result such as Theorem 1.1. This is especially so considering that the universal fibration is a rich construction that involves large and complex spaces whose general structure is not well understood.

For simply connected B, we have $\mathsf{cat}_0(B) := \mathsf{cat}(B_{\mathbb{Q}})$, where cat_0 denotes the rational LS category. Then for simply connected X, from the first inequality of the previous paragraph applied to the rationalization of the universal fibration, we have $\mathsf{secat}_0(u_X) \leq \mathsf{cat}_0(B\mathrm{aut}_1(X))$. Also, because $u_X\colon B\mathrm{aut}_1^*(X) \to B\mathrm{aut}_1(X)$ is *universal*, any fibration $p\colon E \to B$ of simply connected spaces with fibre X is a pullback $f^*(u_X)$ for some classifying map $f\colon B \to B\mathrm{aut}_1(X)$. By the naturality of rationalization with respect to pull-backs, and the second inequality of the previous paragraph, we obtain $\mathsf{secat}_0(p) \leq \mathsf{secat}_0(u_X)$ for any fibration $p\colon E \to B$ of simply connected spaces with fibre X. This explains the inequalities cited above and below the enunciation of Theorem 1.1.

Stanley has given a complete calculation of the rational sectional category of spherical fibrations [20]. His results for the even-dimensional sphere imply (in our normalized notation) that $\mathsf{secat}_0(u_{S^{2n}}) = 1$. Here, we extend Stanley's result from S^{2n} to any F_0-space that satisfies Halperin's Conjecture in rational homotopy theory. We discuss this class of spaces now.

Halperin has conjectured the following generalization of classical results on the rational cohomology of homogeneous spaces.

Conjecture 1 (Halperin) *Let X be an F_0-space and $p\colon E \to B$ any fibration of simply connected spaces with fibre X. Then the rational Serre spectral sequence for p collapses at the E_2-term.*

The conjecture is equivalent to the assertion that $\mathrm{Der}_{<0}(H^*(X;\mathbb{Q})) = 0$ for X an F_0-space, where $\mathrm{Der}_{<0}(A)$ is the graded Lie algebra of degree-lowering derivations of the algebra A [18,22]. The conjecture follows easily from this version for $X = S^{2n}, \mathbb{C}P^n$ and, more generally, for any space with rational cohomology a truncated polynomial algebra. Meier proved the conjecture for flag-manifolds G/T with T a maximal torus, and other homogeneous spaces [18, Th.B]. Shiga and Tezuka extended Meier's result to the general case of homogeneous spaces G/H of equal rank pairs [21]. Halperin's Conjecture has also been confirmed for the cases in which $H^*(X;\mathbb{Q})$ has 3 or fewer generators [15]. Markl has shown that the class of spaces for which Conjecture 1 holds is closed under fibrations, not just products [16]. Our main result, Theorem 1.1, applies in all these cases. We refer the reader to [6, p.516] for a discussion and other references.

Meier made the following connection between Halperin's Conjecture and the rational homotopy of the universal fibration [18, Th.A].

Theorem 2.1 (Meier) *Let X be an F_0-space. Then X satisfies Halperin's Conjecture if and only if $B\mathrm{aut}_1(X)$ is rationally equivalent to a product of even-dimensional Eilenberg–Mac Lane Spaces.* □

Meier's Theorem implies that $\mathsf{cat}_0(B\mathrm{aut}_1(X)) = \infty$ for X an F_0-space that satisfies Halperin's Conjecture. In fact, this is the case for any elliptic space X [7]. We will use Theorem 2.1 to deduce Theorem 1.1 in Sect. 3 as a consequence of Proposition 3.2, a technical result concerning sections of rational fibrations.

3 Rational sectional category of a fibrewise join

We recall a special case of a characterization of secat in terms of the fibrewise join construction (see [12, Sec.8]). Given $p\colon E \to B$ with fibre X, the fibrewise join $p * p$ is a fibre sequence: $X * X \longrightarrow E * E \xrightarrow{p*p} B$. The following is a special case of a result of Schwarz (see [12, Prop.8.1]):

Proposition 3.1 *Let $p\colon E \to B$ be a fibration. Then $\mathsf{secat}(p) \leq 1$ if and only if $p * p$ has a section.* □

Our main result is that the fibrewise join of the universal fibration

$$X * X \longrightarrow B\mathrm{aut}_1^*(X) * B\mathrm{aut}_1^*(X) \xrightarrow{\quad u_X * u_X \quad} B\mathrm{aut}_1(X)$$

has a section after rationalization when X is an F_0-space that satisfies Halperin's Conjecture. We make use of the correspondence between fibre sequences of rational spaces and relative Sullivan models. Although we will recall some basic facts about minimal models, our proofs assume a working familiarity with them. Our reference for rational homotopy is [6].

Let $X \to E \xrightarrow{p} B$ be a fibre sequence of simply connected spaces with B of finite rational type. The *relative Sullivan model* for p is a short exact sequence

$$(\wedge W, d_B) \xrightarrow{\;J\;} (\wedge W \otimes \wedge V, D) \longrightarrow (\wedge V, d_X)$$

of DG algebras, with $(\wedge W, d_B)$ and $(\wedge V, d_X)$ the Sullivan minimal models for B and X, respectively [6, Prop.15.5]. The differential D satisfies

$$D(w) = d_B(w) \text{ for } w \in W \text{ and } D(v) - d_X(v) \in \wedge^+ W \otimes \wedge V \text{ for } v \in V.$$

The inclusion J is a model for p. Applying spatial realization, we obtain that $p_{\mathbb{Q}}\colon E_{\mathbb{Q}} \to B_{\mathbb{Q}}$ admits a section if and only if J has a left-inverse S. That is, $p_{\mathbb{Q}}$ admits a section if and only if there is a DG algebra map $S\colon (\wedge W \otimes \wedge V, D) \to (\wedge W, d_B)$ with $S \circ J = \mathrm{id}_{\wedge W}$. We prove:

Proposition 3.2 *Let $X \to E \xrightarrow{p} B$ be a fibre sequence of simply connected spaces with B of finite rational type. Suppose*

(1) *B has the rational homotopy type of a product of even-dimensional Eilenberg–Mac Lane spaces, and*
(2) *X has the rational homotopy type of a wedge of at least two odd-dimensional spheres.*

Then the rationalization $p_{\mathbb{Q}}\colon E_{\mathbb{Q}} \to B_{\mathbb{Q}}$ *of* p *admits a section.*

Proof In the following, we make use of the identification $V^n \cong \mathrm{Hom}\big(\pi_n(X), \mathbb{Q}\big)$, where $(\wedge V, d_X)$ is the minimal model of X, and also the identification between Samelson products in the rational homotopy Lie algebra $\pi_*(\Omega X)\otimes\mathbb{Q}$ and the quadratic part of the differential in $(\wedge V, d_X)$, for any space X. See [6, Th.15.11, Th.21.6] for details. Hypothesis (1) implies that the minimal model $(\wedge W, d_B)$ for B has trivial differential, $d_B = 0$, with $W^{\mathrm{odd}} = 0$. We deduce two consequences of (2) for the minimal model of the fibre, $(\wedge V, d_X)$. First we see that $V^{\mathrm{even}} = 0$ (a wedge of odd-dimensional spheres has no non-zero rational homotopy in even degrees). Write $V = \langle v_1, v_2, \ldots\rangle$ with $|v_i| \leq |v_j|$ whenever $i < j$ and each $|v_i|$ odd. Then, for degree reasons, we have $d_X(v_1) = d_X(v_2) = 0$. The rational homotopy Lie algebra $\pi_*(\Omega X)\otimes\mathbb{Q}$ is free as graded Lie algebra. Translated to Sullivan models, we deduce, in particular, that there is some v_k for $k \geq 3$, such that $d_X(v_k) = v_1v_2$.

The relative Sullivan model for $X \to E \to B$ is of the form:

$$(\wedge W, 0) \to (\wedge W \otimes \wedge V, D) \to (\wedge V, d_X),$$

Given a subspace $V' \subseteq V$, let $I(V') = \wedge W \otimes \wedge^+ V'$ denote the ideal generated by V' in $\wedge W \otimes \wedge V$. Our goal is to prove that $I(V)$ is a D-stable ideal of $\wedge W \otimes \wedge V$. We may then define a DG algebra map $S\colon (\wedge W \otimes \wedge V, D) \to (\wedge W, 0)$ by $S(w) = w$ and $S(v) = 0$ to obtain the desired section.

We use induction on i to show that, for any v_i, we have $D(v_i) \in I(V)$. First, we show that $D(v_1) = 0$. For suppose that $D(v_1) = P$, for some polynomial $P \in \wedge W$. Since $d_X(v_1) = d_X(v_2) = 0$, we see that $D(v_2) = P_2$ for some $P_2 \in \wedge W$. Then, with v_k chosen as above, we have $D(v_k) = v_1v_2 + P_k$, for some $P_k \in \wedge W$. Notice that there cannot be a term from $\wedge^+ W \otimes \wedge^+ V$ in $D(v_k)$, since V is oddly graded and $D(v_k)$ is of even degree, and also because v_1v_2 is of minimal degree in $\wedge^2 V$. Furthermore, since v_1v_2 is the only term from $\wedge^2 V$ appearing in $d(v_k)$, v_1v_2 is the only such term appearing in $D(v_k)$. Since $D^2(v_k) = 0$, we have

$$0 = D^2(v_k) = D(v_1v_2 + P_k) = Pv_2 - v_1P_2,$$

and it follows that $P = 0$ (also that $P_2 = 0$).

Now suppose that we have $D(v_i) \in I(V)$ for all $i < t$ and for some $t \geq 2$. Then $I(v_1, \ldots, v_{t-1})$ is a D-stable ideal of $\wedge W \otimes \wedge V$. We may take the quotient by this ideal yielding the graded algebra

$$\wedge W \otimes \wedge V_{i\geq t} = (\wedge W \otimes \wedge V)\,/I(v_1, \ldots, v_{t-1}).$$

Since $I(v_1, \ldots, v_{t-1})$ is D-stable, D induces a differential $\overline{D}$ on the quotient. Let $\pi : (\wedge W \otimes \wedge V, D) \to (\wedge W \otimes \wedge V_{i\geq t}, \overline{D})$ denote the projection which is a map of DG algebras.

Next observe that $\wedge(v_1, \ldots, v_{t-1})$ is a d_X-stable sub-algebra of $(\wedge V, d_X)$. Write $\overline{d_X}$ for the induced differential and $\pi' \colon (\wedge V, d_X) \to (\wedge V_{i \geq t}, \overline{d_X})$ for the projection. We claim that $(\wedge V_{i \geq t}, \overline{d_X})$ is the minimal model for a wedge of at least two odd-dimensional spheres. For observe that π' is surjective on generators. Applying the Mapping Theorem [6, Th.29.5], we deduce that

$$\mathsf{cat}(\wedge V_{i \geq t}, \overline{d_X}) \leq \mathsf{cat}(\wedge V, d_X) = \mathsf{cat}_0(X) = 1.$$

Since V is concentrated in odd degrees and nontrivial in infinitely many degrees, the same holds for $V_{i \geq t}$. Our claim follows from [6, Th.28.5].

Now consider the commutative diagram of relative Sullivan models:

$$\begin{array}{ccccc}
(\wedge W, 0) & \longrightarrow & (\wedge W \otimes \wedge V, D) & \longrightarrow & (\wedge V, d_X) \\
\Big\| & & \Big\downarrow \pi & & \Big\downarrow \pi' \\
(\wedge W, 0) & \longrightarrow & (\wedge W \otimes \wedge V_{i \geq t}, \overline{D}) & \longrightarrow & (\wedge V_{i \geq t}, \overline{d_X}).
\end{array}$$

The bottom sequence of this diagram corresponds to a fibre sequence with the original base, say $\overline{X}_{\mathbb{Q}} \to \overline{E}_{\mathbb{Q}} \to B_{\mathbb{Q}}$. We have argued above that the fibre $\overline{X}_{\mathbb{Q}}$, with minimal model $(\wedge V_{i \geq t}, \overline{d_X})$, is a wedge of at least two odd-dimensional spheres. We can now apply the first part of the argument above, to deduce that $\overline{D} v_t = 0$. But this implies that $D(v_t) \in I(v_1, \ldots, v_{t-1}) \subseteq I(V)$. By induction, we conclude that $I(V)$ is D-stable. □

We illustrate the need for the various hypotheses in Proposition 3.2 with the following examples. In these examples, and in the proof of our main result, below, we make use of several standard identifications from (rational) homotopy theory. For the convenience of the reader, we summarize these here and give references for them. A useful representation of the join of two spaces, up to homotopy equivalence, is $X * Y \simeq \Sigma(X \wedge Y)$. A theorem of Ganea yields the following: The homotopy fibre of $X \vee Y \to X \times Y$, the inclusion of the wedge into the product, is $\Omega X * \Omega Y$ [13, Prop.6.63]. Calculations due to Serre show that an odd-dimensional sphere has the rational homotopy type of an Eilenberg–Mac Lane space, $S^{2n+1} \simeq_{\mathbb{Q}} K(\mathbb{Q}, 2n+1)$ [6, p.210]. A result of Berstein implies that a suspension has the rational homotopy type of a wedge of spheres [6, Th.24.5]. Actually, more is true: if a space is a rational co-H-space (has rational category equal to one), then it has the rational homotopy type of a wedge of spheres [6, Th.28.5].

Example 3.3 In which we illustrate that a fibration need not admit a section, if we drop any one of the hypotheses of Proposition 3.2.

(a) We must have that the base space B is rationally a product of only even-dimensional Eilenberg–Mac Lane spaces. For consider the fibre sequence $\Omega S^3 * \Omega S^3 \to S^3 \vee S^3 \xrightarrow{p} S^3 \times S^3$ obtained by converting the inclusion to a fibration. The base is a product of odd-dimensional Eilenberg–Mac Lane spaces, up to rational homotopy type. We deduce that the fibre is a wedge of at least two odd-dimensional spheres, up to rational homotopy type. For $\Omega S^3 \simeq_{\mathbb{Q}} K(\mathbb{Q}, 2)$, and so $H^*(\Omega S^3 \wedge \Omega S^3; \mathbb{Q})$ is

evenly graded. Furthermore, the fibre $\Omega S^3 * \Omega S^3 \simeq \Sigma(\Omega S^3 \wedge \Omega S^3)$, as a suspension, must have the rational homotopy type of a wedge of spheres, and these spheres must be odd-dimensional since we have $H^*(\Omega S^3 * \Omega S^3; \mathbb{Q}) \cong H^*(\Sigma(\Omega S^3 \wedge \Omega S^3); \mathbb{Q}) \cong H^{*-1}(\Omega S^3 \wedge \Omega S^3; \mathbb{Q})$ is oddly graded. Finally, it is clear that there must be two or more spheres in this wedge since, for instance, we have $H^7(\Sigma(\Omega S^3 \wedge \Omega S^3); \mathbb{Q}) \cong \mathbb{Q} \oplus \mathbb{Q}$ (in fact the fibre is a wedge of infinitely many odd-dimensional spheres, rationally). However, $p_{\mathbb{Q}}$ does not admit a section since $p^*\colon H^*(S^3 \times S^3; \mathbb{Q}) \to H^*(S^3 \vee S^3; \mathbb{Q})$ is not injective.
(b) We must have a wedge of at least two odd-dimensional spheres in the fibre. Recall that we have $S^{2k-1} \simeq_{\mathbb{Q}} K(\mathbb{Q}, 2k-1)$. Then, for example, the path-loop fibration $K(\mathbb{Q}, 2k-1) \to PK(\mathbb{Q}, 2k) \to K(\mathbb{Q}, 2k)$ does not have a section. In fact, since the total space is contractible we have $\mathsf{secat}_0(p) = \mathsf{cat}(K(\mathbb{Q}, 2k)) = \infty$.
(c) We cannot have even-dimensional spheres in the fibre. For instance, consider the fibration sequence $X \to S^2 \vee S^3 \xrightarrow{p} K(\mathbb{Q}, 2k)$, in which $p\colon S^2 \vee S^3 \to K(\mathbb{Q}, 2)$ is the composition of the pinch map $S^2 \vee S^3 \to S^2$ followed by the inclusion (of the bottom cell) $S^2 \to K(\mathbb{Q}, 2)$. We deduce that the fibre X is a wedge of at least two spheres, at least one of which is of even dimension. For consider the long exact sequence in rational homotopy groups of this fibration. Since p induces a surjection on rational homotopy groups, it follows that the fibre inclusion $X \to S^2 \vee S^3$ is injective in rational homotopy groups. Now the mapping theorem [6, Th.28.6] implies that we have $\mathsf{cat}_0(X) \leq \mathsf{cat}_0(S^2 \vee S^3) = 1$. It follows that X has the rational homotopy type of a wedge of spheres [6, Th.28.5]. Returning to the long exact sequence in rational homotopy groups, we find that $X \to S^2 \vee S^3$ induces an isomorphism in degrees ≥ 3. Direct computation of the rational homotopy groups of $S^2 \vee S^3$ now shows, for instance, that we have $\pi_3(X) \otimes \mathbb{Q} \cong \mathbb{Q}$ and $\pi_4(X) \otimes \mathbb{Q} \cong \mathbb{Q}$. Since X is 2-connected, it follows that, as a wedge of rational spheres, X must contain at least one odd-dimensional and at least one even-dimensional sphere (in fact X contains infinitely many of each). This fibration does not admit a section, as p^* is not injective in cohomology.

We next observe that the universal fibration u_X for X a rationally non-trivial F_0-space does not admit a section. To prove this, we use the *Gottlieb group* $G_*(X) \subseteq \pi_*(X)$ [10]. Recall that $G_*(X) = \text{Image}\{\omega_\sharp\colon \pi_*(\text{aut}_1(X)) \to \pi_*(X)\}$ where $\omega\colon \text{aut}_1(X) \to X$ is the evaluation map.

Proposition 3.4 *Let X be a rationally non-trivial F_0-space. Then $\mathsf{secat}_0(u_X) \geq 1$.*

Proof First, we have $G_*(X) \otimes \mathbb{Q} \neq 0$ for any such X. To see this, note that in the minimal model $(\wedge V, d_X)$ for X, there is an integer $n > 0$ such that $V^n \neq 0$ and $V^m = 0$ for $m > n$. This is just a translation into minimal model terms of the hypothesis that $\pi_*(X) \otimes \mathbb{Q}$ is finite-dimensional, using the identification $V^n \cong \text{Hom}\big(\pi_n(X), \mathbb{Q}\big)$ mentioned at the start of the proof of Proposition 3.2. Then it follows from the identification of the Gottlieb group in minimal model terms, discussed in [6, §29d], that we have $G_n(X_{\mathbb{Q}}) \cong V^n \neq 0$. But for X a simply connected *finite* complex, we have $G_n(X) \otimes \mathbb{Q} \cong G_n(X_{\mathbb{Q}})$ (see [14]). Hence, we have $G_n(X) \otimes \mathbb{Q} \neq 0$. Next, by [10, §4], $G_*(X)$ corresponds to the image of $\partial\colon \pi_{*+1}(B\text{aut}_1(X)) \to \pi_*(X)$, the linking homomorphism in the long exact sequence of the universal fibration

$u_X\colon B\mathrm{aut}_1(X) \to X$. Thus $G_n(X) \otimes \mathbb{Q} \neq 0$ implies that $\partial\colon \pi_{n+1}(B\mathrm{aut}_1(X)) \to \pi_n(X)$ is non-zero after passing to rational homotopy groups. Therefore, the linking homomorphism in the long exact sequence of the rationalized universal fibration $(u_X)_{\mathbb{Q}}$ is non-zero, and it follows that $(u_X)_{\mathbb{Q}}$ cannot admit a section. □

We apply the preceding to prove our main result:

Proof of Theorem 1.1 Let X be a rationally non-trivial F_0-space. By Proposition 3.4, we have $\mathsf{secat}_0(u_X) \geq 1$. We prove $\mathsf{secat}_0(u_X) \leq 1$.

The fibrewise join of the universal fibration, $u_X * u_X$, has base $B\mathrm{aut}_1(X)$ and fibre $X * X \simeq \Sigma(X \wedge X)$. By Theorem 2.1, since X satisfies Halperin's Conjecture, we have $B\mathrm{aut}_1(X)$ is rationally a product of even-dimensional Eilenberg–Mac Lane spaces. Regarding the fibre, note that, since $H^*(X;\mathbb{Q})$ is evenly graded, $H^*(\Sigma(X\wedge X);\mathbb{Q})$ is oddly graded. As a suspension (of $X \wedge X$), $X * X$ has the rational homotopy type of a wedge of spheres, and these spheres must be odd-dimensional since $H^*(\Sigma(X \wedge X);\mathbb{Q}) \cong H^*(X * X;\mathbb{Q})$ is oddly graded. If $\widetilde{H}^*(X;\mathbb{Q})$ has dimension at least 2, then $X * X$ is rationally a wedge of at least two odd-dimensional spheres. Applying Proposition 3.2, we conclude the rationalization of $u_X * u_X$ has a section and we conclude $\mathsf{secat}_0(u_X) \leq 1$ by Proposition 3.1.

When $\widetilde{H}^*(X;\mathbb{Q})$ has dimension 1, then $X \simeq_{\mathbb{Q}} S^{2n}$ and we can invoke Stanley's result [20, Lem.3.2]. Alternately, we may observe that the fibrewise join $u_{S^{2n}} * u_{S^{2n}}$ has fibre $S^{2n} * S^{2n} \simeq_{\mathbb{Q}} K(\mathbb{Q}, 4n+1)$ and base $B\mathrm{aut}_1(S^{2n}) \simeq_{\mathbb{Q}} K(\mathbb{Q}, 4n)$. It follows easily from degree considerations that $(u_{S^{2n}} * u_{S^{2n}})_{\mathbb{Q}}$ is fibre-homotopically trivial and so, in particular, has a section. □

Theorem 1.1 reduces the computation of secat_0 for fibrations with fibre X satisfying Halperin's Conjecture to the question of the existence of a section. Stanley expressed the obstruction to a section in cohomological terms when $X = S^{2n}$ [20, Th.3.3]. We follow his approach to obtain the following example.

Example 3.5 Write $\mathrm{Fib}_{\mathbb{C}P^m}(\mathbb{C}P^n)$ for the set of fibrations $\mathbb{C}P^m \to E \xrightarrow{p} \mathbb{C}P^n$ modulo rational fibre-homotopy equivalence. We assume $m < n$. The identity $B\mathrm{aut}_1(\mathbb{C}P^m) \simeq_{\mathbb{Q}} \prod_{k=2}^{m+1} K(\mathbb{Q}, 2k)$ follows from [18, Pro.2.6(iii)]. By universality, we have

$$\mathrm{Fib}_{\mathbb{C}P^m}(\mathbb{C}P^n) \equiv [\mathbb{C}P^n, B\mathrm{aut}_1(\mathbb{C}P^m)_{\mathbb{Q}}] \equiv \oplus_{k=2}^{m+1} H^{2k}(\mathbb{C}P^n;\mathbb{Q}) \equiv \mathbb{Q}^m.$$

We associate the rational fibre-homotopy equivalence class of p with an explicit m-tuple $(a_{m-1}, \ldots, a_0) \in \mathbb{Q}^m$ defined as follows. The relative Sullivan model for p is an inclusion $\big(\wedge(x_2, y_{2n+1}), d\big) \to \big(\wedge(x_2, y_{2n+1}) \otimes \wedge(u_2, v_{2m+1}), D\big)$ with $Dx = dx = 0$, $Dy = dy = x^{n+1}$, $Du = 0$. As for Dv, we have

$$Dv = u^{m+1} + a_m u^m x + a_{m-1} u^{m-1} x^2 + \cdots + a_0 x^{m+1}$$

for some $a_i \in \mathbb{Q}$. The basis change $u \mapsto u - \frac{1}{m+1} a_m$ will depress the polynomial and we may assume $a_m = 0$. We observe that the classes $a_k[x^{m+1-k}] \in$

$H^{2(m+1-k)}(\mathbb{C}P^n;\mathbb{Q})$ for $k=0,\ldots,m-1$ correspond to the Chern classes of the fibration p as in [9, Sec.3].

Given a section $S\colon \wedge(x,y)\otimes\wedge(u,v)\to\wedge(x,y)$ write $S(u)=qx$ for $q\in\mathbb{Q}$. Comparing coefficients of x^{m+1} in the equation $S(dv)=DS(v)=0$ we see that $z=q$ is a solution to $z^{m+1}+a_{m-1}z^{m-1}+\cdots+a_1z+a_0=0$. The converse follows similarly and we obtain:

$$\mathsf{secat}_0(p)=\begin{cases}0 & \text{if } z^{m+1}+a_{m-1}z^{m-1}+\cdots+a_1z+a_0 \text{ has a rational root}\\ 1 & \text{otherwise.}\end{cases}$$

We conclude with a question arising from our work. Meier [18] and others have given various equivalent versions of Halperin's Conjecture. It would be interesting to have an equivalent version of the conjecture phrased in terms of the sectional category of the universal fibration. We pose the following:

Question 3.6 *Let X be an F_0-space. Does $\mathsf{secat}_0(u_X)=1$ imply that X satisfies Halperin's Conjecture?*

References

1. Cornea, O., Lupton, G., Oprea, J., Tanré, D.: Lusternik–Schnirelmann Category. Mathematical Surveys and Monographs, vol. 103. American Mathematical Society, Providence (2003)
2. Dold, A.: Halbexakte Homotopiefunktoren. Lecture Notes in Mathematics, vol. 12. Springer, Berlin (1966)
3. Dold, A., Lashof, R.: Principal quasi-fibrations and fibre homotopy equivalence of bundles. Ill. J. Math. **3**, 285–305 (1959)
4. Dror, E., Dwyer, W.G., Kan, D.M.: Self-homotopy equivalences of virtually nilpotent spaces. Comment. Math. Helv. **56**(4), 599–614 (1981)
5. Farber, M.: Topological complexity of motion planning. Discrete Comput. Geom. **29**(2), 211–221 (2003)
6. Félix, Y., Halperin, S., Thomas, J.-C.: Rational Homotopy Theory. Graduate Texts in Mathematics, vol. 205. Springer, New York (2001)
7. Gatsinzi, J.-B.: LS-category of classifying spaces. Bull. Belg. Math. Soc. Simon Stevin **2**(2), 121–126 (1995)
8. Gatsinzi, J.-B.: LS-category of classifying spaces. II. Bull. Belg. Math. Soc. Simon Stevin **3**(2), 243–248 (1996)
9. Gatsinzi, J.-B.: On the genus of elliptic fibrations. Proc. Am. Math. Soc. **132**(2), 597–606 (2004)
10. Gottlieb, D.H.: Evaluation subgroups of homotopy groups. Am. J. Math. **91**, 729–756 (1969)
11. Gottlieb, D.H.: The total space of universal fibrations. Pac. J. Math. **46**, 415–417 (1973)
12. James, I.M.: On category, in the sense of Lusternik–Schnirelmann. Topology **17**(4), 331–348 (1978)
13. James, I.M.: General Topology and Homotopy Theory. Springer, New York (1984)
14. Lang Jr., G.E.: Localizations and evaluation subgroups. Proc. Am. Math. Soc. **50**, 489–494 (1975)
15. Lupton, G.: Note on a Conjecture of Stephen Halperin's, Topology and Combinatorial Group Theory (Hanover, NH, 1986/1987; Enfield, NH, 1988). Lecture Notes in Mathematics, vol. 1440. Springer, Berlin, pp. 148–163 (1990)
16. Markl, M.: Towards one conjecture on collapsing of the Serre spectral sequence. In: Proceedings of the Winter School on Geometry and Physics (Srní, 1989), no. 22, pp. 151–159 (1990)
17. Peter May, J.: Classifying spaces and fibrations. Mem. Am. Math. Soc. **1**(1), 155, xiii+98 (1975)
18. Meier, W.: Rational universal fibrations and flag manifolds. Math. Ann. **258**(3), 329–340 (1981/82)
19. Schön, R.: Fibrations over a CWh-base. Proc. Am. Math. Soc. **62** (1976), no. 1, 165–166 (1977)
20. Stanley, D.: The sectional category of spherical fibrations. Proc. Am. Math. Soc. **128**(10), 3137–3143 (2000)

21. Stasheff, J.: A classification theorem for fibre spaces. Topology **2**, 239–246 (1963)
22. Thomas, J.-C.: Rational homotopy of Serre fibrations. Ann. Inst. Fourier (Grenoble) **31**(3), v, 71–90 (1981)