Revista Matemática Complutense (2019) 32:799–822
https://doi.org/10.1007/s13163-018-0288-z

Hankel matrices acting on the Hardy space H^1 and on Dirichlet spaces

Daniel Girela[1] · Noel Merchán[1]

Received: 6 April 2018 / Accepted: 27 November 2018 / Published online: 13 December 2018

Abstract
If μ is a finite positive Borel measure on the interval $[0, 1)$, we let $\mathcal{H}_\mu$ be the Hankel matrix $(\mu_{n,k})_{n,k\geq 0}$ with entries $\mu_{n,k} = \mu_{n+k}$, where, for $n = 0, 1, 2, \dots$, μ_n denotes the moment of order n of μ. This matrix induces formally the operator $\mathcal{H}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \mu_{n,k} a_k\right) z^n$ on the space of all analytic functions $f(z) = \sum_{k=0}^{\infty} a_k z^k$, in the unit disc $\mathbb{D}$. When μ is the Lebesgue measure on $[0, 1)$ the operator $\mathcal{H}_\mu$ is the classical Hilbert operator $\mathcal{H}$ which is bounded on H^p if $1 < p < \infty$, but not on H^1. J. Cima has recently proved that $\mathcal{H}$ is an injective bounded operator from H^1 into the space $\mathscr{C}$ of Cauchy transforms of measures on the unit circle. The operator $\mathcal{H}_\mu$ is known to be well defined on H^1 if and only if μ is a Carleson measure and in such a case we have that $\mathcal{H}_\mu(H^1) \subset \mathscr{C}$. Furthermore, it is bounded from H^1 into itself if and only if μ is a 1-logarithmic 1-Carleson measure. In this paper we prove that when μ is a 1-logarithmic 1-Carleson measure then $\mathcal{H}_\mu$ actually maps H^1 into the space of Dirichlet type $\mathcal{D}_0^1$. We discuss also the range of $\mathcal{H}_\mu$ on H^1 when μ is an α-logarithmic 1-Carleson measure $(0 < \alpha < 1)$. We study also the action of the operators $\mathcal{H}_\mu$ on Bergman spaces and on Dirichlet spaces.

Keywords Hankel matrix · Generalized Hilbert operator · Hardy spaces · Cauchy transforms · Weighted Bergman spaces · Dirichlet spaces · Duality

Mathematics Subject Classification 47B35 · 30H10

This research is supported in part by a Grant from "El Ministerio de Economía y Competitividad", Spain (MTM2014-52865-P) and by a Grant from la Junta de Andalucía (FQM-210). The second author is also supported by a Grant from "El Ministerio de Educación, Cultura y Deporte", Spain (FPU2013/01478).

✉ Daniel Girela
girela@uma.es

Noel Merchán
noel@uma.es

[1] Análisis Matemático, Universidad de Málaga, Campus de Teatinos, 29071 Málaga, Spain

1 Introduction and main results

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ denote the open unit disc in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ will be the unit circle. The space of all analytic functions in $\mathbb{D}$ will be denoted by $\mathcal{H}ol(\mathbb{D})$. We also let H^p $(0 < p \leq \infty)$ be the classical Hardy spaces. We refer to [11] for the notation and results regarding Hardy spaces.

For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space A^p_α consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \stackrel{\text{def}}{=} \left((\alpha+1)\int_{\mathbb{D}}(1-|z|^2)^\alpha |f(z)|^p\, dA(z)\right)^{1/p} < \infty.$$

Here, dA stands for the area measure on $\mathbb{D}$, normalized so that the total area of $\mathbb{D}$ is 1. Thus $dA(z) = \frac{1}{\pi}\, dx\, dy = \frac{1}{\pi}\, r\, dr\, d\theta$. The unweighted Bergman space A^p_0 is simply denoted by A^p. We refer to [12,18,29] for the notation and results about Bergman spaces.

The space of Dirichlet type $\mathcal{D}^p_\alpha$ $(0 < p < \infty$ and $\alpha > -1)$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that $f' \in A^p_\alpha$. In other words, a function $f \in \mathcal{H}ol(\mathbb{D})$ belongs to $\mathcal{D}^p_\alpha$ if and only if

$$\|f\|_{\mathcal{D}^p_\alpha} \stackrel{\text{def}}{=} |f(0)| + \left((\alpha+1)\int_{\mathbb{D}}(1-|z|^2)^\alpha |f'(z)|^p\, dA(z)\right)^{1/p} < \infty.$$

The Hilbert matrix is the infinite matrix $\mathcal{H} = \left(\frac{1}{k+n+1}\right)_{k,n\geq 0}$. It induces formally an operator, called the Hilbert operator, on spaces of analytic functions as follows:

If $f \in \mathcal{H}ol(\mathbb{D})$, $f(z) = \sum_{n=0}^\infty a_n z^n$, then we set

$$\mathcal{H}f(z) = \sum_{n=0}^\infty \left(\sum_{k=0}^\infty \frac{a_k}{n+k+1}\right) z^n, \quad z \in \mathbb{D}, \tag{1}$$

whenever the right-hand side of (1) makes sense for all $z \in \mathbb{D}$ and the resulting function is analytic in $\mathbb{D}$. We define also

$$\mathcal{I}f(z) = \int_0^1 \frac{f(t)}{1-tz}\, dt, \quad z \in \mathbb{D}, \tag{2}$$

if the integrals in the right-hand side of (2) converge for all $z \in \mathbb{D}$ and the resulting function $\mathcal{I}f$ is analytic in $\mathbb{D}$. It is clear that the correspondences $f \mapsto \mathcal{H}f$ and $f \mapsto \mathcal{I}f$ are linear.

If $f \in H^1$, $f(z) = \sum_{n=0}^\infty a_n z^z$, then by the Fejér-Riesz inequality [11, Theorem 3. 13, p. 46] and Hardy's inequality [11, p. 48], we have

$$\int_0^1 |f(t)|\, dt \leq \pi \|f\|_{H^1} \quad \text{and} \quad \sum_{n=0}^\infty \frac{a_n}{n+1} \leq \pi \|f\|_{H^1}.$$

This immediately yields that if $f \in H^1$ then $\mathcal{H}f$ and $\mathcal{I}f$ are well defined analytic functions in $\mathbb{D}$ and that, furthermore, $\mathcal{H}f = \mathcal{I}f$.

Diamantopoulos and Siskakis [9] proved that $\mathcal{H}$ is a bounded operator from H^p into itself if $1 < p < \infty$, but this is not true for $p = 1$. In fact, they proved that $\mathcal{H}\left(H^1\right) \not\subseteq H^1$. Cima [6] has recently proved the following result.

Theorem A (i) *The operator $\mathcal{H}$ maps H^1 into the space $\mathscr{C}$ of Cauchy transforms of measures on the unit circle $\partial\mathbb{D}$.*
(ii) *$\mathcal{H} : H^1 \to \mathscr{C}$ is injective.*

We recall that if σ is a finite complex Borel measure on $\partial\mathbb{D}$, the Cauchy transform $C\sigma$ is defined by

$$C\sigma(z) = \int_{\partial\mathbb{D}} \frac{d\sigma(\xi)}{1 - \overline{\xi}\, z}, \quad z \in \mathbb{D}.$$

We let $\mathscr{M}$ be the space of all finite complex Borel measure on $\partial\mathbb{D}$. It is a Banach space with the total variation norm. The space of Cauchy transforms is $\mathscr{C} = \{C\sigma : \sigma \in \mathscr{M}\}$. It is a Banach space with the norm $\|C\sigma\| \stackrel{\text{def}}{=} \inf\{\|\tau\| : C\tau = C\sigma\}$. We mention [7] as an excellent reference for the main results about Cauchy transforms. We let $\mathcal{A}$ denote the disc algebra, that is, the space of analytic functions in $\mathbb{D}$ with a continuous extension to the closed unit disc, endowed with the $\|\cdot\|_{H^\infty}$-norm. It turns out [7, Chapter 4] that $\mathcal{A}$ can be identified with the pre-dual of $\mathscr{C}$ via the pairing

$$\langle g, C\sigma\rangle \stackrel{\text{def}}{=} \lim_{r\to 1} \frac{1}{2\pi} \int_0^{2\pi} g(re^{i\theta})\overline{C\sigma(re^{i\theta})}\, d\theta. \tag{3}$$

This is the basic ingredient used by Cima to prove the inclusion $\mathcal{H}(H^1) \subset \mathscr{C}$.

Now we turn to consider a class of operators which are natural generalizations of the operators $\mathcal{H}$ and $\mathcal{I}$. If μ is a finite positive Borel measure on $[0, 1)$ and $n = 0, 1, 2, \ldots$, we let μ_n denote the moment of order n of μ, that is, $\mu_n = \int_{[0,1)} t^n\, d\mu(t)$, and we define $\mathcal{H}_\mu$ to be the Hankel matrix $(\mu_{n,k})_{n,k\ge 0}$ with entries $\mu_{n,k} = \mu_{n+k}$. The measure μ induces formally the operators $\mathcal{I}_\mu$ and $\mathcal{H}_\mu$ on spaces of analytic functions as follows:

$$\mathcal{I}_\mu f(z) = \int_{[0,1)} \frac{f(t)}{1 - tz}\, d\mu(t), \quad \mathcal{H}_\mu f(z) = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} a_k \mu_{n+k} \right) z^n, \quad z \in \mathbb{D},$$

for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}ol(\mathbb{D})$ being such that the terms on the right-hand sides make sense for all $z \in \mathbb{D}$, and the resulting functions are analytic in $\mathbb{D}$. If μ is the Lebesgue measure on $[0, 1)$ the matrix $\mathcal{H}_\mu$ reduces to the classical Hilbert matrix and the operators $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ are simply the operators $\mathcal{H}$ and $\mathcal{I}$.

If $I \subset \partial\mathbb{D}$ is an interval, $|I|$ will denote the length of I. The *Carleson square* $S(I)$ is defined as $S(I) = \{re^{it} : e^{it} \in I, \quad 1 - \frac{|I|}{2\pi} \le r < 1\}$.

If $s > 0$ and μ is a positive Borel measure on $\mathbb{D}$, we shall say that μ is an s-Carleson measure if there exists a positive constant C such that

$$\mu(S(I)) \leq C|I|^s, \quad \text{for any interval } I \subset \partial\mathbb{D}.$$

A 1-Carleson measure will be simply called a Carleson measure. We recall that Carleson [4] proved that $H^p \subset L^p(d\mu)$ $(0 < p < \infty)$ if and only if μ is a Carleson measure (see also [11, Chapter 9]).

For $0 \leq \alpha < \infty$ and $0 < s < \infty$ we say that a positive Borel measure μ on $\mathbb{D}$ is an α-logarithmic s-Carleson measure if there exists a positive constant C such that

$$\frac{\mu(S(I))\left(\log \frac{2\pi}{|I|}\right)^{\alpha}}{|I|^s} \leq C, \quad \text{for any interval } I \subset \partial\mathbb{D}.$$

A positive Borel measure μ on $[0, 1)$ can be seen as a Borel measure on $\mathbb{D}$ by identifying it with the measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(A) = \mu(A \cap [0, 1)), \quad \text{for any Borel subset } A \text{ of } \mathbb{D}.$$

In this way a positive Borel measure μ on $[0, 1)$ is an s-Carleson measure if and only if there exists a positive constant C such that

$$\mu([t, 1)) \leq C(1 - t)^s, \quad 0 \leq t < 1.$$

We have a similar statement for α-logarithmic s-Carleson measures.

The action of the operators $\mathcal{I}_\mu$ and $\mathcal{H}_\mu$ on distinct spaces of analytic functions have been studied in a number of articles (see, e. g., [2,5,14–16,22,25,27]).

Combining results of [14] and of [16] we can state the following result.

Theorem B *Let μ be a finite positive Borel measure on $[0, 1)$.*

(i) *The operator $\mathcal{I}_\mu$ is well defined on H^1 if and only if μ is a Carleson measure.*
(ii) *If μ is a Carleson measure, then the operator $\mathcal{H}_\mu$ is also well defined on H^1 and $\mathcal{I}_\mu f = \mathcal{H}_\mu f$ for all $f \in H^1$.*
(iii) *The operator $\mathcal{H}_\mu$ is a bounded operator from H^1 into itself if and only if μ is a 1-logarithmic 1-Carleson measure.*

Galanopoulos and Peláez [14, Theorem 2. 2] proved the following.

Theorem C *Let μ be a positive Borel measure on $[0, 1)$. If μ is a Carleson measure then $\mathcal{H}_\mu(H^1) \subset \mathscr{C}$.*

This result is stronger than Theorem A(i). In view of these results, the following question arises naturally.

Question 1 *Suppose that μ is a 1-logarithmic 1-Carleson measure on $[0, 1)$. What can we say about the image $\mathcal{H}_\mu(H^1)$ of H^1 under the action of the operator $\mathcal{H}_\mu$?*

To answer Question 1, let us start noticing that it is known that, for $0 < p \le 2$, the space of Dirichlet type $\mathcal{D}^p_{p-1}$ is continuously included in H^p (see [26, Lemma 1.4]). In particular, the space $\mathcal{D}^1_0$ is continuously included in H^1. In fact, the estimates obtained by Vinogradov in the proof of his lemma easily yield the inequality

$$\|f\|_{H^1} \le 2\|f\|_{\mathcal{D}^1_0}, \quad f \in \mathcal{D}^1_0.$$

We shall prove that if μ is a 1-logarithmic 1-Carleson measure on $[0,1)$ then $\mathcal{H}_\mu(H^1)$ is contained in the space $\mathcal{D}^1_0$. Actually, we have the following stronger result.

Theorem 1 *Let μ be a positive Borel measure on $[0,1)$. Then the following conditions are equivalent.*

(i) *μ is a 1-logarithmic 1-Carleson measure.*
(ii) *$\mathcal{H}_\mu$ is a bounded operator from H^1 into itself.*
(iii) *$\mathcal{H}_\mu$ is a bounded operator from H^1 into $\mathcal{D}^1_0$.*
(iv) *$\mathcal{H}_\mu$ is a bounded operator from $\mathcal{D}^1_0$ into $\mathcal{D}^1_0$.*

There is a gap between Theorem C and Theorem 1 and so it is natural to discuss the range of H^1 under the action of $\mathcal{H}_\mu$ when μ is an α-logarithmic 1-Carleson measure with $0 < \alpha < 1$. We shall prove the following result.

Theorem 2 *Let μ be a positive Borel measure on $[0,1)$. Suppose that $0 < \alpha < 1$ and that μ is an α-logarithmic 1-Carleson measure. Then $\mathcal{H}_\mu$ maps H^1 into the space $\mathcal{D}^1(\log^{\alpha-1})$ defined as follows:*

$$\mathcal{D}^1(\log^{\alpha-1}) = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)| \left(\log \frac{2}{1-|z|}\right)^{\alpha-1} dA(z) < \infty \right\}.$$

These results will be proved in Sect. 2. Since the space of Dirichlet type $\mathcal{D}^1_0$ has showed up in a natural way in our work, it seems natural to study the action of the operators $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ on the Bergman spaces A^p_α and the Dirichlet spaces $\mathcal{D}^p_\alpha$ for general values of the parameters p and α. This will be done in Sect. 3.

Throughout this paper the letter C denotes a positive constant that may change from one step to the next. Moreover, for two real-valued functions E_1, E_2 we write $E_1 \lesssim E_2$, or $E_1 \gtrsim E_2$, if there exists a positive constant C independent of the arguments such that $E_1 \le C E_2$, respectively $E_1 \ge C E_2$. If we have $E_1 \lesssim E_2$ and $E_1 \gtrsim E_2$ simultaneously then we say that E_1 and E_2 are equivalent and we write $E_1 \asymp E_2$.

2 Proofs of the theorems 1 and 2

Proof of Theorem 1 We already know that (i) and (ii) are equivalent by Theorem B.

To prove that (i) implies (iii) we shall use some results about the Bloch space. We recall that a function $f \in \mathcal{H}ol(\mathbb{D})$ is said to be a Bloch function if

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The space of all Bloch functions will be denoted by $\mathcal{B}$. It is a non-separable Banach space with the norm $\|\cdot\|_{\mathcal{B}}$ just defined. A classical source for the theory of Bloch functions is [1]. The closure of the polynomials in the Bloch norm is the *little Bloch space* $\mathcal{B}_0$ which consists of those $f \in \mathcal{H}ol(\mathbb{D})$ with the property that

$$\lim_{|z|\to 1}(1-|z|^2)|f'(z)| = 0.$$

It is well known that (see [1, p. 13])

$$|f(z)| \lesssim \|f\|_{\mathcal{B}} \log\frac{2}{1-|z|}. \tag{4}$$

The basic ingredient to prove that (i) implies (iii) is the fact that the dual $(\mathcal{B}_0)^*$ of the little Bloch space can be identified with the Bergman space A^1 via the integral pairing

$$\langle h, f\rangle = \int_{\mathbb{D}} h(z)\,\overline{f(z)}\,dA(z), \quad h \in \mathcal{B}_0,\ f \in A^1. \tag{5}$$

(See [29, Theorem 5. 15]).

Let us proceed to prove the implication (i) $\Rightarrow$ (iii). Assume that μ is a 1-logarithmic 1-Carleson measure and take $f \in H^1$. We have to show that $\mathcal{I}_\mu f \in \mathcal{D}^1_0$ or, equivalently, that $\left(\mathcal{I}_\mu f\right)' \in A^1$. Since $\mathcal{B}_0$ is the closure of the polynomials in the Bloch norm, it suffices to show that

$$\left|\int_{\mathbb{D}} h(z)\,\overline{\left(\mathcal{I}_\mu f\right)'(z)}\,dA(z)\right| \lesssim \|h\|_{\mathcal{B}}\|f\|_{H^1}, \quad \text{for any polynomial } h. \tag{6}$$

So, let h be a polynomial. We have

$$\begin{aligned}\int_{\mathbb{D}} h(z)\,\overline{\left(\mathcal{I}_\mu f\right)'(z)}\,dA(z) &= \int_{\mathbb{D}} h(z)\,\overline{\left(\int_{[0,1)}\frac{t\,f(t)}{(1-tz)^2}\,d\mu(t)\right)}\,dA(z)\\ &= \int_{\mathbb{D}} h(z)\int_{[0,1)}\frac{t\,\overline{f(t)}}{(1-t\,\overline{z})^2}\,d\mu(t)\,dA(z)\\ &= \int_{[0,1)} t\,\overline{f(t)}\int_{\mathbb{D}}\frac{h(z)}{(1-t\,\overline{z})^2}\,dA(z)\,d\mu(t).\end{aligned}$$

Because of the reproducing property of the Bergman kernel [29, Proposition 4.23], $\int_{\mathbb{D}} \frac{h(z)}{(1-t\,\overline{z})^2}\,dA(z) = h(t)$. Then it follows that

$$\int_{\mathbb{D}} h(z)\,\overline{\left(\mathcal{I}_\mu f\right)'(z)}\,dA(z) = \int_{[0,1)} t\,\overline{f(t)}\,h(t)\,d\mu(t). \tag{7}$$

Since μ is a 1-logarithmic 1-Carleson measure, the measure ν defined by

$$d\nu(t) = \log\frac{2}{1-t}\,d\mu(t)$$

is a Carleson measure [15, Proposition 2.5]. This implies that

$$\int_{[0,1)} |f(t)|\log\frac{2}{1-t}\,d\mu(t) \lesssim \|f\|_{H^1}.$$

This and (4) yield

$$\int_{[0,1)} \left|t\,\overline{f(t)}\,h(t)\right|\,d\mu(t) \lesssim \|h\|_{\mathcal{B}}\|f\|_{H^1}.$$

Using this and (7), (6) follows.

Since $\mathcal{D}_0^1 \subset H^1$, the implication (iii) $\Rightarrow$ (iv) is trivial. To prove that (iv) implies (i) we shall use the following result of Pavlović [23, Theorem 3.2].

Theorem D *Let $f \in \mathcal{H}ol(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and suppose that the sequence $\{a_n\}$ is a decreasing sequence of non-negative real numbers. Then $f \in \mathcal{D}_0^1$ if and only if $\sum_{n=0}^{\infty} \frac{a_n}{n+1} < \infty$, and we have*

$$\|f\|_{\mathcal{D}_0^1} \asymp \sum_{n=0}^{\infty} \frac{a_n}{n+1}.$$

Now we turn to prove the implication (iv)$\Rightarrow$(i). Assume that $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{D}_0^1$ into $\mathcal{D}_0^1$. We argue as in the proof of Theorem 1.1 of [16]. For $\frac{1}{2} < b < 1$ set

$$f_b(z) = \frac{1-b^2}{(1-bz)^2}, \quad z \in \mathbb{D}.$$

We have $f_b'(z) = \frac{2b(1-b^2)}{(1-bz)^3}$ $(z \in \mathbb{D})$. Then, using Lemma 3.10 of [29] with $t = 0$ and $c = 1$, we see that

$$\|f_b\|_{\mathcal{D}_0^1} \asymp \int_{\mathbb{D}} \frac{1-b^2}{|1-bz|^3}\,dA(z) \asymp 1.$$

Since $\mathcal{H}_\mu$ is bounded on $\mathcal{D}^1_0$, this implies that

$$1 \gtrsim \|\mathcal{H}_\mu(f_b)\|_{\mathcal{D}^1_0}. \tag{8}$$

We also have,

$$f_b(z) = \sum_{k=0}^{\infty} a_{k,b} z^k, \quad \text{with } a_{k,b} = (1-b^2)(k+1)b^k.$$

Since the $a_{k,b}$'s are positive, it is clear that the sequence $\{\sum_{k=0}^{\infty} \mu_{n+k} a_{k,b}\}_{n=0}^{\infty}$ of the Taylor coefficients of $\mathcal{H}_\mu(f_b)$ is a decreasing sequence of non-negative real numbers. Using this, Theorem D, (8), and the definition of the $a_{k,b}$'s, we obtain

$$\begin{aligned}
1 &\gtrsim \|\mathcal{H}_\mu(f_b)\|_{\mathcal{D}^1_0} \gtrsim \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{\infty} \mu_{n+k} a_{k,b} \right) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=0}^{\infty} a_{k,b} \int_{[0,1)} t^{n+k} \, d\mu(t) \right) \\
&\gtrsim (1-b^2) \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=1}^{\infty} k b^k \int_{[b,1)} t^{n+k} \, d\mu(t) \right) \\
&\gtrsim (1-b^2) \sum_{n=1}^{\infty} \frac{1}{n} \left(\sum_{k=1}^{\infty} k b^{n+2k} \, \mu([b,1)) \right) \\
&= (1-b^2)\mu([b,1)) \sum_{n=1}^{\infty} \frac{b^n}{n} \left(\sum_{k=1}^{\infty} k b^{2k} \right) \\
&= (1-b^2)\mu([b,1)) \left(\log \frac{1}{1-b} \right) \frac{b^2}{(1-b^2)^2}.
\end{aligned}$$

Then it follows that

$$\mu([b,1)) = \mathrm{O}\left(\frac{1-b}{\log \frac{1}{1-b}} \right), \quad \text{as } b \to 1.$$

Hence, μ is a 1-logarithmic 1-Carleson measure. □

Before embarking on the proof of Theorem 2 we have to introduce some notation and results. Following [24], for $\alpha \in \mathbb{R}$ the weighted Bergman space $A^1(\log^\alpha)$ consists of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{A^1(\log^\alpha)} \stackrel{\text{def}}{=} \int_{\mathbb{D}} |f(z)| \left(\log \frac{2}{1-|z|} \right)^{\alpha} dA(z) < \infty.$$

This is a Banach space with the norm $\|\cdot\|_{A^1(\log^\alpha)}$ just defined and the polynomials are dense in $A^1(\log^\alpha)$. Likewise, we define

$$\mathcal{D}^1(\log^\alpha) = \{f \in \mathcal{H}ol(\mathbb{D}) : f' \in A^1(\log^\alpha)\}.$$

We define also the Bloch-type space $\mathcal{B}(\log^\alpha)$ as the space of those $f \in \mathcal{H}ol(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}(\log^\alpha)} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)\left(\log\frac{2}{1-|z|}\right)^{-\alpha}|f'(z)| < \infty,$$

and

$$\mathcal{B}_0(\log^\alpha) = \left\{ f \in \mathcal{H}ol(\mathbb{D}) : |f'(z)| = \mathrm{o}\left(\frac{\left(\log\frac{2}{1-|z|}\right)^\alpha}{1-|z|}\right), \quad \text{as } |z|\to 1 \right\}.$$

The space $\mathcal{B}(\log^\alpha)$ is a Banach space and $\mathcal{B}_0(\log^\alpha)$ is the closure of the polynomials in $\mathcal{B}(\log^\alpha)$.

We remark that the spaces $\mathcal{D}^1(\log^\alpha)$, $\mathcal{B}(\log^\alpha)$, and $\mathcal{B}_0(\log^\alpha)$ were called $\mathfrak{B}^1_{\log^\alpha}$, $\mathfrak{B}_{\log^\alpha}$, and $\mathfrak{b}_{\log^\alpha}$ in [24]. Pavlović identified in [24, Theorem 2.4] the dual of the space $\mathcal{B}_0(\log^\alpha)$.

Theorem E *Let $\alpha \in \mathbb{R}$. Then the dual of $\mathcal{B}_0(\log^\alpha)$ is $A^1(\log^\alpha)$ via the pairing*

$$\langle h, g\rangle = \int_{\mathbb{D}} f(z)\,\overline{g(z)}\,dA(z), \quad h \in \mathcal{B}_0(\log^\alpha), \quad g \in A^1(\log^\alpha).$$

Actually, Pavlović formulated the duality theorem in another way but it is a simple exercise to show that his formulation is equivalent to this one which is better suited to our work.

Proof of Theorem 2 Let μ be a positive Borel measure on $[0, 1)$ and $0 < \alpha < 1$. Suppose that μ is an α-logarithmic 1-Carleson measure. Take $f \in H^1$. We have to show that $\mathcal{I}_\mu f \in \mathcal{D}^1(\log^{\alpha-1})$ or, equivalently, that $\left(\mathcal{I}_\mu f\right)' \in A^1(\log^{\alpha-1})$. Bearing in mind Theorem E and the fact that $\mathcal{B}_0(\log^{\alpha-1})$ is the closure of the polynomials in $\mathcal{B}(\log^{\alpha-1})$, it suffices to show that

$$\left|\int_{\mathbb{D}} h(z)\,\overline{\left(\mathcal{I}_\mu f\right)'(z)}\,dA(z)\right| \lesssim \|h\|_{\mathcal{B}(\log^{\alpha-1})}\|f\|_{H^1}, \quad \text{for any polynomial } h. \tag{9}$$

So, let h be a polynomial. Arguing as in the proof of the implication (i) $\Rightarrow$ (iii) in Theorem 1 we obtain

$$\int_{\mathbb{D}} h(z)\,\overline{\left(\mathcal{I}_\mu f\right)'(z)}\,dA(z) = \int_{[0,1)} t\,\overline{f(t)}\,h(t)\,d\mu(t). \tag{10}$$

Now, it is clear that

$$|h(z)| \lesssim \|h\|_{\mathcal{B}(\log^{\alpha-1})} \left(\log \frac{2}{1-|z|}\right)^{\alpha},$$

and then it follows that

$$\int_{[0,1)} \left|t\,\overline{f(t)}\,h(t)\right|\,d\mu(t) \lesssim \|h\|_{\mathcal{B}(\log^{\alpha-1})} \int_{[0,1)} |f(t)| \left(\log \frac{2}{1-t}\right)^{\alpha} d\mu(t).$$

Using the fact that the measure $\left(\log \frac{2}{1-t}\right)^{\alpha} d\mu(t)$ is a Carleson measure [15, Proposition 2. 5], this implies that

$$\int_{[0,1)} \left|t\,\overline{f(t)}\,h(t)\right|\,d\mu(t) \lesssim \|h\|_{\mathcal{B}(\log^{\alpha-1})} \|f\|_{H^1}.$$

This and (10) give (9). □

3 The operators $\mathcal{H}_\mu$ acting on Bergman spaces and on Dirichlet spaces

Jevtić and Karapetrović [20] have recently proved the following result.

Theorem F *The Hilbert operator* $\mathcal{H}$ *is a bounded operator from* $\mathcal{D}^p_\alpha$ *into itself if and only if* $\max(-1, p-2) < \alpha < 2p-2$.

Now, it is well known that $A^p_\alpha = \mathcal{D}^p_{\alpha+p}$ (see [29, Theorem 4. 28]). Hence, regarding Bergman spaces Theorem F says the following.

Corollary G *The Hilbert operator* $\mathcal{H}$ *is a bounded operator from* A^p_α *into itself if and only if* $-1 < \alpha < p-2$.

Let us recall that Diamantopoulos [8] had proved before that the Hilbert operator is bounded on A^p for $p > 2$, but not on A^2. The situation on A^2 is even worse. Dostanić, Jevtić, and Vukotić [10] proved that the Hilbert operator is not well defined on A^2. Indeed, they considered the function f defined by

$$f(z) = \sum_{n=1}^{\infty} \frac{1}{\log(n+1)} z^n, \quad z \in \mathbb{D}, \tag{11}$$

which belongs to A^2. However, the series defining $\mathcal{H}f(0)$ is $\sum_{n=1}^{\infty} \frac{1}{(n+1)\log(n+1)} = \infty$ and the integral defining $\mathcal{I}f(0)$ is $\int_0^1 f(t)\,dt = \infty$. Hence neither $\mathcal{H}$ nor $\mathcal{I}$ are defined on A^2.

This result can be extended. We can assert that $\mathcal{H}$ is not well defined on A^p_{p-2} for any $p > 1$. Indeed, let f be the function defined in (11). Notice that the sequence

$\{\frac{1}{(n+1)\log(n+1)}\}$ is decreasing and that $\sum_{n=1}^{\infty}\frac{1}{n(\log(n+1))^p} < \infty$. Then (see Proposition 1 below) it follows that $f \in A^p_{p-2}$, and we have already seen that $\mathcal{H}f$ and $\mathcal{I}f$ are not defined. Since $\alpha \geq p-2 \Rightarrow A^p_{p-2} \subset A^p_\alpha$, it follows that the Hilbert operator $\mathcal{H}$ is not defined on A^p_α if $\alpha \geq p-2$.

In this section we shall obtain extensions of the mentioned results of Jevtić and Karapetrović considering the generalized Hilbert operators $\mathcal{H}_\mu$.

Theorem 3 *Suppose that* $\max(-1, p-2) < \alpha < 2p-2$ *and let* μ *be a finite positive Borel measure on* $[0, 1)$. *If* μ *is a Carleson measure then the operators* $\mathcal{H}_\mu$ *and* $\mathcal{I}_\mu$ *are well defined on* $\mathcal{D}^p_\alpha$. *Furthermore,* $\mathcal{I}_\mu f = \mathcal{H}_\mu f$, *for all* $f \in \mathcal{D}^p_\alpha$.

When dealing with Bergman spaces Theorem 3 reduces to the following.

Corollary 1 *Suppose that* $p > 1$ *and* $-1 < \alpha < p-2$, *and let* μ *be a finite positive Borel measure on* $[0, 1)$. *If* μ *is a Carleson measure then the operators* $\mathcal{H}_\mu$ *and* $\mathcal{I}_\mu$ *are well defined on* A^p_α. *Furthermore,* $\mathcal{I}_\mu f = \mathcal{H}_\mu f$, *for all* $f \in A^p_\alpha$.

Proof of Theorem 3 Suppose that μ is a Carleson measure and take $f \in \mathcal{D}^p_\alpha$. Set $\beta = \frac{2+\alpha}{p} - 1$. Observe that $0 < \beta < 1$. Using [29, Theorem 4.14], we see that $|f'(z)| \lesssim \frac{1}{(1-|z|)^{(2+\alpha)/p}}$ and, hence, $|f(z)| \lesssim \frac{1}{(1-|z|)^\beta}$. Then it follows that

$$\int_{[0,1)} |f(t)|\, d\mu(t) \lesssim \int_{[0,1)} \frac{d\mu(t)}{(1-t)^\beta}.$$

Integrating by parts, using that μ is a Carleson measure, and that $0 < \beta < 1$, we obtain

$$\begin{aligned}
\int_{[0,1)} \frac{d\mu(t)}{(1-t)^\beta} &= \mu([0,1)) - \lim_{t\to 1} \frac{\mu([t,1))}{(1-t)^\beta} + \beta \int_0^1 \frac{\mu([t,1))}{(1-t)^{\beta+1}}\, dt \\
&= \mu([0,1)) + \beta \int_0^1 \frac{\mu([t,1))}{(1-t)^{\beta+1}}\, dt \\
&\lesssim \mu([0,1)) + \int_0^1 \frac{1}{(1-t)^\beta}\, dt \\
&< \infty.
\end{aligned}$$

Consequently, we obtain that

$$\int_{[0,1)} |f(t)|\, d\mu(t) < \infty. \tag{12}$$

Clearly, this implies that the integral

$$\int_{[0,1)} \frac{f(t)\, d\mu(t)}{1-tz} \quad \text{converges absolutely and uniformly on compact subsets of } \mathbb{D}. \tag{13}$$

This gives that $\mathcal{I}_\mu f$ is a well defined analytic function in $\mathbb{D}$ and that

$$\mathcal{I}_\mu f(z) = \sum_{n=0}^{\infty} \left(\int_{[0,1)} t^n \, f(t) \, d\mu(t) \right) z^n, \quad z \in \mathbb{D}. \tag{14}$$

Using [19, Theorem 2. 1] (see also [20, Theorem 2. 1]) we see that for these values of p and α we have that if $f \in A^p_\alpha$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then $\sum_{k=0}^{\infty} \frac{|a_k|}{k+1} < \infty$. Now, since μ is a Carleson measure we have that $|\mu_n| \lesssim \frac{1}{n+1}$ ([5, Proposition 1]). Then it follows that

$$\sum_{k=0}^{\infty} |\mu_{n+k} a_k| \lesssim \sum_{k=0}^{\infty} \frac{|a_k|}{k+n+1} \lesssim \sum_{k=0}^{\infty} \frac{|a_k|}{k+1}, \quad \text{for all } n.$$

Clearly, this implies that $\mathcal{H}_\mu f$ is a well defined analytic function in $\mathbb{D}$ and that $\int_{[0,1)} t^n \, f(t) \, d\mu(t) = \sum_{k=0}^{\infty} \mu_{n+k} a_k$ for all n. This and (13) give that $\mathcal{I}_\mu f = \mathcal{H}_\mu f$. □

Our next result is an extension of Corollary G.

Theorem 4 *Suppose that* $-1 < \alpha < p - 2$ *and let* μ *be a finite positive Borel measure on* $[0, 1)$.

The operator $\mathcal{H}_\mu$ *is well defined on* A^p_α *and it is a bounded operator from* A^p_α *to itself if and only if* μ *is a Carleson measure.*

A number of results will be needed to prove this theorem. We start with a characterization of the functions $f \in \mathcal{H}ol(\mathbb{D})$ whose sequence of Taylor coefficients is decreasing which belong to A^p_α.

Proposition 1 *Let* $f \in \mathcal{H}ol(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$. *Suppose that* $1 < p < \infty$, $\alpha > -1$, *and that the sequence* $\{a_n\}_{n=0}^{\infty}$ *is a decreasing sequence of non-negative real numbers. Then*

$$f \in A^p_\alpha \iff \sum_{n=1}^{\infty} n^{p-3-\alpha} a_n^p < \infty.$$

Furthermore, $\|f\|^p_{A^p_\alpha} \asymp |a_0|^p + \sum_{n=1}^{\infty} n^{p-3-\alpha} a_n^p < \infty$.

This result can be proved with arguments similar to those used in the proofs of [15, Theorem 3.10] and [23, Theorem 3. 1] where the analogous results for the Besov spaces $B^p = \mathcal{D}^p_{p-2}$ $(p > 1)$ and for the spaces $\mathcal{D}^p_{p-1}$ $(p > 1)$ were proved. The case $\alpha = 0$ is proved in [3, Proposition 2. 4]. Consequently, we omit the details.

The following lemma is a generalization of [13, Lemma 3 (ii)].

Lemma 1 *Let* μ *be a positive Borel measure on* $[0, 1)$ *which is a Carleson measure. Assume that* $0 < p < \infty$ *and* $\alpha > -1$. *Then there exists a positive constant*

$C = C(p,\alpha,\mu)$ *such that for any* $f \in A^p_\alpha$

$$\int_{[0,1)} M^p_\infty(r,f)(1-r)^{\alpha+1}\,d\mu(r) \le C\|f\|^p_{A^p_\alpha}.$$

Of course, $M_\infty(r,f) = \sup_{|z|=r}|f(z)|$.

Proof Take $f \in A^p_\alpha$ and set

$$g(r) = M^p_\infty(r,f)(1-r)^{\alpha+1},$$
$$F(r) = \mu([0,r)) - \mu([0,1)) = -\mu([r,1)),\ 0 < r < 1.$$

Integrating by parts, we have

$$\begin{aligned}\int_{[0,1)} M^p_\infty(r,f)(1-r)^{\alpha+1}\,d\mu(r) &= \int_{[0,1)} g(r)\,d\mu(r)\\ &= \lim_{r\to 1} g(r)F(r) - g(0)F(0) - \int_0^1 g'(r)F(r)\,dr\\ &= |f(0)|^p\mu([0,1)) - \lim_{r\to 1} M^p_\infty(r,f)(1-r)^{\alpha+1}\mu([r,1))\\ &\quad + \int_0^1 g'(r)\mu([r,1))\,dr. \end{aligned} \tag{15}$$

Since $f \in A^p_\alpha$ we have that $M^p_\infty(r,f) = \mathrm{o}\left((1-r)^{-2-\alpha}\right)$, as $r \to 1$ (see, e.g., [18, p. 54]). This and the fact that μ is a Carleson measure imply that

$$\lim_{r\to 1} M^p_\infty(r,f)(1-r)^{\alpha+1}\mu([r,1)) = 0. \tag{16}$$

Using again that μ is a Carleson measure and integrating by parts we see that

$$\begin{aligned}\int_0^1 g'(r)\mu([r,1))\,dr &\lesssim \int_0^1 g'(r)(1-r)\,dr\\ &= \lim_{r\to 1} g(r)(1-r) - g(0) + \int_0^1 g(r)\,dr\\ &\le \lim_{r\to 1} M^p_\infty(r,f)(1-r)^{\alpha+2} + \int_0^1 M^p_\infty(r,f)(1-r)^{\alpha+1}\,dr\\ &= \int_0^1 M^p_\infty(r,f)(1-r)^{\alpha+1}\,dr.\end{aligned}$$

Then, using [13, Lemma 3. (ii)], it follows that

$$\int_0^1 g'(r)\mu([r,1))\,dr \lesssim \|f\|^p_{A^p_\alpha}.$$

Using this and (16) in (15) readily yields $\int_{[0,1)} M_\infty^p(r,f)(1-r)^{\alpha+1}\,d\mu(r) \lesssim \|f\|_{A^p_\alpha}^p$. □

We shall also need the following characterization of the dual of the spaces A^q_β $(q>1)$. It is a special case of [21, Theorem 2. 1].

Lemma 2 *If* $1<q<\infty$ *and* $\beta>-1$, *then the dual of* A^q_β *can be identified with* A^p_α *where* $\frac{1}{p}+\frac{1}{q}=1$ *and* α *is any number with* $\alpha>-1$, *under the pairing*

$$\langle h,f\rangle_{A_{q,\beta,\alpha}} = \int_{\mathbb{D}} h(z)\overline{f(z)}(1-|z|^2)^{\frac{\beta}{q}+\frac{\alpha}{p}}\,dA(z), \quad h\in A^q_\beta, \quad f\in A^p_\alpha. \tag{17}$$

Finally, we recall the following result from [13, (5. 2), p. 242] which is a version of the classical Hardy's inequality [17, pp. 244–245].

Lemma 3 *Suppose that* $k>0$, $q>1$, *and* h *is a non-negative function defined in* $(0,1)$, *then*

$$\int_0^1\left(\int_{1-r}^1 h(t)\,dt\right)^q(1-r)^{k-1}\,dr \le \left(\frac{q}{k}\right)^q\int_0^1 (h(1-r))^q(1-r)^{q+k-1}\,dr.$$

Proof of Theorem 4 Suppose first that $\mathcal{H}_\mu$ is a bounded operator from A^p_α into itself. For $0<b<1$, set

$$f_b(z) = \frac{(1-b^2)^{1-\frac{\alpha}{p}}}{(1-bz)^{\frac{2}{p}+1}}, \quad z\in\mathbb{D}.$$

Recall that $p-\alpha>2$. Then using [29, Lemma 3. 10] with $t=\alpha$ and $c=p-\alpha$, we obtain

$$\|f_b\|_{A^p_\alpha}^p = (1-b^2)^{p-\alpha}\int_{\mathbb{D}}\frac{(1-|z|^2)^\alpha}{|1-bz|^{2+p}}\,dA(z) \asymp 1.$$

Since $\mathcal{H}_\mu$ is bounded on A^p_α, this implies

$$1 \gtrsim \|\mathcal{H}\mu(f_b)\|_{A^p_\alpha}. \tag{18}$$

We also have

$$f_b(z) = \sum_{k=0}^\infty a_{k,b}z^k, \quad (z\in\mathbb{D}), \quad \text{with } a_{k,b} \asymp (1-b^2)^{1-\frac{\alpha}{p}}k^{\frac{2}{p}}b^k.$$

Since the $a_{k,b}$'s are positive, it is clear that the sequence $\left\{\sum_{k=0}^\infty \mu_{n+k}a_{k,b}\right\}_{n=0}^\infty$ of the Taylor coefficients of $\mathcal{H}_\mu(f_b)$ is a decreasing sequence of non-negative real numbers.

Using this, Proposition 1, (18), and the definition of the $a_{k,b}$'s, we obtain

$$
\begin{aligned}
1 &\gtrsim \|\mathcal{H}_\mu(f_b)\|^p_{A^p_\alpha} \gtrsim \sum_{n=1}^{\infty} n^{p-\alpha-3}\left(\sum_{k=1}^{\infty}\mu_{n+k}a_{k,b}\right)^p \\
&= \sum_{n=1}^{\infty} n^{p-\alpha-3}\left(\sum_{k=1}^{\infty} a_{k,b}\int_{[0,1)} t^{n+k}\,d\mu(t)\right)^p \\
&\gtrsim (1-b^2)^{p-\alpha}\sum_{n=1}^{\infty} n^{p-\alpha-3}\left(\sum_{k=1}^{\infty} k^{\frac{2}{p}} b^k \int_{[b,1)} t^{n+k}\,d\mu(t)\right)^p \\
&\geq (1-b^2)^{p-\alpha}\sum_{n=1}^{\infty} n^{p-\alpha-3}\left(\sum_{k=1}^{\infty} k^{\frac{2}{p}} b^{n+2k}\mu([b,1))\right)^p \\
&= (1-b^2)^{p-\alpha}\mu([b,1))^p\sum_{n=1}^{\infty} n^{p-\alpha-3}b^{np}\left(\sum_{k=1}^{\infty} k^{\frac{2}{p}} b^{2k}\right)^p \\
&\asymp (1-b^2)^{p-\alpha}\mu([b,1))^p\frac{1}{(1-b^2)^{2+p}}\sum_{n=1}^{\infty} n^{p-\alpha-3}b^{np} \\
&\asymp (1-b^2)^{p-\alpha}\mu([b,1))^p\frac{1}{(1-b^2)^{2+p}}\cdot\frac{1}{(1-b^2)^{p-\alpha-2}} \\
&\asymp \mu([b,1))^p\frac{1}{(1-b)^p}.
\end{aligned}
$$

Then it follows that

$$
\mu\left([b,1)\right) = \mathrm{O}\left(1-b\right), \quad \text{as } b\to 1,
$$

and, hence, μ is a Carleson measure.

We turn to prove the other implication. So, suppose that μ is a Carleson measure and take $f\in A^p_\alpha$. Let q be defined by the relation $\frac{1}{p}+\frac{1}{q}=1$ and take $\beta=\frac{-\alpha q}{p}=\frac{-\alpha}{p-1}$. Observe that $\beta>-1$ and that with this election of β the weight in the pairing (17) is identically equal to 1. We have to show that $\mathcal{H}_\mu f\in A^p_\alpha$ which is equal to $\left(A^q_\beta\right)^*$ under the pairing $\langle\cdot,\cdot\rangle_{q,\beta,\alpha}$. So take $h\in A^q_\beta$.

$$
\begin{aligned}
\langle h,\mathcal{H}_\mu f\rangle_{q,\beta,\alpha} &= \int_{\mathbb{D}} h(z)\,\overline{\mathcal{H}_\mu f(z)}\,dA(z) \\
&= \int_{[0,1)}\overline{f(t)}\left(\int_{\mathbb{D}}\frac{h(z)}{1-t\,\overline{z}}\,dA(z)\right)d\mu(t) \\
&= \int_{[0,1)}\overline{f(t)}\left(\int_0^1\frac{r}{\pi}\int_0^{2\pi}\frac{h(re^{i\theta})}{1-tre^{-i\theta}}\,d\theta\,dr\right)d\mu(t) \\
&= \int_{[0,1)}\overline{f(t)}\left(\int_0^1\left(\frac{r}{\pi i}\int_{|\xi|=1}\frac{h(r\xi)}{\xi-tr}\,d\xi\right)dr\right)d\mu(t) \\
&= 2\int_{[0,1)}\overline{f(t)}\left(\int_0^1 rh(r^2t)\,dr\right)d\mu(t).
\end{aligned}
$$

Thus,

$$\left|\langle h, \mathcal{H}_\mu f\rangle_{q,\beta,\alpha}\right| \le 2\int_0^1 |f(t)|G(t)\,d\mu(t),$$

where $G(t) = \int_0^1 r|h(r^2t)|\,dr$. Using Hölder's inequality we obtain,

$$\begin{aligned}\int_{[0,1)} f(t)G(t)\,d\mu(t) &= \int_{[0,1)} |f(t)|(1-t)^{\frac{\alpha+1}{p}} G(t)(1-t)^{-\frac{\alpha+1}{p}}\,d\mu(t)\\ &\le \left(\int_{[0,1)} |f(t)|^p(1-t)^{\alpha+1}\,d\mu(t)\right)^{1/p} \cdot \left(\int_{[0,1)} G(t)^q(1-t)^{-\frac{q(\alpha+1)}{p}}\,d\mu(t)\right)^{1/q}.\end{aligned}$$

Lemma 1 implies that

$$\left(\int_{[0,1)} |f(t)|^p(1-t)^{\alpha+1}\,d\mu(t)\right)^{1/p} \lesssim \|f\|_{A^p_\alpha}.$$

Next we will show that

$$\int_{[0,1)} G(t)^q(1-t)^{-\frac{q(\alpha+1)}{p}}\,d\mu(t) \lesssim \|h\|^q_{A^q_\beta}. \tag{19}$$

This will give that

$$\left|\langle h, \mathcal{H}_\mu f\rangle_{q,\beta,\alpha}\right| \lesssim \|f\|_{A^p_\alpha} \cdot \|h\|^q_{A^q_\beta}.$$

By the duality theorem, this implies that $\mathcal{H}_\mu f \in A^p_\alpha$.

Let us prove (19). Observe first that if $0 < t < 1/2$ then $|h(r^2t)| \le M_\infty(\frac{1}{2}, h)$ for each $r \in (0,1)$, thus

$$G(t) = \int_0^1 |h(r^2t)|r\,dr \le M_\infty\left(\frac{1}{2}, h\right), \quad 0 < t < 1/2.$$

Clearly, this implies

$$\int_{[0,1/2)} G(t)^q(1-t)^{-\frac{q(\alpha+1)}{p}}\,d\mu(t) \lesssim M_\infty^q\left(\frac{1}{2}, h\right) \lesssim \|h\|^q_{A^q_\beta}. \tag{20}$$

Notice that $-\frac{q(\alpha+1)}{p} = \frac{p-2-\alpha}{p-1} - 1 > -1$. Making the change of variables $r^2t = s$, we obtain $\int_0^1 r|h(r^2t)|\,dr = \frac{1}{2t}\int_0^t |h(s)|\,ds$ and, hence,

$$\int_{[1/2,1)} G(t)^q(1-t)^{-\frac{q(\alpha+1)}{p}}\,d\mu(t)$$

$$
\begin{aligned}
&= \int_{[1/2,1)} \left(\int_0^1 |h(r^2 t)| r\, dr \right)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\, d\mu(t) \\
&= \int_{[1/2,1)} \frac{1}{(2t)^q} \left(\int_0^t |h(s)|\, ds \right)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\, d\mu(t) \\
&\leq \int_{[1/2,1)} \left(\int_0^t M_\infty(s,h)\, ds \right)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\, d\mu(t) \\
&\leq \int_{[0,1)} \left(\int_{1-t}^1 M_\infty(1-s,h)\, ds \right)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\, d\mu(t) \qquad (21)
\end{aligned}
$$

Let us call $H(t) = \left(\int_{1-t}^1 M_\infty(1-s,h)\, ds \right)^q (1-t)^{-\frac{q(\alpha+1)}{p}}$ for $0 \leq t < 1$. Integrating by parts we obtain the following

$$
\int_{[0,1)} H(t)\, d\mu(t) = H(0)\mu([0,1)) - \lim_{t\to 1^-} H(t)\mu([t,1)) + \int_0^1 \mu([t,1)) H'(t)\, dt. \qquad (22)
$$

The first term is equal to 0. Using the fact that μ is a Carleson measure we have that

$$
\begin{aligned}
H(t)\mu([t,1)) &\lesssim (1-t)H(t) \\
&= \left(\int_{1-t}^1 M_\infty(1-s,h)\, ds \right)^q (1-t)^{1-\frac{q(\alpha+1)}{p}} \\
&= \left(\int_0^t M_\infty(s,h)\, ds \right)^q (1-t)^{1-\frac{q(\alpha+1)}{p}}.
\end{aligned}
$$

Since $h \in A^q_\beta$ we have $M_\infty(t,h) = o\left((1-t)^{-\frac{\beta+2}{q}}\right)$, as $t \to 1$. Then, bearing in mind that $\frac{\beta+2}{q} > 1$, it follows that

$$
H(t)\mu([t,1)) = \mathrm{o}\left((1-t)^{-\beta-2+q} \cdot (1-t)^{1-\frac{q(\alpha+1)}{p}} \right) = \mathrm{o}(1), \quad \text{as } t \to 1. \qquad (23)
$$

Actually, we have also proved that

$$
(1-t)H(t) = \mathrm{o}(1), \quad \text{as } t \to 1. \qquad (24)
$$

Using that μ is a Carleson measure, integrating by parts, and using the definition of H and (24), we obtain

$$
\begin{aligned}
\int_0^1 \mu([t,1)) H'(t)\, dt &\lesssim \int_0^1 (1-t) H'(t)\, dt \\
&= \lim_{t\to 1} (1-t)H(t) - H(0) + \int_0^1 H(t)\, dt
\end{aligned}
$$

$$= \int_0^1 \left(\int_{1-t}^1 M_\infty(1-s, h)\, ds \right)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\, dt. \quad (25)$$

Now, using Lemma 3 and [13, Lemma 3], we see that

$$\int_0^1 \left(\int_{1-t}^1 M_\infty(1-s, h)\, ds \right)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\, dt$$
$$\lesssim \int_0^1 M_\infty^q(t, h)(1-t)^{\alpha+1}\, dt \lesssim \|h\|^q_{A^q_\beta}.$$

Using this, (25), (23), (22), and (21), it follows that

$$\int_{[1/2,1)} G(t)^q (1-t)^{-\frac{q(\alpha+1)}{p}}\, d\mu(t) \lesssim \|h\|^q_{A^q_\beta}.$$

This and (20) yield (19). □

Our final aim in this article is to find the analogue of Theorem 4 for Dirichlet spaces. In other words, we wish give an answer to the following question.

Question 2 *If* $\max(-1, p-2) < \alpha < 2p-2$, *is it true that* $\mathcal{H}_\mu$ *is a bounded operator from* $\mathcal{D}^p_\alpha$ *into itself if and only if* μ *is a Carleson measure?*

Since $p-1 < \alpha < 2p-2$ implies that $\mathcal{D}^p_\alpha = A^p_{\alpha-p}$, Theorem 4 answers the question affirmatively for these values of p and α. It remains to consider the case $\max(-1, p-2) < \alpha \le p-1$. We shall prove the following result which gives a positive answer to Question 2 in the case $p > 1$.

Theorem 5 *Suppose that* $p > 1$ *and* $p-2 < \alpha \le p-1$, *and let* μ *be a finite positive Borel measure on* $[0, 1)$.

The operator $\mathcal{H}_\mu$ *is well defined on* $\mathcal{D}^p_\alpha$ *and it is a bounded operator from* $\mathcal{D}^p_\alpha$ *into itself if and only if* μ *is a Carleson measure.*

The following two lemmas will be needed in the proof of Theorem 5. The first one follows trivially from Proposition 1.

Lemma 4 *Let* $f \in \mathcal{H}ol(\mathbb{D})$, $f(z) = \sum_{n=0}^\infty a_n z^n$ $(z \in \mathbb{D})$. *Suppose that* $1 < p < \infty$ *and* $p-2 < \alpha \le p-1$, *and that the sequence* $\{a_n\}_{n=0}^\infty$ *is a decreasing sequence of non-negative real numbers. Then*

$$f \in \mathcal{D}^p_\alpha \;\Leftrightarrow\; \sum_{n=0}^\infty (n+1)^{2p-\alpha-3} a_n^p < \infty.$$

The following lemma is a generalization of [13, Lemma 4].

Lemma 5 *Let* μ *be a positive Borel measure on* $[0,1)$ *which is a Carleson measure. Assume that* $0<p<\infty$ *and* $\alpha>-1$. *Then there exists a positive constant* $C=C(p,\alpha,\mu)$ *such that for any* $f\in\mathcal{D}^p_\alpha$

$$\int_{[0,1)} M_\infty^p(r,f)(1-r)^{\alpha-p+1}\,d\mu(r)\;\le C\|f\|^p_{\mathcal{D}^p_\alpha}.$$

Proof We argue as in the proof of Lemma 1. Take $f\in\mathcal{D}^p_\alpha$ and set

$$\begin{aligned} g(r)&=M_\infty^p(r,f)(1-r)^{\alpha-p+1},\\ F(r)&=\mu([0,r))-\mu([0,1))=-\mu([r,1)),\;\;0<r<1. \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} &\int_{[0,1)} M_\infty^p(r,f)(1-r)^{\alpha-p+1}\,d\mu(r)\;=\int_{[0,1)} g(r)\,d\mu(r)\\ &=\lim_{r\to1} g(r)F(r)\;-\;g(0)F(0)\;-\int_0^1 g'(r)F(r)\,dr\\ &=|f(0)|^p\mu([0,1))\;-\;\lim_{r\to1} M_\infty^p(r,f)(1-r)^{\alpha-p+1}\mu([r,1))\\ &\quad+\int_0^1 g'(r)\mu([r,1))\,dr. \end{aligned}\tag{26}$$

Since $f\in\mathcal{D}^p_\alpha$ we have that $M_\infty^p(r,f')=\mathrm{o}\left((1-r)^{-2-\alpha}\right)$, as $r\to1$. Hence, $M_\infty^p(r,f)=\mathrm{o}\left((1-r)^{-2-\alpha+p}\right)$, as $r\to1$. This and the fact that μ is a Carleson measure imply that

$$\lim_{r\to1} M_\infty^p(r,f)(1-r)^{\alpha-p+1}\mu([r,1))\;=\;0.\tag{27}$$

Using again that μ is a Carleson measure and integrating by parts we see that

$$\begin{aligned} \int_0^1 g'(r)\mu([r,1))\,dr &\lesssim\int_0^1 g'(r)(1-r)\,dr\\ &=\lim_{r\to1} g(r)(1-r)\;-g(0)\;+\int_0^1 g(r)\,dr\\ &\le\lim_{r\to1} M_\infty^p(r,f)(1-r)^{\alpha-p+2}\\ &\quad+\int_0^1 M_\infty^p(r,f)(1-r)^{\alpha-p+1}\,dr\\ &=\int_0^1 M_\infty^p(r,f)(1-r)^{\alpha-p+1}\,dr. \end{aligned}$$

Then, using [13, Lemma 3], it follows that

$$\int_0^1 g'(r)\mu([r,1))\,dr \lesssim \|f\|^p_{\mathcal{D}^p_\alpha}.$$

Using this and (27) in (26) readily yields $\int_{[0,1)} M^p_\infty(r,f)(1-r)^{\alpha-p+1}\,d\mu(r) \lesssim \|f\|^p_{\mathcal{D}^p_\alpha}$. □

Proof of Theorem 5 Suppose first that $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{D}^p_\alpha$ into itself. For $1/2 < b < 1$ we set

$$f_b(z) = \frac{\left(1-b^2\right)^{1-\frac{\alpha}{p}}}{(1-bz)^{2/p}}, \quad z \in \mathbb{D}.$$

We have $\|f_b\|_{\mathcal{D}^p_\alpha} \asymp 1$. Then arguing as in the proof of the correspondent implication in Theorem 4 we obtain that μ is a Carleson measure. We omit the details.

To prove the other implication, suppose that μ is a Carleson measure and take $f \in \mathcal{D}^p_\alpha$. Since $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ coincide on $\mathcal{D}^p_\alpha$, we have to prove that $\mathcal{I}_\mu f \in \mathcal{D}^p_\alpha$ and that $\|\mathcal{I}_\mu f\|_{\mathcal{D}^p_\alpha} \lesssim \|f\|_{\mathcal{D}^p_\alpha}$ or, equivalently, that $\left(\mathcal{I}_\mu f\right)' \in A^p_\alpha$ and

$$\|\left(\mathcal{I}_\mu f\right)'\|_{A^p_\alpha} \lesssim \|f\|_{A^p_\alpha}. \tag{28}$$

We shall distinguish two cases.

First case: $\alpha < p-1$. Let q be defined by the relation $\frac{1}{p}+\frac{1}{q} = 1$ and take $\beta = \frac{-\alpha q}{p}$. In view of Lemma 2, (28) is equivalent to

$$\left|\int_{\mathbb{D}} h(z)\,\overline{\left(\mathcal{I}_\mu f\right)'(z)}\,dA(z)\right| \lesssim \|f\|_{\mathcal{D}^p_\alpha}\|h\|_{A^q_\beta}, \quad h \in A^q_\beta. \tag{29}$$

So, take $h \in A^q_\beta$. Just as in the proof of Theorem 1, we have

$$\int_{\mathbb{D}} h(z)\,\overline{\left(\mathcal{I}_\mu f\right)'(z)}\,dA(z) = \int_{[0,1)} t\,\overline{f(t)}\,h(t)\,d\mu(t). \tag{30}$$

Set $s = -1+\frac{\alpha+1}{p}$. Observe that $ps = \alpha-p+1$ and $-qs = \beta+1$. Then, using (30), Hölder's inequality, Lemma 1, and Lemma 5, we obtain

$$\begin{aligned}
\left|\int_{\mathbb{D}} h(z)\,\overline{\left(\mathcal{I}_\mu f\right)'(z)}\,dA(z)\right| &\le \int_{[0,1)} |f(t)|(1-t)^s\,|h(t)|(1-t)^{-s}\,d\mu(t)\\
&\le \left(\int_{\mathbb{D}} |f(t)|^p(1-t)^{\alpha-p+1}\,d\mu(t)\right)^{1/p}\left(\int_{[0,1)} |h(t)|^q(1-t)^{\beta+1}\,d\mu(t)\right)^{1/q}\\
&\le \left(\int_{\mathbb{D}} M^p_\infty(t,f)(1-t)^{\alpha-p+1}\,d\mu(t)\right)^{1/p}
\end{aligned}$$

$$\times\left(\int_{[0,1)} M_\infty^q(t,h)(1-t)^{\beta+1}\,d\mu(t)\right)^{1/q}$$
$$\leq \|f\|_{\mathcal{D}^p_\alpha}\|h\|_{A^q_\beta}.$$

Thus, (29) holds.

Second case: $\alpha = p-1$. We let again q be defined by the relation $\frac{1}{p}+\frac{1}{q}=1$ and take $\beta = q-1$. Using Lemma 2 and arguing as in the preceding case, we have to show that

$$\left|\int_{\mathbb{D}}(1-|z|^2)\,h(z)\,\overline{\left(\mathcal{I}_\mu f\right)'(z)}\,dA(z)\right| \lesssim \|f\|_{\mathcal{D}^p_{p-1}}\|h\|_{A^q_{q-1}},\quad h\in A^q_{q-1}. \tag{31}$$

We have

$$\int_{\mathbb{D}}(1-|z|^2)\,h(z)\,\overline{\left(\mathcal{I}_\mu f\right)'(z)}\,dA(z) = \int_{[0,1)} t\,\overline{f(t)}\int_{\mathbb{D}}\frac{(1-|z|^2)h(z)}{(1-t\,\overline{z})^2}\,dA(z)\,d\mu(t). \tag{32}$$

Now, $\int_{\mathbb{D}}\frac{h(z)}{(1-t\,\overline{z})^2}\,dA(z) = h(t)$ and

$$\int_{\mathbb{D}}\frac{|z|^2\,h(z)}{(1-t\,\overline{z})^2}\,dA(z) = \int_0^1\frac{r^3}{\pi}\int_0^{2\pi}\frac{h(re^{i\theta})\,d\theta}{(1-tre^{-i\theta})^2}\,dr$$
$$= \int_0^1\frac{2r^3}{2\pi i}\int_0^{2\pi}\frac{e^{i\theta}h(re^{i\theta})ie^{i\theta}\,d\theta}{(e^{i\theta}-tr)^2}\,dr = \int_0^1\frac{2r^3}{2\pi i}\int_{|z|=1}\frac{zh(rz)}{(z-tr)^2}\,dz\,dr$$
$$= \int_0^1 2r^3\left[h(r^2t)+r^2th'(r^2t)\right]dr.$$

Then it is clear that $\left|\int_{\mathbb{D}}\frac{(1-|z|^2)\,h(z)}{(1-t\,\overline{z})^2}\,dA(z)\right| \lesssim M_\infty(t,h)$. Using this, (32), Hölder's inequality, Lemma 1, and Lemma 5, we obtain

$$\left|\int_{\mathbb{D}}(1-|z|^2)\,h(z)\,\overline{\left(\mathcal{I}_\mu f\right)'(z)}\,dA(z)\right| \lesssim \int_{[0,1)} M_\infty(t,f)\,M_\infty(t,h)\,d\mu(t)$$
$$\leq\left(\int_{[0,1)} M_\infty^p(t,f)\,d\mu(t)\right)^{1/p}\left(\int_{[0,1)} M_\infty^q(t,h)\,d\mu(t)\right)^{1/q}$$
$$\leq \|f\|_{\mathcal{D}^p_{p-1}}\|h\|_{A^q_{q-1}}.$$

This is (31). □

We shall close the article with some comments about the case $p=1$ in Question 2. We have the following result.

Theorem 6 *Let μ be a finite positive Borel measure on $[0,1)$ and $-1<\alpha<0$. If μ is a Carleson measure then the operator $\mathcal{H}_\mu$ is a bounded operator form $\mathcal{D}^1_\alpha$ to itself.*

Proof Using [29, Theorem 5. 15, p. 113], we see that A^1_α can be identified as the dual of the little Bloch space under the pairing

$$\langle h, g\rangle = \int_{\mathbb{D}} (1-|z|^2)^\alpha\, h(z)\, \overline{g(z)}\, dA(z), \quad h \in \mathcal{B}_0, \quad g \in A^1_\alpha. \tag{33}$$

Suppose that μ is a Carleson measure. Using this duality relation and the fact that $\mathcal{H}_\mu = \mathcal{I}_\mu$ on $\mathcal{D}^1_\alpha$, showing that $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{D}^1_\alpha$ to itself is equivalent to showing that

$$\left|\int_{\mathbb{D}} (1-|z|^2)^\alpha\, h(z)\, \overline{\left(\mathcal{I}_\mu f\right)'(z)}\, dA(z)\right| \lesssim \|h\|_{\mathcal{B}} \cdot \|f\|_{\mathcal{D}^1_\alpha}, \quad h \in \mathcal{B}_0, \quad f \in \mathcal{D}^1_\alpha. \tag{34}$$

Let us prove (34). Take $h \in \mathcal{B}_0$ and $f \in \mathcal{D}^1_\alpha$. We have

$$\begin{aligned} &\int_{\mathbb{D}} (1-|z|^2)^\alpha\, h(z)\, \overline{\left(\mathcal{I}_\mu f\right)'(z)}\, dA(z) \\ &\quad = \int_{[0,1)} t\, \overline{f(t)} \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha h(z)}{(1-t\,\overline{z})^2}\, dA(z)\, d\mu(t). \end{aligned} \tag{35}$$

Using [29, Lemma 5. 14, pp. 113-114] we have that the operator T defined by

$$T\phi(\xi) = (1-|\xi|^2)^{-\alpha} \int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha \phi(z)}{(1-\xi\,\overline{z})^2}\, dA(z)$$

is a bounded operator from $\mathcal{B}$ into $L^\infty(\mathbb{D})$. Then it follows that

$$\left|\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha h(z)}{(1-t\,\overline{z})^2}\, dA(z)\right| \lesssim \|h\|_{\mathcal{B}} (1-t^2)^\alpha, \quad t \in [0,1).$$

Using this in (35), we obtain

$$\left|\int_{\mathbb{D}} (1-|z|^2)^\alpha\, h(z)\, \overline{\left(\mathcal{I}_\mu f\right)'(z)}\, dA(z)\right| \lesssim \|h\|_{\mathcal{B}} \int_{\mathbb{D}} (1-t)^\alpha |f(t)|\, d\mu(t). \tag{36}$$

The fact that μ is a Carleson measure readily implies that the measure ν defined by $d\nu(t) = (1-t)^\alpha\, d\mu(t)$ is a $(1-\alpha)$-Carleson measure. Using Theorem 1 of [28] we see that then ν is a Carleson measure for $\mathcal{D}^1_\alpha$, that is,

$$\int_{[0,1)} (1-t)^\alpha |g(t)|\, d\mu(t) \lesssim \|g\|_{\mathcal{D}^1_\alpha}, \quad g \in \mathcal{D}^1_\alpha.$$

Using this in (36), (34) follows. □

We do not know whether the converse of Theorem 6 is true. This is due to the fact that we do not know whether Lemma 4 remains true for $p = 1$. The inequality

$$\sum_{n=0}^{\infty} |a_n|(n+1)^{-(1+\alpha)} \lesssim \|f\|_{\mathcal{D}^1_\alpha}. \tag{37}$$

is certainly true with no assumption on the sequence $\{a_n\}$. Indeed, by Hardy's inequality [11, p. 48], $\sum_{n=1}^{\infty} |a_n| r^{n-1} \lesssim \int_0^{2\pi} |f'(re^{i\theta})| d\theta$. Hence

$$\begin{aligned}\|f\|_{\mathcal{D}^1_\alpha} &\asymp \int_0^1 (1-r)^\alpha \int_0^{2\pi} |f'(re^{i\theta})| d\theta dr \\ &\gtrsim \sum_{n=1}^{\infty} |a_n| \int_0^1 (1-r)^\alpha r^{n-1}\, dr = \sum_{n=1}^{\infty} |a_n| B(\alpha+1, n),\end{aligned}$$

where $B(\cdot,\cdot)$ is the Beta function. Stirling's formula gives $B(\alpha+1, n) \asymp n^{-(\alpha+1)}$ and then (37) follows.

However, the proof of Theorem D in [23] does not seen to work to prove the opposite inequality when $\{a_n\}$ is decreasing.

Acknowledgements The authors wish to express their gratitude to the referees who made several suggestions for improvement.

References

1. Anderson, J.M., Clunie, J., Pommerenke, Ch.: On Bloch functions and normal functions. J. Reine Angew. Math. **270**, 12–37 (1974)
2. Bao, G., Wulan, H.: Hankel matrices acting on Dirichlet spaces. J. Math. Anal. Appl. **409**(1), 228–235 (2014)
3. Buckley, S.M., Koskela, P., Vukotić, D.: Fractional integration, differentiation, and weighted Bergman spaces. Math. Proc. Camb. Philos. Soc. **126**(2), 369–385 (1999)
4. Carleson, L.: Interpolation by bounded analytic functions and the corona problem. Ann. Math. **76**, 547–559 (1962)
5. Chatzifountas, Ch., Girela, D., Peláez, J.A.: A generalized Hilbert matrix acting on Hardy spaces. J. Math. Anal. Appl. **413**(1), 154–168 (2014)
6. Cima, J.A.: A type of volterra operator. Complex Anal. Synerg. **2**(1), 4 (2016)
7. Cima, J.A., Matheson, A.L., Ross, W.T.: The Cauchy Transform. Mathematical Surveys and Monographs, 125. American Mathematical Society, Providence (2006)
8. Diamantopoulos, E.: Hilbert matrix on Bergman spaces. Ill. J Math. **48**(3), 1067–1078 (2004)
9. Diamantopoulos, E., Siskakis, A.G.: Composition operators and the Hilbert matrix. Stud. Math. **140**(2), 191–198 (2000)
10. Dostanić, M., Jevtić, M., Vukotić, D.: Norm of the Hilbert matrix on Bergman and Hardy spaces and a theorem of Nehari type. J. Funct. Anal. **254**(11), 2800–2815 (2008)
11. Duren, P.L.: Theory of H^p Spaces. Academic Press, New York (1970). Reprint: Dover, Mineola-New York (2000)
12. Duren, P.L., Schuster, A.P.: Bergman Spaces, Mathematical Surveys and Monographs, vol. 100. American Mathematical Society, Providence (2004)
13. Galanopoulos, P., Girela, D., Peláez, J.A., Siskakis, A.: Generalized Hilbert operators. Ann. Acad. Sci. Fenn. Math. **39**(1), 231–258 (2014)

14. Galanopoulos, P., Peláez, J.A.: A Hankel matrix acting on Hardy and Bergman spaces. Stud. Math. **200**(3), 201–220 (2010)
15. Girela, D., Merchán, N.: A generalized Hilbert operator acting on conformally invariant spaces. Banach J. Math. Anal. **12**(2), 374–398 (2018)
16. Girela, D., Merchán, N.: A Hankel matrix acting on spaces of analytic functions. Integral Equ. Oper. Theory **89**(4), 581–594 (2017)
17. Hardy, G.H., Littlewood, J.E., Polya, G.: Inequalities, Reprint of the 1952 edition. Cambridge Mathematical Library. Cambridge University Press, Cambridge (1988)
18. Hedenmalm, H., Korenblum, B., Zhu, K.: Theory of Bergman Spaces, Graduate Texts in Mathematics 199. Springer, New York (2000)
19. Jevtić, M., Karapetrović, B.: Libera operator on mixed norm spaces $H_\nu^{p,q,\alpha}$ when $0<p<1$. Filomat **31**(14), 4641–4650 (2017)
20. Jevtić, M., Karapetrović, B.: Hilbert matrix on spaces of Bergman-type. J. Math. Anal. Appl. **453**(1), 241–254 (2017)
21. Luecking, D.H.: Representation and duality in weighted spaces of analytic functions. Indiana Univ. Math. J. **34**(2), 319–336 (1985)
22. Merchán, N.: Mean Lipschitz spaces and a generalized Hilbert operator. To appear in Collect. Math. (2018). https://doi.org/10.1007/s13348-018-0217-y
23. Pavlović, M.: Analytic functions with decreasing coefficients and Hardy and Bloch spaces. Proc. Math. Soc. Ser. 2 **56**(2), 623–635 (2013)
24. Pavlović, M.: Logarithmic Bloch space and its predual. Publ. Inst. Math. (Beograd) (N.S.) **100**(114), 1–16 (2016)
25. Power, S.C.: Vanishing Carleson measures. Bull. Lond. Math. Soc. **12**, 207–210 (1980)
26. Vinogradov, S.A.: Multiplication and division in the space of analytic functions with area integrable derivative, and in some related spaces (in russian), Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **222** (1995), Issled. po Linein. Oper. i Teor. Funktsii **23**, 45–77, 308; translation in J. Math. Sci. (New York) **87**(5), 3806–3827 (1997)
27. Widom, H.: Hankel matrices. Trans. Am. Math. Soc. **121**, 1–35 (1966)
28. Wu, Z.: Carleson measures and multipliers for Dirichlet spaces. J. Funct. Anal. **169**(1), 148–163 (1999)
29. Zhu, K.: Operator Theory in Function Spaces. Mathematical Surveys and Monographs, vol. 138, 2nd edn. American Mathematical Society, Providence (2007)