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Algebras of symmetric holomorphic functions of several complex variables

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Abstract Given a proper holomorphic mapping $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^n$ and an algebra of holomorphic functions $\mathcal{B}$ (e.g. $\mathscr{P}(K)$ where $K \subset \Omega$ is a compact set, $\mathcal{H}(U)$, $A(U)$ or $\mathcal{H}^\infty(U)$ where U is an open and bounded set with $\overline{U} \subset \Omega$), we study the subalgebra $\mathcal{B}_g$ of all functions compatible with the equivalence relation defined by the proper mapping g. We provide alternative representations of these algebras and describe the fibers in their spectra. Among other examples we relate the algebras of functions that are invariant under permutations and the algebras of functions defined on the symmetrized polydisk.

Keywords Symmetric holomorphic functions · Fibers · Algebras of holomorphic functions · Several complex variables · Symmetrized polydisk

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1 Introduction

Let Ω and Ω' be regions in $\mathbb{C}^n$, $n \in \mathbb{N}$. A mapping $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^n$ naturally defines an equivalence relation on any subset U of Ω by $z \sim w$ if and only if $g(z) = g(w)$, $z, w \in U$. If $\mathcal{B}(U)$ is an algebra of functions defined on a subset U of Ω, then we can consider the subalgebra of functions that are compatible with the equivalence relation defined by g. We denote this subalgebra by

$$\mathcal{B}_g(U) = \{f \in \mathcal{B}(U) : \text{ if } z, w \in U \text{ with } g(z) = g(w) \text{ then } f(z) = f(w)\}.$$

It is a common situation in mathematics that one needs to identify the abstract set $U/\sim$ in terms of a more concrete set V, ideally by exhibiting a bijection between them. In our particular case $V = g(U)$.

The idea of studying the set of holomorphic functions compatible with the equivalence relation defined by a proper holomorphic mapping has been explored in a number of recent papers, including [13,22]. We continue this study here. Without going into details right now, our interest is in the study of algebras of holomorphic functions f (in one or several complex variables) having the property that for some fixed proper holomorphic mapping g, whenever $g(z) = g(w)$ then necessarily $f(z) = f(w)$.

In Sect. 2 we study continuity properties of the inverse mapping of the proper function g. These results will allow us to provide alternative representations of these algebras $\mathcal{B}_g(U)$ and study their spectra. We will study the algebras $\mathcal{B}_g(U)$ by establishing an isomorphism between $\mathcal{B}_g(U)$ and the algebras $\mathcal{B}(g(U))$. We concentrate on algebras of polynomials in Sect. 3 and applications to the algebra $A_g(K)$ are given. Section 4 is devoted to the study of the algebras of holomorphic functions $\mathcal{H}_g(U)$ and $\mathcal{H}_g^\infty(U)$. In the final section we focus on algebras of symmetric functions defined on the polydisk. It turns out that these algebras coincide with the algebras of holomorphic functions defined on the well studied symmetrized polydisk.

It is worth remarking that the approach that we use here can be used in two directions. On the one hand, we can transform functions satisfying a compatibility condition to functions defined on a usually more complicated domain. In this way, we are transforming a condition of the functions to a condition on the domain. On the other hand, some holomorphic functions defined on abstract domains can be related to functions defined on usually simpler domains under the condition that the functions satisfy some kind of compatibility condition. In this case, we ameliorate the conditions on the domain by adding conditions to the functions.

2 Proper holomorphic mappings

Given two regions $\Omega \subset \mathbb{C}^n$ and $\Omega' \subset \mathbb{C}^k$, $n, k \in \mathbb{N}$, a continuous mapping $g : \Omega \to \Omega'$ is said to be *proper* if for every compact set $X \subset \Omega'$, $g^{-1}(X)$ is compact in Ω. Most of the properties of proper mappings have been studied for the particular case of g holomorphic and $k = n$. See [18, Chapter 15] for an introduction to the theory of proper holomorphic mappings. Before we continue we present some properties of proper holomorphic mappings that we will use in the rest of the paper.

Naturally, n is less than or equal to k. Recall that for $k = n$ any proper holomorphic mapping is open, closed and onto. In this case, if we denote by J_g the Jacobian of g, an element $z \in \Omega$ such that $J_g(z) \neq 0$ is called a *regular point* of g and every $z \in \Omega$ with $J_g(z) = 0$ is called a *critical point* of g. The set of critical points of g is denoted by $M = J_g^{-1}(0)$ and every point in $g(M)$ is called a *critical value* of g. Every point in $g(\Omega) \setminus g(M)$ is called a *regular value* of g. By [18, Theorem 15.1.9], there is an integer m, called the *multiplicity* of g, such that for every regular value x of g, the set $g^{-1}(x)$ has exactly m elements and for every critical value x of g, the set $g^{-1}(x)$ has less than m elements. Also, the set of regular points is dense in Ω and the set of regular values is dense in Ω'.

Definition 1 Let us consider a proper holomorphic mapping $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^n$ of multiplicity m and a point $z \in \Omega$. Denote by $\{z_1, \ldots, z_k\} = g^{-1}(g(z)) \subset \Omega$. Consider pairwise disjoint open sets $U_{z_i} \subset \Omega$, for $i = 1, \ldots, k$, such that $z_i \in U_{z_i}$ and the restriction $g|_{U_{z_i}} : U_{z_i} \longrightarrow g(U_{z_i})$ is a proper holomorphic mapping (see the proof of [18, Theorem 15.1.9] for a possible construction of the sets U_i, $i = 1, \ldots, k$). We define the *multiplicity of* z as the multiplicity of the proper holomorphic mapping $g|_{U_z}$ and we denote it by $mult(z)$.

Notice that the definition of the multiplicity of z is independent of the set U_z and the mapping $g|_{U_z}$, and automatically every regular point has multiplicity one. Furthermore, for every $z \in \Omega$,

$$\sum_{z_i \in g^{-1}(g(z))} mult(z_i) = m. \tag{1}$$

In the following proposition we will consider subsets of Ω of cardinality less than or equal to a fixed natural number m. The set of subsets of Ω of cardinality less than or equal to m is denoted by $P_{<m+1}(\Omega)$. Naturally $P_{<m+1}(\Omega)$ can be considered as a metric space when endowed with the Hausdorff distance between sets. Recall that the Hausdorff distance between finite subsets X and Y of a metric space (M, d) is given by

$$d_{\mathrm{H}}(X, Y) = \max\{\max_{x \in X} \min_{y \in Y} d(x, y), \max_{y \in Y} \min_{x \in X} d(x, y)\}.$$

For a point z in $\mathbb{C}^n$ and a positive number ε, $B(z, \varepsilon)$ will denote the open euclidean ball centered at z with radius ε.

The following proposition shows that the inverse of a proper holomorphic mapping is a continuous mapping when considered as a multivalued mapping.

Proposition 1 *Given a proper holomorphic mapping $g : \Omega \subset \mathbb{C}^n \longrightarrow \Omega' \subset \mathbb{C}^n$ of multiplicity m, the mapping*

$$\begin{aligned} g^{-1} : \Omega' &\longrightarrow P_{<m+1}(\Omega) \\ x &\rightsquigarrow \{z \in \Omega : g(z) = x\} \end{aligned}$$

is continuous.

Proof Let us fix $x \in \Omega'$. If x is a regular value of the mapping g, then the set $\{z \in \Omega : g(z) = x\}$ contains exactly m different elements that we denote by $\{z_1, \ldots, z_m\}$. Then, for $i = 1, \ldots, m$, $J_g(z_i) \neq 0$. As a consequence of the inverse function theorem for functions of several complex variables there exist pairwise disjoint open neighborhoods $U_1, \ldots, U_m$ of $z_1, \ldots, z_m$ respectively and an open neighborhood V_x of x such that the mapping $g|_{U_j}: U_j \longrightarrow V_x$ has a continuous inverse that we denote by $g_j^{-1} : V_x \longrightarrow U_j$, $j = 1, \ldots, m$. Then, if $\{x_i\}_{i=1}^{\infty}$ is a sequence of points in V_x convergent to x, by continuity we have that $\{g_j^{-1}(x_i)\}_{i=1}^{\infty}$ converges to z_j for $j = 1, \ldots, m$. Therefore, $\{g^{-1}(x_i)\}_{i=1}^{\infty}$ converges to $g^{-1}(x)$.

Let us assume now that x is a critical value of the mapping g and the set $g^{-1}(x)$ is $\{z_1, \ldots, z_s\}$ with $1 \leq s < m$. To show that g^{-1} is continuous at x we proceed by contradiction. Assume that g^{-1} is not continuous at x. Then, there exists a sequence $\{x_i\}_{i=1}^{\infty}$ convergent to x such that $\{g^{-1}(x_i)\}_{i=1}^{\infty}$ is not convergent to $g^{-1}(x)$. Since $\{g^{-1}(x_i)\}_{i=1}^{\infty}$ is not convergent to $g^{-1}(x)$ there exist a positive number ε and a strictly increasing sequence of natural numbers $\{i_r\}_{r=1}^{\infty}$ with

$$d_H(g^{-1}(x), g^{-1}(x_{i_r})) > \varepsilon.$$

Without loss of generality, we can assume that $\varepsilon < 1/2 \min_{1 \leq j < k \leq s} \|z_j - z_k\|$ and $B(z_j, \varepsilon) \subset U_{z_j}$ for $j = 1, \ldots, s$, where U_{z_j} are the disjoint open sets that appear in Definition 1.

For each $i = i_r$, we denote by $\{z_1^i, \ldots, z_{m_i}^i\}$ the set $g^{-1}(x_i)$.

Note that $\max_{1 \leq j \leq s} \min_{1 \leq t \leq m_i} d(z_j, z_t^i)$ cannot be bigger than ε since $m = \sum_{j=}^{s} mult(z_j) = \sum_{t=1}^{m_i} mult(z_t^i)$. So we can assume without lost of generality that $\max_{1 \leq t \leq m_i} \min_{1 \leq j \leq s} d(z_j, z_t^i) > \varepsilon$ for all natural number i. Then, since $\max_{1 \leq t \leq m_i} \min_{1 \leq j \leq s} d(z_j, z_t^i) > \varepsilon$, for each natural number i there exists t_i with $\min_{1 \leq j \leq s} d(z_j, z_{t_i}^i) > \varepsilon$. Therefore, we can find a sequence of points in Ω, $\{z_{t_i}^i\}_{i=1}^{\infty}$, with $z_{t_i}^i \in \{z_1^i, \ldots, z_{m_i}^i\}$ such that

$$\min_{j=1,\ldots,s} \|z_j - z_{t_i}^i\| > \varepsilon. \tag{2}$$

Put $N = g\big(\cap_{j=1}^{s} (\Omega \setminus B(z_j, \varepsilon))\big)$. Since g is proper and holomorphic, g is a closed mapping. Hence, N is closed in Ω' and does not contain x. Let $\delta > 0$ be such that $B(x, \delta) \subset \Omega' \setminus N$.

Since the sequence $\{x_i\}_{i=1}^{\infty}$ converges to the point x, for i big enough the point x_i belongs to $B(x, \delta)$. As x_i does not belong to N, we have that $g^{-1}(x_i) \cap \big(\cap_{j=1}^{s} (\Omega \setminus B(z_j, \varepsilon))\big) = \emptyset$. Therefore $g^{-1}(x_i) \subset \cup_{j=1}^{s} B(z_j, \varepsilon)$. In particular, $z_{t_i}^i \in \cup_{j=1}^{s} B(z_j, \varepsilon)$ for i big enough. But this contradicts Eq. (2). Therefore, g^{-1} is continuous at x. □

Corollary 1 *Let $g : \Omega \subset \mathbb{C}^n \longrightarrow \Omega' \subset \mathbb{C}^n$ be a proper holomorphic mapping and $K = \overline{U} \subset \Omega$ where U is a bounded non-empty open subset of Ω. If for every regular point z in K the set $g^{-1}(g(z))$ is contained in K, then $g^{-1}(g(K)) = K$.*

Proof The inclusion $K \subset g^{-1}(g(K))$ is trivial. Let us show the other inclusion. Let z be a point in K. Since $K = \overline{U}$ there exists a sequence of regular points $\{z_i\}_{i=1}^{\infty} \subset U$ convergent to z. By the continuity of the mappings g and g^{-1} we have that $g^{-1}(g(z_i))$ is convergent to $g^{-1}(g(z))$. Since $g^{-1}(g(z_i)) \subset K$ then $g^{-1}(g(z)) \subset K$.

3 Banach algebras of polynomials

Given a proper holomorphic mapping $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^k$ and a compact set $K \subset \Omega$, we denote by

$$\mathcal{P}_g(K) = \{P|_K : P \in \mathcal{P}(\mathbb{C}^n) \text{ and if } z, w \in K \text{ with } g(z) = g(w) \text{ then } P(z) = P(w)\},$$

where $\mathcal{P}(\mathbb{C}^n)$ stands for the algebra of all polynomials defined on $\mathbb{C}^n$.

Roughly speaking $\mathcal{P}_g(K)$ is the set of polynomials in $\mathbb{C}^n$ that are compatible with the equivalence relation defined by the mapping g on the set K.

We denote by $\mathscr{P}_g(K)$ the Banach algebra defined as the closure of $\mathcal{P}_g(K)$ with the topology of uniform convergence on K.

Notice, that if g is the identity mapping form $\mathbb{C}^n$ to $\mathbb{C}^n$, then the algebra $\mathscr{P}_g(K)$ is the classical algebra $\mathscr{P}(K)$ defined as the closure in $C(K)$ of $\mathcal{P}(\mathbb{C}^n)$ with the topology of uniform convergence on K. More examples of algebras $\mathscr{P}_g(K)$ will be seen in Sect. 5.

If K is a compact subset of $\mathbb{C}^n$, the *polynomially convex hull* of K is the set

$$\widehat{K} = \{z \in \mathbb{C}^n : |P(z)| \leq \|P\|_K \text{ for every polynomial } P \in \mathcal{P}(\mathbb{C}^n)\}.$$

A compact set K of $\mathbb{C}^n$ is said *polynomially convex* if $\widehat{K} = K$.

We denote by

$$\widehat{K}^g = \{z \in \mathbb{C}^n : |P(z)| \leq \|P\|_K \text{ for every polynomial } P \in \mathcal{P}_g(K)\}.$$

A mapping $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^k$ is a *proper polynomial mapping* whenever g is a polynomial from $\mathbb{C}^n$ to $\mathbb{C}^k$ that is also a proper mapping when restricted to Ω. This is equivalent to saying that g is a proper mapping that can be written as $(g_1, \ldots, g_k)$ with $g_i \in \mathcal{P}(\mathbb{C}^n)$ for $i = 1, \ldots, k$.

The following result establishes the relation between the sets $\widehat{K}$ and $\widehat{K}^g$.

Proposition 2 *For any proper polynomial mapping $g : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ and any compact set $K \subset \mathbb{C}^n$ with $g^{-1}(g(\widehat{K})) = \widehat{K}$ we have that*

$$\widehat{K}^g = \widehat{K}.$$

Proof Clearly $\widehat{K} \subseteq \widehat{K}^g$ so we will check the other inclusion. Let consider $z \notin \widehat{K}$. Then, since $\widehat{K}$ is polynomially convex, $g^{-1}(g(\widehat{K})) = \widehat{K}$ and g is a proper holomorphic mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$, by [21, Theorem 1.6.24] the set $g(\widehat{K})$ is polynomially convex and $g(z) \notin g(\widehat{K})$. Consequently, there exists a polynomial $Q \in \mathcal{P}(\mathbb{C}^n)$ with $|Q(g(z))| > \|Q\|_{g(\widehat{K})}$.

Then the polynomial $P = Q \circ g$ is an element of $\mathcal{P}_g(K)$ with

$$|P(z)| > \|P\|_{\widehat{K}} = \|P\|_K,$$

so $z \notin \widehat{K}^g$.

As a consequence we obtain the following extension of [1, Theorem 3.1].

Corollary 2 *Let $g : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a proper polynomial mapping and let $K \subset \mathbb{C}^n$ be a polynomially convex set with $g^{-1}(g(\widehat{K})) = \widehat{K}$. Then, for every $z \notin K$ there exists a polynomial $P \in \mathcal{P}_g(K)$ with*

$$|P(z)| > \sup_{w \in K} |P(w)|.$$

Definition 2 Given a subset K of $\mathbb{C}^n$, consider an algebra of polynomials $\mathcal{B}(K) \subset \mathcal{P}(\mathbb{C}^n)$. It is said that $\{g_i\}_{i=1}^k \subset \mathcal{B}(K)$ is *algebraically independent* in $\mathcal{B}(K)$ if the only polynomial $Q \in \mathcal{P}(\mathbb{C}^k)$ that satisfies $Q(g_1, \ldots, g_k) = 0$ is $Q = 0$. We recall that $\{g_i\}_{i=1}^k \subset \mathcal{B}(K)$ is a *basis* of $\mathcal{B}(K)$ if for every polynomial $P \in \mathcal{B}(K)$ there exists a unique polynomial $Q \in \mathcal{P}(\mathbb{C}^k)$ such that $P(z) = Q(g_1(z), \ldots, g_k(z))$ for all $z \in K$.

Remark 1 Given a proper polynomial mapping $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^k$, $g = (g_1, \ldots, g_k)$, and a compact subset K of Ω, the polynomials $\{g_i\}_{i=1}^k$ are not always a basis of the algebra $\mathcal{P}_g(K)$, even if K has nonempty interior. Indeed, $g : \mathbb{C}^2 \longrightarrow \mathbb{C}^3$ defined by $g(x, y) = (x^2, y^2, xy)$ is a proper polynomial mapping. However, for any compact subset K of $\mathbb{C}^2$, the polynomial $P(x, y) = x^2 y^2$ cannot be written in a unique way as an algebraic combination of $g_1(x, y) = x^2$, $g_2(x, y) = y^2$ and $g_3(x, y) = xy$.

Before we continue, let us relate the algebra $\mathscr{P}_g(K)$, that we have just defined, to the classical algebra of group invariant polynomials. Let us denote by $GL(n, \mathbb{C})$ the general linear group of degree n consisting of the set of all $n \times n$ complex invertible matrices. We consider the natural group action of $GL(n, \mathbb{C})$ on $\mathbb{C}^n$. For a subgroup $G \leq GL(n, \mathbb{C})$ and a set $K \subset \mathbb{C}^n$ we denote by

$$\langle G, K \rangle = \{gw : g \in G, w \in K\}$$

the action of the group G on the set K and it is said that K is *invariant* under the action of G if $\langle G, K \rangle = K$. Given a finite subgroup $G \leq GL(n, \mathbb{C})$ and a polynomial $P \in \mathcal{P}(\mathbb{C}^n)$, we recall that P is an *invariant polynomial* under the group G or *G-invariant* if $P(w) = P(gw)$ for all $g \in G$ and all $w \in \mathbb{C}^n$. For more details about the theory of invariant polynomials under the action of finite groups we recommend [10, Chapter 7]. Given a set K that is invariant under the action of a group $G \leq GL(n, \mathbb{C})$ we denote by

$$\mathcal{P}_G(K) = \{P \in \mathcal{P}(\mathbb{C}^n) : P \circ \sigma = P \text{ on } K \text{ for all } \sigma \in G\}$$

and by $\mathscr{P}_G(K)$ the Banach algebra defined as the closure in $C(K)$ of $\mathcal{P}_G(K)$ with the topology of uniform convergence on K.

This setting will be discussed in a more general scenario at the beginning of Sect. 4.

Recall that a *finite unitary reflection group* G is a finite subgroup of $GL(n, \mathbb{C}^n)$ of unitary transformations that is generated by the reflections that it contains. By *reflection* we understand a linear transformation $T : \mathbb{C}^n \to \mathbb{C}^n$ that fixes pointwise only a hyperplane of dimension $n-1$. For example, $T(x, y, z) = (x, y, 2z)$ is such a T. The finite unitary groups were studied and classified by Shephard and Todd [20] and Flatto [12]. Hilbert proved that for any finite unitary reflection group G there exist finitely many polynomials $p_1, \ldots, p_r$ that are G-invariant and form a basis of the algebra $\mathcal{P}_G(\mathbb{C}^n)$. Chevalley showed that in fact this basis can be chosen to be exactly n homogeneous G-invariant polynomials. See [10, § 2.5 Theorem 4] and [9, Theorem (A)] for the details.

Here we show that for finite unitary reflection groups the algebras $\mathscr{P}_g(K)$ and $\mathscr{P}_G(K)$ coincide.

Proposition 3 *Let $G \subset GL(n, \mathbb{C})$ be a finite unitary reflection group, let $\{g_1, \ldots, g_n\}$ be a basis of homogeneous polynomials of $\mathcal{P}_G(\mathbb{C}^n)$ and put $g = (g_1, \ldots, g_n)$. Then for any compact set K containing zero as an interior point that is invariant under the action of the group G, the set $\{g_i\}_{i=1}^n$ is a basis of the algebra $\mathcal{P}_g(K)$ and*

$$\mathscr{P}_G(K) = \mathscr{P}_g(K).$$

Proof First we show that $\{g_i\}_{i=1}^n$ is a basis of the algebra $\mathcal{P}_G(K)$. Let P be a polynomial in $\mathcal{P}_G(K)$. By [10, §7.2 Proposition 10] P is G-invariant if and only if its homogeneous parts are G-invariant. Thus, we can assume that P is homogeneous. Since zero is an interior point of K, by homogeneity, P is an element of $\mathcal{P}_G(\mathbb{C}^n)$. Therefore, $\mathcal{P}_G(K) \subseteq \mathcal{P}_G(\mathbb{C}^n)$. Clearly the other inclusion holds, hence $\mathcal{P}_G(K) = \mathcal{P}_G(\mathbb{C}^n)$ and $\{g_1, \ldots, g_n\}$ is a basis of polynomials of $\mathcal{P}_G(K)$.

To see that $\mathscr{P}_G(K) = \mathscr{P}_g(K)$ it is enough to show that $\mathcal{P}_G(K) = \mathcal{P}_g(K)$. If P is a polynomial in $\mathcal{P}_G(K)$ then P can be written as $P = Q \circ g$ for some polynomial $Q \in \mathcal{P}(\mathbb{C}^n)$. Thus $P \in \mathcal{P}_g(K)$. Therefore $\mathcal{P}_G(K) \subset \mathcal{P}_g(K)$. Let us assume now that P is a polynomial in $\mathcal{P}_g(K)$. Let us also assume that $z \in K$ and $\sigma \in G$. Since the mappings g_i, $i = 1, \ldots, n$, are G-invariant, $g_i(\sigma(z)) = g_i(z)$ for all i. Thus $g(\sigma(z)) = g(z)$ for all $z \in K$ and all $\sigma \in G$. Hence, $P(\sigma(z)) = P(z)$ for all $z \in K$ and all $\sigma \in G$ so $P \in \mathcal{P}_G(K)$.

For a complex Banach algebra $\mathcal{B}$, $\mathcal{M}(\mathcal{B})$ stands for the spectrum of $\mathcal{B}$, which is the set of complex non-zero homomorphisms. In what follows, we give a complete description of the spectrum of the algebras $\mathscr{P}_g(K)$.

Let $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^k$ be a proper polynomial mapping, $g = (g_1, \ldots, g_k)$ with $g_i \in \mathcal{P}(\mathbb{C}^n)$ and K be a compact subset of Ω. Assume that the polynomials $\{g_i\}_{i=1}^k$ are a basis of the algebra $\mathcal{P}_g(K)$. We define

$$\begin{aligned} \pi : \mathcal{M}(\mathscr{P}_g(K)) &\longrightarrow \mathbb{C}^k \\ \psi &\rightsquigarrow (\psi(g_1), \ldots, \psi(g_k)). \end{aligned} \tag{3}$$

Notice that the mapping π is well defined since the polynomials $g_1, \ldots, g_k$ belong automatically to the algebra $\mathcal{P}_g(K) \subseteq \mathscr{P}_g(K)$.

If we consider g to be the identity mapping, then the mappings $g_1, \dots, g_n$ are the projection mappings $p_1, \dots, p_n$ that are a basis of the algebra of polynomials $\mathcal{P}(\mathbb{C}^n)$. Also in this particular case the mapping π coincides with the classical mapping $\pi : \mathcal{M}(\mathscr{P}(K)) \longrightarrow \mathbb{C}^n$ defined as $\pi(\psi) = (\psi(p_1), \dots, \psi(p_n))$ for any compact subset K of $\mathbb{C}^n$. As usual, for $z \in \mathbb{C}^n$, we denote by $\mathcal{M}_z(\mathscr{P}(K))$ the fiber of the spectrum of the algebra $\mathscr{P}(K)$ at the point z defined as

$$\mathcal{M}_z(\mathscr{P}(K)) = \{\psi \in \mathscr{P}(K) : \pi(\psi) = z\} = \pi^{-1}(z).$$

It is a classical result that every character of $\mathscr{P}(K)$ is the evaluation at a unique point $z \in \widehat{K}$, δ_z (see for instance [21, Theorem 1.2.9]). The following theorem is a generalization of this result.

Theorem 2 *Let $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^k$ be a proper polynomial mapping, $g = (g_1, \dots, g_k)$. Let K be a compact subset of Ω. Assume that the polynomials $\{g_i\}_{i=1}^k$ are a basis of the algebra $\mathcal{P}_g(K)$. Then,*

$$\pi(\mathcal{M}(\mathscr{P}_g(K))) = \widehat{g(K)}.$$

Furthermore, for every $x \in \widehat{g(K)} \cap g(\Omega)$,

$$\mathcal{M}_x(\mathscr{P}_g(K)) = \{\delta_w : w \in g^{-1}(x)\}. \tag{4}$$

In particular, if $k = n$, then (4) holds for every $x \in \widehat{g(K)} \cap \Omega'$.

Proof First we show that $\pi(\mathcal{M}(\mathscr{P}_g(K))) \subset \widehat{g(K)}$. Let us fix a homomorphism $\psi \in \mathcal{M}(\mathscr{P}_g(K))$ and consider $z := \pi(\psi) = (\psi(g_1), \dots, \psi(g_k)) \in \mathbb{C}^k$. By hypothesis, every polynomial $P \in \mathcal{P}_g(K)$ can be uniquely written as $P = Q \circ g$ for some polynomial $Q \in \mathcal{P}(\mathbb{C}^k)$. Since ψ is linear and multiplicative,

$$\psi(P) = \psi(Q \circ g) = Q(\psi(g_1), \dots, \psi(g_k)) = Q(z).$$

Now, for every polynomial $S \in \mathcal{P}(\mathbb{C}^k)$, we have that $R = S \circ g \in \mathcal{P}_g(K)$. Therefore,

$$|S(z)| = |\psi(R)| \le \|R\|_K = \|S\|_{g(K)}$$

where the above inequality follows from the fact that the norm of the character is bounded by 1. Hence, $\pi(\psi) = z \in \widehat{g(K)}$.

To see the other inclusion, take $x \in \widehat{g(K)}$. Consider the function

$$\begin{aligned} \psi : \mathcal{P}_g(K) &\longrightarrow \mathbb{C} \\ P &\rightsquigarrow Q(x) \end{aligned}$$

where Q is the unique polynomial in $\mathcal{P}(\mathbb{C}^k)$ with $P = Q \circ g$. This function ψ is clearly a continuous homomorphism as $x \in \widehat{g(K)}$. By the density of $\mathcal{P}_g(K)$ in $\mathscr{P}_g(K)$, ψ

can be extended to a homomorphism in $\mathcal{M}(\mathscr{P}_g(K))$ that we denote in the same way. Obviously, $\pi(\psi) = x$ and the inclusion follows.

To finish, let $x \in \widehat{g(K)} \cap g(\Omega)$ and $\psi \in \mathcal{M}(\mathscr{P}_g(K))$ with $\pi(\psi) = x$. Fix $w \in g^{-1}(x)$. For every $P \in \mathcal{P}_g(K)$,

$$\psi(P) = \psi(Q \circ g) = Q(x) = (Q \circ g)(w) = P(w),$$

where Q is the unique polynomial in $\mathcal{P}(\mathbb{C}^k)$ with $P = Q \circ g$. Hence, $\psi = \delta_w$. Therefore,

$$\mathcal{M}_x(\mathscr{P}_g(K)) = \{\delta_w : w \in g^{-1}(x)\}$$

for any $x \in \widehat{g(K)} \cap g(\Omega)$. □

By [21, Theorem 1.6.24] if $g : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is a proper mapping then $g(K)$ is a polynomially convex compact subset of $\mathbb{C}^n$ for any polynomially convex compact subset K of $\mathbb{C}^n$. Hence we have the following corollary of Theorem 2.

Corollary 3 *Let $g : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a proper polynomial mapping, $g = (g_1, \dots, g_n)$. Let K be a polynomially convex compact subset of $\mathbb{C}^n$. Assume that the polynomials $\{g_i\}_{i=1}^n$ are a basis of the algebra $\mathcal{P}_g(K)$. Then,*

$$\pi(\mathcal{M}(\mathscr{P}_g(K))) = g(K).$$

Moreover, for every $x \in g(K)$,

$$\mathcal{M}_x(\mathscr{P}_g(K)) = \{\delta_w : w \in g^{-1}(x)\}.$$

Given a compact set $K \subset \mathbb{C}^n$ with non-empty interior we denote by $A(K)$ the algebra of functions from K into $\mathbb{C}$ that are continuous on K and holomorphic in the interior of K. For a fixed proper polynomial mapping $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^k$, if $K \subset \Omega$ is a compact set with non-empty interior we denote by

$$A_g(K) = \{f \in A(K) : \text{ if } z, w \in K \text{ with } g(z) = g(w) \text{ then } f(z) = f(w)\}.$$

Remark 3 Let K be a compact balanced set with $0 \in int(K)$, $K \subset \Omega \subseteq \mathbb{C}^n$. The following are equivalent:

1. for every function $f \in A_g(K)$, its Taylor polynomials at the origin, $P_m f$, $m = 0, 1, 2, \dots$, belong to $A_g(K)$,
2. if $g(x) = g(y)$ with $x, y \in K$ then $f(\lambda x) = f(\lambda y)$ for every complex number λ with $|\lambda| = 1$ and every $f \in A_g(K)$,
3. if $g(x) = g(y)$ with $x, y \in K$ then $g(\lambda x) = g(\lambda y)$ for every complex number λ with $|\lambda| = 1$.

It is said that a mapping g is *circular* if it satisfies any of the conditions in the remark above.

Remark 4 If K is a compact balanced set in $\mathbb{C}^n$ which coincides with the closure of its interior and $0 \in int(K)$, then $A(K) = \mathscr{P}(K)$. Indeed, given $f \in A(K)$, since the interior of K is balanced $f(x) = \sum_{m=0}^{\infty} P_m f(x)$ for every x in the interior of K, where $P_m f$ are the Taylor polynomials of f at the origin. Hence, since f is uniformly continuous on K, given $\epsilon > 0$ there exists $0 < r < 1$ such that $|f(x) - f(rx)| < \epsilon$ for all $x \in K$. Thus, $|f(x) - \sum_{m=0}^{\infty} r^m P_m f(x)| \le \epsilon$ for all $x \in K$. Therefore, there exists a natural number m_0 such that $|f(x) - \sum_{m=0}^{m_0} r^m P_m f(x)| \le 2\epsilon$ for all $x \in K$.

If we want to know the relation between $A_g(K)$ and $\mathscr{P}_g(K)$ where g is a proper polynomial mapping, by Remark 4 we would need that for any $f \in A_g(K)$ all the Taylor polynomials of f at the origin are elements in $\mathcal{P}_g(K)$. This happens when g is circular.

Corollary 4 *Let $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^k$ be a circular and proper polynomial mapping, $g = (g_1, \dots, g_k)$. Let K be a compact balanced subset of Ω which coincides with the closure of its interior and $0 \in int(K)$. Assume that the polynomials $\{g_i\}_{i=1}^k$ are a basis of the algebra $\mathcal{P}_g(K)$. Then,*

$$\pi(\mathcal{M}(A_g(K))) = \widehat{g(K)}.$$

Moreover, for every $x \in \widehat{g(K)} \cap g(\Omega)$,

$$\mathcal{M}_x(A_g(K)) = \{\delta_w : w \in g^{-1}(x)\}. \tag{5}$$

In particular, if $k = n$, then (5) holds for every $x \in \widehat{g(K)} \cap \Omega'$.

4 Algebras of holomorphic functions

Given a proper mapping $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^k$, an open subset $U \subset \Omega$ and a subalgebra $\mathcal{B}(U)$ of $\mathcal{H}(U)$, we define the algebra

$$\mathcal{B}_g(U) = \{f \in \mathcal{B}(U) : \text{ if } z, w \in U \text{ with } g(z) = g(w) \text{ then } f(z) = f(w)\}.$$

Naturally the mapping g defines an equivalence relation on U by $z \sim w$ if and only if $g(z) = g(w)$ for $z, w \in U$. Roughly speaking the algebra $\mathcal{B}_g(U)$ is the set of holomorphic functions on $\mathcal{B}(U)$ that are compatible with respect to the equivalence relation defined by g.

Recently, several authors have studied algebras of holomorphic functions that are invariant under the action of a given group or semigroup of operators. A lot of attention has been devoted to the study of algebras of symmetric functions (see, e.g., [2,3] and the references therein). In particular, if U is an open subset of Ω and G is a subgroup of $GL(n, \mathbb{C})$ leaving U fixed, i.e. U is invariant under the action of G, we can consider the action of G on $\mathcal{H}(U)$ defined by

$$\begin{array}{rcl} G \times \mathcal{H}(U) & \longrightarrow & \mathcal{H}(U) \\ (\sigma, f) & \rightsquigarrow & f \circ \sigma|_U. \end{array} \tag{6}$$

To simplify the notation, from now on, we will write $f \circ \sigma$ instead of $f \circ \sigma|_U$.

We can then consider the subalgebra of $\mathcal{H}(U)$ consisting of all the G-invariant holomorphic functions, i.e.,

$$\mathcal{H}_G(U) = \{f \in \mathcal{H}(U) : f \circ \sigma = f \text{ for all } \sigma \in G\}.$$

Naturally the definition of $\mathcal{H}_G(U)$ can be extended to any subalgebra $\mathcal{B}(U)$ of $\mathcal{H}(U)$ whenever the mapping (6) is well defined, i.e. for every $\sigma \in G$ and every $f \in \mathcal{B}(U)$ then $f \circ \sigma \in \mathcal{B}(U)$. We define

$$\mathcal{B}_G(U) = \{f \in \mathcal{B}(U) : f \circ \sigma = f \text{ for all } \sigma \in G\}.$$

Recall that for a group G acting on a set X, the orbit of a point $z \in X$ is the set

$$orb(z) = \{\sigma z : \sigma \in G\}.$$

The following result extends Proposition 3 to algebras of holomorphic functions.

Proposition 4 *Let $G \leq GL(n, \mathbb{C})$ be a finite unitary reflection group, $\{g_1, \ldots, g_n\}$ a basis of G-invariant polynomials and $g = (g_1, \ldots, g_n)$. Let U be an open set that is invariant under the action of G and $\mathcal{B}(U)$ a subalgebra of $\mathcal{H}(U)$. If $f \circ \sigma \in \mathcal{B}(U)$ for every $\sigma \in G$ and every $f \in \mathcal{B}(U)$, then we have that*

$$\mathcal{B}_g(U) = \mathcal{B}_G(U).$$

Proof The fact that the polynomials $g_1, \ldots, g_n$ are homogeneous and a combination of [19, Proposition 2.1] and [19, Theorem 5.1] show that the mapping $g : \mathbb{C}^n \to \mathbb{C}^n$ is a proper polynomial mapping. To prove the result it is enough to show that for every point $z \in U$ we have that $orb(z) = g^{-1}(g(z))$. This equality is a direct consequence of [19, Proposition 2.2 (i)] where it is proved that $w \in orb(z)$ if and only if $g(z) = g(w)$.

Proposition 5 *Given a proper holomorphic mapping $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^n$ of multiplicity m, and an open set $U \subset \Omega$ with $g^{-1}(g(U)) = U$, the linear operator*

$$\begin{aligned} \rho_g : \mathcal{H}(U) &\longrightarrow \mathcal{H}_g(U), \\ f &\rightsquigarrow f_g \end{aligned}$$

where $f_g(z) = \dfrac{1}{m} \displaystyle\sum_{z_i \in g^{-1}(g(z))} f(z_i) mult(z_i)$ for every $z \in U$, is well defined. Moreover,

(a) *the mapping ρ_g is continuous,*
(b) *if $f \in \mathcal{H}_g(U)$ then $\rho_g(f) = f$,*
(c) *if $f \in \mathcal{H}_g(U)$ and $h \in \mathcal{H}(U)$ then $\rho_g(fh) = f\rho_g(h)$,*
(d) *if f is bounded by a positive constant M then so is $\rho_g(f)$.*

Proof First we show that f_g is continuous. Let us fix $z \in U$. Put $\{z_1, \ldots, z_k\} = g^{-1}(g(z)) \subset U$. For simplicity we assume that $z = z_1$. We show now that f_g is continuous at z_1.

Following the idea of the proof of [18, Theorem 15.1.9], we can find neighborhoods $U_{z_1}, \ldots, U_{z_k} \subset U$ and $V_{g(z_1)} \subset g(U)$ of the points $z_1, \ldots, z_k$ and $g(z_1)$ respectively such that the mappings

$$G_i = g|_{U_{z_i}} : U_{z_i} \longrightarrow V_{g(z_1)}$$

are proper holomorphic mappings with multiplicity m_i, $i = 1, \ldots, k$. Without loss of generality, we can assume that $U_{z_i} \subset B(z_i, r/2)$, with $r = \min_{1 \le i < j \le k} \|z_i - z_j\|$.

Since f is continuous at $z_1, \ldots, z_k$, given $\varepsilon > 0$ there exists a positive number $\delta < r/2$ such that if $w \in U_{z_i}$ and $\|w - z_i\| < \delta$ then $|f(w) - f(z_i)| < \varepsilon$, for $i = 1, \ldots, k$. By Proposition 1, the set-valued mapping g^{-1} is continuous on Ω' so $g^{-1} \circ g$ is continuous at z_1. Thus, there exists $\gamma > 0$ such that if $w \in U_{z_i}$ and $\|w - z_1\| < \gamma$ then $d_H(g^{-1}(g(w)), g^{-1}(g(z_1))) < \delta$.

For any $w \in U_{z_1}$ with $\|w - z_1\| < \gamma$, we denote by $\{w_1^i, \ldots, w_{m_i}^i\}$ the set $g^{-1}(g(w)) \cap U_{z_i}$ where we are repeating the points according to their multiplicity. Then, for each $i = 1, \ldots, k$ and each $j = 1, \ldots, m_i$ we have that

$$\delta > d_H(g^{-1}(g(w)), g^{-1}(g(z_1))) \ge d(w_j^i, g^{-1}(g(z_1))) = \min_{s=1,\ldots,k} \|w_j^i - z_s\|.$$

Since $r/2 > \delta$, $w_j^i \in U_{z_i} \subset B(z_i, r/2)$ and $z_s \in U_{z_s} \subset B(z_s, r/2)$ we obtain that $\min_{s=1,\ldots,k} \|w_j^i - z_s\| = \|w_j^i - z_i\|$. Hence, for $j = 1, \ldots, m_i$, $i = 1, \ldots, k$, $\|w_j^i - z_i\| < \delta$, so $|f(w_j^i) - f(z_i)| < \varepsilon$. By the triangle inequality, for $i = 1, \ldots, k$,

$$\left|\left(\sum_{j=1}^{m_i} f(w_j^i)\right) - f(z_i)m_i\right| \le \sum_{j=1}^{m_i} |f(w_j^i) - f(z_i)| < m_i\varepsilon.$$

Therefore

$$|f_g(w) - f_g(z_1)| = \left|\frac{1}{m}\sum_{i=1}^{k}\sum_{j=1}^{m_i} f(w_j^i) - \frac{1}{m}\sum_{i=1}^{k} f(z_i)m_i\right|$$
$$\le \frac{1}{m}\sum_{i=1}^{k}\left|\sum_{j=1}^{m_i} f(w_j^i) - f(z_i)m_i\right| < \varepsilon.$$

Thus, f_g is continuous at $z = z_1$.

Now we are going to show that f_g is holomorphic. If z is a regular point of g in U, then $g^{-1}(g(z)) = \{z_1, \ldots, z_m\}$ and $J_g(z) \neq 0$. To simplify our notation we assume that $z = z_1$. Notice that $g(z_1)$ is a regular value of the mapping g and every point $z_1, \ldots, z_m$ is a regular point of g in U. By the inverse function theorem and using that g is holomorphic at z_1 there exist pairwise disjoint neighborhoods $U_{z_1}, \ldots, U_{z_m} \subset U$

and $V_{g(z_1)} \subset g(U)$ of the points $z_1, \ldots, z_m$ and $g(z_1)$ respectively such that the mappings

$$G_i = g|_{U_{z_i}} : U_{z_i} \longrightarrow V_{g(z_1)}$$

are biholomorphic for $i = 1, \ldots, m$.

For $i = 1, \ldots, m$, put,

$$F_i = f|_{U_{z_i}} \circ G_i^{-1} \circ G_1 : U_{z_1} \longrightarrow \mathbb{C}.$$

Then, each function F_i is the composition of holomorphic functions on the neighborhood U_{z_1} of the point z_1. Since

$$f_g|_{U_{z_1}} = \frac{1}{m}\sum_{i=1}^{m} F_i,$$

the function f_g is holomorphic on U_{z_1}, whenever z_1 is a regular point of g in U.

Let us now fix a critical point $z \in J_g^{-1}\{0\} \cap U$. Consider U_z a bounded neighborhood of z with $\overline{U}_z \subset U$. Since f_g is continuous on $\overline{U}_z$, then f_g is bounded on $\overline{U}_z$.

As f_g is holomorphic on $U \setminus J_g^{-1}\{0\}$ and bounded on a neighborhood of every point of $J_g^{-1}\{0\} \cap U$, by [15, Theorem 7.3.3] there exists a holomorphic extension F of f_g to U. By the density of the set $U \setminus J_g^{-1}\{0\}$ in U, the functions F and f_g coincide in U. Thus, f_g is holomorphic in U.

To continue we see that f_g is compatible with the equivalence relation defined by g. Let us consider $z, w \in U$ with $g(z) = g(w)$. Then, $g^{-1}(g(w)) = g^{-1}(g(z))$. Thus,

$$f_g(z) = \frac{1}{m}\sum_{z_i \in g^{-1}(g(z))} f(z_i) mult(z_i) = \frac{1}{m}\sum_{z_i \in g^{-1}(g(w))} f(z_i) mult(z_i) = f_g(w).$$

Therefore, the linear operator ρ_g is well defined.

Let us show that ρ_g is continuous. For a compact set $K \subset U$ and $z \in K$,

$$\begin{aligned} |\rho_g(f)(z)| &= |\frac{1}{m}\sum_{z_i \in g^{-1}(g(z))} f(z_i) mult(z_i)| \\ &\leq \frac{1}{m}\sum_{z_i \in g^{-1}(g(z))} |f(z_i)| mult(z_i) \\ &\leq \frac{1}{m}\sum_{z_i \in g^{-1}(g(z))} \|f\|_{g^{-1}(g(K))} mult(z_i) \\ &= \|f\|_{g^{-1}(g(K))}, \quad (\text{ by (1)}). \end{aligned}$$

Therefore $\|\rho_g(f)\|_K \leq \|f\|_{g^{-1}(g(K))}$. Hence ρ_g is continuous. Properties (b) and (d) follow from the definition of f_g. To prove (c), let us consider $f \in \mathcal{H}_g(U)$ and $h \in \mathcal{H}(U)$. Then, for every $z \in U$,

$$\begin{aligned}\rho_g(fh)(z) &= \frac{1}{m}\sum_{z_i\in g^{-1}(g(z))}(f(z_i)h(z_i))mult(z_i)\\ &= \frac{1}{m}\sum_{z_i\in g^{-1}(g(z))}(f(z)h(z_i))mult(z_i) \quad (\text{since } f\in\mathcal{H}_{g(U)})\\ &= f(z)\frac{1}{m}\sum_{z_i\in g^{-1}(g(z))}h(z_i)mult(z_i)\\ &= f(z)\rho_g(h)(z).\end{aligned}$$

Remark 5 An alternative version of Proposition 5 can be done without giving the explicit definition of the function f_g at every point in U. For this, we can define the function f_g only at the regular points as $f_g(z) = \frac{1}{m}\sum_{z_i\in g^{-1}(g(z))} f(z_i)$ and then use the density of the regular points and [15, Theorem 7.3.3] to extend the function f_g to U. However, the version provided here has the advantage that it gives us a precise description of the function f_g at the singular points of the mapping g.

Proposition 6 *Let $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^n$ be a proper holomorphic mapping of multiplicity m and let U be an open subset of Ω with $g^{-1}(g(U)) = U$. If $\mathcal{B}(U)$ is a Banach subalgebra of $\mathcal{H}(U)$ such that that $\rho_g(f)\in\mathcal{B}_g(U)$ for every $f\in\mathcal{B}(U)$, then the mapping*

$$\begin{aligned}\mathrm{P}_g : \mathcal{M}(\mathcal{B}(U)) &\longrightarrow \mathcal{M}(\mathcal{B}_g(U))\\ \psi &\rightsquigarrow \psi|_{\mathcal{B}_g(U)}\end{aligned}$$

is surjective.

Proof We need to show that every non-zero complex homomorphism

$$\psi : \mathcal{B}_g(U) \longrightarrow \mathbb{C}$$

extends to a non-zero complex homomorphism $\Psi : \mathcal{B}(U) \longrightarrow \mathbb{C}$. Naturally, the kernel of ψ, $Ker(\psi)$, is a subset of $\mathcal{B}(U)$. Let I be the ideal of $\mathcal{B}(U)$ generated by $Ker(\psi)$. We show that I is a proper ideal of $\mathcal{B}(U)$. Indeed, if this were not the case, then 1 would be an element of I, so we could find elements $f_1,\dots,f_m\in Ker(\psi)$ and $h_1,\dots,h_m\in\mathcal{B}(U)$ with

$$1 = f_1h_1+\cdots+f_mh_m.$$

Therefore, by Proposition 5,

$$\begin{aligned}1 = \psi(\rho_g(1)) &= \psi(\rho_g(f_1h_1+\cdots+f_mh_m))\\ &= \psi(f_1\rho_g(h_1)+\cdots+f_m\rho_g(h_m))\\ &= \psi(f_1)\psi(\rho_g(h_1))+\cdots+\psi(f_m)\psi(\rho_g(h_m))\\ &= 0\psi(\rho_g(h_1))+\cdots+0\psi(\rho_g(h_m)) = 0,\end{aligned}$$

a contradiction.

Thus I is a proper ideal. Since $\mathcal{B}(U)$ is a Banach algebra, $I \subseteq Ker(\Psi)$ for some non-trivial complex homomorphism $\Psi : \mathcal{B}(U) \longrightarrow \mathbb{C}$. Thus, Ψ is an extension of ψ and the proof is completed.

Theorem 6 *Let $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^n$ be a proper holomorphic mapping of multiplicity m. Let U be an open bounded subset of Ω with $\overline{U} \subset \Omega$ and $g^{-1}(g(U)) = U$. Then there is an algebra isomorphism between $\mathcal{H}_g(U)$ and $\mathcal{H}(g(U))$*

$$\begin{aligned} \eta : \mathcal{H}(g(U)) &\longrightarrow \mathcal{H}_g(U) \\ h &\rightsquigarrow \eta(h) = h \circ g \end{aligned}$$

when both algebras are endowed with the compact-open topology.

Proof Consider the mapping η from $\mathcal{H}(g(U))$ to $\mathcal{H}_g(U)$ defined by $\eta(h) = h \circ g = \tilde{h}$. The mapping $\tilde{h}$ is holomorphic since it is the composition of two holomorphic functions. Also if z, w are two elements of U with $g(z) = g(w)$, then $\tilde{h}(z) = \tilde{h}(w)$. Hence $\tilde{h} \in \mathcal{H}_g(U)$. Clearly, the mapping η is linear, multiplicative and continuous when both algebras are endowed with the compact-open topology.

Let us show that this mapping η is a bijection from $\mathcal{H}(g(U))$ onto $\mathcal{H}_g(U)$. First we show that the mapping is one-to-one. If $h_1 \circ g = h_2 \circ g$, then $(h_1 \circ g)(z) = (h_2 \circ g)(z)$ for every z in U. Hence, $h_1(x) = h_2(x)$ for every $x \in g(U)$. Therefore, $h_1 = h_2$. Now we need to show that the mapping η is onto, i.e. given $f \in \mathcal{H}_g(U)$ there exists a holomorphic function h in $\mathcal{H}(g(U))$ such that $\tilde{h} = f$. Define the function

$$\begin{aligned} h : g(U) &\longrightarrow \mathbb{C} \\ x &\rightsquigarrow h(x) = f(z), \end{aligned} \tag{7}$$

where z is some point in U such that $x = g(z)$. Notice that since $f \in \mathcal{H}_g(U)$ the definition of the function h is independent of the point x in the set $g^{-1}(z)$. Hence, the function h is well defined.

We show that the function h is continuous. Let $\{x_i\}_{i=1}^{\infty} \subset g(U)$ be a sequence convergent to a point $x \in g(U)$. For each natural number i, let us fix a point $z_i \in U$ such that $x_i = g(z_i)$. Let us fix also a point $z \in g^{-1}(x)$. Notice that since $g^{-1}(g(U)) = U$, z is an element of U. We claim that the sequence $\{f(z_i)\}_{i=1}^{\infty}$ is convergent to the point $f(z)$. If this were not the case, we could find a positive number ε and a subsequence $\{f(z_{i_k})\}_{k=1}^{\infty}$ of the sequence $\{f(z_i)\}_{i=1}^{\infty}$ with $|f(z_{i_k}) - f(z)| > \varepsilon$. Since $\overline{U}$ is compact there would exist a subsequence of the sequence $\{z_{i_k}\}_{k=1}^{\infty}$, that we denote in the same way, and a point $z_0 \in \overline{U}$, such that $\{z_{i_k}\}_{k=1}^{\infty} \to z_0 \in \overline{U}$. By the continuity of the the function f,

$$\{f(z_{i_k})\}_{k=1}^{\infty} \to f(z_0)$$

which is different from $f(z)$, since $|f(z_{i_k}) - f(z)| > \varepsilon$ for all natural numbers k.

By the continuity of the mapping g we have that

$$x = \lim_{k\to\infty} x_{i_k} = \lim_{k\to\infty} g(z_{i_k}) = g(z_0).$$

Since $g(z_0) = x = g(z)$ and $f \in \mathcal{H}_g(U)$ we have that $f(z_0) = f(z)$ which is a contradiction. Hence, the sequence $\{f(z_i)\}_{i=1}^{\infty}$ is convergent to the point $f(z)$. But, $h(x_i) = f(z_i)$ for all natural numbers i and $h(x) = h(z)$. Therefore, $\{h(x_i)\}_{i=1}^{\infty} \to h(x)$ and h is continuous.

Finally, let us show that the function h is holomorphic. For every regular value $x \in g(U)$, let us fix a point $z \in U$ so that $g(z) = x$. Since z is a regular point of g and g is holomorphic at z, there exist neighborhoods U_z and V_x of z and x respectively such that the mapping $g|_{U_z} : U_z \longrightarrow V_x$ is biholomorphic. Since $h = f \circ g^{-1}$ on the set V_x, h is holomorphic at the point x. Thus h is holomorphic on the set of regular values of the mapping g. For every singular value $x \in g(U)$, consider V_x a bounded neighborhood of x with $\overline{V}_x \subset g(U)$. Since h is continuous on $\overline{V}_x$, h is bounded on $\overline{V}_x$. By [15, Theorem 7.3.3] there exists a holomorphic extension H of h to $g(U)$ that coincides with h on a dense subset of $g(U)$. Thus, h is holomorphic on $g(U)$.

Since η is a linear, multiplicative, continuous and bijective mapping between two Fréchet algebras, η is an algebra isomorphism between $\mathcal{H}_g(U)$ and $\mathcal{H}(g(U))$.

In Theorem 2 and Corollary 4 we have a proper polynomial mapping $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^n$, $g = (g_1, \ldots, g_n)$, and we assume the hypothesis that $\{g_i\}_{i=1}^{n}$ is a basis of $\mathcal{P}_g(K)$. Now we are going to provide some cases where this hypothesis is fulfilled.

Corollary 5 *If $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^n$, $g = (g_1, \ldots, g_n)$, is a proper polynomial mapping and K is a compact subset of Ω with non-empty interior, then $\{g_i\}_{i=1}^{n}$ is algebraically independent in the algebra $\mathcal{P}_g(K)$. Moreover if the polynomials $\{g_i\}_{i=1}^{n}$ are homogeneous then it is a basis of $\mathcal{P}_g(K)$.*

Proof Let $Q \in \mathcal{P}(\mathbb{C}^n)$ such that $Q(g_1, \ldots, g_n) = 0$. Let U be the interior of K. By Theorem 6 we have an isomorphism η between $\mathcal{H}_g(U)$ and $\mathcal{H}(g(U))$ defined by $\eta(h) = h \circ g$, where $g = (g_1, \ldots, g_n)$. Since $Q(g_1, \ldots, g_n) \in \mathcal{H}_g(U)$, there exists a unique $F \in \mathcal{H}(g(U))$ such that $Q(g_1, \ldots, g_n)(z) = F(g_1, \ldots, g_n)(z)$ for all $z \in U$. But $\eta(0) = 0$, hence $F = 0$. As the restriction $Q|_{g(U)}$ is in $\mathcal{H}(g(U))$, then $Q|_{g(U)} = 0$. If we now apply the Identity Principle to a non-empty connected component of $g(U)$ we obtain that $Q = 0$ on $\mathbb{C}^n$. Therefore $\{g_i\}_{i=1}^{n}$ is algebraically independent in the algebra $\mathcal{P}_g(K)$.

Moreover, assume now that g_j is homogeneous of degree $\deg(g_j) = m_j$, $j = 1, \ldots, n$. Let $P \in \mathcal{P}_g(K)$. Thus P is also an element of $\mathcal{H}_g(U)$. By Theorem 6 there exists a unique $F \in \mathcal{H}(g(U))$ such that $P(x) = F(g_1, \ldots, g_n)(x)$ for all $x \in U$. Without loss of generality we may assume $0 \in U$ and $0 \in g(U)$. So there are $r, s > 0$ such that $\overline{B_\infty}(0, r) \subset U$, $\overline{B_\infty}(0, s) \subset g(U)$ and $\overline{B_\infty}(0, s) \subset g(\overline{B_\infty}(0, r))$, where $\overline{B_\infty}(0, t)$ denotes the closed polydisk centered at the origin with radius $t = r, s$. Let us denote

$$F(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha(F) z^\alpha$$

the monomial expansion of F, which is absolutely and uniformly convergent on $\overline{B_\infty}(0, r)$. Thus we can make the following rearrangement $F(z) = R(z) + S(z)$ with

$$R(z) = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ \deg(g^\alpha)\le k}} c_\alpha(F) z^\alpha \quad \text{and} \quad S(z) = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ \deg(g^\alpha)> k}} c_\alpha(F) z^\alpha ,$$

for all $z \in \overline{B_\infty}(0, r)$, where $k = \deg(P)$ and $g^\alpha = \prod_{i=1}^n g_i^{\alpha_i}$. Therefore

$$P(x) = R(g(x)) + S(g(x)) = \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ \deg(g^\alpha)\le k}} c_\alpha(F) g^\alpha(x) + \sum_{\substack{\alpha\in\mathbb{N}_0^n \\ \deg(g^\alpha)> k}} c_\alpha(F) g^\alpha(x) , \quad (8)$$

for all $x \in \overline{B_\infty}(0, s)$.

The first summand $R(g)$ of Eq. (8) is a polynomial since the set $\{\alpha \in \mathbb{N}_0^n : \deg(g^\alpha) \le k\}$ is finite, and it belongs to $\mathcal{P}_g(K)$. Indeed, if we denote $|\alpha| = \alpha_1 + \ldots + \alpha_n$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}_0^n$, we have that $\alpha_j \ge \frac{|\alpha|}{n}$ for some j, so $\deg(g^\alpha) = \deg(g_1^{\alpha_1} \cdots g_n^{\alpha_n}) = \alpha_1 m_1 + \ldots + \alpha_n m_n \ge \alpha_j m_j \ge \frac{|\alpha|}{n} m$, where $m = \min\{m_1, \ldots, m_n\} \ge 1$ and $\frac{|\alpha|}{n} m > k$ if and only if $|\alpha| > \frac{nk}{m}$. So

$$\{\alpha \in \mathbb{N}_0^n : \deg(g^\alpha) \le k\} \subset \left\{\alpha \in \mathbb{N}_0^n : |\alpha| \le \frac{nk}{m}\right\} .$$

The second summand $S(g)$ of Eq. (8) is a holomorphic function on some open set containing $\overline{B_\infty}(0, r)$. Clearly $\int_{r\mathbb{T}^n} \frac{g^\alpha(x)}{x^{\alpha+\mathbf{1}}} dx = 0$ for all $|\alpha| \le k < \deg(g^\alpha)$, where $\mathbb{T}$ stands for the torus and $\mathbf{1} = (1, \ldots, 1)$. Hence, by the Cauchy integral formula, we have that $c_\alpha(S(g)) = 0$ for all $|\alpha| \le k$. As a consequence, $c_\alpha(P - R(g)) = c_\alpha(S(g)) = 0$ for all $|\alpha| \le k$, and since $\deg(P - R(g)) \le k$ we obtain that $P = R(g) = \eta(R)$ (and $S(g) = 0$). This completes the proof. □

By [19, Theorem 5.1], every mapping $g = (g_1, \ldots, g_n)$, where g_j is a homogeneous polynomial, $j = 1, \ldots, n$, and $g^{-1}(0) = \{0\}$ is a proper polynomial mapping from $\mathbb{C}^n$ to $\mathbb{C}^n$. Hence these mappings are examples of proper polynomial mappings g satisfying the hypothesis in above corollary, i.e., $\{g_1, \ldots, g_n\}$ is a basis of $\mathcal{P}_g(K)$.

Theorem 7 *Let $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^n$ be a proper holomorphic mapping of multiplicity m. Let U be an open bounded subset of Ω with $\overline{U} \subset \Omega$ and $g^{-1}(g(U)) = U$. Then there is an algebra isometric isomorphism between the Banach algebras $\mathcal{H}_g^\infty(U)$ and $\mathcal{H}^\infty(g(U))$.*

Proof Let η be the algebra isomorphism from $\mathcal{H}(g(U))$ onto $\mathcal{H}_g(U)$ used in the proof of Theorem 6. To prove the result we show that if $h \in \mathcal{H}(g(U))$ is bounded then so is $\eta(h)$ and $\|h\|_{g(U)} = \|\eta(h)\|_U$. For fixed $h \in \mathcal{H}^\infty(g(U))$,

$$\|h\| = \sup_{x\in g(U)} |h(x)| = \sup_{z\in U} |(h \circ g)(z)| = \sup_{z\in U} |(\eta(h))(z)| = \|\eta(h)\|$$

Let $g : \Omega \subseteq \mathbb{C}^n \longrightarrow \Omega' \subseteq \mathbb{C}^n$ be a proper holomorphic mapping, $g = (g_1, \dots, g_n)$ and U be an open subset of Ω. We define

$$\begin{aligned} \pi : \mathcal{M}(\mathcal{H}_g(U)) &\longrightarrow \mathbb{C}^n \\ \psi &\rightsquigarrow (\psi(g_1), \dots, \psi(g_n)). \end{aligned} \tag{9}$$

Notice that the mapping π is well defined since the mappings $g_1, \dots, g_n$ belong automatically to the algebra $\mathcal{H}_g(U)$.

If we consider g to be the identity mapping, then the mapping π coincides with the classical mapping $\pi : \mathcal{M}(\mathcal{H}(U)) \longrightarrow \mathbb{C}^n$ defined as $\pi(\psi) = (\psi(p_1), \dots, \psi(p_n))$ for any open subset U of $\mathbb{C}^n$, where $p_1, \dots, p_n$ are the projection mappings. Also, if g is a proper polynomial mapping and U is bounded, the mapping π restricted to the Banach algebra $\mathscr{P}_g(\overline{U})$ coincides with the mapping π introduced in Eq. (3).

In the following theorem we will need open sets that *solve Gleason's problem*, which we now review. Let $\mathcal{B}(U)$ be some class of holomorphic functions on an open set $U \subset \mathbb{C}^n$. Gleason's problem is to determine the open sets U such that for each $w \in U$ and $f \in \mathcal{B}(U)$ there exist functions $h_1, \dots, h_n \in \mathcal{B}(U)$ such that

$$f(z) - f(w) = \sum_{k=1}^{n} (z_k - w_k) h_k(z) \tag{10}$$

for all $z \in U$.

In [18, Section 6.6.1] Gleason's problem is solved for the Euclidean unit ball B_n of $\mathbb{C}^n$ and several algebras of holomorphic functions such as, for example, $\mathcal{H}^\infty(B_n)$. References to the solution in strictly pseudo-convex domains is given in [18, Section 6.6]. This problem has been studied by many authors (see, e.g., [5,16,17]).

Theorem 8 *Let $g : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ be a proper polynomial mapping $g = (g_1, \dots, g_n)$. Let $U \subset \mathbb{C}^n$ be an open bounded set with $\overline{U}$ polynomially convex and $g^{-1}(g(U)) = U$. Assume that Gleason's problem is solved for $\mathcal{H}^\infty(U)$ and that the polynomials $\{g_i\}_{i=1}^n$ are a basis of the algebra $\mathcal{P}_g(\overline{U})$. Then,*

$$\pi(\mathcal{M}(\mathcal{H}_g^\infty(U))) = \overline{g(U)}.$$

Moreover, for every $x \in g(U)$,

$$\mathcal{M}_x(\mathcal{H}_g^\infty(U)) = \{\delta_w : w \in g^{-1}(x)\}.$$

Proof For the first part, note that $g(U) \subset \pi(\mathcal{M}(\mathcal{H}_g^\infty(U)))$. Indeed, for $x \in g(U)$, consider the mapping $\check{\delta}_x \in \mathcal{M}(\mathcal{H}_g^\infty(U))$ defined for every $h \in \mathcal{H}_g^\infty(U)$ as $\check{\delta}_x(h) = \delta_x(f) = f(x)$ where f is the unique function in $\mathcal{H}^\infty(g(U)))$ with $f \circ g = h$. Then, $\pi(\check{\delta}_x) = x$. On the other hand, every $\phi \in \mathcal{M}(\mathcal{H}_g^\infty(U))$ can be considered as $\phi \in \mathcal{M}(\mathscr{P}_g(\overline{U}))$ and by Corollary 3,

$$\pi(\mathcal{M}(\mathcal{H}_g^\infty(U))) \subset \pi(\mathcal{M}(\mathscr{P}_g(\overline{U}))) = g(\overline{U}) \subset \overline{g(U)}.$$

Since $\pi(\mathcal{M}(\mathcal{H}_g^\infty(U)))$ is compact and $g(U) \subset \pi(\mathcal{M}(\mathcal{H}_g^\infty(U))) \subset \overline{g(U)}$, it follows that $\pi(\mathcal{M}(\mathcal{H}_g^\infty(U))) = \overline{g(U)}$.

Now we show that for every $x \in g(U)$, $\mathcal{M}_x(\mathcal{H}_g^\infty(U)) = \{\delta_w : w \in g^{-1}(x)\}$. Let $\phi \in \mathcal{M}_x(\mathcal{H}_g^\infty(U))$. By Proposition 6, there exists $\Psi \in \mathcal{M}(\mathcal{H}^\infty(U))$ with $\Psi(f) = \phi(f)$ for all $f \in \mathcal{H}_g^\infty(U)$. Therefore, since g_i is a polynomial and Ψ is linear and multiplicative we have that

$$\begin{aligned} g(\Psi(p_1), \dots, \Psi(p_n)) &= \big(g_1(\Psi(p_1), \dots, \Psi(p_n)), \dots, g_n(\Psi(p_1), \dots, \Psi(p_n))\big) \\ &= \big(\Psi(g_1), \dots, \Psi(g_n)\big) = \big(\phi(g_1), \dots, \phi(g_n)\big) = x, \end{aligned}$$

where $p_1, \dots, p_n$ are the projection mappings.

Hence, $\big(\Psi(p_1), \dots, \Psi(p_n)\big) \in g^{-1}(x) \in U$. Therefore, the classical mapping $\pi : \mathcal{M}(\mathcal{H}^\infty(U)) \longrightarrow \mathbb{C}^n$ defined as $\pi(\Phi) = (\Phi(p_1), \dots, \Phi(p_n))$ for every $\Phi \in \mathcal{M}(\mathcal{H}^\infty(U))$ satisfies that $\pi(\Psi) \in U$. Thus, since Gleason's problem is solved for $\mathcal{H}^\infty(U)$, $\Psi = \delta_{(\Psi(p_1),\dots,\Psi(p_n))}$, hence $\phi = \delta_{(\Psi(p_1),\dots,\Psi(p_n))}$.

To finish, notice that for every $w \in g^{-1}(x)$ and every $f \in \mathcal{H}_g^\infty(U)$, we have that

$$\delta_w(f) = f(w) = f(\Psi(p_1), \dots, \Psi(p_n)) = \delta_{(\Psi(p_1),\dots,\Psi(p_n))}(f).$$

Hence,

$$\mathcal{M}_x(\mathcal{H}_g^\infty(U)) = \{\delta_w : w \in g^{-1}(x)\}.$$

5 Algebras of symmetric holomorphic functions

During the last few years the study of algebras of symmetric holomorphic functions has received a lot of attention, see [2,3,6–8,14]. We present in this section the results obtained in Sects. 3 and 4 for the particular case of algebras of symmetric functions.

The symmetric group of order n, consisting of the group of all permutations of the set $\{1, \dots, n\}$ and denoted by S_n is the classical example of a finite unitary reflection group. S_n can be naturally considered as a subgroup of $GL(n, \mathbb{C})$ with the natural action of a permutation $\sigma \in S_n$ on a point $(w_1, \dots, w_n) \in \mathbb{C}^n$ being $\sigma(w_1, \dots, w_n) = (w_{\sigma(1)}, \dots, w_{\sigma(n)})$. For this group, a basis of S_n-invariant polynomials is given by the set of polynomials

$$g_j(w_1, \dots, w_n) = \sum_{1 \le i_1 < \dots < i_j \le n} w_{i_1} \cdots w_{i_j}, \quad j = 1, \dots, n. \tag{11}$$

Another basis of S_n-invariant polynomials is given by the set of polynomials

$$g_j(w_1, \dots, w_n) = \sum_{i=1}^n w_i^j.$$

Let us consider the mapping $g = (g_1, \dots, g_n)$ where g_i are the mappings defined in Eq. (11). The *symmetrized polydisk* is the image of the polydisk of center zero and radius one under the mapping g, usually denoted by $\mathbb{G}_n = g(\mathbb{D}^n)$. The properties of the set $\mathbb{G}_n$ have been intensively studied during the last years. For instance for $n = 2$ an implicit representation of the symmetrized bidisk is as follows

$$\mathbb{G}_2 = \{(s, p) \in \mathbb{C}^2 : |s - s\overline{p}| + |p|^2 < 1\}.$$

The symmetrized bidisk can be naturally identified with the quotient space of the bidisk $\mathbb{D}^2$ under the obvious action of the group of two elements S_2. However the geometries of the sets $\mathbb{D}^2$ and $\mathbb{G}_2$ are very different. For instance, the symmetrized bidisk is not convex though it is polynomially convex, hypoconvex and starlike about (0, 0). It also has as distinguished boundary the symmetrized torus, which is topologically a Möbius band. For a complete description of the geometrical properties of the set $\mathbb{G}_n$ see, e.g., [4] and [11].

Here we study the algebras of symmetric functions on $\mathbb{D}^n$ and relate these algebras with the respective algebras of functions on $\mathbb{G}_n$. Let us fix an open set $U \subset \mathbb{C}^n$ with $(w_{\sigma(1)}, \ldots, w_{\sigma(n)}) \in U$ for all $(w_1, \ldots, w_n) \in U$ and all $\sigma \in S_n$ and an algebra $\mathcal{B}(U)$ of functions defined on U. We denote by

$$\mathcal{B}_s(U) = \{f \in \mathcal{B}(U) : f(w_1, \ldots, w_n) = f(w_{\sigma(1)}, \ldots, w_{\sigma(n)}) \text{ for all } \sigma \in S_n \text{ and all } (w_1, \ldots, w_n) \in U\}$$

the subalgebra of $\mathcal{B}(U)$ consisting of all the symmetric functions of $\mathcal{B}(U)$.

As a consequence of Propositions 3 and 4 we obtain that the algebra $\mathcal{B}_s$ coincides with the algebra $\mathcal{B}_g$.

Proposition 7 *If $g = (g_1, \ldots, g_n)$ where g_i is defined as in Eq. (11), then*

$$\mathscr{P}_s(\overline{\mathbb{D}}^n) = \mathscr{P}_{S_n}(\overline{\mathbb{D}}^n) = \mathscr{P}_g(\overline{\mathbb{D}}^n)$$

and for any subalgebra $\mathcal{B}(\mathbb{D}^n)$ of $\mathcal{H}(\mathbb{D}^n)$,

$$\mathcal{B}_s(\mathbb{D}^n) = \mathcal{B}_{S_n}(\mathbb{D}^n) = \mathcal{B}_g(\mathbb{D}^n).$$

By Corollary 3 and using that the polydisk is polynomially convex and $\overline{\mathbb{G}}_n = g(\overline{\mathbb{D}}^n)$ we obtain the following description of the spectrum of $\mathcal{M}(\mathscr{P}_g(\overline{\mathbb{D}}^n))$.

Theorem 9 *If $g = (g_1, \ldots, g_n)$ where g_i is defined as in Eq. (11), then*

$$\pi(\mathcal{M}(\mathscr{P}_s(\overline{\mathbb{D}}^n))) = \overline{\mathbb{G}}_n.$$

Moreover, for every $x \in \overline{\mathbb{G}}_n$,

$$\mathcal{M}_x(\mathscr{P}_g(\overline{\mathbb{D}}^n)) = \{\delta_w : w \in g^{-1}(z)\}.$$

By using the results obtained in Sect. 4 we obtain a description of the algebras of holomorphic symmetric functions on $\mathbb{D}^n$.

Theorem 10 *If $g = (g_1, \dots, g_n)$ where g_i is defined as in Eq. (11), then $\mathcal{H}_s(\mathbb{D}^n)$ is algebra isomorphic to $\mathcal{H}(\mathbb{G}_n)$ and $\mathcal{H}_s^\infty(\mathbb{D}^n)$ is algebra isometrically isomorphic to $\mathcal{H}^\infty(\mathbb{G}_n)$.*

Moreover,

$$\pi(\mathcal{M}(\mathcal{H}_s^\infty(\mathbb{D}^n))) = \overline{\mathbb{G}}_n$$

and, for every $x \in \mathbb{G}_n$,

$$\mathcal{M}_x(\mathcal{H}_s^\infty(\mathbb{D}^n)) = \{\delta_w : w \in g^{-1}(x)\}.$$

For the particular case of $n = 2$, if we denote by (x, y) an element of $\overline{\mathbb{G}}_2$, then

$$\begin{cases} x = z + w \\ y = zw. \end{cases} \tag{12}$$

for some $(z, w) \in \overline{\mathbb{D}}^2$. Therefore, by solving the system of equations with respect to the variables z and w we obtain that

$$\{\delta_{(z,w)} : (z, w) \in g^{-1}(z, y)\} = \left\{ \delta_{\left(\frac{x+\sqrt{x^2-4y}}{2}, \frac{x-\sqrt{x^2-4y}}{2}\right)}, \delta_{\left(\frac{x-\sqrt{x^2-4y}}{2}, \frac{x+\sqrt{x^2-4y}}{2}\right)} \right\}.$$

To finish it is worth remarking that the techniques used here can be applied to study subalgebras of these Banach algebras. In the following example we study a proper subalgebra of $\mathscr{P}_s(\overline{\mathbb{D}}^n)$. This example also answers in the negative the following natural question: Given a proper holomorphic mapping g, does the algebra $\mathscr{P}_g(\overline{\mathbb{D}}^n)$ coincide with $\mathscr{P}_G(\overline{\mathbb{D}}^n)$ for some subgroup G of S_n?

Example 1 Let $g = (g_1, \dots, g_n)$ where g_i is defined as in Eq. (11). We consider the mapping $g^2(z) = (g_1^2(z), \dots, g_n^2(z))$, where $g_i^2(z)$ stands for the composition of the mapping g_i and the squaring mapping, for $i = 1, \dots, n$. As a consequence of [19, Theorem 5.1] and Corollary 5, $g^2 : \mathbb{C}^n \longrightarrow \mathbb{C}^n$ is a proper holomorphic mapping such that $\{g_1^2, \dots, g_n^2\}$ is a basis of the algebra $\mathscr{P}_{g^2}(\overline{\mathbb{D}}^n)$. This algebra is a proper subalgebra of $\mathscr{P}_g(\overline{\mathbb{D}}^n) = \mathscr{P}_{S_n}(\overline{\mathbb{D}}^n)$ since $g_1 \in \mathscr{P}_g(\overline{\mathbb{D}}^n) \setminus \mathscr{P}_{g^2}(\overline{\mathbb{D}}^n)$. Furthermore, for $n = 2$, the only subgroups of S_2 are the trivial subgroup Id, consisting of the identity permutation, and the whole group S_2. Therefore, the algebra $\mathscr{P}_{g^2}(\overline{\mathbb{D}}^2)$ associated to the mapping $g^2(z) = ((z_1+z_2)^2, (z_1z_2)^2)$ does not coincide with the algebra $\mathscr{P}_G(\overline{\mathbb{D}}^2)$ for any subgroup G of S_2. Indeed,

$$\mathscr{P}_{g^2}(\overline{\mathbb{D}}^2) \subsetneq \mathscr{P}_{S_2}(\overline{\mathbb{D}}^2) \subsetneq \mathscr{P}_{Id}(\overline{\mathbb{D}}^2) = \mathscr{P}(\overline{\mathbb{D}}^2).$$

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