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Local Cauchy theory for the nonlinear Schrödinger equation in spaces of infinite mass

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Abstract We consider the Cauchy problem for the nonlinear Schrödinger equation on $\mathbb{R}^d$, where the initial data is in $\dot{H}^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$. We prove local well-posedness for large ranges of p and discuss some global well-posedness results.

Keywords Nonlinear Schrödinger equation · Local well-posedness · Global well-posedness

Mathematics Subject Classification 35Q55 · 35A01

1 Introduction

In this work, we consider the classical nonlinear Schrödinger equation over $\mathbb{R}^d$:

$$iu_t+\Delta u+\lambda|u|^\sigma u = 0, \quad u = u(t, x), \quad (t, x) \in \mathbb{R}\times\mathbb{R}^d,\ \lambda \in \mathbb{R},\ 0 < \sigma < 4/(d-2)^+$$

and focus on the corresponding Cauchy problem $u(0) = u_0 \in E$, where E is a suitable function space. This model equation is the subject of more than fifty years of intensive research, which makes us unable to give a complete list of important references (we simply refer the monographs [3,11,12] and references therein). The usual framework one considers is $E = H^1(\mathbb{R}^d)$, the so-called energy space, or more generally, $E = H^s(\mathbb{R}^d)$. A common property of these spaces is that they are L^2-based. The reason for this constraint comes from the fact that the linear group is bounded in L^2, but not in any other L^p.

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In the sense of lifting the L^2 constraint, various approaches have been considered: in [7], one considers local well-posedness on Zhidkov spaces

$$E = X^k(\mathbb{R}^d) = \{u \in L^\infty(\mathbb{R}^d) : \nabla u \in H^{k-1}(\mathbb{R}^d)\}.$$

In [8], one takes the Gross–Pitaevskii equation and looks for local well-posedness on

$$E = \{u \in H^1_{loc}(\mathbb{R}^d) : \nabla u \in L^2(\mathbb{R}^d), |u|^2 - 1 \in L^2(\mathbb{R}^d)\}.$$

In [1], one takes E as a modulation space, which allows to measure in a more precise way the concentration of a tempered distribution in both space and frequency. This leads to some local well-posedness results which require less regularity on the initial condition. In [15], one deals with the case $E = W^{s,p}(\mathbb{R}^d)$, $1 < p < 2$ (where the local well-posedness is proven for a weaker notion of solution). In [13], one takes the one-dimensional cubic (NLS) and considers

$$E = \left\{u \in \mathcal{S}'(\mathbb{R}^d) : \|e^{it\partial_{xx}} u\|_{L^3([-1,1];L^6(\mathbb{R}))} < \infty\right\}.$$

Finally, in [5], one considers $E = H^1(\mathbb{R}^2) + X$, where X is either a particular space of bounded functions with no decay or a subspace of $L^4(\mathbb{R}^2)$ (and not of $L^2(\mathbb{R}^2)$).

The aim of this paper is to look for local well-posedness results over another class of spaces, namely

$$E = X_p(\mathbb{R}^d) = \dot{H}^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d), \quad 2 < p < 2d/(d-2)^+.$$

In particular, we obtain local well-posedness in the most general energy space $X_{\sigma+2}(\mathbb{R}^d)$ and obtain global well-posedness over $X_p(\mathbb{R}^d)$ in the defocusing case $\lambda < 0$ for all $p \leqslant \sigma + 2$.

Remark 1.1 Our results can be extended to more general nonlinearities $f(u)$ as in the H^1 framework. We present our results for $f(u) = |u|^\sigma u$ so not to complicate unnecessarily the proofs and deviate from the main ideas.

We briefly explain the structure of this work: in Sect. 2, we derive the required group estimates and show that the Schrödinger group is well-defined over $X_p(\mathbb{R}^d)$. In Sect. 3, we show local well-posedness for $p \leqslant 2\sigma + 2$, where the use of Strichartz estimates is available. We also prove global well-posedness for small σ (cf. Proposition 3.6). In Sect. 4, we deal with the complementary case $p > 2\sigma + 2$ in dimensions $d = 1, 2$.

Notation The norm over $L^p(\mathbb{R}^d)$ will be denoted as $\|\cdot\|_p$ or $\|\cdot\|_{L^p}$, whichever is more convenient. The spatial domain $\mathbb{R}^d$ will often be ommited. The free Schrödinger group in $H^1(\mathbb{R}^d)$ is written as $\{S(t)\}_{t\in\mathbb{R}}$. We write $p^* = dp/(d-p)^+$. To avoid repetition, we hereby set $2 < p < 2^*$ and $0 < \sigma < 4/(d-2)^+$.

2 Linear estimates

We recall the essential Strichartz estimates. We say that (q, r) is an admissible pair if

$$2 \leqslant r \leqslant 2^*, \quad \frac{2}{q} = d\left(\frac{1}{2} - \frac{1}{r}\right), \quad r \neq \infty \text{ if } d = 2.$$

Lemma 2.1 (Strichartz estimates) *Given two admissible pairs (q, r) and (γ, ρ), we have, for all sufficiently regular u_0 and f and for any interval $I \subset \mathbb{R}$,*

$$\|S(\cdot)u_0\|_{L^q(I;L^r(\mathbb{R}^N))} \lesssim \|u_0\|_2 \tag{2.0}$$

and

$$\left\| \int_{0<s<t} S(t-s)f(s)ds \right\|_{L^q(I;L^r(\mathbb{R}^N))} \lesssim \|f\|_{L^{\gamma'}(I;L^{\rho'}(\mathbb{R}^N))}. \tag{2.1}$$

Remark 2.1 The estimate (2.1) may be extended to other sets of admisssible pairs: see [6] and [14]. However, the linear estimate (2.0) is not valid for any other pairs and for $u_0 \notin L^2(\mathbb{R}^d)$.

Proposition 2.2 (Group estimates with loss of derivative) *Define k so that (k, p) is admissible. Then*

- *(Linear estimate) For $\phi \in \mathcal{S}(\mathbb{R}^d)$,*

$$\|S(t)\phi\|_p^2 \lesssim \|\phi\|_p^2 + |t|^{1-\frac{2}{k}}\|\nabla\phi\|_2^2, \quad t \in \mathbb{R}.$$

- *(Non-homogeneous estimate) For $f \in C([0, T]; \mathcal{S}(\mathbb{R}^d))$ and any (q, r) admissible,*

$$\left\| \int_0^{\cdot} S(\cdot - s)f(s)ds \right\|_{L^\infty((0,T);L^p(\mathbb{R}^d))} \lesssim C(T)\left(\|f\|_{L^2((0,T);L^p(\mathbb{R}^d))} + \|\nabla f\|_{L^{q'}((0,T);L^{r'}(\mathbb{R}^d))}\right), \tag{2.2}$$

where $C(\cdot)$ is a increasing bounded function over bounded intervals of $\mathbb{R}$.

Notice that, due to the scaling invariance of the Schrödinger equation, the polynomial growth in time in the linear estimate is unavoidable.

Proof For the linear estimate, write $u = S(t)\phi$. Then $u \in C^1(\mathbb{R}; H^2(\mathbb{R}^d))$ satisfies

$$iu_t + \Delta u = 0, \quad u(0) = \phi.$$

Multiplying the equation by $|u|^{p-2}\bar{u}$, integrating over $\mathbb{R}^d$ and taking the imaginary part, we obtain

$$\frac{1}{p}\frac{d}{dt}\|u(t)\|_p^p \leqslant \left|\mathrm{Im}\int |u|^{p-2}\bar{u}\Delta u\right| \leqslant \frac{p-2}{2}\int |u|^{p-2}|\nabla u|^2$$
$$\leqslant \frac{p-2}{2}\|u(t)\|_p^{p-2}\|\nabla u(t)\|_p^2$$

Thus we have

$$\frac{d}{dt}\|u(t)\|_p^2 \leqslant (p-2)\|\nabla u(t)\|_p^2.$$

An integration between 0 and $t \in \mathbb{R}$ and the linear Strichartz estimate yield

$$\|u(t)\|_p^2 \lesssim \|\phi\|_p^2 + \int_0^t \|\nabla u(s)\|_p^2 ds \lesssim \|\phi\|_p^2 + |t|^{1-\frac{2}{k}}\left(\int_0^t \|\nabla u(s)\|_p^k ds\right)^{\frac{2}{k}}$$
$$\lesssim \|\phi\|_p^2 + |t|^{1-\frac{2}{k}}\|\nabla\phi\|_2^2.$$

For the non-homogeneous estimate, set $v(t) = -i\int_0^t S(t-s)f(s)ds$. Then $v \in C^1([0,T]; H^1(\mathbb{R}^d))$ satisfies

$$iv_t + \Delta v = f, \quad v(0) = 0.$$

As for the previous estimate, we have

$$\frac{1}{p}\frac{d}{dt}\|v(t)\|_p^p \lesssim \|v(t)\|_p^{p-2}\|\nabla v(t)\|_p^2 + \|v(t)\|_p^{p-1}\|f(t)\|_p$$

and so

$$\frac{d}{dt}\|v(t)\|_p^2 \lesssim \|\nabla v(t)\|_p^2 + \|v(t)\|_p\|f(t)\|_p \lesssim \|\nabla v(t)\|_p^2 + \|v(t)\|_p^2 + \|f(t)\|_p^2.$$

The required estimate now follows by direct integration in $(0,t)$, $0 < t < T$, and by the non-homogeneous Strichartz estimate. □

Lemma 2.3 (Local Strichartz estimate without loss of derivatives) *Given* $f \in C([0,T], \mathcal{S}(\mathbb{R}^d))$,

$$\left\|\int_0^\cdot S(\cdot - s)f(s)ds\right\|_{L^\infty((0,T),L^p(\mathbb{R}^d))} \lesssim C(T,q)\|f\|_{L^{q'}((0,T),L^{p'}(\mathbb{R}^d))}, \quad \frac{1}{q} > d\left(\frac{1}{2}-\frac{1}{p}\right). \tag{2.3}$$

Proof This estimate follows easily from the decay estimates of the Schrödinger group: indeed, given $0 < t < T$,

$$\begin{aligned}\left\|\int_0^t S(t-s)f(s)ds\right\|_{L^p(\mathbb{R}^d)} &\lesssim \int_0^T \|S(t-s)f(s)\|_{L^p(\mathbb{R}^d)}ds \\ &\lesssim \int_0^T \frac{1}{|t-s|^{d\left(\frac{1}{2}-\frac{1}{p}\right)}}\|f(s)\|_{L^{p'}(\mathbb{R}^d)}ds \\ &\lesssim \left(\int_0^T \frac{1}{|t-s|^{qd\left(\frac{1}{2}-\frac{1}{p}\right)}}ds\right)^{\frac{1}{q}} \|f\|_{L^{q'}((0,T),L^{p'}(\mathbb{R}^d))}.\end{aligned}$$

□

We set

$$X_p(\mathbb{R}^d) = L^p(\mathbb{R}^d) \cap \dot{H}^1(\mathbb{R}^d).$$

Remark 2.2 From the Gagliardo–Nirenberg inequality, we have $H^1(\mathbb{R}^d) \hookrightarrow X_p(\mathbb{R}^d)$.

Proposition 2.4 *The Schrödinger group* $\{S(t)\}_{t\in\mathbb{R}}$ *over* $H^1(\mathbb{R}^d)$ *defines, by continuous extension, a one-parameter continuous group on* $X_p(\mathbb{R}^d)$.

Proof Given any $\phi \in \dot{H}^1(\mathbb{R}^d)$, we have $\|S(t)\nabla\phi\|_2 = \|\nabla\phi\|_2$. Together with Proposition 2.2, this implies that

$$\|S(t)\phi\|_{X_p} \lesssim (1+|t|^{1-\frac{2}{k}})^{1/2}\|\phi\|_{X_p}, \ t \in \mathbb{R}.$$

Therefore, for each fixed $t \in \mathbb{R}$, $S(t)$ may be extended continuosly to X_p. By density, it follows easily that $S(t+s) = S(t)S(s)$, $t, s \in \mathbb{R}$, and $S(0) = I$ on X_p. Finally, we prove continuity at $t = 0$: given $\phi \in X_p(\mathbb{R}^d)$ and $\epsilon > 0$, take $\phi_\epsilon \in H^1(\mathbb{R}^d)$ such that

$$\|\phi_\epsilon - \phi\|_{X_p} < \epsilon.$$

Then

$$\begin{aligned}\limsup_{t\to 0}\|S(t)\phi - \phi\|_{X_p} &\leqslant \limsup_{t\to 0}\big(\|S(t)(\phi-\phi_\epsilon)\|_{X_p} + \|S(t)\phi_\epsilon \\ &\quad - \phi_\epsilon\|_{X_p} + \|\phi_\epsilon - \phi\|_{X_p}\big) \\ &\lesssim \limsup_{t\to 0}\Big((1+|t|^{1-\frac{2}{k}})^{1/2}\|\phi-\phi_\epsilon\|_{X_p} \\ &\quad + \|S(t)\phi_\epsilon - \phi_\epsilon\|_{H^1}\Big) \lesssim \epsilon.\end{aligned}$$

□

Remark 2.3 Fix $d = 1$. Using the same ideas, one may easily observe that the Schrödinger group is well-defined on the Zhidkov space

$$X^2(\mathbb{R}) = \{u \in L^\infty(\mathbb{R}) : \nabla u \in H^1(\mathbb{R})\}.$$

Indeed, for any $2 \leqslant p \leqslant \infty$ a direct integration of the equation gives

$$\frac{d}{dt}\|u(t)\|_p \leqslant \|\Delta u(t)\|_p.$$

Hence, choosing k so that (k, p) is an admissible pair,

$$\begin{aligned}\|u(t)\|_p &\leqslant \|u_0\|_p + \int_0^t \|\Delta u(s)\|_p ds \leqslant \|u_0\|_p + Ct^{1-\frac{1}{k}}\|\Delta u\|_{L^k((0,t),L^p)}\\ &\leqslant \|u_0\|_p + Ct^{1-\frac{1}{k}}\|\Delta u_0\|_2,\end{aligned}$$

where C is a constant independent on p (this comes from the fact that such a constant may be obtained via the interpolation between $L_t^\infty L_x^2$ and $L_t^4 L_x^\infty$). Then, taking the limit $p \to \infty$, we obtain

$$\|u(t)\|_\infty \lesssim \|u_0\|_\infty + t^{\frac{3}{4}}\|\Delta u_0\|_2, \quad t > 0,\ u_0 \in H^2(\mathbb{R}).$$

For higher dimensions, a similar procedure may be applied, at the expense of some derivatives (one must use Sobolev injection to control L^p, with p large). As one might expect, this argument does not provide the best possible estimate: in [7], one may see that

$$\|u(t)\|_\infty \lesssim (1 + t^{\frac{1}{4}})\left(\|u_0\|_\infty + \|\nabla u_0\|_2\right), \quad t > 0, u_0 \in H^1(\mathbb{R}).$$

Remark 2.4 One may ask if the required regularity is optimal: can we define the Schrödinger group on $X_p^s(\mathbb{R}^d) := \dot{H}^s(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$? What is the optimal s? Taking into consideration the previous remark, we conjecture that it should be possible to lower the regularity assumption. This entails a deeper analysis of the Schrödinger group, as it was done in [7].

3 Local well-posedness for $p \leqslant 2\sigma + 2$

In order to clarify what do we mean by a solution of (NLS), we give the following

Definition 3.1 (*Solution over* $X_p(\mathbb{R}^d)$) Given $u_0 \in X_p(\mathbb{R}^d)$, we say that $u \in C([0, T], X_p(\mathbb{R}^d))$ is a solution of (NLS) with initial data u_0 if the Duhamel formula is valid:

$$u(t) = S(t)u_0 + i\lambda \int_0^t S(t-s)|u(s)|^\sigma u(s) ds, \quad t \in [0, T].$$

Throughout this section, let (γ, ρ) and (q, r) be admissible pairs such that

$$r = (\sigma + 1)\rho' = \max\{\sigma + 2, p\}. \tag{3.1}$$

It is easy to check that such pairs are well-defined for $p \leqslant 2\sigma + 2$.

Proposition 3.2 (Uniqueness over $X_p(\mathbb{R}^d)$) *Suppose that $p \leqslant 2\sigma + 2$. Let $u_1, u_2 \in C([0,T], X_p(\mathbb{R}^d))$ be two solutions of (NLS) with initial data $u_0 \in X_p(\mathbb{R}^d)$. Then $u_1 \equiv u_2$.*

Proof Taking the difference between the Duhamel formula for u_1 and u_2,

$$u_1(t) - u_2(t) = i\lambda \int_0^t S(t-s)\left(|u_1(s)|^\sigma u_1(s) - |u_2(s)|^\sigma u_2(s)\right) ds$$

Then, for any interval $J = [0,t] \subset [0,T]$, since $X_p(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$,

$$\begin{aligned}
\|u_1 - u_2\|_{L^q(J,L^r)} &\lesssim \| |u_1|^\sigma u_1 - |u_2|^\sigma u_2 \|_{L^{\gamma'}(J,L^{\rho'})} \\
&\lesssim \left\| (\|u_1\|_r^\sigma + \|u_2\|_r^\sigma)\|u_1 - u_2\|_r \right\|_{L^{\gamma'}(J)} \\
&\lesssim \Big(\|u_1\|_{L^\infty([0,T],X_p(\mathbb{R}^d))} \\
&\quad + \|u_2\|_{L^\infty([0,T],X_p(\mathbb{R}^d))} \Big) \|u_1 - u_2\|_{L^{\gamma'}(J,L^r)} \\
&\lesssim C(T)\|u_1 - u_2\|_{L^{\gamma'}(J,L^r)}
\end{aligned}$$

The claimed result now follows from [3, Lemma 4.2.2]. □

Theorem 3.3 (Local well-posedness on $X_p(\mathbb{R}^d)$, $p \leqslant 2\sigma + 2$) *Given $u_0 \in X_p(\mathbb{R}^d)$, there exists $T = T(\|u_0\|_{X_p}) > 0$ and an unique solution*

$$u \in C([0,T), X_p(\mathbb{R}^d)) \cap L^\gamma((0,T), \dot{W}^{1,\rho}(\mathbb{R}^d)) \cap L^q((0,T), \dot{W}^{1,r}(\mathbb{R}^d))$$

of (NLS) with initial data u_0. One has

$$u - S(\cdot)u_0 \in C([0,T].L^2(\mathbb{R}^d)) \cap L^q((0,T), L^r(\mathbb{R}^d)) \cap L^\gamma((0,T), L^\rho(\mathbb{R}^d)). \quad (3.2)$$

Moreover, the solution depends continuously on the initial data and may be extended in an unique way to a maximal time interval $[0, T^*(u_0))$. If $T^*(u_0) < \infty$, then

$$\lim_{t \to T^*(u_0)} \|u(t)\|_{X_p} = +\infty.$$

Remark 3.1 The property (3.2) is a type of nonlinear "smoothing" effect: the integral term in Duhamel's formula turns out to have more integrability than the solution itself (a similar property was seen in [8,13]). This insight allows the use of Strichartz estimates at the zero derivatives level. Without this possibility, one would be restricted to the estimate (2.3) and the possible ranges of σ and p would be significantly smaller.

Proof *Step 1* Define

$$\mathcal{S}_0 = L^\infty((0,T), L^2) \cap L^q((0,T), L^r) \cap L^\gamma((0,T), L^\rho).$$

and

$$\mathcal{S}_1 = L^\infty((0,T), H^1) \cap L^q((0,T), W^{1,r}) \cap L^\gamma((0,T), W^{1,\rho}).$$

Consider the space

$$\mathcal{E} = \Big\{ u \in L^\infty((0,T), X_p) \cap L^\gamma((0,T), \dot{W}^{1,\rho}) \cap L^q((0,T), \dot{W}^{1,r}) : \\ \|u\| := \|u\|_{L^\infty((0,T),L^p)} + \|u - S(\cdot)u_0\|_{\mathcal{S}_1} \leqslant M \Big\}.$$

endowed with the distance

$$d(u,v) = \|u - v\|_{\mathcal{S}_0}.$$

It is not hard to check that $(\mathcal{E}, d)$ is a complete metric space: indeed, if $\{u_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{E}$, then $\{u_n - S(\cdot)u_0\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{S}_0$. Then there exists $u \in \mathcal{D}'([0,T]\times\mathbb{R}^d)$ such that $u_n - S(\cdot)u_0 \to u - S(\cdot)u_0$ in $\mathcal{S}_0$. By [3, Theorem 1.2.5], this convergence implies that

$$u - S(\cdot)u_0 \in \mathcal{S}_1, \quad \|u - S(\cdot)u_0\|_{\mathcal{S}_1} \leqslant \liminf \|u_n - S(\cdot)u_0\|_{\mathcal{S}_1}$$

Finally, it follows from the Gagliardo–Nirenberg inequality that, for some $0 < \theta < 1$,

$$\|u_n - u\|_{L^\infty((0,T),L^p)} \lesssim \|u_n - u\|^{1-\theta}_{L^\infty((0,T),L^2)} \|\nabla u_n - \nabla u\|^\theta_{L^\infty((0,T),L^2)} \to 0$$

and so $u_n \to u$ in $L^\infty((0,T), L^p)$.
Step 2 Define, for any $u \in \mathcal{E}$,

$$(\Phi u)(t) = S(t)u_0 + i\lambda \int_0^t S(t-s)|u(s)|^\sigma u(s)ds, \quad 0 < t < T.$$

It follows from the definition of r (see (3.1)) that $X_p(\mathbb{R}^d) \hookrightarrow L^r(\mathbb{R}^d)$. Then

$$\begin{aligned}
\|\Phi u - S(\cdot)u_0\|_{\mathcal{S}_1} &\lesssim \||u|^\sigma u\|_{L^{\gamma'}((0,T),W^{1,\rho'})} \\
&\lesssim \Big\| \|u\|_r^\sigma (\|u\|_r + \|\nabla u\|_r) \Big\|_{L^{\gamma'}(0,T)} \\
&\lesssim \Big\| \|u\|_{X_p}^\sigma (\|u\|_{X_p} + \|\nabla(u - S(\cdot)u_0)\|_r + \|S(\cdot)\nabla u_0\|_r \Big\|_{L^{\gamma'}(0,T)} \\
&\lesssim T^{\frac{1}{\gamma'}} \|u\|^{\sigma+1}_{L^\infty((0,T),X_p)} \\
&\quad + T^{\frac{1}{\gamma'}-\frac{1}{q}} \|u\|^\sigma_{L^\infty((0,T),X_p)} \|\nabla(u - S(\cdot)u_0)\|_{L^q((0,T),L^r)} \\
&\quad + T^{\frac{1}{\gamma'}-\frac{1}{q}} \|u\|^\sigma_{L^\infty((0,T),X_p)} \|S(\cdot)\nabla u_0\|_{L^q((0,T),L^r)} \\
&\lesssim \left(T^{\frac{1}{q'}} + T^{\frac{1}{q'}-\frac{1}{q}} \right) (M^{\sigma+1} + \|u_0\|^{\sigma+1}_{X_p}).
\end{aligned}$$

It follows that, for $M \sim 2\|u_0\|_{X_p}$ and T sufficiently small, we have $\Phi : \mathcal{E} \mapsto \mathcal{E}$.
Step 3 Now we show a contraction estimate: given $u, v \in \mathcal{E}$,

$$\begin{aligned} d(\Phi(u), \Phi(v)) &\lesssim \||u|^\sigma u - |v|^\sigma v\|_{L^{\gamma'}(L^{\rho'})} \\ &\lesssim \left\|(\|u\|_r^\sigma + \|v\|_r^\sigma)\|u - v\|_r\right\|_{L^{\gamma'}(0,T)} \\ &\lesssim T^{\frac{1}{\gamma'} - \frac{1}{q}} \Big(\|u\|^\sigma_{L^\infty((0,T),X_p(\mathbb{R}^d)} \\ &\quad + \|u\|^\sigma_{L^\infty((0,T),X_p(\mathbb{R}^d)}\Big) \|u - v\|_{L^q((0,T),L^r)} \\ &\lesssim T^{\frac{1}{q'} - \frac{1}{q}} \left(M^\sigma + \|u_0\|^\sigma_{X_p}\right) d(u, v). \end{aligned}$$

Therefore, for $T = T(\|u_0\|_{X_p})$ small enough, the mapping $\Phi : \mathcal{E} \mapsto \mathcal{E}$ is a strict contraction and so, by Banach's fixed point theorem, Φ has a unique fixed point over $\mathcal{E}$. This gives the local existence of a solution $u \in C([0, T], X_p(\mathbb{R}^d))$ of (NLS) with initial data u_0. From the uniqueness result, such a solution can then be extended to a maximal interval of existence $(0, T^*(u_0))$. If such an interval is bounded, then necessarily one has $\|u(t)\|_{X_p} \to \infty$ as $t \to T^*(u_0)$. The continuous dependence on the initial data follows as in the H^1 case (see, for example, the proof of [3, Theorem 4.4.1]). □

Remark 3.2 The condition $p \leqslant 2\sigma + 2$ is necessary for one to use Strichartz estimates with no derivatives. Indeed, when one applies Strichartz to the integral term of the Duhamel formula, one has

$$\left\| \int_0^\cdot S(\cdot - s)|u(s)|^\sigma u(s) ds \right\|_{L^q((0,T),L^r)} \lesssim \|u\|^{\sigma+1}_{L^{\gamma'(\sigma+1)}((0,T),L^{\rho'(\sigma+1)})}$$

for any admissible pairs (q, r) and (γ, ρ). Since the solution u only lies on spaces with spatial integrability larger or equal than p, one must have $p \leqslant \rho'(\sigma + 1) \leqslant 2\sigma + 2$ (because $\rho \geqslant 2$).

Proposition 3.4 (Persistence of integrability) *Suppose that $\tilde{p} < p \leqslant 2\sigma + 2$. Given $u_0 \in X_{\tilde{p}}(\mathbb{R}^d)$, consider the $X_p(\mathbb{R}^d)$-solution $u \in C([0, T^*(u_0)), X_p)$ of (NLS) with initial data u_0. Then $u \in C([0, T^*(u_0)), X_{\tilde{p}})$.*

Proof By the local well-posedness result over $X_{\tilde{p}}(\mathbb{R}^d)$ and by the uniqueness over $X_p(\mathbb{R}^d)$, there exists a time $T_0 > 0$ such that $u \in C([0, T_0], X_{\tilde{p}}(\mathbb{R}^d))$. Thus the statement of the proposition is equivalent to saying that u does not blow-up in $X_{\tilde{p}}(\mathbb{R}^d)$ at a time $T_0 < T < T^*(u_0)$. Since u is bounded in X_p over $[0, T]$, it follows from the local existence theorem that

$$\|u - S(\cdot)u_0\|_{L^\infty((0,T),H^1)} < \infty.$$

Then, by Proposition 2.2, for any $0 < t < T$,

$$\begin{aligned}\|u\|_{L^\infty((0,t),X_{\tilde{p}})} &\lesssim \|S(\cdot)u_0\|_{L^\infty((0,t),X_{\tilde{p}})} + \|u - S(\cdot)u_0\|_{L^\infty((0,t),X_{\tilde{p}})}\\ &\lesssim \|u_0\|_{X_{\tilde{p}}} + \|u - S(\cdot)u_0\|_{L^\infty((0,t),H^1)} < \infty,\end{aligned}$$

which implies that u does not blow-up at time $t = T$. □

Proposition 3.5 (Conservation of energy) *Suppose that $p \leqslant \sigma + 2$. Given $u_0 \in X_p(\mathbb{R}^d)$, the corresponding solution u of (NLS) with initial data u_0 satisfies*

$$E(u(t)) = E(u_0) := \frac{1}{2}\|\nabla u_0\|_2^2 - \frac{\lambda}{\sigma+2}\|u_0\|_{\sigma+2}^{\sigma+2}, \ 0 < t < T(u_0).$$

Consequently, if $\lambda < 0$, then $T^*(u_0) = \infty$. Moreover, if $\lambda > 0$ and $T^*(u_0) < \infty$, then

$$\lim_{t\to T^*(u_0)} \|\nabla u(t)\|_2 = \lim_{t\to T^*(u_0)} \|u(t)\|_{\sigma+2} = \infty. \tag{3.3}$$

Proof Since the conservation law is valid for $u_0 \in H^1(\mathbb{R}^d)$, through a regularization argument, the same is true for any $u_0 \in X_p(\mathbb{R}^d)$. If $\lambda < 0$, one has

$$\|u(t)\|_{X_{\sigma+2}(\mathbb{R}^d)} \lesssim E(u_0), \quad 0 < t < T^*(u_0).$$

By the blow-up alternative, this implies that u, as a $X_{\sigma+2}(\mathbb{R}^d)$ solution, is globally defined. By persistence of integrability, this implies that u is global in $X_p(\mathbb{R}^d)$. If $\lambda > 0$, suppose by contradiction that (3.3) is not true. Then, by conservation of energy, u is bounded in $X_{\sigma+2}(\mathbb{R}^d)$ and therefore it is globally defined (as an $X_{\sigma+2}(\mathbb{R}^d)$ solution, but also as an $X_p(\mathbb{R}^d)$ solution, by persistence of integrability). □

Proposition 3.6 *Fix $\lambda > 0$. If*

$$2\sigma^2 + (d+2)\sigma \leqslant 4, \tag{3.4}$$

then, for any $u_0 \in X_{\sigma+2}(\mathbb{R}^d)$, the corresponding solution u of (NLS) is globally defined.

Remark 3.3 Notice that the condition on σ implies that $\sigma < \min\{\sqrt{2}, 4/(d+2)\} < 4/d$.

Proof Using a suitable scaling, one may reduce the case $\lambda > 0$ to $\lambda = 1$. By contradiction, assume that u blows-up at time $t = T$. The previous proposition then implies that

$$\lim_{t\to T} \|\nabla u(t)\|_2 = \infty.$$

The first step is to obtain a corrected mass conservation estimate: indeed, by direct integration of the equation,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)-S(t)u_0\|_2^2 &= -\operatorname{Im}\int |u(t)|^\sigma u(t)\overline{(u(t)-S(t)u_0)}\\ &= \operatorname{Im}\int |u(t)|^\sigma u(t)\overline{S(t)u_0}\\ &\lesssim \|u(t)\|_{\sigma+2}^{\sigma+1}\|S(t)u_0\|_{\sigma+2}.\end{aligned}$$

Integrating on $(0,t)$,

$$\begin{aligned}\|u(t)-S(t)u_0\|_2^2 &\lesssim \int_0^t \|u(s)\|_{\sigma+2}^{\sigma+1}\|S(s)u_0\|_{\sigma+2}ds\\ &\lesssim \|S(\cdot)u_0\|_{L^\infty((0,T),L^{\sigma+2})}\int_0^t \|u(s)\|_{\sigma+2}^{\sigma+1}ds\\ &\lesssim \int_0^t \|u(s)\|_{\sigma+2}^{\sigma+1}ds.\end{aligned}$$

All of these formal computations can be justified by a suitable regularization and approximation argument. The next step is to use the conservation of energy and the Gagliardo–Nirenberg inequality to obtain a bound on $\|\nabla u(t)\|_2$.

$$\begin{aligned}\frac{1}{2}\|\nabla u(t)\|_2^2 &= E(u_0)+\frac{1}{\sigma+2}\|u(t)\|_{\sigma+2}^{\sigma+2}\\ &\lesssim 1+\|u(t)-S(t)u_0\|_{\sigma+2}^{\sigma+2}+\|S(t)u_0\|_{\sigma+2}^{\sigma+2}\\ &\lesssim 1+\|\nabla(u(t)-S(t)u_0)\|_2^{\frac{d\sigma}{2}}\|u(t)-S(t)u_0\|_2^{\frac{4-(d-2)\sigma}{2}}\\ &\lesssim 1+\|\nabla(u(t)-S(t)u_0)\|_2^{\frac{d\sigma}{2}}\left(\int_0^t \|u(s)\|_{\sigma+2}^{\sigma+1}ds\right)^{\frac{4-(d-2)\sigma}{4}}\end{aligned}$$

For t close to T, $\|\nabla(u(t)-S(t)u_0)\|_2 \sim \|\nabla u(t)\|_2$ and, by conservation of energy,

$$\|u(t)\|_{\sigma+2}^{\sigma+1} \lesssim \|\nabla u(t)\|_2^{\frac{2\sigma+2}{\sigma+2}}$$

Thus

$$\|\nabla u(t)\|_2^{\frac{4-d\sigma}{2}} \lesssim 1+\left(\int_0^t \|\nabla u(s)\|_2^{\frac{2\sigma+2}{\sigma+2}}ds\right)^{\frac{4-(d-2)\sigma}{4}}$$

which, together with the condition on σ, implies that

$$g(t):=\|\nabla u(t)\|_2^{\frac{2\sigma+2}{\sigma+2}} \leqslant \|\nabla u(t)\|_2^{\frac{2(4-d\sigma)}{4-(d-2)\sigma}} \lesssim 1+\int_0^t \|\nabla u(s)\|_2^{\frac{2\sigma+2}{\sigma+2}}ds \lesssim 1+\int_0^t g(s)ds.$$

The desired contradiction now follows from a standard application of Gronwall's lemma. □

Remark 3.4 (*Global existence in the focusing L^2-subcritical regime*) As it is well-known, the global existence in $H^1(\mathbb{R}^d)$ for $\sigma < 4/d$ follows easily from the conservation of mass and energy and from the Gagliardo–Nirenberg inequality. In Proposition 3.6, we managed to perform a similar argument by using the corrected mass

$$M(t) = \|u(t) - S(t)u_0\|_2^2.$$

However, the range of exponents for which the result is valid still leaves much to be desired. We are left with some questions: Is there another choice for "corrected mass" that allows a larger range of exponents? Is it possible that the large tails of the initial data contribute to blow-up behaviour?

Remark 3.5 (*Blow-up in the L^2-(super)critical regime*) One may ask whether the known blow-up results for $\sigma \geqslant 4/d$ can be extended to initial data in $X_p(\mathbb{R}^d)$ which do not lie in $L^2(\mathbb{R}^d)$. First of all, notice that

$$X_p(\mathbb{R}^d) \cap L^2(|x|^2 dx) \hookrightarrow L^2(\mathbb{R}^d).$$

Thus, in order to obtain blow-up outside L^2, one must first show blow-up in H^1 without the finite variance assumption. This is an open problem, which has been solved in [10] under radial hypothesis and relying heavily on the conservation of mass (which is unavailable on $X_p(\mathbb{R}^d)$). For the nonradial case, recent works (see, for example, [9]) only manage to prove unboundedness of solutions of negative energy. The problem of blow-up solutions strictly in $X_p(\mathbb{R}^d)$ is an even harder problem, requiring a better control on the tails of the solution.

Remark 3.6 (*Scaling invariance*) It is useful to understand how scalings affect the $X_p(\mathbb{R}^d)$ norm: recalling that the (NLS) is invariant under the scaling $u_\lambda(t,x) = \lambda^{2/\sigma}u(\lambda^2 t, \lambda x)$, we have

$$\|u_\lambda(t)\|_p = \lambda^{\frac{2}{\sigma}-\frac{d}{p}}\|u(\lambda^2 t)\|_p, \quad \|\nabla u_\lambda(t)\|_2 = \lambda^{\frac{2}{\sigma}+1-\frac{d}{2}}\|\nabla u(\lambda^2 t)\|_2.$$

Thus the (NLS) is $X_p(\mathbb{R}^d)$-subcritical for $\sigma < 2p/d$. In this situation, global existence for small data is equivalent to global existence for any data. Recall, however, that, for $\sigma \geqslant 4/d$, existence of blow-up phenomena is known for special initial data in $H^1(\mathbb{R}^d) \hookrightarrow X_p(\mathbb{R}^d)$. Therefore, it is impossible to obtain a global existence result for small data for $4/d \leqslant \sigma < 2p/d$. Notice that in the energy case $p = \sigma + 2$, one has $\sigma < 2p/d$ for any $\sigma + 2 < 2^*$.

Remark 3.7 (*Global existence for small data*) The main obstacle in proving global existence for small data turns out to be the linear part of the Duhamel formula $S(t)u_0$, since there isn't, to our knowledge, a way to bound uniformly this term over $X_p(\mathbb{R}^d)$.

The other possibility is to leave the linear term with a space-time norm: indeed, for some powers $\sigma > 2/d$, it is well-known that, if $u_0 \in H^1(\mathbb{R}^d)$ is such that

$$\|S(\cdot)u_0\|_{L^a((0,\infty),L^{\sigma+2}(\mathbb{R}^d))} \text{ is small,} \quad a = \frac{2\sigma(\sigma+2)}{4-\sigma(d-2)},$$

then the corresponding solution of (NLS) is globally defined (see [4]). It is not hard to check that the result can be extended to $u_0 \in X_{\sigma+2}(\mathbb{R}^d)$.

4 Local well-posedness for $p > 2\sigma + 2$

As it was observed in Remark 3.2, the condition $p \leqslant 2\sigma + 2$ was necessary in order to use Strichartz estimates with no loss in regularity. For $p > 2\sigma + 2$, in order to estimate $L_t^\infty L_x^p$, one must turn to estimate (2.2), which has a loss of one derivative. Therefore the distance one defines for the fixed-point argument must include norms with derivatives. This implies the need of a local Lipschitz condition

$$||u|^\sigma \nabla u - |v|^\sigma \nabla v| \lesssim C(|u|, |v|, |\nabla u|, |\nabla v|)\left(|u-v| + |\nabla(u-v)|\right),$$

which we can only accomplish for $\sigma \geqslant 1$.

Because of the restriction $\sigma \geqslant 1$, one must have $4 < p < 2^*$, which excludes any dimension greater than three. For $d = 3$, it turns out that no range of $p > 2\sigma + 2$ can be considered. Indeed, if one uses (2.2) with $f = |u|^\sigma u$,

$$\begin{aligned}&\left\| \int_0^\cdot S(\cdot - s)|u(s)|^\sigma u(s)ds \right\|_{L^\infty((0,T),L^p)}\\ &\quad\lesssim \|u\|^{\sigma+1}_{L^{2\sigma+2}((0,T),L^{p(\sigma+1)})} + \|\nabla(|u|^\sigma u)\|_{L^{q'}((0,T),L^{r'})}.\end{aligned}$$

We focus on the first norm on the right hand side. To control such a term, either $X_p \hookrightarrow L_x^{p(\sigma+1)}$ and

$$\|u\|_{L^{2\sigma+2}((0,T),L^{p(\sigma+1)})} \lesssim T^{\frac{1}{2\sigma+2}} \|u\|_{L^\infty((0,T),X_p)},$$

or, setting $r \geqslant 2$ so that

$$1 - \frac{3}{r} = -\frac{3}{p(\sigma+1)},$$

one estimates

$$\|u\|_{L^{2\sigma+2}((0,T),L^{p(\sigma+1)})} \lesssim \|\nabla u\|_{L^{2\sigma+2}((0,T),L^r)} \lesssim T^{\frac{1}{2\sigma+2}-\frac{1}{q}} \|\nabla u\|_{L^q((0,T),L^r)}.$$

In the first case, one needs $8 < p(\sigma+1) < 2^* = 6$. In the second, one must impose $2\sigma + 2 < q$. A simple computation yields $p(3\sigma+1) < 6$, which is again impossible, since $p(3\sigma+1) > 16$.

Theorem 4.1 (Local well-posedness on $X_p(\mathbb{R}^d)$ for $d = 1, 2$) *Given $u_0 \in X_p(\mathbb{R}^d)$, there exists $T = T(\|u_0\|_{X_p}) > 0$ and an unique solution*

$$u \in C([0, T], X_p(\mathbb{R}^d))$$

of (NLS) with initial data u_0. The solution depends continuously on the initial data and may be extended uniquely to a maximal interval $[0, T^*(u_0))$. If $T^*(u_0) < \infty$, then

$$\lim_{t \to T^*(u_0)} \|u(t)\|_{X_p} = +\infty.$$

Proof Consider the space

$$\mathcal{E} = \left\{ u \in L^\infty((0, T), X_p) : \|\!|u|\!\| := \|u\|_{L^\infty((0,T),X_p)} \leqslant M \right\}.$$

endowed with the natural distance

$$d(u, v) = \|\!|u - v|\!\|.$$

The space $(\mathcal{E}, d)$ is clearly a complete metric space. If $u, v \in \mathcal{E}$, then

$$\||u|^\sigma u - |v|^\sigma v\|^2_{L^2((0,t),L^p)} \lesssim \int_0^t \left(\|u\|^{2\sigma}_{p(\sigma+1)} + \|v\|^{2\sigma}_{p(\sigma+1)} \right) \|u - v\|^2_{p(\sigma+1)} ds$$

Since $X_p(\mathbb{R}^d) \hookrightarrow L^{p(\sigma+1)}(\mathbb{R}^d)$,

$$\begin{aligned} &\||u|^\sigma u - |v|^\sigma v\|^2_{L^2((0,t),L^p)} \\ &\lesssim T \left(\|u\|^{2\sigma}_{L^\infty((0,t).X_p)} + \|v\|^{2\sigma}_{L^\infty((0,t).L^p)} \right) \|u - v\|^2_{L^\infty((0,t).X_p)}. \end{aligned} \tag{4.1}$$

Choose an admissible pair (γ, ρ) with ρ sufficiently close to 2. We have

$$\begin{aligned} \|\nabla \left(|u|^\sigma u - |v|^\sigma v \right) \|_{L^{\gamma'}((0,T),L^{\rho'})} \lesssim \Big\| &\left(|u|^{\sigma-1} + |v|^{\sigma-1} \right) (|u - v||\nabla v| \\ &+ |v||\nabla(u - v)|) \Big\|_{L^{\gamma'}((0,T),L^{\rho'})} . \end{aligned}$$

As an example, we treat the term $|u|^{\sigma-1}|u - v||\nabla v|$:

$$\begin{aligned} \||u|^{\sigma-1}|u - v||\nabla v|\|_{\rho'} &\lesssim \|u\|^{\sigma-1}_{\frac{2\sigma\rho'}{2-\rho'}} \|u - v\|_{\frac{2\sigma\rho'}{2-\rho'}} \|\nabla v\|_2 \\ &\lesssim \|u\|^\sigma_{X_p} \|u - v\|_{X_p}, \end{aligned}$$

Therefore

$$\begin{aligned}
&\|\nabla\left(|u|^\sigma u-|v|^\sigma v\right)\|_{L^{\gamma'}((0,T),L^{\rho'})}\\
&\quad\lesssim T^{\frac{1}{\gamma'}}\left(\|u\|^\sigma_{L^\infty((0,T),X_p)}+\|v\|^\sigma_{L^\infty((0,T),X_p)}\right)\|u-v\|_{L^\infty((0,T),X_p)}\\
&\quad\lesssim T^{\frac{1}{\gamma'}}M^\sigma d(u,v).
\end{aligned}\tag{4.2}$$

For $u\in\mathcal{E}$, define

$$\Phi(u)(t)=S(t)u_0+i\lambda\int_0^t S(t-s)|u(s)|^\sigma u(s)ds,\quad 0\leqslant t\leqslant T.$$

The estimates (4.1) and (4.2), together with (2.2) and Strichartz's estimates then imply that

$$\begin{aligned}
|\!|\!|\Phi(u)|\!|\!|\lesssim&\|u_0\|_{X_p}+\left\|\int_0^\cdot S(\cdot-s)|u(s)|^\sigma u(s)ds\right\|_{L^\infty((0,T),L^p)}\\
&+\left\|\int_0^\cdot S(\cdot-s)|u(s)|^\sigma u(s)ds\right\|_{L^\infty((0,T),\dot{H}^1)}\\
&\lesssim\|u_0\|_{X_p}+\left(\||u|^\sigma u\|_{L^2((0,T);L^p)}+\|\nabla(|u|^\sigma u)\|_{L^{\gamma'}((0,T);L^{\rho'})}\right)\\
&\lesssim\|u_0\|_{X_p}+\left(T^{\frac12}+T^{\frac{1}{\gamma'}}\right)M^{\sigma+1}
\end{aligned}$$

and

$$\begin{aligned}
d(\Phi(u),\Phi(v))\lesssim&\left(\||u|^\sigma u-|v|^\sigma v\|_{L^2((0,T);L^p)}+\|\nabla(|u|^\sigma u)\right.\\
&\left.-\nabla(|v|^\sigma v)\|_{L^{\gamma'}((0,T);L^{\rho'})}\right)\\
&\lesssim\left(T^{\frac12}+T^{\frac{1}{\gamma'}}\right)M^\sigma d(u,v).
\end{aligned}\tag{4.3}$$

Choosing $M\sim 2\|u_0\|_{X_p}$, for $T=T(\|u_0\|_{X_p})$ small enough, it follows that $\Phi:\mathcal{E}\mapsto\mathcal{E}$ is a strict contraction. Banach's fixed point theorem now implies that Φ has a unique fixed point over $\mathcal{E}$, which is the unique solution u of (NLS) with initial data u_0 on the interval $(0,T)$. This solution may then be extended uniquely to a maximal interval of existence $(0,T(u_0))$. The blow-up alternative follows by a standard continuation argument. Finally, if u,v are two solutions with initial data $u_0,v_0\in X_p(\mathbb{R}^d)$, as in (4.3), one has

$$d(u,v) = d(\Phi(u),\Phi(v)) \lesssim \|u_0 - v_0\|_{X_p} + \left(T^{\frac{1}{2}} + T^{\frac{1}{\gamma'}}\right) M^\sigma d(u,v)$$
$$\lesssim \|u_0 - v_0\|_{X_p}$$
$$+ \left(T^{\frac{1}{2}} + T^{\frac{1}{\gamma'}}\right) \left(\max\{\|u_0\|_{X_p}, \|v_0\|_{X_p}\}\right)^\sigma d(u,v)$$

Thus, for $T_0 = T_0(\|u_0\|_{X_p}, \|v_0\|_{X_p})$ small,

$$d(u,v) \lesssim \|u_0 - v_0\|_{X_p},$$

and continuous dependence follows. □

Proposition 4.2 (Persistence of integrability) *Fix $d = 1, 2$ and $p > \tilde{p}$. Given $u_0 \in X_{\tilde{p}}(\mathbb{R}^d)$, consider the $X_p(\mathbb{R}^d)$-solution $u \in C([0, T^*(u_0)), X_p)$ of (NLS) with initial data u_0. Then $u \in C([0, T^*(u_0)), X_{\tilde{p}})$.*

Proof As in the proof of Proposition 3.4, given $T < T^*(u_0)$, one must prove that the $L^{\tilde{p}}$ norm of u is bounded over $(0, T)$. Applying (2.2) to the Duhamel formula of u,

$$\|u\|_{L^\infty((0,T),L^{\tilde{p}})} \lesssim \|u_0\|_{X_{\tilde{p}}} + \||u|^\sigma u\|_{L^2((0,T),L^{\tilde{p}})} + \||u|^\sigma |\nabla u|\|_{L^{\gamma'}((0,T),L^{\rho'})},$$

for any admissible pair (γ, ρ). The penultimate term is treated using the injection $X_p(\mathbb{R}^d) \hookrightarrow L^{\tilde{p}(\sigma+1)}$:

$$\||u|^\sigma u\|_{L^2((0,T),L^{\tilde{p}})} = \|u\|^{\sigma+1}_{L^{2\sigma+2}((0,T),L^{\tilde{p}(\sigma+1)})} \lesssim T^{\frac{1}{2}} \|u\|^{\sigma+1}_{L^\infty((0,T),X_p(\mathbb{R}^d))} < \infty.$$

Choose ρ sufficiently close to 2 so that $X_p(\mathbb{R}^d) \hookrightarrow L^{\frac{2\sigma\rho'}{2-\rho'}}(\mathbb{R}^d)$. Then

$$\||u|^\sigma |\nabla u|\|_{L^{\gamma'}((0,T),L^{\rho'})} \lesssim \left\| \|u\|^\sigma_{\frac{2\sigma\rho'}{2-\rho'}} \|\nabla u\|_2 \right\|_{L^{\gamma'}(0,T)} \lesssim T^{\frac{1}{\gamma'}} \|u\|^{\sigma+1}_{L^\infty((0,T),X_p)} < \infty.$$

Therefore $\|u\|_{L^\infty((0,T),L^{\tilde{p}})}$ is finite and the proof is finished. □

5 Further comments

In light of the results we have proven, we highlight some new questions that have risen:

1. Local well-posedness: In dimensions $d \geqslant 3$, the local well-posedness in the case $p > 2\sigma + 2$ remains open. Is this optimal? As we have argued in Remark 3.2, this case requires new estimates for the Schrödinger group.
2. Global well-posedness: this problem is completely open for $p > \sigma + 2$. Even if the energy is well-defined, there are still several cases where global well-posedness (even for small data) remains unanswered.

3. New blow-up behaviour: in the opposite perspective, is it possible to exhibit new blow-up phenomena? This would be especially interesting either for the defocusing case or for the L^2-subcritical case, where blow-up behaviour in H^1 is impossible.
4. Stability of ground-states: in the H^1 framework, the work of [2] has shown that the ground-states are orbitally stable under H^1 perturbations. Does the result still hold if we consider X_p perturbations?

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