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On the algebraicity of Puiseux series

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Abstract We deal with the algebraicity of a formal Puiseux series in terms of the properties of its coefficients. We show that the algebraicity of a Puiseux series for given bounded degrees is determined by a finite number of explicit universal polynomial formulas. Conversely, given a vanishing polynomial, there is a closed-form formula for the coefficients of the series in terms of the coefficients of the polynomial and of an initial part of the series.

Keywords Bivariate polynomials · Algebraic Puiseux series · Implicitization · Closed form for coefficients

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1 Introduction

Let K be a field, $\overline{K}$ its algebraic closure, and x an indeterminate. We consider $K[[x]]$, the domain of formal power series with coefficients in K, and its fraction field $K((x))$. We denote by $K((\hat{x})) := \bigcup_{n=1}^{\infty} K((x^{1/n}))$ the field of formal Puiseux series (with coefficients in K). If K is of characteristic zero, by the Newton–Puiseux theorem (see e.g. [18, Theorem 3.1] and [13, p. 314, Proposition]), an algebraic closure of $K((x))$ is given by $\mathcal{P}_K := \bigcup_L L((\hat{x}))$ where L ranges over the finite extensions of K in $\overline{K}$. In

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particular, if $K = \overline{K}$, then $\mathcal{P}_K = K((\hat{x}))$. Among Puiseux series, we are interested in algebraic ones, say the Puiseux series which verify a polynomial equation $P(x, y) = 0$ with coefficients which are themselves polynomials in x: $P(x, y) \in K[x][y]$. In fact, for the problems that we will investigate, it is sufficient to consider power series without fractional exponents (see Sect. 2). Therefore our article will mainly deal with algebraic power series.

Among the numerous works concerning algebraic Puiseux or power series [3,9,17], we deal with the following questions:

- **Reconstruction of a vanishing polynomial for a given algebraic Puiseux series.** Generically, a vanishing polynomial of a given algebraic power series can be computed as a Hermite-Padé approximant [1, Chap. 7]. In fact, the algebraicity of a Puiseux series can be encoded by the vanishing of certain determinants derived from the coefficients of the series. We extend this approach by showing how to reconstruct the coefficients of a vanishing polynomial by means of some minors of these determinants (see Sect. 3). More precisely, we show that, for given bounded degrees, there are finitely many universal polynomials allowing to check the algebraicity of a series and to perform this reconstruction (see Theorem 2). Note that this result holds for K of arbitrary characteristic.
- **Description of the coefficients of an algebraic Puiseux series in terms of the coefficients of a vanishing polynomial.** An approach consists in considering that the series coefficients verify a linear recurrence relation, which allows an asymptotic computation of the coefficients. This property follows classically from the fact that an algebraic Puiseux series is *differentiably finite (D-finite)*, that is, it satisfies a linear differential equation with coefficients in $K[x]$ [2,4–6,14–16]. Another approach consists in determining a closed-form expression in terms of the coefficients of a vanishing polynomial. In this direction, P. Flajolet and M. Soria (see the habilitation thesis of M. Soria (1990) and [8]) proposed a formula in the case of a series satisfying a reduced Henselian equation (see Definition 2 for this terminology) with complex coefficients. This formula extends to coefficients in an arbitrary field of characteristic zero K via a work of Henrici [12]. Here we complete this approach to the case of a Puiseux series which satisfies a general polynomial equation $P(x, y) = 0$, by showing that the coefficients of such series can be computed applying Flajolet–Soria's formula to a polynomial naturally derived from P (see Sect. 4). In this section, the field K is required to be of zero characteristic.

2 Preliminaries

Let us denote $\mathbb{N} := \mathbb{Z}_{\geq 0}$ and $\mathbb{N}^* := \mathbb{N}\backslash\{0\} = \mathbb{Z}_{>0}$. For any set $\mathcal{E}$, we will write $|\mathcal{E}| := \text{Card}(\mathcal{E})$. For any vector of natural numbers $L = (l_1, \ldots, l_n)$, we set $L! := \prod_{i=1}^{n} l_i!$, $|L| := \sum_{i=1}^{n} l_i$ and $\|L\| := \sum_{i=1}^{n} i\, l_i$. The floor function will be written $\lfloor q \rfloor$ for $q \in \mathbb{Q}$. Let K be a (commutative) field.

We will need the following elementary lemma of linear algebra:

Lemma 1 *Let $V_1, \dots, V_p$ be a family of infinite vectors, $V_j = (v_{i,j})_{i\in\mathbb{N}^*} \in K^{\mathbb{N}^*}$. Let M denote the matrix whose columns are the V_j's. The rank of $V_1, \dots, V_p$ is less than p if and only if all the minors of order p of M vanish.*

Proof If M has rank less than p, then there is a nonzero vector $(a_j)_{j=1,\dots,p} \in K^p$ such that:

$$\sum_{j=1}^{p} a_j V_j = 0.$$

This implies that the same nontrivial linear combination vanishes for the columns of any submatrix of size $p \times p$ of M. Hence, the determinant of any of such matrix has to vanish.

Conversely, suppose that M has rank p. In particular, all the vectors V_j are nonzero. Let us consider the first nonzero coefficient of V_1. Up to a permutation of the rows (which does not change the rank of M), we may assume for simplicity that $v_{1,1} \neq 0$. Then for any $j = 2, \dots, p$, let us replace V_j by $V_j - \frac{v_{1,j}}{v_{1,1}} V_1$, which does not change the rank of M either. We obtain a new matrix M_1 whose first row is the p-tuple $(v_{1,1}, 0, \dots, 0)$, and whose columns are still all nonzero. The same process can be done with the next column, with nonzero coefficient taken without loss of generality at the second row etc. The process stops after p iterations and we obtain a matrix M_p with the p first lines consisting of a lower triangular matrix with nonzero coefficients in the diagonal. The determinant of the latter—which is equal up to the sign to a minor of order p of M—is nonzero.

Let $\tilde{y}_0 = \sum_{n\geq n_0} \tilde{c}_n x^{n/p} \in K((\hat{x})),\ \tilde{c}_{n_0} \neq 0$, be a Puiseux series. We denote

$$\tilde{y}_0 = x^{(n_0-1)/p} \sum_{n\geq 1} c_n x^{n/p} = x^{(n_0-1)/p} \tilde{z}_0 \quad \text{with } c_1 \neq 0,$$

where $c_n = \tilde{c}_{n-n_0+1}$ for $n \in \mathbb{N}^*$. The series $\tilde{y}_0$ is a root of a polynomial $\tilde{P}(x, y) = \sum_{i,j} \tilde{a}_{i,j} x^i y^j \in K[x, y]$ of degree d_y in y if and only if the series $y_0 = \sum_{n\geq 1} c_n x^n$ is a root of $x^m \tilde{P}(x^p, x^{n_0-1} y)$, the latter being a polynomial for $m = \max\{0; (1-n_0)d_y\}$. The existence of a polynomial $\tilde{P}$ cancelling $\tilde{y}_0$ is equivalent to the one of a polynomial $P(x, y) = \sum_{i,j} a_{i,j} x^i y^j$ cancelling y_0, such that, for (i, j) belonging to the support of P, one has $i \equiv (n_0 - 1) j \bmod p$ if $n_0 - 1 \geq 0$, and $i \equiv (1 - n_0)(d_y - j) \bmod p$ otherwise. Thus the algebraicity of $\tilde{y}_0$ is equivalent to that of y_0 but *with constraints* on the support of P. This leads us to the following definition:

Definition 1 Let $\mathcal{F}$ and $\mathcal{G}$ be two strictly increasing finite sequences of ordered pairs $(i, j) \in \mathbb{N}^2$ ordered anti-lexicographically:

$$(i_1, j_1) \leq (i_2, j_2) \Leftrightarrow j_1 < j_2 \text{ or } (j_1 = j_2 \text{ and } i_1 \leq i_2).$$

We suppose additionally that $\mathcal{F} \geq (0, 1) > \mathcal{G} > (0, 0)$ (thus the elements of $\mathcal{G}$ are ordered pairs of the form $(i, 0), i \in \mathbb{N}^*$, and those of $\mathcal{F}$ are of the form $(i, j),\ j \geq 1$).

We say that a series $y_0 = \sum_{n\geq 1} c_n x^n \in K[[x]]$, $c_1 \neq 0$, is **algebraic relatively to** $(\mathcal{F}, \mathcal{G})$ if there exists a polynomial $P(x, y) = \sum_{(i,j)\in\mathcal{F}\cup\mathcal{G}} a_{i,j} x^i y^j \in K[x, y]$ such that $P(x, y_0) = 0$.

Flajolet and Soria (see the habilitation thesis of M. Soria (1990) and [8]) gave a closed-form expression to compute the coefficients of a formal solution of a reduced Henselian equation in the following sense:

Definition 2 We call **reduced Henselian equation** any equation of the following type:

$$y = Q(x, y) \text{ with } Q(x, y) \in K[x, y],$$

such that $Q(0, 0) = \frac{\partial Q}{\partial y}(0, 0) = 0$ and $Q(x, 0)\not\equiv 0$.

Theorem 1 (Flajolet–Soria's formula) *Let $y = Q(x, y) = \sum_{i,j} a_{i,j} x^i y^j$ be a reduced Henselian equation. Then the coefficients of the unique solution $\sum_{n\geq 1} c_n x^n$ are given by:*

$$c_n = \sum_{m=1}^{2n-1} \frac{1}{m} \sum_{|\underline{k}|=m,\ ||\underline{k}||_1=n,\ ||\underline{k}||_2=m-1} \frac{m!}{\prod_{i,j} k_{i,j}!} \prod_{i,j} a_{i,j}^{k_{i,j}},$$

where $\underline{k} = (k_{i,j})_{i,j}$, $|\underline{k}| = \sum_{i,j} k_{i,j}$, $||\underline{k}||_1 = \sum_{i,j} i\, k_{i,j}$ and $||\underline{k}||_2 = \sum_{i,j} j\, k_{i,j}$.

Remark 1 Let us consider the particular case where the coefficients of Q verify $a_{0,j} = 0$ for all j. So, for any $\underline{k}$ such that $|\underline{k}| = m$ and $\prod_{i,j} a_{i,j}^{k_{i,j}} \neq 0$, we must have $\|\underline{k}\|_1 \geq m$. Thus, to have $\|\underline{k}\|_1 = n$, one needs to have $m \leq n$. Flajolet–Soria's Formula can be written:

$$c_n = \sum_{m=1}^{n} \frac{1}{m} \sum_{|\underline{k}|=m,\ ||\underline{k}||_1=n,\ ||\underline{k}||_2=m-1} \frac{m!}{\prod_{i,j} k_{i,j}!} \prod_{i,j} a_{i,j}^{k_{i,j}}.$$

3 Characterizing the algebraicity of a formal power series

Here we resume the remarks from [19]. The purpose of the following discussion is to translate the vanishing of a polynomial P at a formal series y_0 in terms of the vanishing of minors of an infinite matrix. Let us consider a series $Y_0 = \sum_{n\geq 1} C_n x^n \in K[(C_n)_{n\in\mathbb{N}^*}][[x]]$ where x and the C_n's are variables. We denote the multinomial expansion of the jth power Y_0^j of Y_0 by:

$$Y_0^j = \sum_{n\geq 1} C_n^{(j)} x^n$$

where $C_n^{(j)} = C_n^{(j)}(C_1, \ldots, C_{n-j+1}) \in K[C_1, \ldots, C_{n-j+1}]$. Of course, one has that $C_n^{(j)} \equiv 0$ for $n < j$ and $C_j^{(j)} = C_1^j$. For $j = 0$, let $Y_0^0 := 1$. We remark that for any n and any $j \leq n$, $C_n^{(j)}$ is a homogeneous polynomial of degree j in the C_m's for $m \leq n - j + 1$, with natural number coefficients (indeed, each monomial occurring in $C_n^{(j)}$ is of the form $C_{i_1} \dots C_{i_j}$ with $i_k \geq 1$ and $i_1 + \dots + i_j = n$, so $i_k \leq n - j + 1$ for any k).

Now suppose we are given a series $y_0 = \sum_{n \geq 1} c_n x^n \in K[[x]]$ with $c_1 \neq 0$. For any $j \in \mathbb{N}$, we denote the multinomial expansion of y_0^j by:

$$y_0^j = \sum_{n \geq 1} c_n^{(j)} x^n.$$

So, $c_n^{(j)} = C_n^{(j)}(c_1, \dots, c_{n-j+1})$.

Definition 3 (i) Given an ordered pair $(i, j) \in \mathbb{N} \times \mathbb{N}$, we call **Wilczynski vector** $V_{i,j}$ the infinite vector with components:

- if $j \geq 1$, a sequence of i zeros followed by the coefficients of the multinomial expansion y_0^j:

$$V_{i,j} := \left(0, \dots, 0, c_1^{(j)}, c_2^{(j)}, \dots, c_n^{(j)}, \dots\right);$$

- otherwise, 1 in the ith position and 0 for the other coefficients

$$V_{i,0} := (0, \dots, 1, 0, 0, \dots, 0, \dots).$$

(ii) Let $\mathcal{F}$ and $\mathcal{G}$ be two sequences as in Definition 1. We associate to $\mathcal{F}$ and $\mathcal{G}$ the **(infinite) Wilczynski matrix** whose columns are the corresponding vectors $V_{i,j}$:

$$M_{\mathcal{F},\mathcal{G}} := (V_{i,j})_{(i,j) \in \mathcal{F} \cup \mathcal{G}},$$

$\mathcal{F} \cup \mathcal{G}$ being ordered anti-lexicographically. We define also the **reduced Wilczynski matrix**, $M_{\mathcal{F},\mathcal{G}}^{red}$: it is the matrix obtained from $M_{\mathcal{F},\mathcal{G}}$ by removing the columns indexed in $\mathcal{G}$, and also removing the corresponding rows (suppress the ith row for any $(i, 0) \in \mathcal{G}$). This amounts exactly to remove the rows containing the coefficient 1 for some Wilczynski vector indexed in $\mathcal{G}$.

Lemma 2 (Wilczynski) *The series y_0 is algebraic relatively to $(\mathcal{F}, \mathcal{G})$ if and only if all the minors of order $|\mathcal{F} \cup \mathcal{G}|$ of the Wilczynski matrix $M_{\mathcal{F},\mathcal{G}}$ vanish, or also if and only if all the minors of order $|\mathcal{F}|$ of the reduced Wilczynski matrix $M_{\mathcal{F},\mathcal{G}}^{red}$ vanish.*

Proof Given a nontrivial polynomial $P(x, y) = \sum_{(i,j) \in \mathcal{F} \cup \mathcal{G}} a_{i,j} x^i y^j$, we compute:

$$P(x, y_0) = \sum_{(i,j) \in \mathcal{F}} a_{i,j} x^i \left(\sum_{n \geq 1} c_n^{(j)} x^n \right) + \sum_{(i,0) \in \mathcal{G}} a_{i,0} x^i.$$

The coefficients of the expansion of $P(x, y_0)$ with respect to the powers of x in increasing order are exactly the components of the infinite vector resulting from the following operation:

$$M_{\mathcal{F},\mathcal{G}} \cdot (a_{i,j})_{(i,j)\in\mathcal{F}\cup\mathcal{G}}.$$

The series y_0 is a root of a nonzero polynomial with support included into $\mathcal{F} \cup \mathcal{G}$ if and only if there is a non zero solution $(a_{i,j})_{(i,j)\in\mathcal{F}\cup\mathcal{G}}$ of the following equation:

$$M_{\mathcal{F},\mathcal{G}} \cdot (a_{i,j})_{(i,j)\in\mathcal{F}\cup\mathcal{G}} = 0.$$

This means that the rank of $M_{\mathcal{F},\mathcal{G}}$ is less than $|\mathcal{F} \cup \mathcal{G}|$, the number of columns of $M_{\mathcal{F},\mathcal{G}}$. The latter condition is characterized as in finite dimension by the vanishing of all the minors of maximal order (see Lemma 1).

Let us now remark that, in the infinite vector $M_{\mathcal{F},\mathcal{G}} \cdot (a_{i,j})_{(i,j)\in\mathcal{F}\cup\mathcal{G}}$, if we remove the components of number i for $(i, 0) \in \mathcal{G}$, then we get exactly the infinite vector $M^{red}_{\mathcal{F},\mathcal{G}} \cdot (a_{i,j})_{(i,j)\in\mathcal{F}}$. The vanishing of the latter means precisely that the rank of $M^{red}_{\mathcal{F},\mathcal{G}}$ is less than $|\mathcal{F}|$. Conversely, if the columns of $M^{red}_{\mathcal{F},\mathcal{G}}$ are dependent for certain $\mathcal{F}$ and $\mathcal{G}$, we denote by $(a_{i,j})_{(i,j)\in\mathcal{F}}$ a corresponding sequence of coefficients of a nontrivial vanishing linear combination of the column vectors. Then it suffices to note that the remaining coefficients $a_{k,0}$ for $(k, 0) \in \mathcal{G}$ are each uniquely determined as follows:

$$a_{k,0} = - \sum_{(i,j)\in\mathcal{F}, i<k} a_{i,j} c^{(j)}_{k-i}\,. \tag{1}$$

We deal with the implicitization problem for algebraic power series: for fixed bounded degrees in x and y, given the expression of an algebraic series, can we reconstruct a vanishing polynomial? if yes, how?

Definition 4 Let us consider the abstract version $\mathbf{M}_{\mathcal{F},\mathcal{G}}$ and $\mathbf{M}^{red}_{\mathcal{F},\mathcal{G}}$ associated to the abstract series $Y_0 = \sum_{n\geq 1} C_n x^n \in K[(C_n)_{n\in\mathbb{N}^*}][[x]]$ and to two sequences $\mathcal{F}$ and $\mathcal{G}$ of ordered pairs (i, j) as in Definition 1, of the Wilczynski matrices. We call **Wilczynski polynomial** any polynomial in the variables C_n obtained as a minor of $\mathbf{M}^{red}_{\mathcal{F},\mathcal{G}}$. We denote such Wilczynski polynomial by $Q_{\underline{k},\underline{I}}$, where $\underline{I} := ((i_1, j_1), \ldots, (i_l, j_l))$ is a subsequence of $\mathcal{F}$ indicating the l columns of $\mathbf{M}^{red}_{\mathcal{F},\mathcal{G}}$, and $\underline{k} := (k_1, k_2, \ldots, k_l)$ a strictly increasing sequence of natural numbers indicating the l rows of $\mathbf{M}^{red}_{\mathcal{F},\mathcal{G}}$ used to form the minor of $\mathbf{M}^{red}_{\mathcal{F},\mathcal{G}}$. One has that $l \in \mathbb{N}^*, l \leq |\mathcal{F}|$, l being the order of that minor, that we will also call the **order** of the Wilczynski polynomial $Q_{\underline{k},\underline{I}}$. Note also that a Wilczynski polynomial $Q_{\underline{k},\underline{I}}$ is either homogeneous of degree equal to $\sum_{(i,j)\in\underline{I}} j$ or identically 0 (indeed, the multinomial coefficients $C^{(j)}_k$ in a column indexed by (i, j) of $\mathbf{M}^{red}_{\mathcal{F},\mathcal{G}}$ are either homogenous of degree j (case $k \geq j$) or identically 0 (case $k < j$)). By convention, we call **Wilczynski polynomial of order 0** any nonzero constant polynomial.

By Lemma 2, the algebraicity of y_0 for certain $\mathcal{F}$ and $\mathcal{G}$ is equivalent to the vanishing of all the $Q_{\underline{k},\mathcal{F}}$ of order $l = |\mathcal{F}|$, at the specific values of the given coefficients c_n of y_0.

Example 1 Let $y_0 = \sum_{n\geq 1} c_n x^n \in K[[x]]$ be a series with $c_1 \neq 0$. We consider the conditions for y_0 to be a root of a polynomial of type:

$$P(x, y) = a_{2,0}x^2 + a_{2,1}x^2 y + (a_{0,2} + a_{2,2}x^2)y^2.$$

Thus, $\mathcal{F} = \{(2, 1), (0, 2), (2, 2)\}$ and $\mathcal{G} = \{(2, 0)\}$. The corresponding Wilczynski matrix is:

$$M := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & c_1^2 & 0 \\ 0 & c_1 & 2\cdot c_1 \cdot c_2 & 0 \\ 0 & c_2 & c_2^2 + 2\, c_1\, c_3 & c_1^2 \\ 0 & c_3 & 2\, c_1\, c_4 + 2\, c_2\, c_3 & 2\cdot c_1\cdot c_2 \\ 0 & c_4 & 2\, c_2\, c_4 + c_3^2 + 2\, c_1\, c_5 & c_2^2 + 2\, c_1\, c_3 \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix},$$

and the reduced matrix is:

$$M^{red} := \begin{bmatrix} 0 & 0 & 0 \\ c_1 & 2\cdot c_1 \cdot c_2 & 0 \\ c_2 & c_2^2 + 2\, c_1\, c_3 & c_1^2 \\ c_3 & 2\, c_1\, c_4 + 2\, c_2\, c_3 & 2\cdot c_1\cdot c_2 \\ c_4 & 2\, c_2\, c_4 + c_3^2 + 2\, c_1\, c_5 & c_2^2 + 2\, c_1\, c_3 \\ \vdots & \vdots & \vdots \end{bmatrix}.$$

We give the four first nontrivial Wilczynski polynomials of maximal order 3, which are equal to 3×3 minors of $\mathbf{M}^{red}$. So one has that $\underline{I} = \mathcal{F}$ as index for $Q_{\underline{k},\underline{I}}$:

$$\begin{aligned} Q_{\underline{k},\mathcal{F}} &:= -2\, C_1{}^2 \left(C_2{}^3 - 2\, C_3 C_1 C_2 + C_1{}^2 C_4\right) \quad \text{for } \underline{k} = (2, 3, 4),\\ Q_{\underline{k},\mathcal{F}} &:= -C_1 \left(C_2{}^4 - 3\, C_1{}^2 C_3{}^2 + 2\, C_1{}^3 C_5\right) \quad \text{for } \underline{k} = (2, 3, 5),\\ Q_{\underline{k},\mathcal{F}} &:= -2\, C_1{}^2 \left(-C_4 C_2{}^2 - 2\, C_1 C_4 C_3 + C_2 C_3{}^2 + 2\, C_1 C_2 C_5\right) \quad \text{for } \underline{k} = (2, 4, 5),\\ Q_{\underline{k},\mathcal{F}} &:= 8\, C_2 C_1{}^2 C_4 C_3 + C_2{}^4 C_3 - 2\, C_2{}^2 C_3{}^2 C_1 - 4\, C_1{}^2 C_2{}^2 C_5 \\ &\quad - 3\, C_1{}^2 C_3{}^3 + 2\, C_3 C_1{}^3 C_5 - 2\, C_1{}^3 C_4{}^2 \quad \text{for } \underline{k} = (3, 4, 5). \end{aligned}$$

The series y_0 is a root of a polynomial $P(x, y)$ as above if and only if all the Wilczynski polynomials of order 3 vanish at the c_n's. This implies in particular that:

$$c_4 = -\frac{c_2\left(c_2{}^2 - 2\, c_1 c_3\right)}{c_1{}^2} \quad \text{and} \quad c_5 = -\frac{c_2{}^4 - 3\, c_1{}^2 c_3{}^2}{2\, c_1{}^3}.$$

Theorem 2 *Let $\mathcal{F}$ and $\mathcal{G}$ be two finite sequences of ordered pairs as in Definition 1. We set $d_y := \max\{j,\ (i, j) \in \mathcal{F}\}$, $d_x := \max\{i,\ (i, j) \in \mathcal{F} \cup \mathcal{G}\}$ and $N := 2d_x d_y$.*

Then there exist a finite set Λ and a finite number of homogeneous polynomials $a_{i,j}^{(\lambda)} \in \mathbb{Z}[C_1, \ldots, C_N]$, $(i, j) \in \mathcal{F} \cup \mathcal{G}$, $\lambda \in \Lambda$, of total degree $\deg a_{i,j}^{(\lambda)} \leq \frac{1}{2}d_y(d_y+1)(d_x+1)-1$ for $(i, j) \in \mathcal{F}$, and $\deg a_{i,0}^{(\lambda)} \leq \frac{1}{2}d_y(d_y+1)(d_x+1)-1+i$ for $(i, 0) \in \mathcal{G}$, such that, for any series $y_0 = \sum_{n\geq 1} c_n x^n \in K[[x]]$ with $c_1 \neq 0$ algebraic relatively to $(\mathcal{F}, \mathcal{G})$, there is $\lambda \in \Lambda$ such that the polynomial:

$$P^{(\lambda)}(x, y) = \sum_{(i,j)\in\mathcal{F}} a_{i,j}^{(\lambda)}(c_1, \ldots, c_N)x^i y^j + \sum_{(i,0)\in\mathcal{G}} a_{i,0}^{(\lambda)}(c_1, \ldots, c_N)x^i \in K[x, y] \tag{2}$$

is nonzero and vanishes at y_0.

First, we give the reconstruction process. Then we will show its finiteness.

Proof Let $y_0 = \sum_{n\geq 1} c_n x^n \in K[[x]]$ with $c_1 \neq 0$ be algebraic relatively to $(\mathcal{F}, \mathcal{G})$. We show how to reconstruct a nonzero vanishing polynomial $P(x, y)$ of y_0.

We consider a minimal family $\mathcal{F}' \subseteq \mathcal{F}$ such that y_0 is algebraic relatively to $(\mathcal{F}', \mathcal{G})$. Let $Q(x, y) = \sum_{(i,j)\in\mathcal{F}'} b_{i,j}x^i y^j + \sum_{(i,0)\in\mathcal{G}} b_{i,0}x^i$ be a nonzero polynomial that vanishes at y_0. Let $m := |\mathcal{F}'|$. If $m = 1$, $Q(x, y)$ is of the form:

$$Q(x, y) = b_{i,j}x^i y^j + \sum_{(i,0)\in\mathcal{G}} b_{i,0}x^i,$$

with $b_{i,j} \neq 0$. So we must have $b_{n,0} = 0$ for $n < i + j$, and the series y_0 verifies:

$$\sum_{(n,0)\in\mathcal{G}} b_{n,0}x^n = -b_{i,j}x^i y_0^j = \sum_{n>i} -b_{i,j}c_{n-i}^{(j)}x^n.$$

By Lemma 2, the minors of order 1 of $M_{(i,j),\mathcal{G}}^{red}$, being equal to $c_{n-i}^{(j)}$ for $(n, 0) \notin \mathcal{G}$, are all zero. We fix the coefficient $a_{i,j}$ arbitrarily in $\mathbb{Z}\backslash\{0\}$: it is a constant Wilczynski polynomial. Then the other coefficients are uniquely determined in accordance with Relation (1) by the equation:

$$a_{n,0}(c_1, c_2, \ldots) := -a_{i,j}c_{n-i}^{(j)}, \quad (n, 0) \in \mathcal{G}.$$

Thus $a_{n,0}$ is a polynomial of degree j in the C_k, $k \leq n - i - j + 1$, which verifies indeed that $j \leq d_y \leq \frac{1}{2}d_y(d_y+1)(d_x+1) \leq \frac{1}{2}d_y(d_y+1)(d_x+1)-1+n$.

Suppose now that $m = |\mathcal{F}'| \geq 2$. By Lemma 2, the minors of order m of $M_{\mathcal{F}',\mathcal{G}}^{red}$ all vanish, and, because $\mathcal{F}'$ is minimal, there exists a nonzero minor of order $m-1$ of this matrix, i.e. a Wilczynski polynomial evaluated at the c_n's

$$Q_{\underline{k}_0,\underline{I}_0}(c_1, c_2, \ldots) \neq 0. \tag{3}$$

Let $(i_0, j_0) \in \mathcal{F}'$ be such that $\mathcal{F}' = \underline{I}_0 \cup \{(i_0, j_0)\}$ and p_0 be the position of (i_0, j_0) in $\mathcal{F}'$. Denote by $M_{\underline{k}_0,\underline{I}_0}$ the square matrix whose determinant is $Q_{\underline{k}_0,\underline{I}_0}(c_1, c_2, \ldots)$, and

$W_{\underline{k}_0,(i_0,j_0)}$ the truncated p_0-th column that has been removed from $M^{red}_{\mathcal{F}',\mathcal{G}}$ to form this minor. We get a system of equations with a non-vanishing determinant and $b_{i_0,j_0} \neq 0$:

$$M_{\underline{k}_0,\underline{I}_0} \cdot (b_{i,j})_{(i,j)\neq(i_0,j_0)} = -b_{i_0,j_0} W_{\underline{k}_0,(i_0,j_0)}. \tag{4}$$

Let us build polynomials $a_{i,j}$ verifying:

$$M_{\underline{k}_0,\underline{I}_0} \cdot (a_{i,j}(c_1, c_2, \ldots))_{(i,j)\neq(i_0,j_0)} = -a_{i_0,j_0}(c_1, c_2, \ldots) W_{\underline{k}_0,(i_0,j_0)}, \tag{5}$$

by taking $a_{i_0,j_0}(c_1, c_2, \ldots) := (-1)^{p_0} Q_{\underline{k}_0,\underline{I}_0}(c_1, c_2, \ldots) \neq 0$ and by computing the other $a_{i,j}(c_1, c_2, \ldots)$ by Cramer's rule. Thus the $a_{i,j}(c_1, c_2, \ldots)$ are all minors of order $m-1$ of $M^{red}_{\mathcal{F}',\mathcal{G}}$, and so, up to the sign, evaluations at the c_n's of Wilczynski polynomials $Q_{\underline{k}_0,\underline{I}}$ of order $m-1$. If $\underline{k}_0 = (k_{0,1}, \ldots, k_{0,m-1})$, we set:

$$N_{y_0} := k_{0,m-1}. \tag{6}$$

The $a_{i,j}$ are homogeneous polynomials of $\mathbb{Z}[C_1, \ldots, C_{N_{y_0}}]$. The degree of a Wilczynski polynomial $Q_{\underline{k}_0,\underline{I}}$ verifies:

$$\begin{aligned}
\deg Q_{\underline{k}_0,\underline{I}} &= \sum_{(i,j)\in\underline{I},\ c_k^{(j)} \not\equiv 0} j \\
&\leq -1 + \sum_{(i,j)\in\mathcal{F}'} j \\
&\leq -1 + (d_x + 1) \sum_{j=1}^{d_y} j \\
&= \frac{1}{2} d_y(d_y + 1)(d_x + 1) - 1.
\end{aligned}$$

The coefficients $a_{n,0}(c_1, c_2, \ldots)$ for $(n, 0) \in \mathcal{G}$ are obtained via relations (1):

$$a_{n,0}(c_1, c_2, \ldots) = - \sum_{(i,j)\in\mathcal{F}',n>i} a_{i,j}(c_1, c_2, \ldots) c^{(j)}_{n-i}. \tag{7}$$

Note that the coefficients $b_{n,0}$ for $(n, 0) \in \mathcal{G}$ of Q also satisfy:

$$b_{n,0} = - \sum_{(i,j)\in\mathcal{F}',n>i} b_{i,j} c^{(j)}_{n-i}. \tag{8}$$

Let us set $a_{i,j} := 0$ for $(i, j) \in \mathcal{F}\backslash\mathcal{F}'$. Knowing that $C^{(j)}_{n-i} \not\equiv 0 \Rightarrow n - i \geq j$, and in this case $\deg C^{(j)}_{n-i} = j$, we deduce that $\deg a_{n,0} \leq n + \max_{(i,j)\in\mathcal{F}}(\deg a_{i,j})$ as desired. As the right-hand sides of Systems (4) and (5) are proportional, there is $\mu := \dfrac{a_{i_0,j_0}(c_1, c_2, \ldots)}{b_{i_0,j_0}} \in K\backslash\{0\}$ such that $a_{i,j}(c_1, c_2, \ldots) = \mu b_{i,j}$ for any $(i, j) \in \mathcal{F}'$.

By Systems (7) and (8), one has also $a_{n,0}(c_1, c_2, \ldots) = \mu b_{n,0}$ for $(n, 0) \in \mathcal{G}$. The polynomial

$$P(x, y) = \sum_{(i,j)\in\mathcal{F}'\cup\mathcal{G}} a_{i,j}(c_1, c_2, \ldots)x^i y^j$$

is proportional to Q (i.e. $P = \mu Q$), so it is nonzero and vanishes at y_0.

To obtain Theorem 2, it suffices now to show that there exists a uniform bound N_{d_x,d_y} for the depth in $M^{red}_{\mathcal{F},\mathcal{G}}$ to which we get the reconstruction process, that is, the depth at which we find a first nonzero minor. We reach this in the two following lemmas.

Lemma 3 *Let $d_x, d_y \in \mathbb{N}^*$. For any series $y_0 = \sum_{n\geq 1} c_n x^n \in K[[x]]$ with $c_1 \neq 0$, verifying an equation $P(x, y_0) = 0$ where $P(x, y) \in K[x, y]\backslash\{0\}$, $\deg_x P \leq d_x$, $\deg_y P \leq d_y$, and for any polynomial $Q(x, y) \in K[x, y]$, $\deg_x Q \leq d_x$, $\deg_y Q \leq d_y$, such that $Q(x, y_0) \neq 0$, one has that $\operatorname{ord}_x Q(x, y_0) \leq 2d_x d_y$.*

Proof Let y_0 be a series as in the statement of Lemma 3. We consider the ideal $I_0 := \{R(x, y) \in K[x, y] \mid R(x, y_0) = 0\}$. By assumption, it is a nontrivial prime ideal, so its height is one or two. If it were equal to 2, then it would be a maximal ideal. But I_0 is included into the ideal $\{R(x, y) \in K[x, y] \mid R(0, 0) = 0\}$, so:

$$I_0 = \{R(x, y) \in K[x, y] \mid R(0, 0) = 0\} = (x, y)$$

which is absurd because $x \notin I_0$. So, I_0 is a height one prime ideal of the factorial ring $K[x, y]$. It is generated by an irreducible polynomial $P_0(x, y) \in K[x, y]$. We set $d_{0,x} := \deg_x P_0$ and $d_{0,y} := \deg_y P_0$. Note also that, by factoriality of $K[x, y]$, P_0 is also irreducible as an element of $K(x)[y]$.

Let P be as in the statement of Lemma 3. One has that $P = SP_0$ for some $S \in K[x, y]$. Hence $d_{0,x} \leq d_x$ and $d_{0,y} \leq d_y$. Let $Q \in K[x, y]$ be such that $Q(x, y_0) \neq 0$ with $\deg_x Q \leq d_x$, $\deg_y Q \leq d_y$. So P_0 and Q are coprime in $K(x)[y]$. Their resultant $r(x)$ is nonzero. One has the following Bézout relation in $K[x][y]$:

$$A(x, y)P_0(x, y) + B(x, y)Q(x, y) = r(x).$$

We evaluate at $y = y_0$:

$$0 + B(x, y_0)Q(x, y_0) = r(x).$$

So $\operatorname{ord}_x Q(x, y_0) \leq \deg_x r(x)$. But, the resultant is a determinant of order at most $d_y + d_{0,y} \leq 2\, d_y$ whose entries are polynomials in $K[x]$ of degree at most $\max\{d_x, d_{0,x}\} = d_x$. So, $\deg_x r(x) \leq 2\, d_x d_y$. Hence, one has that: $\operatorname{ord}_x Q(x, y_0) \leq 2\, d_x d_y$.

Lemma 4 *Let $\mathcal{F}'' \subsetneq \mathcal{F}$. If y_0 is not algebraic relatively to $(\mathcal{F}'', \mathcal{G})$, we denote $l := |\mathcal{F}''|$ and $q := \min\left\{k_l \mid Q_{\underline{k},\mathcal{F}''}(c_1, c_2, \ldots) \neq 0,\ \underline{k} = (k_1, \ldots, k_l)\right\}$. We denote by p the index in $M_{\mathcal{F}'',\mathcal{G}}$ of the q^{th} row of $M^{red}_{\mathcal{F}'',\mathcal{G}}$. Then, for any polynomial*

$$Q(x, y) = \sum_{(i,j)\in\mathcal{F}''\cup\mathcal{G}} b_{i,j} x^i y^j$$

with $b_{i,j} \neq 0$ *for some* $(i, j) \in \mathcal{F}''$*, we have:*

$$\operatorname{ord}_x Q(x, y_0) \leq p \leq 2\, d_x d_y,$$

and the value p *is reached for a certain polynomial* Q_0 *of this type.*

Proof By the definition of q, for any $\underline{k} = (k_1, \ldots, k_l)$ with $k_l < q$, we have $Q_{\underline{k},\mathcal{F}''}(c_1, c_2, \ldots) = 0$. This means that the rank of the column vectors $V_{i,j,q-1}$ that are the restrictions of those of $M^{red}_{\mathcal{F}'',\mathcal{G}}$ up to the row $q-1$, is less than $l = |\mathcal{F}''|$. There are coefficients $(a_{i,j})_{(i,j)\in\mathcal{F}''}$ not all zero such that $\sum_{(i,j)\in\mathcal{F}''} a_{i,j} V_{i,j,q-1} = 0$. By computing the coefficients $a_{n,0}$ for $(n, 0) \in \mathcal{G}$ via relations (1):

$$a_{n,0} = -\sum_{(i,j)\in\mathcal{F}'',n>i} a_{i,j} c^{(j)}_{n-i}, \tag{9}$$

we obtain the vanishing of the $p-1$ first terms of $Q_0(x, y_0) := \sum_{(i,j)\in\mathcal{F}''\cup\mathcal{G}} a_{i,j} x^i (y_0)^j$. Thus, $\operatorname{ord}_x Q_0(x, y_0) \geq p$, and so $p \leq 2\, d_x d_y$. On the other hand, again by the definition of q, the column vectors $V_{i,j,q}$, $(i, j) \in \mathcal{F}''$, up to the row q are, in turn, of rank $l = |\mathcal{F}''|$. This means that the rank of the matrix $M^{red}_{\mathcal{F}'',\mathcal{G},q}$ consisting of the q first rows of $M^{red}_{\mathcal{F}'',\mathcal{G}}$ is l. Thus, for any nonzero vector $(b_{i,j})_{(i,j)\in\mathcal{F}''}$, we have:

$$M^{red}_{\mathcal{F}'',\mathcal{G},q} \cdot (b_{i,j})_{(i,j)\in\mathcal{F}''} \neq 0.$$

But the components of this nonzero vector, up to a change of indexes, are exactly the coefficients e_k, $(k, 0) \notin \mathcal{G}$ and $k \leq p$, of the expansion of $\sum_{(i,j)\in\mathcal{F}''} b_{i,j}\, x^i\, (y_0)^j$. Now, these terms of the latter series do not overlap with the terms of $\sum_{(i,0)\in\mathcal{G}} b_{i,0}\, x^i$. Therefore, for a given polynomial $Q(x, y) = \sum_{(i,j)\in\mathcal{F}''\cup\mathcal{G}} b_{i,j} x^i y^j$ with $b_{i,j} \neq 0$ for some $(i, j) \in \mathcal{F}''$, the series $Q(x, y_0)$ has a nonzero term e_k with $k \leq p$, $(k, 0) \notin \mathcal{G}$. Hence, $\operatorname{ord}_x Q(x, y_0) \leq p$.

We achieve the proof of Theorem 2 via Lemmas 3 and 4 by considering for a given algebraic series y_0 a family $\mathcal{F}' \subset \mathcal{F}$ minimal among the families such that y_0 is algebraic relatively to $(\mathcal{F}', \mathcal{G})$. We consider an associated nonzero Wilczynski polynomial $Q_{\underline{k}_0,\underline{I}_0}$ as in (3) with N_{y_0} as defined in Formula (6) minimal. Taking $\mathcal{F}'' = \underline{I}_0$, Lemma 4 applies and $N_{y_0} = q$. So $N_{y_0} \leq p \leq N = 2\, d_x d_y$.

Recall that the coefficients $a_{i,j}$ constructed in the first part of the proof are homogenous polynomials in $\mathbb{Z}[C_1, \ldots, C_{N_{y_0}}] \subseteq \mathbb{Z}[C_1, \ldots, C_N]$. To complete the proof of Theorem 2, let us describe a finite set Λ which enumerates all possible reconstruction formulas. Let M_N be the matrix obtained from $M^{red}_{\mathcal{F},\mathcal{G}}$ by taking its first $N = 2d_x d_y$ rows. So M_N is a $N \times |\mathcal{F}|$-matrix. Let ν be the number of minors of order less or equal to $\min\{N; |\mathcal{F}| - 1\}$ of M_N. We fix a finite set Λ of cardinality $|\mathcal{F}| + \nu$. Its first $|\mathcal{F}|$ elements are the indexes of reconstruction formulas (2) as built in the first part of the proof of Theorem 2 (case $m = |\mathcal{F}'| = 1$). The other ν elements are used to enumerate reconstruction formulas (2) in the case described in the second part of the proof (case $m = |\mathcal{F}'| \geq 2$).

Construction of the coefficients $a_{i,j}^{(\lambda)}(c_1, c_2, \ldots)$ for a given y_0.
Let y_0 be algebraic relatively to $(\mathcal{F}, \mathcal{G})$ as in Definition 1. Let $N = 2\, d_x d_y$ as in Theorem 2. Recall that M_N denotes the matrix consisting in the N first rows of $M^{red}_{\mathcal{F},\mathcal{G}}$. Let r be the rank of M_N, and $m := r + 1$. The minors of M_N of order m are all zero and there exists a minor of order $m - 1 = r$ which is nonzero. There are two cases. If $r = 0$, we choose $(i, j) \in \mathcal{F}$ and we fix the coefficients $a_{i,j} := 1$ and $a_{l,m} = 0$ for $(l, m) \in \mathcal{F}$, $(l, m) \neq (i, j)$. Then we derive the coefficients $a_{i,0}(c_1, c_2, \ldots)$ for $(i, 0) \in \mathcal{G}$ from Relations (1). The polynomials P thus obtained are all annihilators of y_0. If $r \geq 1$, we consider all the Wilczynski polynomials $Q_{\underline{k},\underline{I}}$ of order r that do not vanish when evaluated at $c_1, \ldots, c_N$. Each of them allows to reconstruct coefficients $a_{i,j}^{(\lambda)}(c_1, c_2, \ldots)$, $(i, j) \in \mathcal{F}$, and subsequently coefficients $a_{i,0}(c_1, c_2, \ldots)$, $(i, 0) \in \mathcal{G}$, via Relations (1). The corresponding polynomials $P^{(\lambda)}$ are annihilators of y_0 if and only if $\mathrm{ord}_x P^{(\lambda)}\left(x, \sum_{k=1}^{N} c_k x^k\right) > N$.

Remark 2 With the hypothesis and notations of Theorem 2 and its proof, let us denote $f := |\mathcal{F}| \leq (d_x + 1)d_y$ and $g := \min\{N, f - 1\}$. Then $|\Lambda|$ is bounded by $f + \sum_{t=1}^{g} \binom{f}{t}\binom{N}{t}$, which is itself roughly bounded by $f + (2^f - 1)(2^N - 1)$.

Example 2 We resume Example 1, and note that, for $\underline{I} = ((2, 1), (2, 2))$ and $\underline{k} = (2, 3)$, we have that:

$$Q_{\underline{k},\underline{I}} = \begin{vmatrix} C_1 & 0 \\ C_2 & C_1^2 \end{vmatrix} = C_1^3 \neq 0.$$

which does not vanish at $c_1 \neq 0$. So we set $a_{0,2} := (-1)^2 C_1^3 = C_1^3$ and, applying Cramer's rule:

$$\begin{cases} a_{2,1} := (-1)^1 \begin{vmatrix} 2 \cdot C_1 \cdot C_2 & 0 \\ C_2^2 + 2\, C_1\, C_3 & C_1^2 \end{vmatrix} = -2C_1^3 C_2 \\ a_{2,2} := (-1)^3 \begin{vmatrix} C_1 & 2 \cdot C_1 \cdot C_2 \\ C_2 & C_2^2 + 2\, C_1\, C_3 \end{vmatrix} = C_1\left(C_2^2 - 2C_1C_3\right). \end{cases}$$

We deduce from formulas (1) that:

$$a_{2,0} = -a_{2,1} \cdot 0 - a_{0,2} \cdot C_1^2 - a_{2,2} \cdot 0 = -C_1^5.$$

A vanishing polynomial of a series $y_0 = \sum_{n \geq 1} c_n x^n \in K[[x]]$, $c_1 \neq 0$, algebraic relatively to $\mathcal{F} = ((2, 1), (0, 2), (2, 2))$ and $\mathcal{G} = (2, 0)$ is:

$$\begin{aligned} P(x, y) &= -c_1^5 x^2 - 2c_1^3 c_2\, x^2 y + c_1^3\, y^2 + c_1\left(c_2^2 - 2c_1c_3\right) x^2 y^2 \\ &= c_1\left[-c_1^4 x^2 - 2c_1^2 c_2\, x^2 y + c_1^2\, y^2 + \left(c_2^2 - 2c_1c_3\right) x^2 y^2\right]. \end{aligned}$$

Remark 3 (i) Let y_0 be an algebraic series with vanishing polynomial of degree d_x in x and d_y in y. According to [1, Chap. 7], the method of reconstruction of equation based on Hermite-Padé approximants provides a priori only polynomials $P(x, y) = \sum_{i \le d_x,\ j \le d_y} a_{i,j} x^i y^j$ such that $P(x, y_0) \equiv 0\, [x^\sigma]$ with $\sigma = (d_x + 1)(d_y + 1) - 1$. Subsequently, one has to check whether $P(x, y_0) = 0$ actually. By our Lemma 3, one can always certify that $P(x, y_0) = 0$ just by verifying that $P(x, y_0) \equiv 0\, [x^\tau]$ with $\tau = 2d_x d_y$. Hence this reconstruction method as implemented in the GFUN package in Maple software holds for any equation of degree less than d_x in x and d_y in y, not for only irreducible ones as in [1, Theorem 8, p. 110].

(ii) Theorem 2 provides an alternative to the Hermite-Padé reconstruction process. However, its algorithmic interest may be limited by the a priori big size of the finite set Λ. But, for a given $(\mathcal{F}, \mathcal{G})$, generically, any minor of M_N of order $\min\{|\mathcal{F}| - 1, 2d_x d_y\}$ is nonzero. So any reconstruction formula (2) corresponding to such nonzero minor holds true.

(iii) Let us consider the case where y_0 is a rational fraction:

$$y_0 = \frac{-a_0(x)}{a_1(x)} = \frac{-a_{1,0}x - \cdots - a_{d_0,0}x^{d_0}}{1 + a_{1,1}x + \cdots + a_{d_1,1}x^{d_1}}$$
$$= \sum_{n \ge 1} c_n x^n \quad \text{with} \quad c_1 \ne 0.$$

Thus, y_0 is algebraic relatively to $\mathcal{F} = \{(0, 1), \ldots, (d_1, 1)\}$ and $\mathcal{G} = \{(1, 0), \ldots, (d_0, 0)\}$. The Wilczynski polynomials of order $|\mathcal{F}| = d_1 + 1$ evaluated at the c_n's are all zero. The evaluation of the Wilczynski polynomial $Q_{\underline{k}_0, \underline{I}_0}$ of order d_1 with $\underline{I}_0 = ((1, 1), \ldots, (d_1, 1))$ and $\underline{k}_0 = (1, \ldots, d_1)$ is equal, up to the sign, to the resultant of $a_0(x)$ and $a_1(x)$, by [11, Chap. 12 (1.15), p. 401].

(iv) In the present section, the field K can be of any characteristic.

4 Closed-form expression of an algebraic series

Let us assume from now on that K has characteristic zero. Our purpose is to determine the coefficients of an algebraic series in terms of the coefficients of a vanishing polynomial. We consider the following polynomial of degrees bounded by d_x in x and d_y in y:

$$P(x, y) = \sum_{i=0}^{d_x} \sum_{j=0}^{d_y} a_{i,j} x^i y^j, \quad \text{with } P(x, y) \in K[x, y]$$
$$= \sum_{i=0}^{d_x} \pi_i^P(y) x^i$$
$$= \sum_{j=0}^{d_y} a_j(x) y^j,$$

and a formal power series:

$$y_0 = \sum_{n \geq 1} c_n x^n, \text{ with } y_0 \in K[[x]],\ c_1 \neq 0.$$

The field $K((x))$ is endowed with the x-adic valuation ord_x.

Notation 1 *For any $k \in \mathbb{N}$ and for $Q(x, y) = \sum_{j=0}^{d} a_j(x) y^j \in K((x))[y]$, we denote:*

- $\mathrm{ord}_x Q := \min\{\mathrm{ord}_x a_j(x),\ j = 0, \ldots, d\}$;
- $z_0 := 0$ *and for* $k \geq 1$, $z_k := \sum_{n=1}^{k} c_n x^n$;
- $y_k := y_0 - z_k = \sum_{n \geq k+1} c_n x^n$;
- $Q_k(x, y) := Q(x, z_k + x^{k+1} y) = \sum_{i=i_k}^{d_k} \pi_{k,i}^{Q}(y) x^i$ *where* $i_k = \mathrm{ord}_x Q_k$ *and* $d_k := \deg_x Q_k$. *Note that the sequence* $(i_k)_{k \in \mathbb{N}}$ *is nondecreasing since* $Q_{k+1}(x, y) = Q_k(x, c_{k+1} + xy)$.

Classically (e.g. [18]), reducing to the case where y_0 is a simple root, the resolution of $P = 0$ with the Newton–Puiseux method is algorithmic, with two stages:

(i) a first stage of separation of the branches solutions, which illustrates the following fact: y_0 may share a principal part with other roots of P. This is equivalent to the fact that this principal part is also the principal part of a root of $\partial P / \partial y$.

(ii) a second stage of unique "automatic" resolution: once the branches are separated, the remaining part of y_0 is a root of an equation called Henselian in the formal valued context (y_0 seen as an algebraic formal power series), and called of implicit function type in the context of differentiable functions (y_0 seen as the convergent Taylor's expansion of an algebraic function).

We give here a version of the algebraic content of this algorithmic resolution.

Lemma 5 *(i) The series y_0 is a root of $P(x, y)$ if and only if the sequence $(i_k)_{k \in \mathbb{N}^*}$ is strictly increasing where $i_k = \mathrm{ord}_x P_k$.*

(ii) The series y_0 is a simple root of $P(x, y)$ if and only if the sequence $(i_k)_{k \in \mathbb{N}^*}$ is strictly increasing and there exists a lowest index k_0 such that $i_{k_0+1} = i_{k_0} + 1$. In that case, one has that $i_{k+1} = i_k + 1$ for any $k \geq k_0$.

Proof (i) Note that for any k, $i_k \leq \mathrm{ord}_x P_k(x, 0) = \mathrm{ord}_x P(x, z_k)$. Hence, if the sequence $(i_k)_{k \in \mathbb{N}^*}$ is strictly increasing, it tends to $+\infty$, and so does $\mathrm{ord}_x P(x, z_k)$. The series y_0 is indeed a root of $P(x, y)$. Reciprocally, suppose that there exist $1 \leq k < l$ such that $i_k \geq i_l$. Since the sequence $(i_n)_{n \in \mathbb{N}}$ is nondecreasing, one has that $i_l \geq i_k$, so $i_l = i_k$. We apply Taylor's formula to $P_j(x, y)$ for $j > k$:

$$\begin{aligned} P_j(x, y) &= P_k(x, c_{k+1} + c_{k+2} x + \cdots + x^{j-k} y) \\ &= \pi_{k,i_k}^{P}(c_{k+1}) x^{i_k} + \left[(\pi_{k,i_k}^{P})'(c_{k+1}) c_{k+2} + \pi_{k,i_k+1}^{P}(c_{k+1}) \right] x^{i_k+1} + \cdots . \end{aligned} \tag{10}$$

For $j = l$, we deduce that $\pi^P_{k,i_k}(c_{k+1}) \neq 0$. This implies for any $j > k$, that $i_j = i_k$ and $\operatorname{ord}_x P_j(x, 0) = \operatorname{ord}_x P(x, z_j) = i_k$. Hence $\operatorname{ord}_x P(x, y_0) = i_k \neq +\infty$.
(ii) The series y_0 is a double root of P if and only if it is a root of P and $\partial P/\partial y$. We apply Taylor's formula for certain $k \in \mathbb{N}^*$:

$$\begin{aligned} P_{k+1}(x, y) &= P_k(x, c_{k+1} + xy) \\ &= \pi^P_{k,i_k}(c_{k+1})x^{i_k} + \left[(\pi^P_{k,i_k})'(c_{k+1})y + \pi^P_{k,i_k+1}(c_{k+1})\right] x^{i_k+1} \\ &\quad + \left[\frac{(\pi^P_{k,i_k})''(c_{k+1})}{2} y^2 + (\pi^P_{k,i_k+1})'(c_{k+1})\, y + \pi^P_{k,i_k+2}(c_{k+1})\right] x^{i_k+2} \\ &\quad + \cdots . \end{aligned} \tag{11}$$

Note that:

$$\frac{\partial P_k}{\partial y}(x, y) = x^{k+1} \left(\frac{\partial P}{\partial y}\right)_k (x, y) = \sum_{i=i_k}^{d_k} (\pi^P_{k,i})'(y) x^i$$

One has that $\pi^P_{k,i_k} \not\equiv 0$ and $\pi^P_{k,i_k}(c_{k+1}) = 0$ (see the point (i) above), so $(\pi^P_{k,i_k})'(y) \not\equiv 0$. Thus $\operatorname{ord}_x \left(\frac{\partial P}{\partial y}\right)_k = i_k - k - 1$. We perform the Taylor's expansion of $\left(\frac{\partial P}{\partial y}\right)_{k+1} = \left(\frac{\partial P}{\partial y}\right)_k (x, c_{k+1} + x\, y)$:

$$\begin{aligned} \left(\frac{\partial P}{\partial y}\right)_{k+1} (x, y) &= (\pi^P_{k,i_k})'(c_{k+1}) x^{i_k-k-1} \\ &\quad + \left[(\pi^P_{k,i_k})''(c_{k+1}) y + (\pi^P_{k,i_k+1})'(c_{k+1})\right] x^{i_k-k} + \cdots . \end{aligned}$$

By the point (i) applied to $\dfrac{\partial P}{\partial y}$, if y_0 is a double root of P, we must have $(\pi^P_{k,i_k})'(c_{k+1}) = 0$. Moreover, if $\pi^P_{k,i_k+1}(c_{k+1}) \neq 0$, by formula (11) we would have $i_{k+1} = i_k + 1$ and even $i_{k+j} = i_k + 1$ for every j according to formula (10): y_0 could not be a root of P. So, $\pi^P_{k,i_k+1}(c_{k+1}) = 0$, and, accordingly, $i_{k+1} \geq i_k + 2$.

If y_0 is a simple root of P, from the point (i) and its proof applied to $\dfrac{\partial P}{\partial y}$, there exists a lowest natural number k_0 such that the sequence $(i_k - k - 1)_{k\in\mathbb{N}^*}$ is no longer strictly increasing, or equivalently $i_{k+1} = i_k + 1$, that is, such that $(\pi^P_{k_0,i_{k_0}})'(c_{k_0+1}) \neq 0$. For any $k \geq k_0$, we consider the Taylor's expansion of $\left(\frac{\partial P}{\partial y}\right)_{k+1} (x, y) = \left(\frac{\partial P}{\partial y}\right)_{k_0} (x, c_{k_0+1} + \cdots + x^{k-k_0+1} y)$:

$$\left(\frac{\partial P}{\partial y}\right)_{k+1}(x,y) = (\pi^P_{k_0,i_{k_0}})'(c_{k_0+1})x^{i_{k_0}-k_0-1} + \Big[(\pi^P_{k_0,i_{k_0}})''(c_{k_0+1})c_{k_0+2} + (\pi^P_{k_0,i_{k_0}+1})'(c_{k_0+1})\Big]x^{i_{k_0}-k_0} + \cdots \tag{12}$$

and we get that:

$$\operatorname{ord}_x \frac{\partial P}{\partial y}(x, z_{k+1}) = \operatorname{ord}_x \left(\frac{\partial P}{\partial y}\right)_{k+1}(x,0) = \operatorname{ord}_x \left(\frac{\partial P}{\partial y}\right)_{k+1} = i_{k_0} - k_0 - 1. \tag{13}$$

As $(\pi^P_{k+1,i_{k+1}})'(y)\not\equiv 0$, we obtain that $i_{k+1} = \operatorname{ord}_x P_{k+1} = \operatorname{ord}_x \left(\frac{\partial P_{k+1}}{\partial y}\right) = k+2+\operatorname{ord}_x \left(\frac{\partial P}{\partial y}\right)_{k+1} = i_{k_0}+k-k_0+1$. Hence, the sequence $(i_k)_{k\geq k_0}$ increases one by one.

Resuming the notations of Lemma 5, the natural number k_0 represents the length of the principal part in the stage of separation of the branches. In the following lemma, we bound it using Lemma 3 or the discriminant Δ_P of P.

Lemma 6 *Let $y_0 = \sum_{n\geq 1} c_n x^n,\ c_1 \neq 0$, be a simple root of a polynomial $P(x,y)$ with $\deg_x(P) \leq d_x$ and $\deg_y(P) \leq d_y$. The natural number k_0 of Lemma 5 verifies that:*

$$k_0 \leq 2d_x d_y.$$

In particular, if P has only simple roots:

$$k_0 \leq d_x(2\,d_y - 1).$$

Proof By Lemma 3, since $P(x,y_0)=0$ and $\frac{\partial P}{\partial y}(x,y_0) \neq 0$, one has that:

$$\operatorname{ord}_x \frac{\partial P}{\partial y}(x, y_0) \leq 2d_x d_y.$$

But, by definition of k_0 and by formula (13), we obtain:

$$\operatorname{ord}_x \frac{\partial P}{\partial y}(x, z_{k+1}) = \operatorname{ord}_x \frac{\partial P}{\partial y}(x, z_{k_0+1}) = i_{k_0} - k_0 - 1$$

for any $k \geq k_0$. So, $\operatorname{ord}_x \frac{\partial P}{\partial y}(x, y_0) = \operatorname{ord}_x \frac{\partial P}{\partial y}(x, z_{k_0+1})$. Moreover, by minimality of k_0, the sequence $(i_k - k - 1)_k$ is strictly increasing up to k_0, so:

$$\begin{aligned}\operatorname{ord}_x \frac{\partial P}{\partial y}(x, y_0) &= \operatorname{ord}_x \frac{\partial P}{\partial y}(x, z_{k_0+1}) \\ &= \operatorname{ord}_x \left(\frac{\partial P}{\partial y}\right)_{k_0+1}(x, 0) \geq \operatorname{ord}_x \left(\frac{\partial P}{\partial y}\right)_{k_0+1} \geq k_0.\end{aligned}$$

Hence we obtain as desired that:

$$k_0 \leq 2 d_x d_y.$$

In the case where P has only simple roots, as in the proof of Lemma 3, $\operatorname{ord}_x \frac{\partial P}{\partial y}(x, y_0)$ is bounded by the degree of the resultant of P and $\frac{\partial P}{\partial y}$, say the discriminant Δ_P of P, which is bounded by $d_x(2\,d_y - 1)$.

Notation 2 *Resuming Notation 1 and the content of Lemma 5, we set:*

$$\omega_0 := (\pi^P_{k_0, i_{k_0}})'(c_{k_0+1}).$$

By formula (12), we note that:

$$\left(\frac{\partial P}{\partial y}\right)(x, y_0) = \omega_0 x^{i_{k_0} - k_0 - 1} + \cdots .$$

Thus, ω_0 is the initial coefficient of $\left(\frac{\partial P}{\partial y}\right)(x, y_0)$, hence $\omega_0 \neq 0$.

Theorem 3 *Consider the following polynomial in $K[x, y]$ of given degrees d_x in x and d_y in y:*

$$P(x, y) = \sum_{i=0}^{d_x} \sum_{j=0}^{d_y} a_{i,j} x^i y^j = \sum_{i=0}^{d_x} \pi_i^P(y) x^i,$$

and a formal power series which is a simple root:

$$y_0 = \sum_{n \geq 1} c_n x^n \ \in K[[x]],\ c_1 \neq 0.$$

Resuming Notations 1 and 2, and the content of Lemma 5, we recall that $\omega_0 := (\pi^P_{k_0, i_{k_0}})'(c_{k_0+1}) \neq 0$. Then, for any $k > k_0$:

- *either the polynomial $z_{k+1} = \sum_{n=1}^{k+1} c_n x^n$ is a solution of $P(x, y) = 0$;*

– *or the polynomial* ${}_kR(x,y) := \frac{P_k(x,y+c_{k+1})}{-\omega_0 x^{i_k}} = -y + {}_kQ(x,y)$ *defines a reduced Henselian equation:*

$$y = {}_kQ(x,y)$$

with ${}_kQ(0,y) \equiv 0$ *and satisfied by:*

$$t_{k+1} := \frac{y_0 - z_{k+1}}{x^{k+1}} = c_{k+2}x + c_{k+3}x^2 + \cdots .$$

Proof We show by induction on $k > k_0$ that ${}_kR(x,y) = -y + x\,{}_kT(x,y)$ with ${}_kT(x,y) \in K[x,y]$. The initial step of the induction is for $k = k_0 + 1$. Let us apply Formula (11) with parameter $k = k_0$. Since $i_{k_0+1} = i_{k_0} + 1$, we have that $\pi^P_{k_0,i_{k_0}}(c_{k_0+1}) = 0$ and accordingly:

$$P_{k_0+1}(x,y) = \left[\omega_0\, y + \pi^P_{k_0,i_{k_0}+1}(c_{k_0+1})\right] x^{i_{k_0}+1} + \cdots .$$

Since $i_{k_0+2} = i_{k_0} + 2$, $\pi^P_{k_0+1,i_{k_0}+1}(y) = \omega_0\, y + \pi^P_{k_0,i_{k_0}+1}(c_{k_0+1})$ vanishes at c_{k_0+2}, which implies that $c_{k_0+2} = \dfrac{-\pi^P_{k_0,i_{k_0}+1}(c_{k_0+1})}{\omega_0}$. Computing ${}_{k_0+1}R(x,y) := \dfrac{P_{k_0+1}(x, y + c_{k_0+2})}{-\omega_0 x^{i_k}}$, it follows that:

$${}_{k_0+1}R(x,y) = -y + {}_{k_0+1}Q(x,y),$$

with ${}_{k_0+1}Q(x,y)$ being equal to:

$$\frac{x}{-\omega_0}\left[\frac{(\pi^P_{k,i_{k_0}})''(c_{k_0+1})}{2}(y + c_{k_0+2})^2 \right.$$
$$\left. + (\pi^P_{k_0,i_{k_0}+1})'(c_{k_0+1})\,(y + c_{k_0+2}) + \pi^P_{k_0,i_{k_0}+2}(c_{k_0+1})\right] + \frac{x^2}{-\omega_0}[\cdots].$$

So ${}_{k_0+1}Q(0,y) \equiv 0$.

Suppose that the property holds true at a rank $k \geq k_0 + 1$, which means that ${}_kR(x,y) := \dfrac{P_k(x, y + c_{k+1})}{-\omega_0 x^{i_k}} = -y + x\,{}_kT(x,y)$. Therefore, for ${}_k\tilde{T} := -\omega_0\,{}_kT(x, y - c_{k+1}) \in K[x,y]$, we can write:

$$\begin{aligned} P_k(x,y) &= \omega_0(y - c_{k+1})x^{i_k} + x^{i_k+1}\,{}_k\tilde{T}(x,y) \\ &= \pi^P_{k,i_k}(y)x^{i_k} + \pi^P_{k,i_k+1}(y)x^{i_k+1} + \cdots . \end{aligned}$$

In particular, $\pi^P_{k,i_k}(y) = \omega_0(y - c_{k+1})$. Since $P_{k+1}(x, y) = P_k(x, c_{k+1} + xy)$, we have:

$$P_{k+1}(x, y) = \left[\omega_0\, y + \pi^P_{k,i_k+1}(c_{k+1})\right] x^{i_k+1} + \pi^P_{k+1,i_k+2}(y) x^{i_k+2} + \cdots .$$

Now, $\pi^P_{k+1,i_k+1}(y) = \omega_0\, y + \pi^P_{k,i_k+1}(c_{k+1})$. But $i_k + 2 = i_{k+2} > i_{k+1} = i_k + 1$. So we must have $\pi^P_{k+1,i_k+1}(c_{k+2}) = 0$. So, $c_{k+2} = \dfrac{-\pi^P_{k,i_k+1}(c_{k+1})}{\omega_0}$. It follows that:

$$P_{k+1}(x, y) = \omega_0(y - c_{k+2}) x^{i_k+1} + \pi^P_{k+1,i_k+2}(y) x^{i_k+2} + \cdots .$$

Hence:

$$\begin{aligned} {}_{k+1}R(x, y) &:= \frac{P_{k+1}(x, y + c_{k+2})}{-\omega_0 x^{i_{k+1}}} \\ &= -y - x \frac{\pi^P_{k+1,i_k+2}(y + c_{k+2})}{\omega_0} + x^2[\cdots] + \cdots \\ &= -y + x \cdot {}_{k+1}T(x, y), \quad {}_{k+1}T \in K[x, y], \end{aligned}$$

as desired.

In particular, ${}_kQ(0, 0) = \dfrac{\partial\, {}_kQ}{\partial y}(0, 0) = 0$. So the equation $y = {}_kQ(x, y)$ is reduced Henselian if and only if ${}_kQ(x, 0) \not\equiv 0$, which is equivalent to z_{k+1} not being a root of P.

We will need the following lemma:

Lemma 7 *Let y_0 be a simple root of a nonzero polynomial $P(x, y)$ of degrees $\deg_x(P) \le d_x$ and $\deg_y(P) \le d_y$. For $y_1 \neq y_0$ any other root of P, one has that:*

$$\mathrm{ord}_x(y_0 - y_1) \le 2\, d_x d_y.$$

Proof Note that the hypothesis imply that $d_y \ge 2$. Let us write $y_1 - y_0 = x^k\, \delta_{1,0}$ where $k := \mathrm{ord}_x(y_1 - y_0)$ and $\delta_{1,0} \in K[[x]]\setminus\{0\}$. By Taylor's Formula, we have:

$$\begin{aligned} P(x, y_0 + x^k \delta_{1,0}) &= 0 \\ &= P(x, y_0) + \frac{\partial P}{\partial y}(x, y_0) x^k \delta_{1,0} + \cdots + \frac{1}{d_y!} \frac{\partial^{d_y} P}{\partial y^{d_y}}(x, y_0) x^{k d_y} \delta_{1,0}^{d_y} \\ &= x^k \delta_{1,0} \left(\frac{\partial P}{\partial y}(x, y_0) + \cdots + \frac{1}{d_y!} \frac{\partial^{d_y} P}{\partial y^{d_y}}(x, y_0) x^{k(d_y-1)} \delta_{1,0}^{d_y-1} \right). \end{aligned}$$

Since $x^k \delta_{1,0} \neq 0$ and $\dfrac{\partial P}{\partial y}(x, y_0) \neq 0$, one has that:

$$\frac{\partial P}{\partial y}(x, y_0) = -x^k \delta_{1,0} \left(\frac{1}{2} \frac{\partial^2 P}{\partial y^2}(x, y_0) + \cdots + \frac{1}{d_y!} \frac{\partial^{d_y} P}{\partial y^{d_y}}(x, y_0) x^{k(d_y-2)} \delta_{1,0}^{d_y-2} \right)$$

The order of the right hand side being at least k, we obtain that:

$$\operatorname{ord}_x\left(\frac{\partial P}{\partial y}(x, y_0)\right) \geq k.$$

But, by Lemma 3, we must have $\operatorname{ord}_x\left(\frac{\partial P}{\partial y}(x, y_0)\right) \leq 2d_x d_y$. So $k \leq 2d_x d_y$.

For the courageous reader, in the case where y_0 is a series which is not a polynomial, we deduce from Theorem 3 and from Flajolet–Soria's Formula 1 a closed-form expression for the coefficients of y_0 in terms of the coefficients $a_{i,j}$ of P and of the coefficients of an initial part z_k of y_0 sufficiently large. For any $k \geq 2d_x d_y + 1$, recall that $i_k = \operatorname{ord}_x P_k$.

Corollary 1 *For any $k \geq 2d_x d_y + 1$, for any $p \geq 1$, one has that:*

$$c_{k+1+p} = \sum_{q=1}^{p} \frac{1}{q}\left(\frac{-1}{\omega_0}\right)^q \sum_{|S|=q,\ \|S\|_2 \geq q-1} A^S \left(\sum_{\substack{|T_S|=\|S\|_2-q+1 \\ \|T_S\|=p+qi_k-(q-1)(k+1)-\|S\|_1}} e_{T_S} C^{T_S} \right),$$

where $S = (s_{i,j})$, $A^S = \prod_{i=0,\ldots,d_x,\, j=0,\ldots,d_y} a_{i,j}^{s_{i,j}}$, $T_S = (t_{S,i})$, $C^{T_S} = \prod_{i=1}^{k+1} c_i^{t_{S,i}}$, and $e_{T_S} \in \mathbb{N}$ is of the form:

$$e_{T_S} = \sum_{\left(n_{i,j,L}^{l,m}\right)} \left(\frac{q!}{\prod_{\substack{l=1,\ldots,(k+1)d_y+d_x-i_k \\ m=0,\ldots,m_l}} \prod_{\substack{i=0,\ldots,d_x \\ j=m,\ldots,d_y}} \prod_{\substack{|L|=j-m \\ \|L\|=l+i_k-m(k+1)-i}} n_{i,j,L}^{l,m}!} \prod_{\substack{l=1,\ldots,(k+1)d_y+d_x-i_k \\ m=0,\ldots,m_l}} \prod_{\substack{i=0,\ldots,d_x \\ j=m,\ldots,d_y}} \prod_{\substack{|L|=j-m \\ \|L\|=l+i_k-m(k+1)-i}} \left(\frac{j!}{m!\,L!}\right)^{n_{i,j,L}^{l,m}} \right),$$

where $m_l := \min\left\{\left\lfloor \frac{l+i_k}{k+1} \right\rfloor, d_y\right\}$, $L = L_{i,j}^{l,m} = \left(l_{i,j,1}^{l,m}, \ldots, l_{i,j,k+1}^{l,m}\right)$, and where the sum is taken over the set of sequences $\left(n_{i,j,L}^{l,m}\right)_{\substack{l=1,\ldots,(k+1)d_y+d_x-i_k,\ m=0,\ldots,m_l \\ i=0,\ldots,d_x,\ j=m,\ldots,d_y,\ |L|=j-m,\ \|L\|=l+i_k-m(k+1)-i}}$ such that:

$$\sum_{l,m}\sum_{L} n_{i,j,L}^{l,m} = s_{i,j}, \quad \sum_{l,m}\sum_{i,j}\sum_{L} n_{i,j,L}^{l,m} = q \quad \textit{and} \quad \sum_{l,m}\sum_{i,j}\sum_{L} n_{i,j,L}^{l,m} L = T_S.$$

Remark 4 Note that the coefficients e_{T_S} are indeed natural numbers, since they are sums of products of multinomial coefficients because $\sum_{l,m}\sum_{i,j}\sum_L n_{i,j,L}^{l,m} = q$ and $m + |L| = j$.

Proof We get started by computing the coefficients of $\omega_0 x^{i_k}\,{}_kR$, in order to get those of ${}_kQ$:

$$\begin{aligned}
-\omega_0 x^{i_k}\,{}_kR &= P_k(x,\ y + c_{k+1})\\
&= P(x, z_{k+1} + x^{k+1}y)\\
&= \sum_{i=0,\ldots,d_x\,,\ j=0,\ldots,d_y} a_{i,j}x^i\left(z_{k+1} + x^{k+1}y\right)^j\\
&= \sum_{i=0,\ldots,d_x\,,\ j=0,\ldots,d_y} a_{i,j}x^i \sum_{m=0}^{j} \frac{j!}{m!\,(j-m)!} z_{k+1}^{j-m} x^{m(k+1)} y^m.
\end{aligned}$$

For $L = (l_1, \ldots, l_{k+1})$, we denote $C^L := c_1^{l_1}\cdots c_{k+1}^{l_{k+1}}$. One has that:

$$z_{k+1}^{j-m} = \sum_{|L|=j-m} \frac{(j-m)!}{L!} C^L x^{\|L\|}.$$

So:

$$-\omega_0 x^{i_k}\,{}_kR = \sum_{m=0}^{d_y} \sum_{\substack{i=0,\ldots,d_x\\ j=m,\ldots,d_y}} a_{i,j} \sum_{|L|=j-m} \frac{j!}{m!\,L!} C^L x^{\|L\|+m(k+1)+i} y^m.$$

We set $\hat{l} = \|L\| + m(k+1) + i$, which ranges between $m(k+1)$ and $(k+1)(d_y - m) + m(k+1) + d_x = (k+1)d_y + d_x$. Thus:

$$-\omega_0 x^{i_k}\,{}_kR = \sum_{\substack{m=0,\ldots,d_y\\ \hat{l}=m(k+1),\ldots,(k+1)d_y+d_x}} \sum_{\substack{i=0,\ldots,d_x\\ j=m,\ldots,d_y}} a_{i,j} \sum_{\substack{|L|=j-m\\ \|L\|=\hat{l}-m(k+1)-i}} \frac{j!}{m!\,L!} C^L x^{\hat{l}} y^m.$$

Since ${}_kR(x, y) = -y + {}_kQ(x, y)$ with ${}_kQ(0, y) \equiv 0$, the coefficients of ${}_kQ$ are obtained for $\hat{l} = i_k+1, \ldots, (k+1)d_y+d_x$. We set $l := \hat{l} - i_k$, $m_l := \min\left\{\left\lfloor \frac{l+i_k}{k+1}\right\rfloor, d_y\right\}$ and we obtain:

$${}_kQ(x, y) = \sum_{\substack{l=1,\ldots,(k+1)d_y+d_x-i_k\\ m=0,\ldots,m_l}} b_{l,m} x^l y^m,$$

with:

$$b_{l,m} = \frac{-1}{\omega_0} \sum_{\substack{i=0,\ldots,d_x \\ j=m,\ldots,d_y}} a_{i,j} \sum_{\substack{|L|=j-m \\ \|L\|=l+i_k-m(k+1)-i}} \frac{j!}{m!\,L!} C^L.$$

According to Lemmas 6, 7 and Theorem 3, we are in position to apply the version in Remark 1 of Flajolet–Soria's Formula in order to compute the coefficients of $t_{k+1} = c_{k+2}x + c_{k+3}x^2 + \cdots$. Thus, denoting $B := (b_{l,m})$, $Q := (q_{l,m})$ and $B^Q := \prod_{l,m} b_{l,m}^{q_{l,m}}$ for $l = 1, \ldots, (k+1)d_y + d_x - i_k$ and $m = 0, \ldots, m_l$, we obtain:

$$c_{k+1+p} = \sum_{q=1}^{p} \frac{1}{q} \sum_{|Q|=q,\ \|Q\|_1=p,\ \|Q\|_2=q-1} \frac{q!}{Q!} B^Q.$$

Let us compute:

$$\begin{aligned} b_{l,m}^{q_{l,m}} &= \left(\frac{-1}{\omega_0}\right)^{q_{l,m}} \left(\sum_{\substack{i=0,\ldots,d_x \\ j=m,\ldots,d_y}} a_{i,j} \sum_{\substack{|L|=j-m \\ \|L\|=l+i_k-m(k+1)-i}} \frac{j!}{m!\,L!} C^L \right)^{q_{l,m}} \\ &= \left(\frac{-1}{\omega_0}\right)^{q_{l,m}} \sum_{|M_{l,m}|=q_{l,m}} \left(\frac{q_{l,m}!}{M_{l,m}!} A^{M_{l,m}} \right. \\ &\qquad \left. \prod_{\substack{i=0,\ldots,d_x \\ j=m,\ldots,d_y}} \left(\sum_{\substack{|L|=j-m \\ \|L\|=l+i_k-m(k+1)-i}} \frac{j!}{m!\,L!} C^L \right)^{m_{i,j}^{l,m}} \right) \\ &\quad \text{where } M_{l,m} = (m_{i,j}^{l,m}) \text{ for } i = 0, \ldots, d_x,\ j = 0, \ldots, d_y \text{ and} \\ &\quad m_{i,j}^{l,m} = 0 \text{ for } j < m. \end{aligned} \tag{14}$$

Let us expand the expression $\prod_{\substack{i=0,\ldots,d_x \\ j=m,\ldots,d_y}} \left(\sum_{\substack{|L|=j-m \\ \|L\|=l+i_k-m(k+1)-i}} \frac{j!}{m!\,L!} C^L \right)^{m_{i,j}^{l,m}}$. For each (l, m, i, j), we enumerate the terms $\dfrac{j!}{m!\,L!} C^L$ with $u = 1, \ldots, \alpha_{i,j}^{l,m}$. Subsequently:

$$\left(\sum_{\substack{|L|=j-m \\ l+i_k-m(k+1)-i}} \frac{j!}{m!\,L!} C^L \right)^{m_{i,j}^{l,m}} = \left(\sum_{u=1}^{\alpha_{i,j}^{l,m}} \frac{j!}{m!\,L_{i,j,u}^{l,m}!} C^{L_{i,j,u}^{l,m}} \right)^{m_{i,j}^{l,m}}$$

$$= \sum_{|N_{i,j}^{l,m}|=m_{i,j}^{l,m}} \left(\frac{m_{i,j}^{l,m}!}{N_{i,j}^{l,m}!} \left(\prod_{u=1}^{\alpha_{i,j}^{l,m}} \left(\frac{j!}{m!\, L_{i,j,u}^{l,m}!} \right)^{n_{i,j,u}^{l,m}} \right) \right.$$

$$\left. C^{\sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} L_{i,j,u}^{l,m}} \right),$$

where $N_{i,j}^{l,m} = \left(n_{i,j,u}^{l,m}\right)_{u=1,\ldots,\alpha_{i,j}^{l,m}}$, $N_{i,j}^{l,m}! = \prod_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m}!$. Denoting $U_{l,m} := \sum_{\substack{i=0,\ldots,d_x \\ j=m,\ldots,d_y}} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} L_{i,j,u}^{l,m}$, one computes:

$$\begin{aligned} |U_{l,m}| &= \sum_{\substack{i=0,\ldots,d_x \\ j=m,\ldots,d_y}} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} |L_{i,j,u}^{l,m}| \\ &= \sum_{i=0,\ldots,d_x\, j=m,\ldots,d_y} \left(\sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} \right) (j-m) \\ &= \sum_{i=0,\ldots,d_x\, j=m,\ldots,d_y} m_{i,j}^{l,m} (j-m) \\ &= \|M_{l,m}\|_2 - m\, q_{l,m}. \end{aligned} \tag{15}$$

Likewise, one computes:

$$\begin{aligned} \|U_{l,m}\| &= \sum_{\substack{i=0,\ldots,d_x \\ j=m,\ldots,d_y}} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} \|L_{i,j,u}^{l,m}\| \\ &= \sum_{\substack{i=0,\ldots,d_x \\ j=m,\ldots,d_y}} \left(\sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} \right) (l + i_k - m(k+1) - i) \\ &= \sum_{\substack{i=0,\ldots,d_x \\ j=m,\ldots,d_y}} m_{i,j}^{l,m} (l + i_k - m(k+1) - i) \\ &= q_{l,m}[l + i_k - m(k+1)] - \|M_{l,m}\|_1. \end{aligned} \tag{16}$$

So, according to formula (14) and the new way of writing the expression $\prod_{\substack{i=0,\ldots,d_x \\ j=m,\ldots,d_y}} \left(\sum_{\substack{|L|=j-m \\ \|L\|=l+i_k-m(k+1)-i}} \frac{j!}{m!\,L!} C^L \right)^{m_{i,j}^{l,m}}$, we obtain:

$$b_{l,m}^{q_{l,m}} = \left(\frac{-1}{\omega_0}\right)^{q_{l,m}} \sum_{|M_{l,m}|=q_{l,m}} A^{M_{l,m}} \sum_{\substack{|U_{l,m}|=\|M_{l,m}\|_2-m\,q_{l,m}\\ \|U_{l,m}\|=q_{l,m}[l+i_k-m(k+1)]-\|M_{l,m}\|_1}} d_{U_{l,m}} C^{U_{l,m}}$$

$$\text{with } d_{U_{l,m}} := \sum_{\left(N_{i,j}^{l,m}\right)} \frac{q_{l,m}!}{\prod_{\substack{i=0,\dots,d_x\\ j=m,\dots,d_y}} N_{i,j}^{l,m}!} \prod_{\substack{i=0,\dots,d_x\\ j=m,\dots,d_y}} \prod_{u=1}^{\alpha_{i,j}^{l,m}} \left(\frac{j!}{m!\,L_{i,j,u}^{l,m}!}\right)^{n_{i,j,u}^{l,m}},$$

where the sum is taken over

$$\left\{ \left(N_{i,j}^{l,m}\right)_{\substack{i=0,\dots,d_x\\ j=m,\dots,d_y}} \text{ such that } |N_{i,j}^{l,m}| = m_{i,j}^{l,m} \text{ and } \sum_{\substack{i=0,\dots,d_x\\ j=m,\dots,d_y}} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} L_{i,j,u}^{l,m} = U_{l,m} \right\}.$$

(Note that, if the latter set is empty, then $d_{U_{l,m}} = 0$.)

We deduce that:

$$\begin{aligned} B^Q &= \prod_{l=1,\dots,(k+1)d_y+d_x-i_k,\; m=0,\dots,m_l} b_{l,m}^{q_{l,m}} \\ &= \left(\frac{-1}{\omega_0}\right)^q \prod_{l,m} \left[\sum_{|M_{l,m}|=q_{l,m}} A^{M_{l,m}} \sum_{\substack{U_{l,m}|=\|M_{l,m}\|_2-m\,q_{l,m}\\ \|U_{l,m}\|=q_{l,m}[l+i_k-m(k+1)]-\|M_{l,m}\|_1}} d_{U_{l,m}} C^{U_{l,m}} \right]. \end{aligned}$$

Now, in order to expand the latter product of sums, we consider the corresponding sets:

$$\mathcal{S}_Q := \left\{ \sum_{l,m} M_{l,m} \;\; / \;\; \forall l, m,\; |M_{l,m}| = q_{l,m} \text{ and } m_{i,j}^{l,m} = 0 \text{ for } j < m \right\}$$

and, for any $S \in \mathcal{S}_Q$,

$$\begin{aligned} \mathcal{U}_{Q,S} := \Big\{ \left(U_{l,m}\right) \;\; / \;\; &\exists (M_{l,m}) \text{ s.t. } |M_{l,m}| = q_{l,m} \text{ and } m_{i,j}^{l,m} = 0 \\ &\text{for } j < m,\; \sum_{l,m} M_{l,m} = S,\; |U_{l,m}| = \|M_{l,m}\|_2 - m\,q_{l,m} \text{ and } \|U_{l,m}\| \\ &= q_{l,m}[l + i_k - m(k+1)] - \|M_{l,m}\|_1 \Big\} \end{aligned}$$

and

$$\mathcal{T}_{Q,S} := \left\{ \sum_{l,m} U_{l,m} \ / \ \left(U_{l,m}\right) \in \mathcal{U}_{Q,S} \right\}.$$

We have:

$$\begin{aligned} B^Q &= \left(\frac{-1}{\omega_0}\right)^q \sum_{S \in \mathcal{S}_Q} A^S \sum_{T_S \in \mathcal{T}_{Q,S}} \left(\sum_{\substack{(U_{l,m}) \in \mathcal{U}_{Q,S},\\ \Sigma_{l,m} U_{l,m} = T_S}} \prod_{l,m} d_{U_{l,m}} \right) \mathcal{C}^{T_S} \\ &= \left(\frac{-1}{\omega_0}\right)^q \sum_{S \in \mathcal{S}_Q} A^S \sum_{T_S \in \mathcal{T}_{Q,S}} e_{Q,T_S} C^{T_S}. \end{aligned} \tag{17}$$

where:

$$e_{Q,T_S} := \sum_{\left(N_{i,j}^{l,m}\right)} \frac{\prod_{l,m} q_{l,m}!}{\prod_{l,m} \prod_{i,j} N_{i,j}^{l,m}!} \prod_{l,m} \prod_{i,j} \prod_{u} \left(\frac{j!}{m!\, L_{i,j,u}^{l,m}!} \right)^{n_{i,j,u}^{l,m}}$$

and where the previous sum is taken over:

$$\mathcal{E}_{Q,T_S} := \left\{ \left(N_{i,j}^{l,m}\right)_{\substack{l=1,\dots,(k+1)d_y+d_x-i_k,\ m=0,\dots,m_l \\ i=0,\dots,d_x,\ j=m,\dots,d_y}} \ / \ \sum_{l,m} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} = s_{i,j},\ \forall l, m,\ \sum_{i,j} |N_{i,j}^{l,m}| = q_{l,m}, \text{ and } \sum_{l,m} \sum_{i,j} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} L_{i,j,u}^{l,m} = T_S \right\}.$$

where $S = (s_{i,j})$.

Note that, for any Q and for any $S \in \mathcal{S}_Q$, $|S| = \sum_{l,m} q_{l,m} = q$ and $\|S\|_2 \geq \sum_{l,m} m q_{l,m} = \|Q\|_2 = q - 1$. Moreover, for any $T_S \in \mathcal{T}_{Q,S}$:

$$\begin{aligned} |T_S| &= \sum_{l,m} \|M_{l,m}\|_2 - m\, q_{l,m} \\ &= \|S\|_2 - \|Q\|_2 \\ &= \|S\|_2 - q + 1 \end{aligned}$$

and:

$$\begin{aligned}\|T_S\| &= \sum_{l,m} q_{l,m}[l+i_k-m(k+1)] - \|M_{l,m}\|_1\\ &= \|Q\|_1 + |Q|i_k - \|Q\|_2(k+1) - \|S\|_1\\ &= p + qi_k - (q-1)(k+1) - \|S\|_1.\end{aligned}$$

Now, let us show that:

$$\sum_{|Q|=q,\ \|Q\|_1=p,\ \|Q\|_2=q-1} \frac{q!}{Q!} B^Q = \left(\frac{-1}{\omega_0}\right)^q \left(\sum_{|S|=q,\ \|S\|_2\geq q-1} A^S \sum_{\substack{|T_S|=\|S\|_2-q+1\\ \|T_S\|=p+qi_k-(q-1)(k+1)-\|S\|_1}} e_{T_S} C^{T_S} \right), \quad (18)$$

where $e_{T_S} := \sum_{\left(N_{i,j}^{l,m}\right)} \frac{q!}{\prod_{l,m}\prod_{i,j} N_{i,j}^{l,m}!} \prod_{l,m}\prod_{i,j}\prod_u \left(\frac{j!}{m!\,L_{i,j,u}^{l,m}!}\right)^{n_{i,j,u}^{l,m}}$ and where the sum is taken over

$$\mathcal{E}_{T_S} := \left\{ \begin{array}{c} \left(N_{i,j}^{l,m}\right) \\ {\scriptstyle l=1,\ldots,(k+1)d_y+d_x-i_k,\ m=0,\ldots,m_l} \\ {\scriptstyle i=0,\ldots,d_x,\ j=m,\ldots,d_y} \end{array} \;/\; \sum_{l,m}\sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} = s_{i,j},\ \sum_{l,m}\sum_{i,j} |N_{i,j}^{l,m}| = q \right.$$
$$\left. \text{and } \sum_{l,m}\sum_{i,j}\sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} L_{i,j,u}^{l,m} = T_S \right\}.$$

(Note that, if the latter set is empty, then $e_{T_S} = 0$.)

Recall that $N_{i,j}^{l,m}! = \prod_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m}!$ and that the $L_{i,j,u}^{l,m}$'s enumerate the L's such that $|L| = j-m$ and $\|L\| = l + i_k - m(k+1) - i$ for given l, m, i, j.

Let us consider S and T_S such that $|S| = q$, $\|S\|_2 \geq q-1$, $|T_S| = \|S\|_2 - q + 1$, $\|T_S\| = p + qi_k - (q-1)(k+1) - \|S\|_1$ and such that $\mathcal{E}_{T_S} \neq \emptyset$. Take an element $(n_{i,j,u}^{l,m}) \in \mathcal{E}_{T_S}$. Define $m_{i,j}^{l,m} := \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m}$ for each $i,\ j,\ l,\ m$ with $j \geq m$, and $m_{i,j}^{l,m} := 0$ if $j < m$. Set $M_{l,m} := (m_{i,j}^{l,m})_{i,j}$ for each $l,\ m$. So, $\sum_{l,m} m_{i,j}^{l,m} = \sum_{l,m}\sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} = s_{i,j}$, and $S = \sum_{l,m} M_{l,m}$. Define $q_{l,m} := \sum_{i,j} m_{i,j}^{l,m} = |M_{l,m}|$ for each $l,\ m$, and $Q := (q_{l,m})$. Let us show that $|Q| = q$, $\|Q\|_1 = p$ and $\|Q\|_2 = q-1$. By definition of $\mathcal{E}_{T_S}$,

$$|Q| := \sum_{l,m} q_{l,m} = \sum_{l,m} \sum_{i,j} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} = q.$$

Recall that $\|Q\|_2 := \sum_{l,m} m q_{l,m}$. We have:

$$\begin{aligned}
|T_S| &= \left| \sum_{l,m} \sum_{i,j} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} L_{i,j,u}^{l,m} \right| = \|S\|_2 - q + 1 \\
&\Leftrightarrow \sum_{l,m} \sum_{i,j} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} |L_{i,j,u}^{l,m}| = \sum_{i,j} j s_{i,j} - q + 1 \\
&\Leftrightarrow \sum_{l,m} \sum_{i,j} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} (j - m) = \sum_{i,j} j s_{i,j} - q + 1 \\
&\Leftrightarrow \sum_{i,j} j \sum_{l,m} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} - \sum_{l,m} m \sum_{i,j} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} = \sum_{i,j} j s_{i,j} - q + 1 \\
&\Leftrightarrow \sum_{i,j} j s_{i,j} - \sum_{l,m} m q_{l,m} = \sum_{i,j} j s_{i,j} - q + 1 \\
&\Leftrightarrow \|Q\|_2 = q - 1.
\end{aligned}$$

Recall that $\|Q\|_1 := \sum_{l,m} l q_{l,m}$. We have:

$$\begin{aligned}
\|T_S\| \;=\; &\left\| \sum_{l,m} \sum_{i,j} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} L_{i,j,u}^{l,m} \right\| = p + q i_k - (q-1)(k+1) - \|S\|_1 \\
&\Leftrightarrow \sum_{l,m} \sum_{i,j} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} \|L_{i,j,u}^{l,m}\| = p + q i_k - (q-1)(k+1) - \|S\|_1 \\
&\Leftrightarrow \sum_{l,m} \sum_{i,j} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} (l + i_k - m(k+1) - i) \\
&\quad = p + q i_k - (q-1)(k+1) - \|S\|_1 \\
&\Leftrightarrow \sum_{l,m} l \sum_{i,j} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} + i_k \sum_{l,m} \sum_{i,j} \sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m}
\end{aligned}$$

$$-(k+1)\sum_{l,m} m \sum_{i,j}\sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} - \sum_{i,j} i \sum_{l,m}\sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m}$$
$$= p + qi_k - (q-1)(k+1) - \|S\|_1$$
$$\Leftrightarrow \sum_{l,m} l q_{l,m} + i_k \cdot q$$
$$-(k+1)\sum_{l,m} m q_{l,m} - \sum_{i,j} i s_{i,j} = p + qi_k - (q-1)(k+1) - \|S\|_1$$
$$\Leftrightarrow \|Q\|_1 + qi_k - \|Q\|_2(k+1) - \|S\|_1 = p + qi_k - (q-1)(k+1) - \|S\|_1.$$

Since $\|Q\|_2 = q-1$, we deduce that $\|Q\|_1 = p$ as desired. So, $S \in \mathcal{S}_Q$ for Q as in the left-hand side of (18).

Now, set $U_{l,m} := \sum_{i,j}\sum_{u=1}^{\alpha_{i,j}^{l,m}} n_{i,j,u}^{l,m} L_{i,j,u}^{l,m}$, so $\sum_{l,m} U_{l,m} = T_S$. Let us show that $(U_{l,m}) \in \mathcal{U}_{Q,S}$, which implies that $T_S \in \mathcal{T}_{Q,S}$ as desired. The existence of $(M_{l,m})$ such that $|M_{l,m}| = q_{l,m}$ and $m_{i,j}^{l,m} = 0$ for $j < m$ and $\sum_{l,m} M_{l,m} = S$ follows by construction. Conditions $|U_{l,m}| = \|M_{l,m}\|_2 - m\, q_{l,m}$ and $\|U_{l,m}\| = q_{l,m}[l + i_k - m(k+1)] - \|M_{l,m}\|_1$ are obtained exactly as in (15) and (16). This shows that $(n_{i,j,u}^{l,m}) \in \mathcal{E}_{Q,T_S}$, so:

$$\mathcal{E}_{T_S} \subseteq \dot{\bigcup_{|Q|=q,\ \|Q\|_1=p,\ \|Q\|_2=q-1}} \mathcal{E}_{Q,T_S}.$$

The reverse inclusion holds trivially since $|Q| = q$, so:

$$\mathcal{E}_{T_S} = \dot{\bigcup_{|Q|=q,\ \|Q\|_1=p,\ \|Q\|_2=q-1}} \mathcal{E}_{Q,T_S}.$$

We deduce that:

$$e_{T_S} = \sum_{|Q|=q,\ \|Q\|_1=p,\ \|Q\|_2=q-1} \frac{q!}{Q!} e_{Q,T_S}.$$

We conclude that any term occuring in the right-hand side of (18) comes from a term from the left-hand side.

Conversely, for any Q as in the left-hand side of Formula (18), $S \in \mathcal{S}_Q$ and $T_S \in \mathcal{T}_{Q,S}$ verify the following conditions:

$$|S| = q,\quad \|S\|_2 \geq q-1,\quad |T_S| = \|S\|_2 - q + 1,$$
$$\|T_S\| = p + qi_k - (q-1)(k+1) - \|S\|_1$$

and

$$\mathcal{E}_{T_S} = \dot{\bigcup_{|Q|=q,\ \|Q\|_1=p,\ \|Q\|_2=q-1}} \mathcal{E}_{Q,T_S}, \quad e_{T_S} = \sum_{|Q|=q,\ \|Q\|_1=p,\ \|Q\|_2=q-1} \frac{q!}{Q!} e_{Q,T_S}.$$

Hence, any term occuring in the expansion of B^Q contributes to the right hand side of formula (18).

Thus we obtain formula (18) from which the statement of Corollary 1 follows.

Remark 5 We have seen in Theorem 3 and its proof (see Formula (11) with $k = k_0$) that $\omega_0 = (\pi^P_{k_0,i_{k_0}})'(c_{k_0+1})$ is the coefficient of the monomial $x^{i_{k_0}+1}y$ in the expansion of $P_{k_0+1}(x, y) = P(x, c_1x+\cdots+c_{k_0+1}x^{k_0+1}+x^{k_0+2}y)$, and that $c_{k_0+2} = \dfrac{-\pi^P_{k_0,i_{k_0}+1}(c_{k_0+1})}{\omega_0}$ where $\pi^P_{k_0,i_{k_0}+1}(c_{k_0+1})$ is the coefficient of $x^{i_{k_0}+1}$ in the expansion of $P_{k_0+1}(x, y)$. Expanding $P_{k_0+1}(x, y)$, having done the whole computations, we deduce that:

$$\begin{cases} \omega_0 = \displaystyle\sum_{i=0,\ldots,d_x,\ j=1,\ldots,d_y} \ \sum_{|L|=j-1,\ \|L\|=i_{k_0}-i-k_0-1} \frac{j!}{L!} a_{i,j} C^L \ ; \\ c_{k_0+2} = \displaystyle\frac{-1}{\omega_0} \sum_{i=0,\ldots,d_x,\ j=0,\ldots,d_y} \ \sum_{|L|=j,\ \|L\|=i_{k_0}-i+1} \frac{j!}{L!} a_{i,j} C^L. \end{cases}$$

Example 3 In order to illustrate Corollary 1 and its proof, we resume the polynomial of Example 1 with $a_{0,2} \neq 0$:

$$\begin{aligned} P(x, y) &= a_{0,2}y^2 + \left(a_{2,0} + a_{2,1}y + a_{2,2}y^2\right) x^2 \\ P_0(x, y) &= \left(a_{2,0} + a_{0,2}y^2\right) x^2 + a_{2,1}yx^3 + a_{2,2}y^2x^4 \\ P_1(x, y) &= \left(2a_{0,2}c_1y + a_{2,1}c_1\right) x^3 + \left(a_{0,2}y^2 + a_{2,1}y + a_{2,2}c_1^2\right) x^4 \\ &\quad + 2a_{2,2}c_1yx^5 + a_{2,2}y^2x^6 \\ &\quad \text{with } a_{2,0} + a_{0,2}c_1^2 = 0 \ \Leftrightarrow\ c_1 = \pm\sqrt{\frac{-a_{2,0}}{a_{0,2}}}. \end{aligned}$$

Thus, $i_0 = 2$, $i_1 = 3 = i_0 + 1$, so $k_0 = 0$, $\omega_0 = 2\,a_{0,2}\,c_1$. The coefficient c_2 must verify $2a_{0,2}c_1c_2 + a_{2,1}c_1 = 0 \ \Leftrightarrow\ c_2 = \dfrac{-a_{2,1}}{2a_{0,2}}$. We obtain that:

$$\begin{aligned} -\omega_{0\,1}R &= \frac{P_1(x, y + c_2)}{x^3} \\ &= \omega_0 y + \left(a_{2,2}{c_1}^2 + a_{2,1}c_2 + a_{0,2}{c_2}^2 + \left(a_{2,1} + 2\,a_{0,2}c_2\right) y + a_{0,2}y^2\right) x \\ &\quad + \left(2\,a_{2,2}c_1c_2 + 2\,a_{2,2}c_1y\right) x^2 + \left(a_{2,2}{c_2}^2 + 2\,a_{2,2}c_2y + a_{2,2}y^2\right) x^3. \end{aligned}$$

So the coefficients of the corresponding reduced Henselian equation $y = {}_1Q(x, y)$ are:

$$b_{1,0} = -\left(a_{2,2}{c_1}^2 + a_{2,1}c_2 + a_{0,2}{c_2}^2\right)/\omega_0,\; b_{1,1} = -\left(a_{2,1} + 2\,a_{0,2}c_2\right)/\omega_0 = 0,$$
$$b_{1,2} = -a_{0,2}/\omega_0, \quad b_{2,0} = -2\,a_{2,2}c_1c_2/\omega_0,\; b_{2,1} = -2\,a_{2,2}c_1/\omega_0,$$
$$b_{3,0} = -a_{2,2}{c_2}^2/\omega_0,\; b_{3,1} = -2\,a_{2,2}c_2/\omega_0, \quad b_{3,2} = -a_{2,2}/\omega_0,$$

But, by version 1 of Flajolet–Soria's Formula 1, one has that:

$$c_3 = b_{1,0} = \frac{-a_{2,2}{c_1}^2 - a_{2,1}c_2 - a_{0,2}{c_2}^2}{2\,a_{0,2}\,c_1};$$
$$c_4 = b_{2,0} + b_{1,0}b_{1,1} = b_{2,0} = \frac{-2\,a_{2,2}c_1c_2}{2\,a_{0,2}\,c_1};$$
$$c_5 = b_{3,0} + b_{1,0}b_{2,1} + b_{1,0}^2 b_{1,2} + b_{1,0}b_{1,1}^2 + b_{2,0}b_{1,1} = b_{3,0} + b_{1,0}b_{2,1} + b_{1,0}^2 b_{1,2}$$
$$= \frac{-a_{2,2}\,{c_2}^2}{2\,a_{0,2}\,c_1} + \frac{2\,a_{2,1}a_{2,2}c_1c_2 + 2\,a_{0,2}a_{2,2}c_1{c_2}^2 + 2\,{a_{2,2}}^2{c_1}^3}{\left(2\,a_{0,2}\,c_1\right)^2} - \frac{a_{0,2}{a_{2,1}}^2{c_2}^2 + 2\,{a_{0,2}}^2a_{2,1}{c_2}^3 + 2\,a_{0,2}a_{2,1}a_{2,2}{c_1}^2c_2 + {a_{0,2}}^3{c_2}^4}{\left(2\,a_{0,2}\,c_1\right)^3} - \frac{2\,{a_{0,2}}^2a_{2,2}{c_1}^2{c_2}^2 + a_{0,2}{a_{2,2}}^2{c_1}^4}{\left(2\,a_{0,2}\,c_1\right)^3};$$
$$\vdots$$

Remark 6 Classically, a series $y_0 = \sum_{n\geq 0} c_n x^n \in K[[x]]$ is algebraic if and only if its coefficients c_n are the diagonal coefficients of the power series expansion of a bivariate rational fraction [7,10]. In particular, in the reduced Henselian case $y = Q(x, y)$ (see Definition 2), the rational fraction can be written:

$$y_0 = \mathrm{Diag}\left(\frac{y^2 - y^2\dfrac{\partial Q}{\partial y}(xy, y)}{y - Q(xy, y)}\right).$$

With the computations in the proof of Corollary 1, we can deduce in the general case $P(x, y) = 0$ a formula for the rational fraction having the c_n as diagonal coefficients of its expansion.

As a consequence of Theorem 2 and Corollary 1, we get the following result. Let d_x, d_y be some fixed degrees, and an integer $n > 2d_x d_y + 2$. There is a finite number of universal polynomial formulas which compute the coefficient c_n of any algebraic series y_0 of degrees at most d_x, d_y. These formulas are evaluated at the first $2d_x d_y + 2$ first coefficients of y_0, and their number is independent of n. More precisely:

Corollary 2 *Let $d_x, d_y \in \mathbb{N}^*$. We set $M := \frac{1}{2}d_y(d_y+1)(d_x+1)+d_y-2$ and $\mu := N+2 = 2d_xd_y+2$. There exists a finite set Λ and for any $\lambda \in \Lambda$, there exist a polynomial $\Omega^{(\lambda)}(C_1,\ldots,C_\mu) \in \mathbb{Z}[C_1,\ldots,C_\mu]$, $\deg \Omega^{(\lambda)} \leq M$, and for every $p \in \mathbb{N}^*$, a polynomial $\Psi_p^{(\lambda)}(C_1,\ldots,C_{\mu+1}) \in \mathbb{Z}[C_1,\ldots,C_{\mu+1}]$, $\deg \Psi_p^{(\lambda)} \leq p(M+d_x)+1$, such that for every $y_0 = \sum_{n\geq 1} c_n x^n$, $c_1 \neq 0$, algebraic with vanishing polynomial of degrees bounded by d_x in x and d_y in y, there exists $\lambda \in \Lambda$ such that for every $p \in \mathbb{N}^*$:*

$$c_{\mu+1+p} = \frac{\Psi_p^{(\lambda)}(c_1,\ldots,c_{\mu+1})}{\Omega^{(\lambda)}(c_1,\ldots,c_\mu)^p}.$$

Proof Let $y_0 = \sum_{n\geq 1} c_n x^n, c_1 \neq 0$, be algebraic with vanishing polynomial of degree bounded by d_x in x and d_y in y. According to Theorem 2, there is a finite set Λ and for every $\lambda \in \Lambda$, polynomials $a_{i,j}^{(\lambda)}(C_1,\ldots,C_N) \in \mathbb{Z}[C_1,\ldots,C_N]$ such that:

$$P^{(\lambda)} = \sum_{i\leq d_x, j\leq d_y} a_{i,j}^{(\lambda)}(c_1,\ldots,c_N)x^i y^j$$

is a vanishing polynomial for y_0 for a certain $\lambda \in \Lambda$. Enlarging the finite set Λ by indices corresponding to the various $\frac{\partial^k P^{(\lambda)}}{\partial y^k}$, $k = 1,\ldots,d_y-1$, we can assume that there is λ such that y_0 is a simple root of $P^{(\lambda)}$. So the coefficients of y_0 can be computed as in Corollary 1. More precisely, for any $p \in \mathbb{N}^*$:

$$c_{\mu+1+p} = \sum_{q=1}^{p} \sum_{S\in I_q} \sum_{T_S\in J_S} \frac{m_{S,T_S}}{\omega_0^p} \omega_0^{p-q} A^S C^{T_S} \tag{19}$$

where $I_q = \left\{(s_{i,j}) \mid |S| = q,\ \|S\|_2 \geq q-1\right\}$,

$$J_S = \left\{(t_{S,i}) \mid |T_S| = \|S\|_2 - q + 1,\ \|T_S\| = p + qi_\mu - (q-1)(\mu+1) - \|S\|_1\right\}$$

and $m_{S,T_S} \in \mathbb{Z}$. Note that $C = (c_1,\ldots,c_{\mu+1})$ and $A = (a_{i,j}(c_1,\ldots,c_N))$. It suffices to bound the degrees of the numerator and denominator in the terms of formula (19). By Theorem 2, $\deg a_{i,j}^{(\lambda)} \leq M+d_x+1-d_y$. So by Remark 5 and Theorem 2, we deduce that ω_0 is the evaluation of a polynomial $\Omega^{(\lambda)}(C_1,\ldots,C_\mu)$ such that $\deg \Omega^{(\lambda)} \leq M$. The degree $d_{q,S}$ corresponding to a term $\omega_0^{p-q} A^S C^{T_S}$ is bounded by:

$$\begin{aligned}&(p-q)M + |S|(M+d_x+1-d_y) + |T_S|\\ &\quad = (p-q)M + q(M+d_x+1-d_y) + \|S\|_2 - q + 1.\end{aligned}$$

But, $\|S\|_2 \leq qd_y$ and $1 \leq q \leq p$. So we get that:

$$d_{q,S} \leq (p-q)M + q(M+d_x+1-d_y) + qd_y - q + 1 \leq p(M+d_x)+1$$

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