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On the essential bounded Riesz Φ-variation

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Abstract In Nakamura and Hashimoto (Collect Math 65(3):407–416, 2014), the authors showed that for every $f \in L^1_{\text{loc}}(\mathbb{R})$, the essential p-variation $\text{ess}\, V_p(f, \mathbb{R})$ of f is given by

$$\lim_{h \to 0} \int_{\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} \right|^p dx.$$

In this paper, more generally we treat the following convergence for a function $f \in L^1_{\text{loc}}(\mathbb{R})$ and a convex function $\Phi : \mathbb{R} \to [0, \infty)$;

$$\lim_{h \to 0} \int_{\mathbb{R}} \Phi \left(\frac{f(t+h) - f(t)}{h} \right) dt,$$

and we show that the limit is equivalent to an essential Φ-variation $\text{ess}\, V_\Phi(f)$. Moreover, we obtain a characterization of the class of functions f with $\text{ess}\, V_\Phi(f) < \infty$.

Keywords Bounded variation · Essential bounded variation · Convexity · Sobolev space

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1 Introduction

Let $(\mathbb{R}, \mathfrak{M}, m)$ be the usual Lebesgue measure space on the set $\mathbb{R}$ of real numbers. Let $1 \leq p < +\infty$. The space $L^p(\mathbb{R})$ will denote the $L^p(\mathbb{R}, \mathfrak{M}, m)$, which are made up of the Lebesgue measurable functions f, for which $|f|^p$ is integrable on $\mathbb{R}$.

A function $f \in L^1_{\text{loc}}(\mathbb{R})$ is said to be *locally integrable* on $\mathbb{R}$ if f is integrable on every bounded closed subinterval I of $\mathbb{R}$.

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *locally absolutely continuous* if it is absolutely continuous on every bounded closed subinterval I of $\mathbb{R}$. We denote by $\mathrm{AC}_{\text{loc}}(\mathbb{R})$ the vector space of locally absolutely continuous functions on $\mathbb{R}$.

Let $W^{1,p}(\mathbb{R})$ be a Sobolev space, i.e, $f \in W^{1,p}(\mathbb{R})$ if and only if $f \in L^p(\mathbb{R})$ and the derivative Df of f in the sense of distribution belongs to $L^p(\mathbb{R})$. In particular, if $f \in L^1(\mathbb{R})$ and Df is a Radon measure of bounded variation on $\mathbb{R}$, then f is called a function of bounded variation. The class of all such functions will be denoted by $BV(\mathbb{R})$. Thus, $f \in BV(\mathbb{R})$ if and only if there is a Radon measure μ defined in $\mathbb{R}$ such that $|\mu|(\mathbb{R}) < +\infty$ and

$$\int_{\mathbb{R}} f\varphi' dt = -\int_{\mathbb{R}} \varphi d\mu, \ \ \varphi \in C_0^\infty(\mathbb{R}),$$

where, $|Df|(\mathbb{R}) = |\mu|(\mathbb{R})$ means the total variation of μ. It is obvious that a function f on $\mathbb{R}$ is absolutely continuous and the derivative f' is in $L^1(\mathbb{R})$, then f is of bounded variation, i.e. $W^{1,1}(\mathbb{R}) \subset BV(\mathbb{R})$ (see [2, p. 222]).

Let $\mathcal{N}$ be denote the set of all convex functions $\Phi : \mathbb{R} \to [0, \infty)$ such that $\Phi(0) = 0$ and Φ is not identically zero. It is well-known that every $\Phi \in \mathcal{N}$ is continous on $\mathbb{R}$.

Given $f \in L^1_{\text{loc}}(\mathbb{R})$ and a subset D of $\mathbb{R}$, we write

$$V_\Phi(f, D) = \sup_\pi \sum_{i=1}^n \Phi\left(\frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}}\right)(t_i - t_{i-1}),$$

for the Φ-*variation on* D of the function f in the sense of F. Riesz ([6]), where the supremum is taken over all partitions $\pi : t_0 < t_1 < \cdots < t_k < \cdots < t_n$ and $t_k \in D$. If $D = \mathbb{R}$, we denote the $V_\Phi(f, D)$ simply by $V_\Phi(f)$. If $\Phi(x) = |x|$, $x \in \mathbb{R}$, then it is obvious that $\Phi(x)$, $\Phi^+(x) = x \vee 0$, $\Phi^-(x) = (-x) \vee 0 \in \mathcal{N}$ and $V_\Phi(f) = V_{\Phi^+}(f) + V_{\Phi^-}(f)$. In the case $D = [a, b]$ (a subinterval of $\mathbb{R}$), we refer to [1] and [3], but they are treated under the additional assumption that Φ is even on $\mathbb{R}$.

We define a subset ess $D(f)$ of $\mathbb{R}$ by

$$\text{ess } D(f) = \left\{ t \in \mathbb{R} : \lim_{h \to 0} \frac{1}{h} \int_t^{t+h} f(s)\, ds \text{ converges} \right\}.$$

We shall call the set ess $D(f)$ the *essential domain* of f. We should note that $m(\mathbb{R} \backslash \text{ess } D(f)) = 0$ and ess $D(f)$ does not depend on measurable functions equal to f a.e. on $\mathbb{R}$. We define

$$\widetilde{f}(t) = \begin{cases} \lim\limits_{h\to 0} \dfrac{1}{h} \displaystyle\int_t^{t+h} f(s)\,ds, & t \in \text{ess } D(f) \\ \text{undefined}, & \text{otherwise}. \end{cases}$$

Then we should note that $f = \widetilde{f}$ $(a.e.)$. Put

$$L(f) = \left\{ t \in \text{ess } D(f) : \widetilde{f}(t) = f(t) \right\},$$

then it is obvious that $m(\mathbb{R}\backslash L(f)) = 0$.

For $f \in L^1_{\text{loc}}(\mathbb{R})$, define

$$\text{ess } V_\Phi(f) = \sup \left\{ \sum_{k=1}^{n} \Phi\left(\frac{\widetilde{f}(t_k) - \widetilde{f}(t_{k-1})}{t_k - t_{k-1}} \right) \times (t_k - t_{k-1}) : t_0 < t_1 < \cdots < t_n,\ t_k \in \text{ess } D(f) \right\}$$

In what follows, we call ess $V_\Phi(f)$ the *essential* Φ*-variation* of f. In particular, if $\Phi(x) = |x|^p$, then we call this variation the essential p-variation instead of the essential Φ-variation, we write ess $V_p(f)$ in place of ess $V_\Phi(f)$. Moreover, for $h \neq 0$, we write $f_h(t) = \dfrac{f(t+h) - f(t)}{h}$.

In the case $p = 1$, in [4] we showed that ess $V_1(f) < \infty$ if and only if $f \in BV(\mathbb{R})$. More generally, in [5], we have obtained that the following holds:

Theorem *Let* $1 \leq p < \infty$ *and* $f \in L^1_{\text{loc}}(\mathbb{R})$*. Then we have*

$$\lim_{h\to 0} \int_{\mathbb{R}} |f_h(t)|^p\, dt = \text{ess } V_p(f).$$

In this paper, more generally we treat the following convergence for $f \in L^1_{\text{loc}}(\mathbb{R})$ *and* $\Phi \in \mathcal{N}$*:*

$$\lim_{h\to 0} \int_{\mathbb{R}} \Phi\left(f_h(t)\right) dt,$$

and we show that the limit is equivalent to an essential Φ*-variation ess* $V_\Phi(f)$*. Moreover, we obtain a characterization (Theorem 2.7) of the class of functions* f *with ess* $V_\Phi(f) < \infty$*.*

2 Main results

Theorem 2.1 *Let* $\Phi \in \mathcal{N}$ *and* $f \in L^1_{\text{loc}}(\mathbb{R})$*. Then we have*

(i) $\lim_{h\to 0} \int_{\mathbb{R}} \Phi\left(f_h(t)\right) dt = \textit{ess } V_\Phi(f)$.

(ii) ess $V_\Phi(f) \leq V_\Phi(f)$.

(iii) If $f \in AC_{\mathrm{loc}}(\mathbb{R})$*, we have* $\operatorname{ess} V_\Phi(f) = \int_{\mathbb{R}} \Phi(f'(t))\,dt$*.*

In particular, if $\Phi(x) = |x|^p$ *for* $1 \le p < \infty$*, then we have that*

$(iii)'$ $\operatorname{ess} V_p(f) = \int_{\mathbb{R}} |f'(t)|^p\,dt$.

Proof To prove (i), we first prove that $\liminf_{h\to 0} \int_{\mathbb{R}} \Phi\left(f_h(t)\right)\,dt \ge \operatorname{ess} V_\Phi(f)$. Let $t_0 < t_2 < \cdots < t_n$ be an arbitrary sequence in $\operatorname{ess} D(f)$. Then by Jensen's inequality we have

$$\Phi\left(\frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f_h(t)\,dt\right) \le \frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} \Phi\left(f_h(t)\right)\,dt,$$

and so

$$\begin{aligned}\int_{t_{k-1}}^{t_k} \Phi\left(f_h(t)\right)\,dt &\ge (t_k - t_{k-1})\Phi\left(\frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f_h(t)\,dt\right)\\ &= (t_k - t_{k-1})\Phi\left(\frac{1}{h(t_k - t_{k-1})}\left(\int_{t_k}^{t_k+h} f(t)\,dt \right.\right.\\ &\qquad \left.\left. - \int_{t_{k-1}}^{t_{k-1}+h} f(t)\,dt\right)\right).\end{aligned}$$

Thus for all $h \ne 0$, we have

$$\begin{aligned}\int_{\mathbb{R}} \Phi\left(f_h(t)\right)\,dt &\ge \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Phi\left(f_h(t)\right)\,dt\\ &= \sum_{k=1}^{n} (t_k - t_{k-1})\Phi\left(\frac{1}{h(t_k - t_{k-1})}\left(\int_{t_k}^{t_k+h} f(t)\,dt \right.\right.\\ &\qquad \left.\left. - \int_{t_{k-1}}^{t_{k-1}+h} f(t)\,dt\right)\right).\end{aligned}$$

It is obvious that the right-hand side of the above equation coverges to the following value as $h \to 0$:

$$\sum_{k=1}^{n} (t_k - t_{k-1})\Phi\left(\frac{\widetilde{f}(t_k) - \widetilde{f}(t_{k-1})}{t_k - t_{k-1}}\right).$$

Thus we have

$$\liminf_{h\to 0} \int_{\mathbb{R}} \Phi\left(f_h(t)\right)\,dt \ge \sum_{k=1}^{n} \Phi\left(\frac{\widetilde{f}(t_k) - \widetilde{f}(t_{k-1})}{t_k - t_{k-1}}\right)(t_k - t_{k-1}),$$

and so we see from the definition of ess $V_\Phi(f)$ that

$$\liminf_{h\to 0}\int_{\mathbb{R}}\Phi\left(f_h(t)\right)\,dt \ge \text{ess } V_\Phi(f). \tag{2.1}$$

To show the converse inequality:

$$\limsup_{h\to 0}\int_{\mathbb{R}}\Phi\left(f_h(t)\right)\,dt \le \text{ess } V_\Phi(f),$$

it suffices to show that

$$\int_{\mathbb{R}}\Phi\left(f_h(t)\right)\,dt \le \text{ess } V_\Phi(f) \quad \text{for all } h\neq 0.$$

Without loss of generality, we can assume that $h>0$.

$$\begin{aligned}\int_{\mathbb{R}}\Phi\left(f_h(t)\right)\,dt &= \sum_{k=-\infty}^{\infty}\int_{(k-1)h}^{kh}\Phi\left(f_h(t)\right)\,dt\\ &= \sum_{k=-\infty}^{\infty}\int_0^h \Phi\left(\frac{f(t+kh)-f(t+(k-1)h)}{h}\right)dt\\ &= \int_0^h\sum_{k=-\infty}^{\infty}\Phi\left(\frac{f(t+kh)-f(t+(k-1)h)}{h}\right)dt.\end{aligned}$$

On the other hand, since $\mathbb{R}\backslash L(f)$ is a null set, and also $m\left(\bigcup_{k=-\infty}^{\infty}\{(\mathbb{R}\backslash L(f))-kh\}\right)$ $=0$. Moreover, since $t\notin\bigcup_{k=-\infty}^{\infty}\{(\mathbb{R}\backslash L(f))-kh\}$ implies that $t+kh\in L(f)$ for every integer k, we see that

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}\Phi\left(\frac{f(t+kh)-f(t+(k-1)h)}{h}\right)h\\ &\quad=\sum_{k=-\infty}^{\infty}\Phi\left(\frac{\widetilde{f}(t+kh)-\widetilde{f}(t+(k-1)h)}{h}\right)h\\ &\quad\le \text{ess } V_\Phi(f)\ a.e. \text{ on } [0,h],\end{aligned}$$

and so

$$\begin{aligned}\int_{\mathbb{R}}\Phi\left(f_h(t)\right)\,dt &\le \frac{1}{h}\int_0^h \text{ess } V_\Phi(f)\,dt\\ &= \text{ess } V_\Phi(f) \quad \text{for } h>0.\end{aligned}$$

Hence we have

$$\int_{\mathbb{R}} \Phi\left(f_h(t)\right)\, dt \leq \text{ess } V_\Phi(f) \quad \text{for } h \neq 0. \tag{2.2}$$

Thus we have

$$\limsup_{h\to 0} \int_{\mathbb{R}} \Phi\left(f_h(t)\right)\, dt \leq \text{ess } V_\Phi(f).$$

Combining with (2.1), we have

$$\lim_{h\to 0} \int_{\mathbb{R}} \Phi\left(f_h(t)\right)\, dt = \text{ess } V_\Phi(f),$$

which completes the proof of (i).

Next we show (ii). The inequality ess $V_\Phi(f) \leq V_\Phi(f)$ is obvious from (i) and the following inequality:

$$\begin{aligned}
\int_{\mathbb{R}} \Phi\left(f_h(t)\right)\, dt &= \sum_{k=-\infty}^{\infty} \int_0^h \Phi\left(\frac{f(t+kh) - f(t+(k-1)h)}{h}\right) dt \\
&= \frac{1}{h} \int_0^h \sum_{k=-\infty}^{\infty} \Phi\left(\frac{f(t+kh) - f(t+(k-1)h)}{h}\right) h\, dt \\
&\leq \frac{1}{h} \int_0^h V_\Phi(f)\, dt \\
&= V_\Phi(f).
\end{aligned}$$

To prove (iii), we suppose that $f \in AC_{\mathrm{loc}}(\mathbb{R})$. Let $t_0 < t_1 < \cdots < t_n$ with $t_k \in \text{ess } D(f)$ for $0 \leq k \leq n$. Then we have from $f(t_k) = \widetilde{f}(t_k)$ and Jensen's inequality that

$$\begin{aligned}
\frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} \Phi(f'(t))\, dt &\geq \Phi\left(\frac{1}{t_k - t_{k-1}} \int_{t_{k-1}}^{t_k} f'(t)\, dt\right) \\
&= \Phi\left(\frac{\widetilde{f}(t_k) - \widetilde{f}(t_{k-1})}{t_k - t_{k-1}}\right),
\end{aligned}$$

and so

$$\int_{\mathbb{R}} \Phi(f'(t))\, dt \geq \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \Phi(f'(t))\, dt \geq \sum_{k=1}^{n} \Phi\left(\frac{\widetilde{f}(t_k) - \widetilde{f}(t_{k-1})}{t_k - t_{k-1}}\right)(t_k - t_{k-1}).$$

Thus we have from the definition of ess $V_\Phi(f)$ that

$$\int_{\mathbb{R}} \Phi(f'(t))\, dt \geq \text{ess } V_\Phi(f). \tag{2.3}$$

On the other hand, we have from Fatou's lemma that

$$\int_{\mathbb{R}} \liminf_{n\to\infty} \Phi\left(\frac{f(t+\frac{1}{n})-f(t)}{\frac{1}{n}}\right) dt \le \liminf_{n\to\infty} \int_{\mathbb{R}} \Phi\left(\frac{f(t+\frac{1}{n})-f(t)}{\frac{1}{n}}\right) dt.$$

Thus we have from Theorem 2.1 that

$$\int_{\mathbb{R}} \Phi(f'(t))\, dt \le \operatorname{ess} V_{\Phi}(f). \tag{2.4}$$

Combining this inequality with (2.3) completes the proof of (ii). □

Remark 2.1 In Theorem 2.1, we see easily from (2.2) that

$$\sup_{h\neq 0} \int_{\mathbb{R}} \Phi\left(f_h(t)\right)\, dt = \operatorname{ess} V_{\Phi}(f).$$

Theorem 2.1(ii) has more refined results. To show this, we first prepare the following two lemmas.

Lemma 2.1 *Let $\Phi \in \mathcal{N}$. Then the following (i) or (ii) holds:*

(i) There exist x_0, a, $b \in \mathbb{R}$ with $x_0 > 0$, $a > 0$ such that $\Phi(x_0) > 0$ and $\Phi(x) \ge ax + b$ for all $x \in \mathbb{R}$.
(ii) There exist x_0, a, $b \in \mathbb{R}$ with $x_0 < 0$, $a < 0$ such that $\Phi(x_0) > 0$ and $\Phi(x) \ge ax + b$ for all $x \in \mathbb{R}$.

Proof By the assumption, since Φ is not constant, then there exists a $x_0 \neq 0$ such that $\Phi(x_0) > 0$. We see from the convexity of Φ that there exist $a \in \mathbb{R}$, $b \in \mathbb{R}$ such that $\Phi(x) \ge ax + b$ for all $x \in \mathbb{R}$ and $\Phi(x_0) = ax_0 + b$.

In case of $x_0 > 0$;

$$0 < \frac{\Phi(x_0) - \Phi(0)}{x_0} \le \frac{(ax_0 + b) - b}{x_0} = a.$$

In case of $x_0 < 0$;

$$0 > \frac{\Phi(x_0) - \Phi(0)}{x_0} \ge \frac{(ax_0 + b) - b}{x_0} = a.$$

□

Remark 2.2 For (i) and (ii) in Lemma 2.1. We should note that

(i) If both of (i) and (ii) hold, then $\lim_{x\to\pm\infty} \Phi(x) = \infty$.
(ii) If (i) holds, but (ii) does not hold, then $\lim_{x\to\infty} \Phi(x) = \infty$ and $\Phi(x) = 0$ for $x < 0$.
(iii) If (i) does not hold, but (ii) holds, then $\lim_{x\to-\infty} \Phi(x) = \infty$ and $\Phi(x) = 0$ for $x > 0$.

Let $T = \{t_0, t_1, \ldots, t_n\}$ be a finite subset of $\mathbb{R}$ with at least two elements and $t_0 < t_1 < \cdots < t_n$. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $\Phi \in \mathcal{N}$. We define $V_\Phi[f, T]$ by

$$V_\Phi[f, T] = \sum_{k=1}^{n} \Phi\left(\frac{f(t_k) - f(t_{k-1})}{t_k - t_{k-1}}\right)(t_k - t_{k-1}).$$

Lemma 2.2 *If $S \subseteq T \subset \mathbb{R}$, then we have*

$$V_\Phi[f, S] \leq V_\Phi[f, T].$$

Proof Let $S = \{s_0, s_1, \ldots, s_m\}$ with $s_0 < s_1 < \cdots < s_m$ and $T = \{t_0, t_1, \ldots, t_n\}$ with $t_0 < t_1 < \cdots < t_n$. Without loss of generality, we can suppose $n = m + 1$ and $T \backslash S = \{t_k\}$. In cases of $t_k < s_0$ or $s_m < t_k$, we see easily that $V_\Phi[f, S] \leq V_\Phi[f, T]$. So we suppose that $s_0 < t_k < s_m$ and $s_\ell = \min\{s \in S : t_k < s\}$. Then we have

$$\begin{aligned}
V_\Phi[f, T] - V_\Phi[f, S] &= \Phi\left(\frac{f(t_k) - f(s_{\ell-1})}{t_k - s_{\ell-1}}\right)(t_k - s_{\ell-1}) \\
&\quad + \Phi\left(\frac{f(s_\ell) - f(t_k)}{s_\ell - t_k}\right)(s_\ell - t_k) \\
&\quad - \Phi\left(\frac{f(s_\ell) - f(s_{\ell-1})}{s_\ell - s_{\ell-1}}\right)(s_\ell - s_{\ell-1}) \\
&= (s_\ell - s_{\ell-1})\left\{\Phi\left(\frac{f(t_k) - f(s_{\ell-1})}{t_k - s_{\ell-1}}\right)\frac{t_k - s_{\ell-1}}{s_\ell - s_{\ell-1}}\right. \\
&\quad \left. + \Phi\left(\frac{f(s_\ell) - f(t_k)}{s_\ell - t_k}\right)\frac{s_\ell - t_k}{s_\ell - s_{\ell-1}}\right\} - (s_\ell - s_{\ell-1}) \\
&\quad \times \Phi\left(\frac{f(s_\ell) - f(s_{\ell-1})}{s_\ell - s_{\ell-1}}\right) \\
&\geq (s_\ell - s_{\ell-1})\Phi\left(\frac{f(t_k) - f(s_{\ell-1})}{t_k - s_{\ell-1}} \cdot \frac{t_k - s_{\ell-1}}{s_\ell - s_{\ell-1}}\right. \\
&\quad \left. + \frac{f(s_\ell) - f(t_k)}{s_\ell - t_k} \cdot \frac{s_\ell - t_k}{s_\ell - s_{\ell-1}}\right) \\
&\quad - (s_\ell - s_{\ell-1})\Phi\left(\frac{f(s_\ell) - f(s_{\ell-1})}{s_\ell - s_{\ell-1}}\right) \quad \text{(By convexity of } \Phi) \\
&= (s_\ell - s_{\ell-1})\Phi\left(\frac{f(s_\ell) - f(s_{\ell-1})}{s_\ell - s_{\ell-1}}\right) - (s_\ell - s_{\ell-1}) \\
&\quad \times \Phi\left(\frac{f(s_\ell) - f(s_{\ell-1})}{s_\ell - s_{\ell-1}}\right) \\
&= 0.
\end{aligned}$$

Thus we have $V_\Phi[f, S] \leq V_\Phi[f, T]$. □

Theorem 2.2 *If $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ with ess $V_\Phi(f) < \infty$, then the following properties hold:*

(i) For each $t \in \mathbb{R}$ there exist

$$\alpha(t) = \lim_{\substack{t+h \in \operatorname{ess} D(f) \\ h \to 0^+}} \widetilde{f}(t+h) \text{ and } \beta(t) = \lim_{\substack{t+h \in \operatorname{ess} D(f) \\ h \to 0^-}} \widetilde{f}(t+h) \quad \text{in } \mathbb{R}.$$

(ii) $t \in \operatorname{ess} D(f)$ if and only if $\alpha(t) = \beta(t)$. Then $\widetilde{f}(t) = \alpha(t) = \beta(t)$ holds.
(iii) $V_\Phi(f) = \operatorname{ess} V_\Phi(f)$ if and only if $\alpha(t) \le f(t) \le \beta(t)$ or $\beta(t) \le f(t) \le \alpha(t)$ holds.

Proof Proof of (i): We first show that $\lim_{n\to\infty} \widetilde{f}(t_n)$ exists in $\mathbb{R}$ for every $t \in \mathbb{R}$ and $\{t_n\} \subseteq \operatorname{ess} D(f)$ with $t_0 < t_1 < t_2 < \cdots < t_n \to t$ $(n \to \infty)$. Let $a \in \mathbb{R}$ and $b \in \mathbb{R}$ satisfy the condition (i) or (ii) of Lemma 2.1. Let $\Psi(x) = \Phi(x) - ax - b \ge 0$ for $x \in \mathbb{R}$, and so $x = \frac{1}{a}(\Phi(x) - \Psi(x) - b)$. Then for each $n \in \mathbb{N}$, we have that

$$\begin{aligned}
\widetilde{f}(t_n) &= \widetilde{f}(t_0) + \sum_{k=1}^{n} \frac{\widetilde{f}(t_k) - \widetilde{f}(t_{k-1})}{t_k - t_{k-1}} \cdot (t_k - t_{k-1}) \\
&= \widetilde{f}(t_0) + \sum_{k=1}^{n} \frac{1}{a} \left\{ \Phi\left(\frac{\widetilde{f}(t_k) - \widetilde{f}(t_{k-1})}{t_k - t_{k-1}} \right) \right. \\
&\quad \left. -\Psi\left(\frac{\widetilde{f}(t_k) - \widetilde{f}(t_{k-1})}{t_k - t_{k-1}} \right) - b \right\} (t_k - t_{k-1}) \\
&= \widetilde{f}(t_0) + \frac{1}{a} \left\{ \sum_{k=1}^{n} \Phi\left(\frac{\widetilde{f}(t_k) - \widetilde{f}(t_{k-1})}{t_k - t_{k-1}} \right) (t_k - t_{k-1}) \right. \\
&\quad \left. - \sum_{k=1}^{n} \Psi\left(\frac{\widetilde{f}(t_k) - \widetilde{f}(t_{k-1})}{t_k - t_{k-1}} \right) (t_k - t_{k-1}) \right\} - \frac{b}{a}(t_n - t_0).
\end{aligned}$$

By the assumption $\operatorname{ess} V_\Phi(f) < \infty$, we have

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Phi\left(\frac{\widetilde{f}(t_k) - \widetilde{f}(t_{k-1})}{t_k - t_{k-1}} \right) (t_k - t_{k-1})$$

converges in $[0, \infty)$. For $\Psi \ge 0$ and $t_k - t_{k-1} > 0$,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \Psi\left(\frac{\widetilde{f}(t_k) - \widetilde{f}(t_{k-1})}{t_k - t_{k-1}} \right) (t_k - t_{k-1}) \in [0, \infty],$$

and $\lim_{n\to\infty} t_n = t$. Thus we have

$$\lim_{n\to\infty} \widetilde{f}(t_n) \in [-\infty, \infty].$$

Since $\{t_n\}$ is arbitrary, we have

$$\lim_{h\uparrow 0, t+h\in \operatorname{ess} D(f)} \widetilde{f}(t+h) \in [-\infty, \infty].$$

We suppose that

$$\lim_{h\uparrow 0, t+h\in \operatorname{ess} D(f)} \widetilde{f}(t+h) = \pm\infty.$$

We take any $c, d \in \operatorname{ess} D(f)$ so that $c < t < d$. By the definition of $\operatorname{ess} V_\Phi(f)$,

$$\Phi\left(\frac{\widetilde{f}(t+h)-\widetilde{f}(c)}{t+h-c}\right)(t+h-c) + \Phi\left(\frac{\widetilde{f}(d)-\widetilde{f}(t+h)}{d-(t+h)}\right)(d-(t+h))$$
$$\leq \operatorname{ess} V_\Phi(f) < \infty,$$

for $c < t+h < d$ with $t+h \in \operatorname{ess} D(f)$. On the other hand, by Remark 2.2,

$$\begin{aligned}
&\lim_{h\uparrow 0, t+h\in \operatorname{ess} D(f)} \left\{\Phi\left(\frac{\widetilde{f}(t+h)-\widetilde{f}(c)}{t+h-c}\right)(t+h-c) + \Phi\left(\frac{\widetilde{f}(d)-\widetilde{f}(t+h)}{d-(t+h)}\right)\right.\\
&\qquad \left.\times (d-(t+h))\right\}\\
&= \Phi(\pm\infty)(t-c) + \Phi(\mp\infty)(d-t)\\
&= \infty,
\end{aligned}$$

which is a contradiction. Thus we have that $\lim_{h\uparrow 0, t+h\in \operatorname{ess} D(f)} \widetilde{f}(t+h) \neq \pm\infty$, and so $\lim_{h\uparrow 0, t+h\in \operatorname{ess} D(f)} \widetilde{f}(t+h)$ coverges in $\mathbb{R}$.

Similarly, we can show that for any $t \in \mathbb{R}$, $\lim_{h\downarrow 0, t+h\in \operatorname{ess} D(f)} \widetilde{f}(t+h)$ converges in $\mathbb{R}$.

In fact, put $\bar{\Phi}(x) = \Phi(-x)$, $\bar{f}(t) = f(-t)$ for $x, t \in \mathbb{R}$. Then we see that $\bar{\Phi} : \mathbb{R} \to [0, \infty)$ is a convex, continuous and nonconstant function with $\bar{\Phi}(0) = 0$, and $\bar{f} \in L^1_{\mathrm{loc}}(\mathbb{R})$, $\operatorname{ess} D(\bar{f}) = -\operatorname{ess} D(f)$, $\widetilde{f}(t) = \widetilde{\bar{f}}(-t)$ for $t \in \operatorname{ess} D(f)$ and $\operatorname{ess} V_{\bar{\Phi}}(\bar{f}) = \operatorname{ess} V_\Phi(f) < \infty$. Hence, we see that

$$\lim_{h\downarrow 0, t+h\in \operatorname{ess} D(f)} \widetilde{f}(t+h) = \lim_{h\downarrow 0, -t-h\in \operatorname{ess} D(\bar{f})} \widetilde{\bar{f}}(-t-h) = \lim_{h\uparrow 0, -t+h\in \operatorname{ess} D(\bar{f})} \widetilde{\bar{f}}(-t+h)$$

converges in $\mathbb{R}$.

Proof of (ii) By $f \in L^1_{\mathrm{loc}}(\mathbb{R})$, $f(t) = \widetilde{f}(t)$ *a.e.t*, combining with the result (i) above, we have that for every $t \in \mathbb{R}$,

$$\lim_{h\to 0+} \frac{1}{h}\int_t^{t+h} f(s)\,ds = \lim_{h\to 0+} \frac{1}{h}\int_t^{t+h} \widetilde{f}(s)\,ds = \alpha(t)$$
$$\lim_{h\to 0-} \frac{1}{h}\int_t^{t+h} f(s)\,ds = \lim_{h\to 0-} \frac{1}{h}\int_t^{t+h} \widetilde{f}(s)\,ds = \beta(t)$$

On the other hand, by the definition of $\operatorname{ess} D(f)$, $t \in \operatorname{ess} D(f)$ if and only if $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} f(s)\,ds$ exits in $\mathbb{R}$. Thus we see that (ii) holds.

Proof of (iii) We first assume that $\alpha(t) \le f(t) \le \beta(t)$ or $\beta(t) \le f(t) \le \alpha(t)$ for all $t \in \mathbb{R}$. For any set $T = \{t_0, t_1, t_2, \dots, t_n\}$ with $t_0 < t_1 < \cdots < t_n$, we put $\Lambda = \{0,1\}^{\{0,1,2,\dots,n\}}$. For each $\lambda \in \Lambda$, we denote the function $g_\lambda : \mathbb{R} \to \mathbb{R}$ as follows:

$$g_\lambda(t) = \begin{cases} \alpha(t_i), & \text{if } t = t_i, \lambda(i) = 0, 0 \le i \le n \\ \beta(t_i), & \text{if } t = t_i, \lambda(i) = 1, 0 \le i \le n \\ 0, & \text{if } t \notin T \end{cases}$$

For every $\lambda \in \Lambda$, $g_\lambda(t_i) = \alpha(t_i)$ or $\beta(t_i)$ $(0 \le i \le n)$, and so there exists $\{t_{ij}\}_j \subset \operatorname{ess} D(f)$ for each $0 \le i \le n$ such that $\lim_{j\to\infty} t_{ij} = t_i$, $\lim_{j\to\infty} \widetilde{f}(t_{ij}) = g_\lambda(t_i)$.

For sufficiently large j, we have that $t_{0j} < t_{1j} < t_{2j} < \cdots < t_{nj}$. Let $T_j = \{t_{0j}, t_{1j}, t_{2j}, \dots, t_{nj}\}$. Then we have that

$$V_\Phi[\widetilde{f}, T_j] \le \operatorname{ess} V_\Phi(f), \ \lim_{j\to\infty} V_\Phi[\widetilde{f}, T_j] = V_\Phi[g_\lambda, T],$$

and so

$$V_\Phi[g_\lambda, T] \le \operatorname{ess} V_\Phi(f) \text{ for } \lambda \in \Lambda.$$

According to our assumption that $\alpha(t) \le f(t) \le \beta(t)$ or $\beta(t) \le f(t) \le \alpha(t)$ for all $t \in \mathbb{R}$, we can take $0 \le \theta_i \le 1$ for each $0 \le i \le n$ such that

$$f(t_i) = \theta_i \alpha(t_i) + (1-\theta_i)\beta(t_i).$$

Moreover, for each $\lambda \in \Lambda$ we set

$$\theta_{\lambda,i} = \begin{cases} \theta_i, & \text{if } \lambda(i) = 0 \\ 1-\theta_i, & \text{if } \lambda(i) = 1, \end{cases}$$

and $\theta_\lambda = \prod_{i=0}^n \theta_{\lambda,i}$. Then $\theta_\lambda \ge 0$, $\sum_{\lambda\in\Lambda} \theta_\lambda = \sum_{\lambda\in\Lambda} \prod_{i=0}^n \theta_{\lambda,i} = \prod_{i=0}^n (\theta_i + (1-\theta_i)) = 1$. For each $0 \le j \le n$, we have that

$$\begin{aligned} \sum_{\lambda\in\Lambda} \theta_\lambda g_\lambda(t_j) &= \sum_{\lambda\in\Lambda} \left(\prod_{i=0}^n \theta_{\lambda,i}\right) g_\lambda(t_j) \\ &= \sum_{\substack{\lambda\in\Lambda \\ \lambda(j)=0}} \left(\prod_{i=0}^n \theta_{\lambda,i}\right) g_\lambda(t_j) + \sum_{\substack{\lambda\in\Lambda \\ \lambda(j)=1}} \left(\prod_{i=0}^n \theta_{\lambda,i}\right) g_\lambda(t_j) \end{aligned}$$

$$
\begin{aligned}
&= \theta_j \left(\prod_{\substack{0 \le i \le n \\ i \neq j}} (\theta_i + (1 - \theta_i)) \alpha(t_j) \right) + (1 - \theta_j) \\
&\quad \times \left(\prod_{\substack{0 \le i \le n \\ i \neq j}} (\theta_i + (1 - \theta_i)) \beta(t_j) \right) \\
&= \theta_j \alpha(t_j) + (1 - \theta_j) \beta(t_j) \\
&= f(t_j).
\end{aligned}
$$

$$
\begin{aligned}
V_\Phi[f, T] &= \sum_{j=1}^{n} \Phi \left(\frac{f(t_j) - f(t_{j-1})}{t_j - t_{j-1}} \right) (t_j - t_{j-1}) \\
&= \sum_{j=1}^{n} \Phi \left(\sum_{\lambda \in \Lambda} \theta_\lambda \frac{g_\lambda(t_j) - g_\lambda(t_{j-1})}{t_j - t_{j-1}} \right) (t_j - t_{j-1}) \\
&\le \sum_{j=1}^{n} \left(\sum_{\lambda \in \Lambda} \theta_\lambda \Phi \left(\frac{g_\lambda(t_j) - g_\lambda(t_{j-1})}{t_j - t_{j-1}} \right) \right) (t_j - t_{j-1}) \\
&= \sum_{j=1}^{n} \sum_{\lambda \in \Lambda} \theta_\lambda \Phi \left(\frac{g_\lambda(t_j) - g_\lambda(t_{j-1})}{t_j - t_{j-1}} \right) (t_j - t_{j-1}) \\
&= \sum_{\lambda \in \Lambda} \theta_\lambda \sum_{j=1}^{n} \Phi \left(\frac{g_\lambda(t_j) - g_\lambda(t_{j-1})}{t_j - t_{j-1}} \right) (t_j - t_{j-1}) \\
&= \sum_{\lambda \in \Lambda} \theta_\lambda V_\Phi[g_\lambda, T] \\
&\le \sum_{\lambda \in \Lambda} \theta_\lambda \text{ess } V_\Phi(f) \\
&= \text{ess } V_\Phi(f).
\end{aligned}
$$

Since T is arbitrary, we have that $V_\Phi(f) \le \text{ess } V_\Phi(f)$. Thus it follows from (ii) of Theorem 2.1 that $V_\Phi(f) = \text{ess } V_\Phi(f)$.

To show the converse, assume that $\alpha(s), \beta(s) < f(s)$ for some $s \in \mathbb{R}$. Let g be $g(t) = f(t)$ for $t \neq s$ and $\alpha(s), \beta(s) < g(s) < f(s)$. Then it is obvious that $\text{ess } V_\Phi(f) = \text{ess } V_\Phi(g) \le V_\Phi(g)$. For every n, there exists a finite subset $\{T_n\} \subset \mathbb{R}$ such that

$$
\text{ess } V_\Phi(f) - \frac{1}{n} < V_\Phi[g, T_n]. \tag{2.5}
$$

From Lemma 2.2, by adding more points in T_n if necessary, we can assume that $s \in T_n$ and there exist $s_n, s_n' \in T_n \cap L(f)$ such that $s_n = \max\{(s - 1/n, s) \cap T_n\}$ and $s_n' = \min\{(s, s + 1/n) \cap T_n\}$. Then we have that

$$
\begin{aligned}
V_\Phi[f, T_n] - V_\Phi[g, T_n] &= \left\{\Phi\left(\frac{f(s) - f(s_n)}{s - s_n}\right)(s - s_n)\right.\\
&\quad \left. + \Phi\left(\frac{f(s_n') - f(s)}{s_n' - s}\right)(s_n' - s)\right\}\\
&\quad - \left\{\Phi\left(\frac{g(s) - f(s_n)}{s - s_n}\right)(s - s_n)\right.\\
&\quad \left. + \Phi\left(\frac{f(s_n') - g(s)}{s_n' - s}\right)(s_n' - s)\right\}\\
&= \left\{\Phi\left(\frac{f(s) - f(s_n)}{s - s_n}\right) - \Phi\left(\frac{g(s) - f(s_n)}{s - s_n}\right)\right\}(s - s_n)\\
&\quad + \left\{\Phi\left(\frac{f(s_n') - f(s)}{s_n' - s}\right) - \Phi\left(\frac{f(s_n') - g(s)}{s_n' - s}\right)\right\}\\
&\quad \times (s_n' - s). \qquad (2.6)
\end{aligned}
$$

Since $\lim_{n\to\infty} s_n = s$, $\lim_{n\to\infty} f(s_n) = \lim_{n\to\infty} \widetilde{f}(s_n) = \beta(s)$ and $\lim_{n\to\infty} s_n' = s$, $\lim_{n\to\infty} f(s_n') = \lim_{n\to\infty} \widetilde{f}(s_n') = \alpha(s)$, we have from the assumption that

$$
\lim_{n\to\infty} \frac{g(s) - f(s_n)}{s - s_n} = \infty, \quad \lim_{n\to\infty} \frac{f(s_n') - g(s)}{s_n' - s} = -\infty, \qquad (2.7)
$$

and so for sufficiently large n we have that

$$
0 < \frac{g(s) - f(s_n)}{s - s_n} < \frac{f(s) - f(s_n)}{s - s_n}, \quad \frac{f(s_n') - f(s)}{s_n' - s} < \frac{f(s_n') - g(s)}{s_n' - s} < 0.
$$

For simplicity of notation we write $\hat{\Phi}(t) = \dfrac{\Phi(t)}{t}$ $(t \neq 0)$. Hence we have from the monotonicity of $\hat{\Phi}(t)$ on $(0, \infty)$ that

$$
\begin{aligned}
&\left\{\Phi\left(\frac{f(s) - f(s_n)}{s - s_n}\right) - \Phi\left(\frac{g(s) - f(s_n)}{s - s_n}\right)\right\}(s - s_n)\\
&= \left\{\hat{\Phi}\left(\frac{f(s) - f(s_n)}{s - s_n}\right)\frac{f(s) - f(s_n)}{s - s_n} - \hat{\Phi}\left(\frac{g(s) - f(s_n)}{s - s_n}\right)\frac{g(s) - f(s_n)}{s - s_n}\right\}\\
&\quad \times (s - s_n)\\
&\geq \left\{\hat{\Phi}\left(\frac{g(s) - f(s_n)}{s - s_n}\right)\frac{f(s) - f(s_n)}{s - s_n} - \hat{\Phi}\left(\frac{g(s) - f(s_n)}{s - s_n}\right)\frac{g(s) - f(s_n)}{s - s_n}\right\}\\
&\quad \times (s - s_n)\\
&= \hat{\Phi}\left(\frac{g(s) - f(s_n)}{s - s_n}\right)(f(s) - g(s)). \qquad (2.8)
\end{aligned}
$$

On the other hand, we have from the monotonicity of $\hat{\Phi}(t)$ on $(-\infty, 0)$ that

$$\begin{aligned}
&\left\{\Phi\left(\frac{f(s_n')-f(s)}{s_n'-s}\right)-\Phi\left(\frac{f(s_n')-g(s)}{s_n'-s}\right)\right\}\times(s_n'-s)\\
&\quad=\left\{\hat{\Phi}\left(\frac{f(s_n')-f(s)}{s_n'-s}\right)\frac{f(s_n')-f(s)}{s_n'-s}-\hat{\Phi}\left(\frac{f(s_n')-g(s)}{s_n'-s}\right)\frac{f(s_n')-g(s)}{s_n'-s}\right\}\\
&\qquad\times(s_n'-s)\\
&\quad\geq\left\{\hat{\Phi}\left(\frac{f(s_n')-g(s)}{s_n'-s}\right)\frac{f(s_n')-f(s)}{s_n'-s}-\hat{\Phi}\left(\frac{f(s_n')-g(s)}{s_n'-s}\right)\frac{f(s_n')-g(s)}{s_n'-s}\right\}\\
&\qquad\times(s_n'-s)\\
&\quad=-\hat{\Phi}\left(\frac{f(s_n')-g(s)}{s_n'-s}\right)(f(s)-g(s)).
\end{aligned}\tag{2.9}$$

For sufficiently large n, combining those inequalities with (2.6) gives

$$\begin{aligned}
V_\Phi[f,T_n]-V_\Phi[g,T_n]\geq&\left\{\hat{\Phi}\left(\frac{g(s)-f(s_n)}{s-s_n}\right)-\hat{\Phi}\left(\frac{f(s_n')-g(s)}{s_n'-s}\right)\right\}\\
&\times(f(s)-g(s)).
\end{aligned}$$

We see from (2.7) and the monotonicity of $\dfrac{\Phi(t)}{t}$ on $(-\infty, 0)$ and $(0, \infty)$ that

$$\begin{aligned}
\lim_{n\to\infty}\left\{\hat{\Phi}\left(\frac{g(s)-f(s_n')}{s-s_n'}\right)-\hat{\Phi}\left(\frac{f(s_n')-g(s)}{s_n'-s}\right)\right\}=&\lim_{t\to\infty}\hat{\Phi}(t)\\
&-\lim_{t\to-\infty}\hat{\Phi}(t)\in(0,\infty].
\end{aligned}$$

Thus we have from assumption $f(s)-g(s)>0$ that $\liminf_{n\to\infty}(V_\Phi[f,T_n]-V_\Phi[g,T_n])>0$.

On the other hand, from (2.5) we have that $\liminf_{n\to\infty} V_\Phi[g,T_n]\geq \operatorname{ess} V_\Phi(f)$. Hence we have that

$$\begin{aligned}
\liminf_{n\to\infty} V_\Phi[f,T_n]&\geq\liminf_{n\to\infty}(V_\Phi[f,T_n]-V_\Phi[g,T_n])+\liminf_{n\to\infty}V_\Phi[g,T_n]\\
&>\operatorname{ess} V_\Phi(f).
\end{aligned}$$

Thus it follows from $V_\Phi(f)\geq V_\Phi[f,T_n]$ that $V_\Phi(f)>\operatorname{ess} V_\Phi(f)$.

For the case of $f(s)<\alpha(s), \beta(s)$ for some $s\in\mathbb{R}$, in a similar procedure for $\bar{\Phi}(x)=\Phi(-x)$, we obtain that $V_\Phi(f)=V_{\bar{\Phi}}(-f)>\operatorname{ess} V_{\bar{\Phi}}(-f)=\operatorname{ess} V_\Phi(f)$. Thus we see that the converse holds. □

Corollary 2.1 *For all $f\in L^1_{\mathrm{loc}}(\mathbb{R})$, we have*

$$\operatorname{ess} V_\Phi(f)=\inf\{V_\Phi(g): f=g\ a.e.\}.$$

Proof Thus for $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ with $f = g$ *a.e.*, from Theorem 2.1(ii) the following holds:

$$\mathrm{ess}\, V_\Phi(f) = \mathrm{ess}\, V_\Phi(g) \le V_\Phi(g),$$

and hence we have that

$$\mathrm{ess}\, V_\Phi(f) \le \inf\{V_\Phi(g) : f = g\ a.e.\}.$$

To show the converse inequality, since it is obvious if $\mathrm{ess}\, V_\Phi(f) = \infty$, we suppose that $\mathrm{ess}\, V_\Phi(f) < \infty$. Put $g(t) = \alpha(t)$ or $g(t) = \beta(t)$ for every $t \in \mathbb{R}$, where α and β are obtained in Theorem 2.2, Then we easily that $f = g$ *a.e* and $V_\Phi(g) = \mathrm{ess}\, V_\Phi(g) = \mathrm{ess}\, V_\Phi(f)$. □

Set

$$\mathcal{K}_\Phi(\mathbb{R}) := \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}) : \Phi \circ f (= \Phi(f(\cdot))) \in L^1(\mathbb{R}) \right\}.$$

Then we see easily that $\mathcal{K}_\Phi(\mathbb{R})$ is a convex subset in $L^1_{\mathrm{loc}}(\mathbb{R})$. Moreover, we define

$$\widetilde{\Phi}(x) = \chi_{[0,\infty)}(x)\Phi(x) - \chi_{(-\infty,0]}(x)\Phi(x),\ \ x \in \mathbb{R},$$

where χ_I is the characteristic function of a subset $I \subseteq \mathbb{R}$.

For $f, g \in \mathcal{K}_\Phi(\mathbb{R})$, we define

$$d_\Phi(f, g) = \|\widetilde{\Phi} \circ f - \widetilde{\Phi} \circ g\|_1.$$

Then it is easily checked that the following hold

(1) $d_\Phi(f, g) = d_\Phi(g, f) \ge 0$ and $d_\Phi(f, f) = 0$.
(2) $d_\Phi(f, g) \le d_\Phi(f, h) + d_\Phi(h, g)$.
Moreover, if $\Phi(x) > 0$ for all $x \ne 0$, then $\widetilde{\Phi}$ is strictly increasing and we have
(3) $d_\Phi(f, g) = 0$ if and only if $f = g$ *a.e.*.

Then we have the following result.

Theorem 2.3 *Let $\Phi \in \mathcal{N}$ with $\Phi(x) > 0$ for all $x \ne 0$. Then $(\mathcal{K}_\Phi(\mathbb{R}), d_\Phi)$ is a complete metric space.*

Proof We see easily from the properties of Φ that Φ is continuous and strictly monotonically decreasing on $(-\infty, 0]$ and is continuous and strictly monotonically increasing on $[0, \infty)$ and $\lim_{x\to\pm\infty} \Phi(x) = \infty$. Therefore, $\widetilde{\Phi}, \widetilde{\Phi}^{-1} : \mathbb{R} \to \mathbb{R}$ is continuous, strictly monotonically increasing and onto on $\mathbb{R}$. From this it easily follows that $d_\Phi(f, g) = 0$ implies $f = g$ *a.e.*, so we see that $(\mathcal{K}_\Phi(\mathbb{R}), d_\Phi)$ is a metric space.

To show the completeness of $(\mathcal{K}_\Phi(\mathbb{R}), d_\Phi)$, we should note from convexity of Φ with $\Phi(0) = 0$ that $\Phi(x)/x$ is monotonically increasing. And hence we have that $\Phi(x) \ge \Phi(1)x$ for all $x \ge 1$ and $\Phi(x) \ge -\Phi(-1)x$ for all $x \le -1$, and so

$\Phi(x) \geq \min\{\Phi(1), \Phi(-1)\}|x|$ for all $|x| \geq 1$. Let a be $\min\{\Phi(1), \Phi(-1)\}$, so $a > 0$. Put $b = \inf_{x\in\mathbb{R}}\{\Phi(x) - a|x|\}$, then $b \in \mathbb{R}$, and so $\Phi(x) \geq a|x| + b$ for all $x \in \mathbb{R}$. From this we have that

$$|\widetilde{\Phi}(x)| \geq a|x|+b \text{ for all } x \in \mathbb{R} \text{ and } |\widetilde{\Phi}^{-1}(x)| \leq |x|/a+|b|/a \text{ for all } x \in \mathbb{R}. \quad (2.10)$$

Let $\{f_n\}$ be a Cauchy's sequence in $\mathcal{K}_\Phi(\mathbb{R})$. Then $\{\widetilde{\Phi} \circ f_n\}$ is a Cauchy's sequence in $L^1(\mathbb{R})$ and so there exists $g \in L^1(\mathbb{R})$ such that $\lim_{n\to\infty} \widetilde{\Phi} \circ f_n = g$ in $L^1(\mathbb{R})$. Let $f(x) = \widetilde{\Phi}^{-1}(g(x))$, then we see from (2.10) that f is in $L^1_{\text{loc}}(\mathbb{R})$. Thus we have that $\widetilde{\Phi} \circ f = g \in L^1(\mathbb{R})$ and so $f \in \mathcal{K}_\Phi(\mathbb{R})$ and $\lim_{n\to\infty} d_\Phi(f_n, f) = 0$. □

Put $\Phi(x) = |x|^p$ for $1 \leq p < \infty$, then we see easily that $\mathcal{K}_\Phi(\mathbb{R}) = L^p(\mathbb{R})$. Moreover we have the following result.

Theorem 2.4 *Let $\Phi(x) = |x|^p$ for $1 \leq p < \infty$. Then $(L^p(\mathbb{R}), d_\Phi)$ is isomorphic to $(L^p(\mathbb{R}), \|\cdot\|_p)$.*

Proof Let us see that:

$$|\widetilde{\Phi}(x) - \widetilde{\Phi}(y)| \leq (|x| + |y - x|)^p - |x|^p \quad \text{for } x, y \in \mathbb{R}. \quad (2.11)$$

Assume first that $xy \geq 0$. Then it is obvious that $|\widetilde{\Phi}(x) - \widetilde{\Phi}(y)| = ||x|^p - |y|^p|$ holds. Furthermore, we show that $||x|^p - |y|^p| \leq (|x| + |y - x|)^p - |x|^p$. If $|x| \leq |y|$, then it is obvious that $||x|^p - |y|^p| = |y|^p - |x|^p = (|x| + |y - x|)^p - |x|^p$. If $|y| \leq |x|$, then

$$\left||x|^p - |y|^p\right| = |x|^p - |y|^p = (|y| + |y - x|)^p - |y|^p.$$

Put $u(t) = (t + |y - x|)^p - t^p$ $(t \in [0, \infty))$, then $u(t)$ is monotone increasing with respect to t, and so $u(|y|) \leq u(|x|)$. Thus we have

$$(|y| + |y - x|)^p - |y|^p \leq (|x| + |y - x|)^p - |x|^p,$$

and so

$$\left||x|^p - |y|^p\right| \leq (|x| + |y - x|)^p - |x|^p.$$

Thus if $xy \geq 0$, then we have that (2.11) holds.

On the other hand, if $xy < 0$, then we have from Jensen's inequality and the monotonicity of u that

$$|\widetilde{\Phi}(x) - \widetilde{\Phi}(y)| = |x|^p + |y|^p \leq (|x| + |y|)^p = |y - x|^p \leq (|x| + |y - x|)^p - |x|^p.$$

Thus we see that (2.11) holds.

Let $f, g \in L^p(\mathbb{R})$. From (2.11) we have:

$$d_\Phi(f, g) = \int_{\mathbb{R}} \left|\widetilde{\Phi}(f(t)) - \widetilde{\Phi}(g(t))\right| dt$$

$$\begin{aligned}
&\leq \int_{\mathbb{R}} (|f(t)| + |f(t) - g(t)|)^p dt - \int_{\mathbb{R}} |f(t)|^p dt \\
&= \| |f| + |f-g| \|_p^p - \|f\|_p^p \\
&\leq (\|f\|_p + \|f-g\|_p)^p - \|f\|_p^p. \qquad (2.12)
\end{aligned}$$

On the other hand, if $xy \geq 0$, since

$$\left|\widetilde{\Phi}(x) - \widetilde{\Phi}(y)\right| = \left||x|^p - |y|^p\right| = \begin{cases} u(|x|) & \text{if } |x| \leq |y| \\ u(|y|) & \text{if } |y| \leq |x|, \end{cases}$$

we have that

$$\left|\widetilde{\Phi}(x) - \widetilde{\Phi}(y)\right| \geq u(0) = |y-x|^p \geq \frac{1}{2^{p-1}}|x-y|^p.$$

If $xy < 0$, we have that

$$\left|\widetilde{\Phi}(x) - \widetilde{\Phi}(y)\right| = |x|^p + |y|^p \geq \frac{1}{2^{p-1}}(|x| + |y|)^p = \frac{1}{2^{p-1}}|x-y|^p.$$

Hence we have that

$$|\widetilde{\Phi}(x) - \widetilde{\Phi}(y)| \geq \frac{1}{2^{p-1}}|x-y|^p \quad \text{for } x, y \in \mathbb{R}. \qquad (2.13)$$

Thus we have that

$$\begin{aligned}
d_\Phi(f, g) &= \int_{\mathbb{R}} \left|\widetilde{\Phi}(f(t)) - \widetilde{\Phi}(g(t))\right| dt \\
&\geq \frac{1}{2^{p-1}} \int_{\mathbb{R}} |f(t) - g(t)|^p \, dt \\
&= \frac{1}{2^{p-1}} \|f-g\|_p^p. \qquad (2.14)
\end{aligned}$$

Combining this inequality with (2.12) gives

$$\frac{1}{2^{p-1}}\|f-g\|_p^p \leq d_\Phi(f, g) \leq (\|f\|_p + \|f-g\|_p)^p - \|f\|_p^p.$$

We conclude the proof. □

Theorem 2.5 *Let $f \in AC_{\text{loc}}(\mathbb{R})$ and $f' \in \mathcal{K}_\Phi(\mathbb{R})$. Then we have that*

$$f_h \in \mathcal{K}_\Phi(\mathbb{R}) \text{ for every } h \neq 0 \quad \text{and} \quad \lim_{h \to 0} d_\Phi\left(f_h, f'\right) = 0.$$

In particular, if $\Phi(x) = |x|^p$ for $1 \leq p < \infty$ and $f' \in L_p(\mathbb{R})$, then we have that

$$\lim_{h \to 0} \left\| f_h - f' \right\|_p = 0.$$

Proof Let $f' \in \mathcal{K}_\Phi(\mathbb{R})$ and $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R})$. Then from Theorem 2.1 (iii) we see that $\operatorname{ess} V_\Phi(f) < \infty$. Moreover, we see from Remark 2.1 that

$$\int_{\mathbb{R}} \Phi\left(f_h(t)\right)\, dt \leq \operatorname{ess} V_\Phi(f) < \infty \text{ for every } h \neq 0,$$

which implies that

$$f_h \in \mathcal{K}_\Phi(\mathbb{R}) \text{ for every } h \neq 0.$$

Let $\Phi_1(x) = \chi_{[0,\infty)}(x)\Phi(x)$ and $\Phi_2(x) = \chi_{(-\infty,0]}(x)\Phi(x)$, then these functions from $\mathbb{R}$ to $[0,\infty)$ are also convex functions with $\Phi_i(0) = 0$ $(i = 1, 2)$ and $\widetilde{\Phi}(x) = \Phi_1(x) - \Phi_2(x)$.

Then $0 \leq \Phi_1 \leq \Phi$, $\Phi\left(f_h\right) \in L^1(\mathbb{R})$ implies $\Phi_1\left(f_h\right) \in L^1(\mathbb{R})$. We put $\delta_h = \Phi_1\left(f_h\right) - \Phi_1(f')$, $\delta_h^+ = \left\{\Phi_1\left(f_h\right) - \Phi_1(f')\right\}^+$, $\delta_h^- = \left\{\Phi_1\left(f_h\right) - \Phi_1(f')\right\}^-$.

If $\delta_h^+(t) > 0$, we have that

$$\begin{aligned}
\delta_h^+(t) &= \Phi_1\left(f_h(t)\right) - \Phi_1(f'(t)) \\
&= \Phi_1\left(\frac{1}{h}\int_0^h f'(t+s)\, ds\right) - \Phi_1(f'(t)) \\
&\leq \frac{1}{h}\int_0^h \Phi_1(f'(t+s))\, ds - \Phi_1(f'(t)) \\
&= \frac{1}{h}\int_0^h (\Phi_1(f'(t+s)) - \Phi_1(f'(t)))\, ds \\
&\leq \frac{1}{h}\int_0^h \left|\Phi_1(f'(t+s)) - \Phi_1(f'(t))\right|\, ds.
\end{aligned}$$

Thus we have that

$$0 \leq \delta_h^+(t) \leq \frac{1}{h}\int_0^h \left|\Phi_1(f'(t+s)) - \Phi_1(f'(t))\right|\, ds \text{ for } a.e.,$$

and so we have by Fubini's Theorem that

$$\begin{aligned}
\|\delta_h^+\|_1 &\leq \frac{1}{h}\int_{-\infty}^{\infty}\int_0^h \left|\Phi_1(f'(t+s)) - \Phi_1(f'(t))\right|\, ds\, dt \\
&= \frac{1}{h}\int_0^h\int_{-\infty}^{\infty} \left|\Phi_1(f'(t+s)) - \Phi_1(f'(t))\right|\, dt\, ds \\
&= \frac{1}{h}\int_0^h \left\||\Phi_1(f'(\cdot+s)) - \Phi_1(f'(\cdot))\right\|_1\, ds.
\end{aligned}$$

Since $\Phi_1(f'(\cdot)) \in L^1(\mathbb{R})$, we have

$$\lim_{h\to 0}\|\delta_h^+\|_1 = 0. \tag{2.15}$$

On the other hand, if $\delta_h^-(t) > 0$,

$$\delta_h^-(t) = \Phi_1(f'(t)) - \Phi_1(f_h(t)) \leq \Phi_1(f'(t)).$$

Hence $0 \leq \delta_h^-(t) \leq \Phi_1(f'(t))\ a.e.$.
$\lim_{h\to 0} \Phi_1(f_h(t)) = \Phi_1(f'(t)) a.e.$ implies $\lim_{h\to 0} \delta_h^-(t) = 0\ a.e.$.
Thus we have from $\Phi(f') \in L^1(\mathbb{R})$ that

$$\lim_{h\to 0} \|\delta_h^-\|_1 = 0. \tag{2.16}$$

$$\begin{aligned}\left\|\Phi_1(f_h) - \Phi_1(f')\right\|_1 &= \int_{\mathbb{R}} \left|\Phi_1(f_h(t)) - \Phi_1(f'(t))\right| dt \\ &= \int_{\mathbb{R}} (\delta_h^+(t) + \delta_h^-(t))\, dt \\ &= \|\delta_h^+\|_1 + \|\delta_h^-\|_1.\end{aligned}$$

From (2.15) and (2.16)

$$\lim_{h\to 0} \left\|\Phi_1(f_h) - \Phi_1(f')\right\|_1 = 0.$$

In the same way, we have that

$$\lim_{h\to 0} \left\|\Phi_2(f_h) - \Phi_2(f')\right\|_1 = 0.$$

$$\begin{aligned}d_\Phi\left(f_h, f'\right) &= \int_{\mathbb{R}} \left|\widetilde{\Phi}(f_h(t)) - \widetilde{\Phi}(f'(t))\right| dt \\ &= \int_{\mathbb{R}} \left|\Phi_1(f_h(t)) - \Phi_1(f'(t))\right| dt \\ &\quad + \int_{\mathbb{R}} \left|\Phi_2(f_h(t)) - \Phi_2(f'(t))\right| dt \\ &= \left\|\Phi_1(f_h) - \Phi_1(f')\right\|_1 + \left\|\Phi_2(f_h) - \Phi_2(f')\right\|_1.\end{aligned}$$

Thus we have that

$$\lim_{h\to 0} d_\Phi\left(f_h, f'\right) = 0.$$

□

Let $I \subset \mathbb{R}$ be a closed interval. A function $f \in L^1_{\text{loc}}(\mathbb{R})$ is said to be *essentially absolutely continuous* on I if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\sum_{k=1}^{n} |\widetilde{f}(b_k) - \widetilde{f}(a_k)| \le \epsilon$$

for every finite number of nonoverlapping interval (a_k, b_k), $k = 1, \ldots, n$ with a_k, $b_k \in \operatorname{ess} D(f) \cap I$ and

$$\sum_{k=1}^{n} |b_k - a_k| \le \delta.$$

Note that a function $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ is *locally essentially absolutely continuous* on $\mathbb{R}$ if and only if there exists a *locally absolutely continuous* (and therefore continuous) $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ such that $f = g$ a.e. In this case $\operatorname{ess} D(f) = \mathbb{R}$ and $\operatorname{ess} V_\Phi(f) = V_\Phi(\widetilde{f})$.

Theorem 2.6 *Let us suppose that $\Phi(-a) > 0$, $\Phi(b) > 0$ for some $a,\ b > 0$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. Then the following are equivalent:*

- *(i) $f_h \in \mathcal{K}_\Phi(\mathbb{R})$ for $h \ne 0$ sufficiently small and there exists $g \in \mathcal{K}_\Phi(\mathbb{R})$ such that* $\lim_{h\to 0} d_\Phi(f_h, g) = 0$.
- *(ii) $f_h \in \mathcal{K}_\Phi(\mathbb{R})$ for all $h \ne 0$ and there exists $g \in \mathcal{K}_\Phi(\mathbb{R})$ such that* $\lim_{h\to 0} d_\Phi(f_h, g) = 0$.
- *(iii) f is locally essentially absolutely continuous and* $(\widetilde{f})' \in \mathcal{K}_\Phi(\mathbb{R})$.

Then $\lim_{h\to 0} d_\Phi\left(f_h, (\widetilde{f})'\right) = 0$ *holds.*

Remark 2.3 In this theorem, if Φ satisfies an additional property with $\Phi(x) > 0$ for $x \ne 0$, from Theorem 2.3, since $(\mathcal{K}_\Phi(\mathbb{R}), d_\Phi)$ is a metric space, we see that $\lim_{h\to 0} f_h$ converges to $(\widetilde{f})'$ on $(\mathcal{K}_\Phi(\mathbb{R}), d_\Phi)$ and $(\widetilde{f})' = g$ *a.e.*.

Lemma 2.3 *Let us suppose that $\Phi(-a) > 0$, $\Phi(b) > 0$ for some $a,\ b > 0$. Then there exist p, $q \in (0, \infty)$ such that $|x - y| \le p|\widetilde{\Phi}(x) - \widetilde{\Phi}(y)| + q$ for all x, $y \in \mathbb{R}$.*

Proof There exists $m > 0$ such that

$$\frac{\Phi(-a)}{a}, \frac{\Phi(b)}{b} > m > 0.$$

Define the function

$$\widehat{\Phi}(x) := \widetilde{\Phi}(x) - mx,\ \ x \in \mathbb{R}.$$

Let $V^-(\widehat{\Phi}, I)$ be the negative variation of $\widehat{\Phi}$ on an interval I.

(i) For any $x_1 < x_2 \le a$,

$$\frac{\widetilde{\Phi}(x_2) - \widetilde{\Phi}(x_1)}{x_2 - x_1} = -\frac{\Phi(x_2) - \Phi(x_1)}{x_2 - x_1} \ge -\frac{\Phi(0) - \Phi(-a)}{0 - (-a)} = \frac{\Phi(-a)}{a} \ge m.$$

Hence $\widetilde{\Phi}(x_2) - mx_2 \ge \widetilde{\Phi}(x_1) - mx_1$, and so $\widehat{\Phi}(x_1) \le \widehat{\Phi}(x_2)$. Thus we have that $V^-(\widehat{\Phi}, (-\infty, -a]) = 0$.

(ii) For any $b \le x_3 < x_4 < \infty$,

$$\frac{\widetilde{\Phi}(x_4) - \widetilde{\Phi}(x_3)}{x_4 - x_3} = \frac{\Phi(x_4) - \Phi(x_3)}{x_4 - x_3} \ge \frac{\Phi(b) - \Phi(0)}{b - 0} = \frac{\Phi(b)}{b} \ge m.$$

Hence $\widetilde{\Phi}(x_3) - mx_3 \le \widetilde{\Phi}(x_4) - mx_4$, and so $\widehat{\Phi}(x_3) \le \widehat{\Phi}(x_4)$. Thus we have that $V^-(\widehat{\Phi}, [b, \infty)) = 0$.

(iii) For any $-a \le x_5 < x_6 < \infty$, $\widehat{\Phi}(x_6) - \widehat{\Phi}(x_5) = (\widetilde{\Phi}(x_6) - mx_6) - (\widetilde{\Phi}(x_5) - mx_5) = (\widetilde{\Phi}(x_6) - \widetilde{\Phi}(x_5)) - m(x_6 - x_5) \ge -m(x_6 - x_5)$, and so $V^-(\widehat{\Phi}, [-a, b]) \le m(a + b)$.

From (i), (ii) and (iii) we have that

$$\begin{aligned} V^-(\widehat{\Phi}, (-\infty, \infty)) &= V^-(\widehat{\Phi}, (-\infty, -a]) + V^-(\widehat{\Phi}, [-a, b]) + V^-(\widehat{\Phi}, [b, \infty)) \\ &\le m(a + b). \end{aligned}$$

Consequently, for any $x \le y$, we have that

$$-m(a + b) \le -V^-(\widehat{\Phi}, (-\infty, \infty)) \le \widehat{\Phi}(y) - \widehat{\Phi}(x) = (\widetilde{\Phi}(y) - \widetilde{\Phi}(x)) - m(y - x),$$

and so

$$m(y - x) - m(a + b) \le \widetilde{\Phi}(y) - \widetilde{\Phi}(x),$$

that is,

$$m|x - y| - m(a + b) \le |\widetilde{\Phi}(x) - \widetilde{\Phi}(y)| \text{ for } x, y \in \mathbb{R}.$$

Thus we have that

$$|x - y| \le \frac{1}{m}|\widetilde{\Phi}(x) - \widetilde{\Phi}(y)| + (a + b) \text{ for } x, y \in \mathbb{R},$$

and hence the proof is complete. □

Proof of Theorem 2.6 First, to show that (i) implies (iii), take $p, q > 0$ so that $|x - y| \le p|\widetilde{\Phi}(x) - \widetilde{\Phi}(y)| + q$ as in Lemma 2.3. For any $t_1, t_2 \in \operatorname{ess} D(f)$ with $t_1 < t_2$,

$$\begin{aligned} \widetilde{f}(t_2) - \widetilde{f}(t_1) &= \lim_{h \to 0} \left\{ \frac{1}{h} \int_{t_2}^{t_2 + h} f(t)\, dt - \frac{1}{h} \int_{t_1}^{t_1 + h} f(t)\, dt \right\} \\ &= \lim_{h \to 0} \frac{1}{h} \int_{t_1}^{t_2} (f(t + h) - f(t))\, dt. \end{aligned}$$

$$\begin{aligned}
\left|\widetilde{f}(t_2) - \widetilde{f}(t_1)\right| &= \lim_{h\to 0}\left|\int_{t_1}^{t_2} f_h(t)\,dt\right| \\
&= \lim_{h\to 0}\left|\int_{t_1}^{t_2} \{f_h(t) - g(t)\}\,dt + \int_{t_1}^{t_2} g(t)\,dt\right| \\
&\le \liminf_{h\to 0}\int_{t_1}^{t_2} |f_h(t) - g(t)|\,dt + \int_{t_1}^{t_2} |g(t)|\,dt \\
&\le \liminf_{h\to 0}\int_{t_1}^{t_2} \left\{p\left|\widetilde{\Phi}(f_h(t)) - \widetilde{\Phi}(g(t))\right| + q\right\}dt \\
&\quad + \int_{t_1}^{t_2} |g(t)|\,dt \\
&= p\liminf_{h\to 0} d_\Phi(f_h, g) + q(t_2 - t_1) + \int_{t_1}^{t_2} |g(t)|\,dt \\
&= q(t_2 - t_1) + \int_{t_1}^{t_2} |g(t)|\,dt.
\end{aligned}$$

Thus we see easily from $g \in L^1_{\mathrm{loc}}(\mathbb{R})$ that f is locally essentially absolutely continuous on $\mathbb{R}$. Therefore, as mentioned before Theorem 2.6, $\widetilde{f}$ is extended uniquely from $\operatorname{ess} D(f)$ to $\mathbb{R}$ in such a way that the extended function is still locally absolutely continous on $\mathbb{R}$.

$$\begin{aligned}
\int_{\mathbb{R}} \Phi(\widetilde{f}'(t))\,dt &= \operatorname{ess} V_\Phi(f) \quad \text{(by Theorem 2.1)} \\
&= \lim_{h\to 0}\int_{\mathbb{R}} \Phi(f_h(t))\,dt \quad \text{(by Theorem 2.1)} \\
&= \lim_{h\to 0}\left\|\widetilde{\Phi}(f_h)\right\|_1 \\
&\le \liminf_{h\to 0}\left\|\widetilde{\Phi}(f_h) - \widetilde{\Phi}(g)\right\|_1 + \left\|\widetilde{\Phi}(g)\right\|_1 \\
&= \left\|\widetilde{\Phi}(g)\right\|_1 < \infty.
\end{aligned}$$

Thus we have shown that (iii) holds.

Next, we show that (iii) implies (ii). Assume that f is locally essentially absolutely continuous and $\widetilde{f}' \in \mathcal{K}_\Phi(\mathbb{R})$. Then we see from Temorem 2.5 that (ii) holds. It is obvious that (ii) implies (i), which completes the proof. □

Theorem 2.7 *Let* $\lim_{|x|\to\infty} \dfrac{\Phi(x)}{|x|} = \infty$ *and* $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. *Then the following are equivalent:*

(i) $f_h \in \mathcal{K}_\Phi(\mathbb{R})$ *for* $h \neq 0$ *sufficiently small and there exists* $g \in \mathcal{K}_\Phi(\mathbb{R})$ *such that* $\lim_{h\to 0} d_\Phi(f_h, g) = 0$.

(ii) $f_h \in \mathcal{K}_\Phi(\mathbb{R})$ *for all* $h \neq 0$ *and there exists* $g \in \mathcal{K}_\Phi(\mathbb{R})$ *such that* $\lim_{h\to 0} d_\Phi(f_h, g) = 0$.

(iii) f *is locally essentially absolutely continuous and* $\widetilde{f}' \in \mathcal{K}_\Phi(\mathbb{R})$.
(iv) $\operatorname{ess} V_\Phi(f) < \infty$

Then $\lim_{h\to 0} d_\Phi\left(f_h, \widetilde{f}'\right) = 0$ *holds.*

Proof It is obvious from Theorem 2.6 that the properties (i), (ii) and (iii) are equivalent. Hence it suffices to show that properties (iii) and (iv) are equivalent.

Suppose that (iii) holds, then it follows from the immediate description before Theorems 2.6 and 2.1(iii) that the property (iv) holds.

To prove the converse implication, we first consider the case $\operatorname{ess} V_\Phi(f) = 0$. For any $s < t$ with $s, t \in \operatorname{ess} D(f)$,

$$0 \le \Phi\left(\frac{\widetilde{f}(t) - \widetilde{f}(s)}{t-s}\right)(t-s) \le \operatorname{ess} V_\Phi(f) = 0.$$

Thus we have that $\dfrac{\widetilde{f}(t) - \widetilde{f}(s)}{t-s} \in \Phi^{-1}(0)$ for every $s < t$ with $s, t \in \operatorname{ess} D(f)$. Since $\Phi^{-1}(0)$ is bounded in $\mathbb{R}$ form the hypothesis of Φ, there exists $C > 0$ such that $\Phi^{-1}(0) \subset [-C, C]$, and so we have that $|\widetilde{f}(s) - \widetilde{f}(t)| \le C|s-t|$ for $s, t \in \operatorname{ess} D(f)$. Thus we see that f is locally essentially absolutely continuous on $\mathbb{R}$.

In case of $\operatorname{ess} V_\Phi(f) > 0$: Fix $\varepsilon > 0$, then we have from the hypothesis of Φ that there exists $M > 0$ such that

$$\frac{2}{\varepsilon}\operatorname{ess} V_\Phi(f)|x| \le \Phi(x) \quad \text{for all } |x| \ge M. \tag{2.17}$$

Let $\{I_k : 1 \le k \le n\}$ be any finite number of nonoverlapping intervals with $I_k = [a_k, b_k)$, $a_k, b_k \in \operatorname{ess} D(f)$ and $\sum_{k=1}^{n}(b_k - a_k) < \dfrac{\varepsilon}{2M}$. Let $\{I'_k : 1 \le k \le \ell\}$ and $\{I''_k : 1 \le k \le m\}$ be families of I_k such that

$$\left|\frac{\widetilde{f}(b_k) - \widetilde{f}(a_k)}{b_k - a_k}\right| \ge M \text{ and } \left|\frac{\widetilde{f}(b_k) - \widetilde{f}(a_k)}{b_k - a_k}\right| < M,$$

respectively, and so we denote that $I'_k = [a'_k, b'_k)(1 \le k \le \ell)$, $I''_k = [a''_k, b''_k)(1 \le k \le m)$. Thus we have from (2.17) and the definition of I'_k that

$$\begin{aligned}
\operatorname{ess} V_\Phi(f) &\ge \sum_{k=1}^{\ell} \Phi\left(\frac{\widetilde{f}(b'_k) - \widetilde{f}(a'_k)}{b'_k - a'_k}\right)(b'_k - a'_k) \\
&\ge \sum_{k=1}^{\ell} \frac{2}{\varepsilon}\operatorname{ess} V_\Phi(f)\left|\frac{\widetilde{f}(b'_k) - \widetilde{f}(a'_k)}{b'_k - a'_k}\right|(b'_k - a'_k) \\
&= \frac{2}{\varepsilon}\operatorname{ess} V_\Phi(f)\sum_{k=1}^{\ell}|\widetilde{f}(b'_k) - \widetilde{f}(a'_k)|.
\end{aligned}$$

Since $0 < \operatorname{ess} V_\Phi(f) < \infty$, we have that $\sum_{k=1}^{\ell}|\widetilde{f}(b'_k) - \widetilde{f}(a'_k)| \le \varepsilon/2$.

On the other hand, we have from the definition of I_k'' that

$$\begin{aligned}\sum_{k=1}^{m} |\widetilde{f}(b_k'') - \widetilde{f}(a_k'')| &= \sum_{k=1}^{m} \left| \frac{\widetilde{f}(b_k'') - \widetilde{f}(a_k'')}{b_k'' - a_k''} \right| (b_k'' - a_k'') \leq \sum_{k=1}^{m} M(b_k'' - a_k'') \\ &\leq M \sum_{k=1}^{n} (b_k - a_k) \\ &< \varepsilon/2.\end{aligned}$$

Thus we have

$$\sum_{k=1}^{n} |\widetilde{f}(b_k) - \widetilde{f}(a_k)| = \sum_{k=1}^{\ell} |\widetilde{f}(b_k') - \widetilde{f}(a_k')| + \sum_{k=1}^{m} |\widetilde{f}(b_k'') - \widetilde{f}(a_k'')| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

By the arbitrariness of $\varepsilon > 0$, we see that f is essentially absolutely continuous. Moreover, we have from Theorem 2.1(iii) that $\Phi \circ (\widetilde{f})' \in L^1(\mathbb{R})$, and hence $(\widetilde{f})' \in \mathcal{K}_\Phi(\mathbb{R})$. This concludes the proof. □

References

1. Appell, J., Banaś, J., Merentes, N.: Bounded Variation and Around, De Gruyter Series in Nonlinear Analysis and Applications, 17. De Gruyter, Berlin (2014)
2. Leoni, G.: A first course in Sobolev spaces, Graduate Studies in Mathematics, vol. 105. American Mathematical Society, Providence, RI (2009)
3. Medvedev, Y.T.: Generalization of a theorem of F. Riesz. Uspekhi Mat. Nauk. **8**(6), 115–118 (1953). (Russian)
4. Nakamura, G., Hashimoto, K.: On the linearity of some sets of sequences defined by L^p-functions and L_1-functions determining ℓ_1. Proc. Japan Acad. Ser. A Math Sci, **87**, 77–82 (2011)
5. Nakamura, G., Hashimoto, K.: On the essential bounded variation of $L^p(\mathbb{R}, X)$-functions. Collect. Math. **65**(3), 407–416 (2014)
6. Riesz, F.: Untersuchungen über Systeme integrierbarer Funktionen. Math. Ann. **69**(4), 449–497 (1910)