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Compact embedding for $p(x, t)$-Sobolev spaces and existence theory to parabolic equations with $p(x, t)$-growth

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Abstract In this paper, we establish the compact embedding of $p(x, t)$-Sobolev spaces into $p(x, t)$-Lebesgue spaces. Moreover, we prove some existence results for nonlinear parabolic problems of the form

$$\partial_t u - \operatorname{div} a(x, t, Du) = f - \operatorname{div}\left(|F|^{p(x,t)-2}F\right) \quad \text{in } \Omega_T,$$

where the vector-field $a(x, t, \cdot)$ satisfies certain $p(x, t)$-growth conditions.

Keywords Existence theory · Nonlinear parabolic problems · Nonstandard growth · Nonstandard parabolic Lebesgue and Sobolev spaces · Compactness theorem

Mathematics Subject Classification 35K86 · 35A01 · 46E35 · 54C25

1 Introduction

The first main aim of this paper is to establish the compact embedding of nonstandard $p(x, t)$-Sobolev spaces into nonstandard $p(x, t)$-Lebesgue spaces. This Aubin–Lions type Theorem is important among others for our existence result to general nonlinear parabolic equations with nonstandard $p(x, t)$-growth of the type

$$\partial_t u - \operatorname{div} a(x, t, Du) = f - \operatorname{div}\left(|F|^{p(x,t)-2}F\right) \quad \text{in } \Omega_T. \tag{1.1}$$

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Moreover, the results of this manuscript are also important to prove the existence of solutions to parabolic obstacle problems with $p(x, t)$-growth, see [20].

The motivation of this paper and the study of problems with nonstandard growth is on the one hand based on mathematical interest, on the other hand the consideration of problems in the sense of (1.1) are motivated by issues of life sciences. In general, parabolic problems are often motivated by physical aspects. In particular, evolutionary equations and systems can be used to model physical processes, e.g. heat conduction or diffusion processes. For example the basic equation of fluid mechanics is the Navier–Stokes equation. Some properties of solutions to the system of a modified Navier–Stokes equation describing electro-rheological fluids are studied in [5]. Such fluids are recently of high technological interest, because of their ability to change the mechanical properties under the influence of exterior electro-magnetic field, see [23,27]. Many electro-rheological fluids are suspensions consisting of solid particles and a carrier oil. These suspensions change their material properties dramatically if they are exposed to an electric field, see [28].

Most of the known results concern the stationary models with $p(x)$-growth, see e.g. [1–4]. In the context of parabolic problems with $p(x, t)$-growth applications are e.g. the models for flows in porous media [11,25] or nonlinear parabolic obstacle problems [19]. Moreover, in the last years parabolic problems with $p(x, t)$-growth arouse more and more interest in mathematics, also in regularity theory, cf. [12–15,18–22,31] and [32].

Finally, we want to highlight that in the case of certain parabolic equations with nonstandard growth condition first existence results are available, i.e. by Antontsev and Shmarev in [7–9], Alkutov and Zhikov in [6] and Diening et al. [17].

1.1 General assumptions

We consider a bounded domain $\Omega \subset \mathbb{R}^n$ of dimension $n \geq 2$ and we write $\Omega_T := \Omega \times (0, T)$ for the space–time cylinder over Ω of height $T > 0$. Here, u_t resp. $\partial_t u$ denote the partial derivate with respect to the time variable t and Du denotes the one with respect to the space variable x.

The setting First of all, we should mention that we denote by $\partial_{\mathcal{P}}\Omega_T = (\bar{\Omega} \times \{0\}) \cup (\partial\Omega \times (0, T))$ the parabolic boundary of Ω_T. Furthermore, we write $z = (x, t)$ for points in $\mathbb{R}^{n+1}$. We shall consider vector-fields $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ which are assumed to be Carathéodory functions—i.e. $a(z, w)$ is measurable in the first argument for every $w \in \mathbb{R}^n$ and continuous in the second one for a.e. $z \in \Omega_T$—and satisfy the following nonstandard growth and monotonicity properties, for some growth exponent $p : \Omega_T \to (\frac{2n}{n+2}, \infty)$ and structure constants $0 < \nu \leq 1 \leq L$ and $\mu \in [0, 1]$:

$$|a(z, w)| \leq L(1 + |w|)^{p(z)-1}, \tag{1.2}$$

$$(a(z, w) - a(z, w_0)) \cdot (w - w_0) \geq \nu(\mu^2 + |w|^2 + |w_0|^2)^{\frac{p(z)-2}{2}} |w - w_0|^2 \tag{1.3}$$

for all $z \in \Omega_T$ and $w,\ w_0 \in \mathbb{R}^n$. Furthermore, the growth exponent $p : \Omega_T \to (\frac{2n}{n+2}, \infty)$ satisfies the following conditions: There exist constants $\gamma_1,\ \gamma_2 < \infty$, such that

$$\frac{2n}{n+2} < \gamma_1 \le p(z) \le \gamma_2 \quad \text{and} \quad |p(z_1) - p(z_2)| \le \omega(d_{\mathcal{P}}(z_1, z_2)) \tag{1.4}$$

hold for any choice of $z_1,\ z_2 \in \Omega_T$, where $\omega : [0, \infty) \to [0, 1]$ denotes a modulus of continuity. More precisely, we shall assume that $\omega(\cdot)$ is a concave, non-decreasing function with $\lim_{\rho\downarrow 0} \omega(\rho) = 0 = \omega(0)$. Moreover, the parabolic distance is given by $d_{\mathcal{P}}(z_1, z_2) := \max\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\}$ for $z_1 = (x_1, t_1),\ z_2 = (x_2, t_2) \in \mathbb{R}^{n+1}$. In addition, for the modulus of continuity $\omega(\cdot)$ we assume the following weak logarithmic continuity condition:

$$\limsup_{\rho\downarrow 0} \omega(\rho) \log\left(\frac{1}{\rho}\right) < +\infty. \tag{1.5}$$

By virtue of (1.5) we may assume for a constant $L_1 > 0$ depending on $\omega(\cdot)$ that

$$\omega(\rho) \log\left(\frac{1}{\rho}\right) \le L_1, \quad \text{for all } \rho \in (0, 1]. \tag{1.6}$$

Moreover, we denote by p_1 and p_2 the infimum resp. supremum of $p(\cdot)$ with respect to the domain we are going to deal with, e.g. $p_1 := \inf_{\Omega_T} p(\cdot),\ p_2 := \sup_{\Omega_T} p(\cdot)$. Finally, we point out that (1.3) implies, by using (1.2) and Young's inequality, the coercivity property

$$a(z, w) \cdot w \ge \frac{\nu}{c(\gamma_1, \gamma_2)} |w|^{p(z)} - c(\gamma_1, \gamma_2, \nu, L) \quad \forall\, z \in \Omega_T \text{ and } w \in \mathbb{R}^n. \tag{1.7}$$

1.2 The function spaces

The spaces $L^p(\Omega)$, $W^{1,p}(\Omega)$ and $W_0^{1,p}(\Omega)$ denote the usual Lebesgue and Sobolev spaces.

Parabolic Lebesgue–Orlicz spaces We start by the definition of the nonstandard $p(z)$-Lebesgue space. The space $L^{p(z)}(\Omega_T, \mathbb{R}^k)$ is defined as the set of those measurable functions $v : \Omega_T \to \mathbb{R}^k$ for $k \in \mathbb{N}$, which satisfy $|v|^{p(z)} \in L^1(\Omega_T, \mathbb{R}^k)$, i.e.

$$L^{p(z)}(\Omega_T, \mathbb{R}^k) := \left\{ v : \Omega_T \to \mathbb{R}^k \text{ is measurable in } \Omega_T : \int_{\Omega_T} |v|^{p(z)} \mathrm{d}z < +\infty \right\}.$$

The set $L^{p(z)}(\Omega_T, \mathbb{R}^k)$ equipped with the Luxemburg norm

$$\|v\|_{L^{p(z)}(\Omega_T)} := \inf\left\{ \lambda > 0 : \int_{\Omega_T} \left|\frac{v}{\lambda}\right|^{p(z)} \mathrm{d}z \le 1 \right\}$$

becomes a Banach space. This space is separable and reflexive, see [6,17]. For elements of $L^{p(z)}(\Omega_T, \mathbb{R}^k)$ the generalized Hölder's inequality holds in the following form: If

$f \in L^{p(z)}(\Omega_T, \mathbb{R}^k)$, $g \in L^{p'(z)}(\Omega_T, \mathbb{R}^k)$, where $p'(z) = \frac{p(z)}{p(z)-1}$, we have

$$\left|\int_{\Omega_T} fg\mathrm{d}z\right| \leq \left(\frac{1}{\gamma_1} + \frac{\gamma_2 - 1}{\gamma_2}\right) \|f\|_{L^{p(z)}(\Omega_T)} \|g\|_{L^{p'(z)}(\Omega_T)}, \tag{1.8}$$

see also [6]. Moreover, the norm $\|\cdot\|_{L^{p(z)}(\Omega_T)}$ can be estimated as follows:

$$-1 + \|v\|^{\gamma_1}_{L^{p(z)}(\Omega_T)} \leq \int_{\Omega_T} |v|^{p(z)}\mathrm{d}z \leq \|v\|^{\gamma_2}_{L^{p(z)}(\Omega_T)} + 1. \tag{1.9}$$

Finally, for the right-hand side of (1.1) we assume

$$F \in L^{p(z)}(\Omega_T, \mathbb{R}^n) \quad \text{and} \quad f \in L^{\gamma_1'}(0, T; W^{-1,\gamma_1'}(\Omega)). \tag{1.10}$$

Notice that we will use also the abbreviation $p(\cdot)$ for the exponent $p(z)$.
Parabolic Sobolev–Orlicz spaces First, we introduce nonstandard Sobolev spaces for fixed $t \in (0, T)$. From assumption (1.4), we know that $p(\cdot, t)$ satisfy $|p(x_1, t) - p(x_2, t)| \leq \omega(|x_1 - x_2|)$ for any choice of $x_1,\ x_2 \in \Omega$ and for every $t \in (0, T)$. Next, we define for every fixed $t \in (0, T)$ the Banach space $W^{1,p(\cdot,t)}(\Omega)$ as $W^{1,p(\cdot,t)}(\Omega) := \{u \in L^{p(\cdot,t)}(\Omega, \mathbb{R}) \mid Du \in L^{p(\cdot,t)}(\Omega, \mathbb{R}^n)\}$ equipped with the norm $\|u\|_{W^{1,p(\cdot,t)}(\Omega)} := \|u\|_{L^{p(\cdot,t)}(\Omega)} + \|Du\|_{L^{p(\cdot,t)}(\Omega)}$. In addition, $W_0^{1,p(\cdot,t)}(\Omega) \equiv$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot,t)}(\Omega)$ and denote by $W^{1,p(\cdot,t)}(\Omega)'$ its dual. For every $t \in (0, T)$ the inclusion $W_0^{1,p(\cdot,t)}(\Omega) \subset W_0^{1,\gamma_1}(\Omega)$ holds. Now, we consider more general nonstandard parabolic Sobolev spaces without fixed t. By $W_g^{p(\cdot)}(\Omega_T)$ we denote the Banach space

$$W_g^{p(\cdot)}(\Omega_T) := \left\{u \in [g + L^1(0, T; W_0^{1,1}(\Omega))] \cap L^{p(\cdot)}(\Omega_T) \mid Du \in L^{p(\cdot)}(\Omega_T, \mathbb{R}^n)\right\}$$

equipped by the norm $\|u\|_{W^{p(\cdot)}(\Omega_T)} := \|u\|_{L^{p(\cdot)}(\Omega_T)} + \|Du\|_{L^{p(\cdot)}(\Omega_T)}$. If $g = 0$ we write $W_0^{p(\cdot)}(\Omega_T)$ instead of $W_g^{p(\cdot)}(\Omega_T)$. Here, it is worth to mention that the notion $(u - g) \in W_0^{p(\cdot)}(\Omega_T)$ respectively $u \in g + W_0^{p(\cdot)}(\Omega_T)$ indicate that u agrees with g on the lateral boundary of the cylinder Ω_T, i.e. $u \in W_g^{p(\cdot)}(\Omega_T)$. Now, we are ready to give the definition of a weak solution to the nonstandard parabolic equation (1.1):

Definition 1.1 We identify a function $u \in L^1(\Omega_T)$ as a weak solution of the parabolic equation (1.1), if and only if $u \in C^0([0, T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T)$ and

$$\int_{\Omega_T} [u \cdot \varphi_t - a(z, Du) \cdot D\varphi]\,\mathrm{d}z = -\int_{\Omega_T} \left[f \cdot \varphi + |F|^{p(\cdot)-2} F \cdot D\varphi\right] \mathrm{d}z \tag{1.11}$$

holds, whenever $\varphi \in C_0^\infty(\Omega_T)$.

Remark 1.2 In this paper we will consider certain initial value problems. Therefore, we should mention the meaning when referring to an initial condition of the type $u(\cdot, 0) = g(\cdot, 0)$ a.e. on Ω. Here, we shall always mean

$$\frac{1}{h}\int_0^h \int_\Omega |u - g(\cdot, 0)|^2 \mathrm{d}x\mathrm{d}t \to 0 \quad \text{as } h \downarrow 0. \tag{1.12}$$

In particular, when $u \in C^0([0, T]; L^2(\Omega))$, then (1.12) is obviously equivalent with saying $u(\cdot, 0) = g(\cdot, 0)$. □

Our next aim is to introduce the dual space of $W_0^{p(\cdot)}(\Omega_T)$. Therefore, we denote by $W^{p(\cdot)}(\Omega_T)'$ the dual of the space $W_0^{p(\cdot)}(\Omega_T)$. Assume that $v \in W^{p(\cdot)}(\Omega_T)'$. Then, there exist functions $v_i \in L^{p'(\cdot)}(\Omega_T)$, $i = 0, 1, \dots, n$, such that

$$\langle\langle v, w\rangle\rangle_{\Omega_T} = \int_{\Omega_T}\left(v_0 w + \sum_{i=1}^n v_i D_i w\right)\mathrm{d}z \quad \forall\, w \in W_0^{p(\cdot)}(\Omega_T). \tag{1.13}$$

Here and in the following, we will write $\langle\langle \cdot, \cdot\rangle\rangle_{\Omega_T}$ for the dual pairing between $W^{p(\cdot)}(\Omega_T)'$ and $W_0^{p(\cdot)}(\Omega_T)$. Furthermore, if $v \in W^{p(\cdot)}(\Omega_T)'$, we define the norm

$$\|v\|_{W^{p(\cdot)}(\Omega_T)'} = \sup\{\langle\langle v, w\rangle\rangle_{\Omega_T} \,|\, w \in W_0^{p(\cdot)}(\Omega_T),\ \|w\|_{W_0^{p(\cdot)}(\Omega_T)} \le 1\}.$$

Notice, whenever (1.13) holds, we can write $v = v_0 - \sum_{i=1}^n D_i v_i$, where $D_i v_i$ has to be interpreted as a distributional derivate. By

$$w \in W(\Omega_T) := \left\{w \in W^{p(\cdot)}(\Omega_T) | w_t \in W^{p(\cdot)}(\Omega_T)'\right\}$$

we mean that there exists $w_t \in W^{p(\cdot)}(\Omega_T)'$, such that

$$\langle\langle w_t, \varphi\rangle\rangle_{\Omega_T} = -\int_{\Omega_T} w \cdot \varphi_t \mathrm{d}z \quad \text{for all } \varphi \in C_0^\infty(\Omega_T),$$

see also [17]. The previous equality makes sense due to the inclusions

$$W^{p(\cdot)}(\Omega_T) \hookrightarrow L^2(\Omega_T) \cong (L^2(\Omega_T))' \hookrightarrow W^{p(\cdot)}(\Omega_T)'$$

which allow us to identify w as an element of $W^{p(\cdot)}(\Omega_T)'$. Finally, from the definition of the norm $\|\cdot\|_{W^{p(\cdot)}(\Omega_T)'}$, we can conclude that the following holds: if $f \in W_0^{p(\cdot)}(\Omega_T)$ and $g \in W^{p(\cdot)}(\Omega_T)'$ we have

$$\langle\langle f, g\rangle\rangle_{\Omega_T} \le c(\gamma_1, \gamma_2)\|f\|_{W^{p(\cdot)}(\Omega_T)}\|g\|_{W^{p(\cdot)}(\Omega_T)'}, \tag{1.14}$$

see [19]. Notice also that in the case $p(\cdot) = \text{const.}$, we deal with the standard Lebesgue and Sobolev spaces. This means for example, if $p(\cdot) = \gamma_1$, then we have

$W^{\gamma_1}(\Omega_T) = L^{\gamma_1}(0, T; W^{1,\gamma_1}(\Omega))$. Consequently, the dual space of $W^{\gamma_1}(\Omega_T)$ is given by $W^{\gamma_1}(\Omega_T)' = L^{\gamma_1'}(0, T; W^{-1,\gamma_1'}(\Omega))$.

1.3 Statement of the result and plan of the paper

Here, we mention our main results and briefly describe the strategy of the proof to these results and the technical novelties of the paper. We start with some useful and important preliminary results, see Sect. 2, before we will prove the compact embedding of nonstandard $p(z)$-Sobolev spaces into nonstandard $p(z)$-Lebesgue spaces, see Sect. 3. This will be our first main result, i.e. the compactness theorem in the sense of Aubin–Lions and reads as follows.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$ an open, bounded Lipschitz domain with $n \geq 2$ and $p(\cdot) > \frac{2n}{n+2}$ satisfying (1.4), (1.5). Furthermore, define $\hat{p}(\cdot) := \max\{2, p(\cdot)\}$. Then, the inclusion $W(\Omega_T) \hookrightarrow L^{\hat{p}(\cdot)}(\Omega_T)$ is compact.*

Theorem 1.3 is important for the strong convergence in $p(z)$-Lebesgue spaces and therefore, for our existence results. In Sect. 4, we prove the existence of solutions to the parabolic equation (1.1) under certain boundary and initial data conditions. First of all, we establish the existence of a weak solution to the Dirichlet problem of (1.1), i.e.

$$\begin{cases} \partial_t u - \operatorname{div} a(z, Du) &= f - \operatorname{div}(|F|^{p(\cdot)-2}F) \quad \text{in } \Omega_T, \\ u &= 0 \quad \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) &= g(\cdot, 0) \quad \text{on } \Omega \times \{0\}. \end{cases} \tag{1.15}$$

The approach to prove the existence of solutions to the Dirichlet problem is to construct a solution, which solve the problem (1.15). We start by constructing a sequence of the Galerkin's approximations, where the limit of this sequence is equal to the solution in (1.15). Then, we show that this approximate solution converges to a general solution, where we used some energy bounds, which derive by the proof and finally, the compact embedding of Theorem 1.3 yields the desired convergence of the approximate solutions to general solutions. This yields

Theorem 1.4 *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain and $p : \Omega_T \to [\gamma_1, \gamma_2]$ satisfies (1.4), (1.5). Then, suppose that the vector-field $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function and satisfies, for a given function $v \in C^0([0, T]; L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T)$ with $\partial_t v \in L^{\gamma_1'}(0, T; W^{-1,\gamma_1'}(\Omega))$, the growth condition*

$$|a(z, w)| \leq c(\gamma_2, L)\left((1 + |w|)^{p(\cdot)-1} + |v|^{p(\cdot)-1}\right) \tag{1.16}$$

and the monotonicity property

$$\nu(\mu^2 + |w + v|^2 + |w_0 + v|^2)^{\frac{p(\cdot)-2}{2}} |w - w_0|^2 \leq (a(z, w) - a(z, w_0)) \cdot (w - w_0) \tag{1.17}$$

for all $z \in \Omega_T$ *and* $w, w_0 \in \mathbb{R}^n$. *Moreover, let* (1.10) *and* $g(\cdot,0) \in L^2(\Omega)$ *hold. Furthermore, define*

$$\mathcal{M}_0 := \int_{\Omega_T} 1 + |F|^{p(\cdot)} + |v|^{p(\cdot)} dz + \|f\|^{\gamma_1'}_{L^{\gamma_1'}(0,T;W^{-1,\gamma_1'}(\Omega))} + 1 \geq 1. \tag{1.18}$$

Then, there exists an unique weak solution $u \in C^0([0,T];L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T)$ *with* $\partial_t u \in W^{p(\cdot)}(\Omega_T)'$ *of* (1.15) *and this solution satisfies the following energy estimate*

$$\sup_{0\leq t\leq T} \int_\Omega |u(\cdot,t)|^2 \mathrm{d}x + \int_{\Omega_T} |Du|^{p(\cdot)} \mathrm{d}z \leq c\left(\|g(\cdot,0)\|^2_{L^2(\Omega)} + \mathcal{M}_0\right) \tag{1.19}$$

with $u(\cdot,0) = g(\cdot,0)$ *and a constant* $c = c(n,\gamma_1,\gamma_2,\mathrm{diam}(\Omega))$.

Remark 1.5 We should mention the role of the function v of the preceding theorem. We will need this function in (1.16), (1.17) or in (4.5), (4.6) later for the proof of the next theorem. For the general existence result to the Dirichlet problem (1.15), we would choose v equal to zero. But for the proof of the existence of weak solution to the Cauchy–Dirichlet problem from below, we will re-write the Cauchy–Dirichlet problem into an Dirichlet problem and v will play the role of the boundary value. Then, this existence result derives immediately from Theorem 1.4 because of (1.16), (1.17). □

Finally, the existence of solutions to initial value problem (1.15) can be extend to general boundary problems. Therefore, we consider the Cauchy–Dirichlet problem of the parabolic problem (1.1):

$$\begin{cases} \partial_t u - \mathrm{div}\ a(z,Du) = f - \mathrm{div}\ (|F|^{p(\cdot)-2}F) & \text{in } \Omega_T, \\ u = g \quad \text{on } \partial\Omega \times (0,T), \\ u(\cdot,0) = g(\cdot,0) \quad \text{on } \Omega \times \{0\}, \end{cases} \tag{1.20}$$

where g denotes the boundary data and satisfies

$$g \in C^0([0,T];L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T) \quad \text{and} \quad \partial_t g \in L^{\gamma_1'}(0,T;W^{-1,\gamma_1'}(\Omega)). \tag{1.21}$$

We use the result of Theorem 1.4 to the Cauchy–Dirichlet problem (1.20) to get existence of solutions to (1.1) with general boundary data. Therefore, we have to change the problem (1.20) into a problem comparing to (1.15). Then, we can conclude the existence of solution to the modified problem. Hence, we get the existence result to the primal Cauchy–Dirichlet problem (1.20). This result is stated in the following Theorem.

Theorem 1.6 *Let* $\Omega \subset \mathbb{R}^n$ *be an open, bounded Lipschitz domain and* $p : \Omega_T \to [\gamma_1,\gamma_2]$ *satisfies* (1.4), (1.5). *Then, suppose that the vector-field* $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ *is a Carathéodory function and satisfies the growth condition* (1.2) *and the monotonicity condition* (1.3). *Moreover, let* (1.10) *fulfilled. Furthermore, the boundary data* g *satisfy*

(1.21). *Then, there exists an unique weak solution* $u \in C^0([0,T]; L^2(\Omega)) \cap W_g^{p(\cdot)}(\Omega_T)$ *with* $\partial_t u \in W^{p(\cdot)}(\Omega_T)'$ *of the parabolic Cauchy–Dirichlet problem* (1.20) *and this solution satisfies the following energy estimate*

$$\sup_{0\le t\le T}\int_\Omega |u(\cdot,t)|^2 \mathrm{d}x + \int_{\Omega_T} |Du|^{p(\cdot)} \mathrm{d}z \le c\left(\|g(\cdot,0)\|^2_{L^2(\Omega)} + \|g\|^2_{L^\infty(0,T;L^2(\Omega))} + \mathcal{M}_g\right), \tag{1.22}$$

where $c = c(n, \gamma_1, \gamma_2, \nu, L, \mathrm{diam}(\Omega))$ *and* $\mathcal{M}_g$ *is defined as follows*

$$\mathcal{M}_g := \int_{\Omega_T} 1 + |F|^{p(\cdot)} + |Dg|^{p(\cdot)} \mathrm{d}z + \|f\|^{\gamma_1'}_{W^{\gamma_1}(\Omega_T)'} + \|\partial_t g\|^{\gamma_1'}_{W^{\gamma_1}(\Omega_T)'} + 1.$$

Remark 1.7 Here, we would like to mention that in [20] we need the existence theorems 1.4 and 1.6. But we could also prove the existence of weak solutions to (1.15) and (1.20), if we assume that $a(\cdot)$ satisfies the growth condition (1.2), coercivity condition (1.7) and the monotonicity condition $(a(z,w) - a(z,w_0)) \cdot (w - w_0) \ge 0$ for all $z \in \Omega_T$ and $w,\ w_0 \in \mathbb{R}^n$, see also [17]. □

2 Preliminaries

2.1 Convolution and smoothing

Here, we introduce tools that will allow us to build smooth approximations to given functions. First, we will consider the so-called Friedrich's mollifier. This mollifier can be used to smooth a function in space and time. Therefore, let $\delta \in C^\infty(\mathbb{R}^{n+1})$ be the Friedrich's mollifying kernel

$$\delta(z) := \begin{cases} \kappa \exp\left(-\frac{1}{1-|z|^2}\right) & \text{if } |z| < 1, \\ 0 & \text{if } |z| \ge 1, \end{cases} \quad \text{and} \quad \int_{\mathbb{R}^{n+1}} \delta(z)\mathrm{d}z = 1,$$

where $\kappa = $ const. and $\delta(z) \ge 0$. Furthermore, we extend the given function $v \in W^{p(\cdot)}(\Omega_T)$ by zero to the whole $\mathbb{R}^{n+1}$ and define

$$v_h(z) := \int_{\mathbb{R}^{n+1}} v(s)\delta_h(z-s)ds \quad \text{with } \delta_h(s) = \frac{1}{h^{n+1}}\delta\left(\frac{s}{h}\right), \quad h > 0.$$

The next two propositions are stated in [10] and show some properties of Friedrich's mollified functions.

Proposition 2.1 *If* $u \in W_0^{p(\cdot)}(\Omega_T)$ *with exponent function* $p(z)$ *satisfying* (1.4), (1.5), *then* $\|u_h - u\|_{W^{p(\cdot)}(\Omega_T)} \longrightarrow 0$ *as* $h \downarrow 0$ *and*

$$\|u_h\|_{W^{p(\cdot)}(\Omega_T)} \le c\left(\|u\|_{W^{1,1}(\Omega_T)} + \|u\|_{W^{p(\cdot)}(\Omega_T)}\right).$$

Proposition 2.2 *If $u \in W_0^{p(\cdot)}(\Omega_T)$ with exponent function $p(z)$ satisfying* (1.4), (1.5) *and $u_t \in W^{p(\cdot)}(\Omega_T)'$, then $(u_h)_t \in W^{p(\cdot)}(\Omega_T)'$, and for every $\phi \in W_0^{p(\cdot)}(\Omega_T)$*

$$\langle\langle (u_h)_t, \phi\rangle\rangle_{\Omega_T} \longrightarrow \langle\langle u_t, \phi\rangle\rangle_{\Omega_T} \quad \text{as } h \downarrow 0.$$

Since, $W_0^{1,\gamma_2}(\Omega)$ is separable, it is a span of a countable set of linearly independent functions $\{\phi_k\} \subset W_0^{1,\gamma_2}(\Omega)$. Moreover, we have the dense embedding $W_0^{1,\gamma_2}(\Omega) \subset L^2(\Omega)$ for any $\gamma_2 > \frac{2n}{n+2}$, see e.g. [29,30]. Therefore, without loss of generality, we may assume that this system forms an orthonormal basis of $L^2(\Omega)$.

Moreover, since weak solutions u of parabolic equations possess only weak regularity properties with respect to the time variable t, i.e. they are not assumed to be weakly differentiable, in principle it is not possible to use the solution u itself (also disregarding boundary values) as a test-function in the weak formulation of the parabolic equation. In order to be able to test the equation properly, we smooth the solution u with respect to the time direction t using the so-called Steklov averages. Hence, we introduce the following: the Steklov averages of a function $f \in L^1(\Omega_T)$ are defined as

$$[f]_h(x,t) := \begin{cases} \frac{1}{h}\displaystyle\int_t^{t+h} f(x,s)\mathrm{d}s & \text{for } t \in (0, T-h), \\ 0 & \text{for } t \in [T-h, T), \end{cases} \tag{2.1}$$

for $x \in \Omega$, for all $t \in (0,T)$ and $0 < h < T$. Assuming that $u \in C^0([0,T]; L^2(\Omega)) \cap W^{p(\cdot)}(\Omega_T)$ is a weak solution to the parabolic equation (1.1) the Steklov average $[u]_h$ satisfies the corresponding equation

$$\int_{\Omega\times\{t\}} \partial_t([u]_h)\cdot\varphi + [a(\cdot, Du)]_h \cdot D\varphi \mathrm{d}x = \int_{\Omega\times\{t\}} [f]_h\cdot\varphi + [F^{p(\cdot)-2}F]_h\cdot D\varphi\mathrm{d}x \tag{2.2}$$

for a.e. $t \in (0,T)$ and all $\varphi \in C_0^\infty(\Omega)$.

2.2 Poincaré type estimate

Our next problem is, that we need a Poincaré inequality. It is only possible to use the elliptic Poincaré inequality slicewise for a.e. times t. For parabolic problems with nonstandard growth, it is not allowed to apply such an estimate, not even slicewise. There exists just a "Luxemburg-version", see [7], i.e. $\|u\|_{L^{p(x)}(\Omega)} \le c\|Du\|_{L^{p(x)}(\Omega)}$ for all $u \in W_0^{1,p(x)}(\Omega)$, where $c > 0$. But we need a "classical" Poincaré type inequality. The desired result is given by the following lemma, which is stated in [19].

Lemma 2.3 ([19], Lemma 3.9) *Let $\Omega \subset \mathbb{R}^n$ a bounded Lipschitz domain and $\gamma_2 := \sup_{\Omega_T} p(\cdot)$. Assume that $u \in C^0([0,T]; L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T)$ and the exponent $p(\cdot)$ satisfies the conditions* (1.4), (1.5)*. Then, there exists a constant*

$c = c(n, \gamma_1, \gamma_2, \mathrm{diam}(\Omega), \omega(\cdot))$, *such that the two versions of the Poincaré type estimate are valid:*

$$\int_{\Omega_T} |u|^{p(\cdot)}\,\mathrm{d}z \le c\left(\|u\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{4\gamma_2}{n+2}} + 1\right)\left(\int_{\Omega_T} |Du|^{p(\cdot)} + 1\,\mathrm{d}z\right) \tag{2.3}$$

and

$$\|u\|_{L^{p(z)}(\Omega_T)}^{\gamma_1} \le c\left(\|u\|_{L^\infty(0,T;L^2(\Omega))}^{\frac{4\gamma_2}{n+2}} + 1\right)\left(\int_{\Omega_T} |Du|^{p(\cdot)} + 1\,\mathrm{d}z\right). \tag{2.4}$$

3 Proof of the compact embedding: compactness theorem

In this chapter, we will show some properties concerning distributional nonstandard Sobolev spaces. For usual Sobolev spaces $W^{1,p}(\Omega)$, these results are given in [29, Chapter III], [30, Chapter III]. Moreover, we would like to mention that in [17, Theorem 7.1] the authors proved a very similar assertion to Lemma 3.1.

Since $L^2(\Omega)$ is a Hilbert space which is identified with its dual $L^2(\Omega) \cong (L^2(\Omega))'$ and in which $L^{p(\cdot,t)}(\Omega)$ is dense and continuously embedded $\forall t \in [0, T]$, where $p(\cdot, t) \ge 2$, see [16, Lemma 5.5], we have $L^{p(\cdot,t)}(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow L^{p'(\cdot,t)}(\Omega)$ for all $t \in [0, T]$. The fact that $L^2(\Omega) \cong (L^2(\Omega))'$ can be demonstrated by the Riesz representation theorem. We denote the dual of $W_0^{1,p(\cdot,t)}(\Omega)$ by $W^{1,p(\cdot,t)}(\Omega)'$ and the natural pairing between $W^{1,p(\cdot,t)}(\Omega)'$ and $W_0^{1,p(\cdot,t)}(\Omega)$ by $\langle\cdot,\cdot\rangle$. Moreover, we have the embeddings $W_0^{1,p(\cdot,t)}(\Omega) \subset L^2(\Omega)$ and $(L^2(\Omega))' \subset W^{1,p(\cdot,t)}(\Omega)'$. Therefore, we can conclude that $W_0^{1,p(\cdot,t)}(\Omega) \hookrightarrow L^2(\Omega) \cong (L^2(\Omega))' \hookrightarrow W^{1,p(\cdot,t)}(\Omega)'$, where the injections are compact. This also allows us to identify the duality product $\langle\cdot,\cdot\rangle$ with the inner product between $L^2(\Omega)$ and $W_0^{1,p(\cdot,t)}(\Omega)$, i.e.

$$f(v) = \langle f, v\rangle = \langle f, v\rangle_{L^2(\Omega)} = \int_\Omega f \cdot v\mathrm{d}x$$

whenever $f \in L^2(\Omega) \subset W^{1,p(\cdot,t)}(\Omega)'$ and $v \in W_0^{1,p(\cdot,t)}(\Omega)$ and $t \in [0, T]$. Next, we consider the Banach space

$$W_0(\Omega_T) := \left\{w \in W_0^{p(\cdot)}(\Omega_T) | w_t \in W^{p(\cdot)}(\Omega_T)'\right\}.$$

Let $a < 0 < T < b$. We shall construct an extension of each $u \in W_0(\Omega_T)$ to $\tilde{u} \in W_0(\Omega_I)$, where $I := (a, b)$. First extend u to $(a, 0)$ and (T, b) (e.g. by symmetry). Let $\Theta \in C_0^\infty(a, b)$ with $\Theta = 1$ on $(0, T)$. We define $\tilde{u} = u \cdot \Theta$ and note that $\tilde{u} \in W_0(\Omega_I)$ and $\tilde{u} = u$ on $(0, T)$. Therefore, $\|u\|_{W(\Omega_T)} \le \|\tilde{u}\|_{W(\Omega_I)} \le C(\Theta)\|u\|_{W(\Omega_T)}$, where $C(\Theta)$ depends only on Θ and $\tilde{u} = 0$ in a neighborhood of a and b. Next, we regularize $\tilde{u}$ by the mollifier $u_h(t) = \int_{\mathbb{R}} \tilde{u}(s)\delta_h(t - s)\mathrm{d}s$, where $\delta_h(s) = \frac{1}{h}\delta\left(\frac{s}{h}\right)$, $\delta \in C_0^\infty(-1, 1)$, $\delta \ge 0$ and $\int_{\mathbb{R}} \delta(s)\mathrm{d}s = 1$. It follows that $u_h \in C^\infty(\Omega_I)$ and $u_h(\cdot, a) = 0 = u_h(\cdot, b)$ for sufficiently small $h > 0$. Moreover, we have $u_h \to \tilde{u}$ in $W_0(\Omega_I)$ with

$\|u_h\|_{W(\Omega_I)} \le \|\tilde{u}\|_{W(\Omega_I)}$, cf. Propositions 2.1 and 2.2. By the preceding identification of spaces we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_h(\cdot,t)\|^2_{L^2(\Omega)} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle u_h(\cdot,t), u_h(\cdot,t)\rangle_{L^2(\Omega)} = \int_\Omega \partial_t u_h(\cdot,t)\cdot u_h(\cdot,t)\mathrm{d}x$$

and this yields $\frac{1}{2}\|u_h(\cdot,t)\|^2_{L^2(\Omega)} \le \|\partial_t u_h(\cdot,s)\|_{W^{p(\cdot,t)}(\Omega\times(a,t))'}\|u_h(\cdot,s)\|_{W^{p(\cdot,t)}(\Omega\times(a,t))}$ $\le \|u_h\|^2_{W(\Omega_I)}$ for all $a \le t \le b$. Since $\{u_h\}$ is a Cauchy sequence in $W_0(\Omega_I)$, such an estimate on differences $u_h - u_k$ from the sequence shows that it converges (uniformly) to $\tilde{u}$ in $C^0([0,T];L^2(\Omega))$. Thus, we obtain the following:

Lemma 3.1 *Let $n \ge 2$. Assume that $p : \Omega_T \to [\gamma_1,\gamma_2]$ satisfies (1.4), (1.5). Then $W(\Omega_T)$ is contained in $C^0([0,T];L^2(\Omega))$. Moreover, if $u \in W_0(\Omega_T)$ then $t \mapsto \|u(\cdot,t)\|^2_{L^2(\Omega)}$ is absolutely continuous on $[0,T]$,*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |u(\cdot,t)|^2\mathrm{d}x = 2\langle \partial_t u(\cdot,t), u(\cdot,t)\rangle, \quad \text{for a.e. } t \in [0,T],$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $W^{1,p(\cdot,t)}(\Omega)'$ and $W_0^{1,p(\cdot,t)}(\Omega)$. Moreover, there is a constant c such that $\|u\|_{C^0([0,T];L^2(\Omega))} \le c\|u\|_{W(\Omega_T)}$ for every $u \in W_0(\Omega_T)$.

The proof of the compactness theorem will be based on the following interpolation lemma, which is established in [19]. Here, we consider $p > 2n/(n+2)$ and we will utilize the fact that we have open, bounded Lipschitz domains $\Omega \subseteq \mathbb{R}^n$, the dense embeddings $W^{1,p}(\Omega) \subset L^2(\Omega)$ and $(L^2(\Omega))' \subset W^{-1,p'}(\Omega)$. Moreover, the injection of $W^{1,p}(\Omega)$ into $L^q(\Omega)$ is compact, provided the exponents satisfy $1 \le q < p^*$ if $\frac{2n}{n+2} < p < n$ and $q \ge 1$ if $p \ge n$, where $p^* := np/(n-p)$ is the Sobolev exponent of p. Further, we want to infer from the Interpolation Lemma 3.2 a Gagliardo–Nirenberg inequality. This Gagliardo-Nirenberg inequality we use to conclude the desired compact embedding between $W(\Omega_T)$ and $L^{\hat{p}(\cdot)}(\Omega_T)$ with $\hat{p}(\cdot) := \max\{2, p(\cdot)\}$, see the Aubin–Lions type Theorem 1.3.

Lemma 3.2 ([19], Lemma 3.6) *Let $p, r > 1$ and $\Omega \subset \mathbb{R}^n$ be an open, bounded Lipschitz domain with $n \ge 2$. Moreover, suppose that the exponent q satisfies the following conditions $q \in [1,p^*)$ if $1 < p < n$ or $q \in [1,\infty)$ if $n \le p < \infty$ with the Sobolev exponent $p^* := \frac{np}{n-p}$, if $p < n$. Then, for each $\eta > 0$, there exists some constant C_η depending on η, p, q, r and Ω, such that the following interpolation inequality holds:*

$$\|v\|_{L^q(\Omega)} \le \eta\|v\|_{W^{1,p}(\Omega)} + C_\eta\|v\|_{W^{-1,r}(\Omega)} \quad \forall\, v \in W^{1,p}(\Omega)\cap W^{-1,r}(\Omega). \tag{3.1}$$

Our next aim is to conclude a needed version of (3.1). Therefore, we consider a bounded Lipschitz domain $A \subset \mathbb{R}^n$ and suppose that, $v \in W^{1,p}(A)\cap W^{-1,r}(A)$ for some $p > \frac{2n}{n+2}$ and $r > 1$. Notice that, the Sobolev's embedding implies $v \in L^2(A)$. Next, we consider q and s, such that $2 < s < q < p^*$ with $p^* = \frac{np}{n-p}$ if $p < n$ and

$p^* =$ any number > 1 if $p \geq n$. Now, we apply Hölder's inequality with exponents $\frac{q-s}{q-2}$ and $\frac{s-2}{q-2}$ to $\|v\|^s_{L^s(\Omega)}$. This yields

$$\|v\|^s_{L^s(\Omega)} = \int_\Omega |v|^{2\frac{q-s}{q-2}} |v|^{s-2\frac{q-s}{q-2}} \mathrm{d}x \leq \left(\int_\Omega |v|^2 \mathrm{d}x\right)^{\frac{q-s}{q-2}} \left(\int_\Omega |v|^q \mathrm{d}x\right)^{\frac{s-2}{q-2}}.$$

At this stage, we apply (3.1) to the last term on the right-hand side of the previous estimate. Hence, we have

$$\|v\|_{L^s(\Omega)} \leq \left(\int_\Omega |v|^2 \mathrm{d}x\right)^{\frac{q-s}{q-2}\frac{1}{s}} \left(\eta \|v\|_{W^{1,p}(\Omega)} + C_\eta \|v\|_{W^{-1,r}(\Omega)}\right)^{q\frac{s-2}{q-2}\frac{1}{s}}. \tag{3.2}$$

Moreover, in the case $p \leq 2$, we have again by Lemma 3.2

$$\begin{aligned}\|v\|_{L^2(\Omega)} &= \left(\int_A |v|^2 \mathrm{d}x\right)^{\frac{2-p}{4}} \left(\int_A |v|^2 \mathrm{d}x\right)^{\frac{p}{4}} \\ &\leq \left(\int_A |v|^2 \mathrm{d}x\right)^{\frac{2-p}{4}} \left(\eta \|v\|_{W^{1,p}(\Omega)} + C_\eta \|v\|_{W^{-1,r}(\Omega)}\right)^{\frac{p}{2}}. \end{aligned} \tag{3.3}$$

Now, we are in the situation to prove our compactness theorem 1.3 in the sense of Aubin and Lions.

Proof of Theorem 1.3 Let $\{u_k\}$ be a bounded sequence in $W(\Omega_T)$. $W(\Omega_T)$ is reflexive. This is obvious, since $W^{p(\cdot)}(\Omega_T)$ and $W^{p(\cdot)}(\Omega_T)'$ are reflexive Banach spaces. Hence, there is a subsequence, which we denote again by $\{u_k\}$, such that $u_k \rightharpoonup u$ weakly in $W(\Omega_T)$. Therefore, we have $u_k \rightharpoonup u$ weakly in $W^{p(\cdot)}(\Omega_T)$ and $\partial_t u_k \rightharpoonup \partial_t u$ weakly in $W^{p(\cdot)}(\Omega_T)'$. We shall show that, up to a subsequence, $u_k \to u$ strongly in $L^{\hat{p}(\cdot)}(\Omega_T)$. For this aim, we first have to conclude the strong convergence in $L^{\hat{p}(\cdot)}$ on subdomains of Ω_T with some restriction on the diameter of these subdomains and then, we observe the strong convergence in $L^{\hat{p}(\cdot)}$ on Ω_T by a covering argument. This is necessary, since in the nonstandard case with $p(z)$-growth, there do not exist such global arguments respectively estimates as in the standard case. For this reason, we have to bound the maximal oscillation of the exponent $p(\cdot)$ and use this bound to restrict the diameter of the subdomains of Ω_T. Therefore, we fix $R_0 \in (0, 1]$, such that $\omega(R_0) \leq \frac{1}{4n}$ and we consider an open, bounded Lipschitz domain $A \subset \Omega$ and $0 \leq t_1 < t_2 \leq T$, such that $\mathrm{diam}(A) \leq R_0$ and $t_2 - t_1 \leq R_0^2$. Next, we set $p_1 := \inf_Q p(\cdot)$ and $p_2 := \sup_Q p(\cdot)$, where $Q := A \times (t_1, t_2)$. Then, we define

$$q := \begin{cases} \frac{np_1}{n-\frac{1}{2}p_1} & \text{if } p_1 < n, \\ 2p_2 & \text{if } p_1 \geq n. \end{cases} \tag{3.4}$$

In the following, we will show that in the case $p_1 < n$, there holds

$$p_2 < q < p_1^*. \tag{3.5}$$

The second inequality is obvious from the definition of q, while the first one follows from (1.4) and the choice of R_0:

$$p_2 - q = p_2 - \frac{np_1}{n - \frac{1}{2}p_1} = \frac{n(p_2 - p_1) - \frac{1}{2}p_1 p_2}{n - \frac{1}{2}p_1} \le \frac{n\omega(R_0) - \frac{1}{2}}{n - \frac{1}{2}p_1} \le \frac{\frac{1}{4} - \frac{1}{2}}{n - \frac{1}{2}p_1} < 0.$$

Next, we prove that in the case $p_2 > 2$, there holds

$$\frac{q(p_2 - 2)}{q - 2} \le p_1. \tag{3.6}$$

In the case $p_1 \ge n$, this follows immediately from (1.4) and the fact that $\omega(R_0) \le 1$, since

$$q\frac{(p_2 - 2)}{q - 2} - p_1 \le \frac{2\omega(R_0)p_2 - 4p_2 + 2p_1}{2p_2 - 2} \le \frac{2\omega(R_0)p_2 - 2p_2}{2p_2 - 2} \le 0.$$

In the case $p_1 < n$, we first observe that $q > 2$, which is a consequence of (1.4), the choice of R_0, the fact that $p_2 > 2$ and the following chain of inequalities:

$$\begin{aligned} 2 - q &= 2 - \frac{np_1}{n - \frac{1}{2}p_1} = \frac{2n - p_1 - np_1}{n - \frac{1}{2}p_1} \le \frac{2n - np_2 + n\omega(R_0) - p_1}{n - \frac{1}{2}p_1} \\ &\le \frac{n\omega(R_0) - p_1}{n - \frac{1}{2}p_1} \le \frac{\frac{1}{4} - p_1}{n - \frac{1}{2}p_1} < 0. \end{aligned}$$

Again from (1.4) and the choice of R_0, we obtain

$$\frac{q(p_2 - 2)}{q - 2} - p_1 \le \frac{np_1\omega(R_0) - p_1^2}{np_1 - 2n + p_1} \le \frac{\frac{p_1}{4} - p_1^2}{np_1 - 2n + p_1} < 0,$$

which proves (3.6) also in the case $p_1 < n$. Therefore, we consider the exponent q from (3.4), which satisfies (3.5) and $2 \le \hat{p}(\cdot) \le q$ on Q. Notice that we are allowed to assume that $2 \le \hat{p}(\cdot) \le q$ on Q, since $p_2 < q$ in the case $p_1 < n$ by (3.5) and $2p_2 = q$ in the case $p_1 \ge n$ by (3.4). Finally, we assume without loss of generality that $u = 0$.

First, we want to show that $u_k(\cdot, t) \to 0$ in $W^{-1,p_2'}(A)$ for each $t \in [t_1, t_2]$. This will be done for $t = t_1$, with any other case being similar. For $t \in [t_1, t_2]$ we have that

$$u_k(\cdot, t_1) = u_k(\cdot, t) - \int_{t_1}^{t} \partial_t u_k(\cdot, \tau)\, \mathrm{d}\tau.$$

Integration over $[t_1, s]$ with respect to t yields for any $s \in [t_1, t_2]$ the following

$$u_k(\cdot, t_1) = \frac{1}{s-t_1}\int_{t_1}^{s} u_k(\cdot, t)\mathrm{d}t - \frac{1}{s-t_1}\int_{t_1}^{s}\int_{t_1}^{t} \partial_t u_k(\cdot, \tau)\, d\tau \mathrm{d}t$$
$$= \frac{1}{s-t_1}\int_{t_1}^{s} u_k(\cdot, t)\mathrm{d}t - \frac{1}{s-t_1}\int_{t_1}^{s} (s-t)\partial_t u_k(\cdot, t)\mathrm{d}t = a_k + b_k.$$

Since $\partial_t u_k$ is bounded in $W^{p(\cdot)}(Q)'$ and $W^{p(\cdot)}(Q)' \subset L^1(t_1, t_2; W^{-1,p_2'}(A))$, for any $\varepsilon > 0$, we can choose s close to t_1, such that

$$\|b_k\|_{W^{-1,p_2'}(A)} \le \int_{t_1}^{s} \|\partial_t u_k\|_{W^{-1,p_2'}(A)} \mathrm{d}t \le \frac{\varepsilon}{2}.$$

Next, we conclude from $u_k \rightharpoonup 0$ weakly in $W^{p(\cdot)}(Q)$ that $a_k \rightharpoonup 0$ weakly in $W^{1,p_1}(A)$, as $k \to \infty$. Therefore, $a_k \to 0$ strongly in $L^{\mathfrak{q}}(A)$ for all $\mathfrak{q} < p_1^*$ in the case $p < n$ and $\mathfrak{q} < \infty$ in the case $n \le p$, as $k \to \infty$ by compactness. Since $2 < p_1^* = \frac{np_1}{n-p_1}$ for $p_1 < n$ by (3.5), we can conclude from the fact that $a_k \to 0$ strongly in $L^{\mathfrak{q}}(A)$, also $a_k \to 0$ strongly in $L^2(A)$, as $k \to \infty$. Next, we can infer from the continuous embedding $L^2(A) \hookrightarrow W^{-1,p_2'}(A)$ and the strong convergence of a_k in $L^2(A)$, that $a_k \to 0$ in $W^{-1,p_2'}(A)$, i.e. we have for sufficiently large k that $\|a_k\|_{W^{-1,p_2'}(A)} \le \frac{\varepsilon}{2}$. Finally, we have also shown that $u_k(\cdot, t_1) \to 0$ in $W^{-1,p_2'}(A)$, as $k \to \infty$. Second, observe that the continuity of

$$W(Q) \hookrightarrow C([t_1, t_2]; W^{-1,p_2'}(A))$$

shows that $\{\|u_k\|_{C([t_1,t_2];W^{-1,p_2'}(A))}\}$ is bounded, i.e. $\|u_k\|_{C([t_1,t_2];W^{-1,p_2'}(A))} \le c\|u_k\|_{W(Q)}$. Since u_k is bounded, we can obtain by the Dominated Convergence Theorem, see [24, Theorem 5, p. 648] respectively [29, Theorem 1.4, Chapter III], that

$$u_k \to 0 \text{ in } L^{p_1}(t_1, t_2; W^{-1,p_2'}(A)), \text{ as } k \to \infty.$$

Our next aim is to prove, that $u_k \to 0$ in $L^{\hat{p}(\cdot)}(Q)$, as $k \to \infty$. This will be a consequence of the fact, that $u_k \to 0$ in $L^{\max\{p_2,2\}}(Q)$. We start with the case $p_2 > 2$. Here, we recall the definition of q from (3.4). Due to (3.5), we are allowed to apply (3.2) with (p_1, p_2, p_2') instead of (p, s, r). This yields, for $\eta > 0$ that

$$\int_Q |u_k|^{p_2}\mathrm{d}z \le \int_{t_1}^{t_2} \left(\int_A |u_k(\cdot, t)|^2 \mathrm{d}x\right)^{\frac{q-p_2}{q-2}}$$
$$\times \left(\eta\|u_k(\cdot, t)\|_{W^{1,p_1}(A)} + C_\eta\|u_k(\cdot, t)\|_{W^{-1,p_2'}(A)}\right)^{q\frac{(p_2-2)}{q-2}} \mathrm{d}t$$
$$\le 2^{q-1}\int_{t_1}^{t_2}\Bigg(\eta^{q\frac{(p_2-2)}{q-2}}\|u_k(\cdot, t)\|_{W^{1,p_1}(A)}^{q\frac{(p_2-2)}{q-2}}$$

$$+ C_\eta^{q\frac{(p_2-2)}{q-2}} \|u_k(\cdot,t)\|_{W^{-1,p_2'}(A)}^{q\frac{(p_2-2)}{q-2}} \Bigg) \mathrm{d}t \times \sup_{t_1<t<t_2} \|u_k(\cdot,t)\|_{L^2(A)}^{2\frac{q-p_2}{q-2}}.$$

Now, we recall (3.6), which allows us to apply Hölder's inequality and thus obtain, that

$$\begin{aligned}
\int_{t_1}^{t_2} \|u_k(\cdot,t)\|_{W^{1,p_1}(A)}^{q\frac{(p_2-2)}{q-2}} \mathrm{d}t &\le c\left(\int_{t_1}^{t_2} \|u_k(\cdot,t)\|_{W^{1,p_1}(A)}^{p_1} \mathrm{d}t\right)^{\frac{q(p_2-2)}{p_1(q-2)}} \\
&\le c\left(\int_Q |u_k|^{p(\cdot)} + |Du_k|^{p(\cdot)} + 1\,\mathrm{d}z\right)^{\frac{q(p_2-2)}{p_1(q-2)}} \\
&\le c\left(\|u_k\|_{W^{p(\cdot)}(Q)}^{p_2\frac{q(p_2-2)}{p_1(q-2)}} + 1\right) \le c\left(\|u_k\|_{W^{p(\cdot)}(Q)}^{p_2} + 1\right)
\end{aligned}$$

with a constant $c = c(p_1, p_2, q)$, where we also used (1.9), (3.6) and that $t_2 - t_1 \le R_0^2 \le 1$ and $|A| \le \alpha_n R_0^n \le c$. Plugging this into the previous estimate, we gain

$$\begin{aligned}
\int_Q |u_k|^{p_2} \mathrm{d}z \le \sup_{t_1<t<t_2} \|u_k(\cdot,t)\|_{L^2(A)}^{2\frac{q-p_2}{q-2}} \Bigg[& c(p_1,p_2,q)\eta^{q\frac{p_2-2}{q-2}} \left(\|u_k\|_{W^{p(\cdot)}(Q)}^{p_2} + 1\right) \\
& + c(q) C_\eta^{q\frac{(p_2-2)}{q-2}} \int_{t_1}^{t_2} \|u_k(\cdot,t)\|_{W^{-1,p_2'}(A)}^{q\frac{(p_2-2)}{q-2}} \mathrm{d}t \Bigg].
\end{aligned}$$

Further, since u_k is bounded in $W^{p(\cdot)}(\Omega_T)$ and by Lemma 3.1, we can infer that

$$\int_Q |u_k|^{p_2}\,\mathrm{d}z \le c_1 \cdot \eta^{q\frac{p_2-2}{q-2}} + c_2 \int_{t_1}^{t_2} \|u_k(\cdot,t)\|_{W^{-1,p_2'}(A)}^{q\frac{(p_2-2)}{q-2}} \mathrm{d}t, \tag{3.7}$$

where $c_1 = c_1(n, \gamma_1, \gamma_2, \sup_{k\in\mathbb{N}} \|u_k\|_{L^\infty(t_1,t_2;L^2(A))}, \sup_{k\in\mathbb{N}} \|u_k\|_{W^{p(\cdot)}(Q)})$ and $c_2 = c_2(\eta, n, \gamma_1, \gamma_2, \sup_{k\in\mathbb{N}} \|u_k\|_{L^\infty(t_1,t_2;L^2(A))})$. Since, the dependencies on p_1, p_2 and q is continuous, it can be replaced by a dependencies on γ_1 and γ_2. Next, we consider the case $p_2 \le 2$. Here, we use (3.3) applied with (p, r) replaced by (p_1, p_2') to infer that

$$\begin{aligned}
\int_Q |u_k|^2 \mathrm{d}z \le & \int_{t_1}^{t_2} \left(\int_A |u_k(\cdot,t)|^2 \mathrm{d}x\right)^{1-\frac{p_1}{2}} \\
& \times \left(\eta \|u_k(\cdot,t)\|_{W^{1,p_1}(A)} + C_\eta \|u_k(\cdot,t)\|_{W^{-1,p_2'}(A)}\right)^{p_1} \mathrm{d}t \\
\le & \; 2^{p_1-1}\eta^{p_1} \sup_{t_1<t<t_2} \|u_k(\cdot,t)\|_{L^2(A)}^{1-\frac{p_1}{2}} \left(\|u_k\|_{W^{p(\cdot)}(Q)}^{p_1} + 1\right) \\
& + c C_\eta^{p_1} \sup_{t_1<t<t_2} \|u_k(\cdot,t)\|_{L^2(A)}^{1-\frac{p_1}{2}} \|u_k\|_{L^{p_1}(t_1,t_2;W^{-1,p_2'}(A))}^{p_1}
\end{aligned}$$

with a constant $c = c(p_1, p_2)$, where we utilized $\|u_k\|_{W^{1,p_1}(Q)} \le c\|u_k\|_{W^{p(\cdot)}(Q)}$ by the compact embedding $W^{p(\cdot)}(Q) \hookrightarrow W^{1,p_1}(Q)$. Here, we have also used that $t_2 - t_1 \le R_0^2 \le 1$ and $|A| \le \alpha_n R_0^n \le c$. Thus, since u_k is bounded in $W^{p(\cdot)}(\Omega_T)$ and by Lemma 3.1, we can infer that

$$\int_Q |u_k|^2 \, \mathrm{d}z \le c_1 \cdot \eta^{p_1} + c_2 \|u_k\|^{p_1}_{L^{p_1}(t_1,t_2;W^{-1,p_2'}(A))}, \tag{3.8}$$

where $c_1 = c_1(n, \gamma_1, \gamma_2, \sup_{k\in\mathbb{N}} \|u_k\|_{L^\infty(t_1,t_2;L^2(A))}, \sup_{k\in\mathbb{N}} \|u_k\|_{W^{p(\cdot)}(Q)})$ and the constant c_2 depends on $\eta, n, \gamma_1, \gamma_2, \sup_{k\in\mathbb{N}} \|u_k\|_{L^\infty(t_1,t_2;L^2(A))})$. Since, $u_k \to 0$ in $L^{p_1}(t_1, t_2; W^{-1,p_2'}(A))$ and the last term on the right-hand side of (3.7) and (3.8) converges to zero, so

$$\limsup_{k\to\infty} \int_Q |u_k|^{\max\{2,p_2\}} \, \mathrm{d}z \le c_1 \max\left\{\eta^{q\frac{p_2-2}{q-2}}, \eta^{p_1}\right\},$$

where we also used the fact $L^{p_1}(t_1, t_2; W^{-1p_2'}(A)) \hookrightarrow L^{q\frac{p_2-2}{q-2}}(t_1, t_2; W^{-1p_2'}(A))$ by (3.6). Since, $\eta > 0$ is chosen arbitrary, this upper limit is 0. This shows $u_k \to 0$ strongly in $L^{\max\{2,p_2\}}(Q)$, as $k \to \infty$ for every open, bounded Lipschitz domain $Q = A \times (t_1, t_2)$, provided the condition $\mathrm{diam}(A) \le R_0 = R_0(n, \omega(\cdot))$ and $t_2 - t_1 \le R_0^2$ holds. Moreover, the compact embedding of L^{p_2} into $L^{p(\cdot)}$ implies that $u_k \to 0$ strongly in $L^{\hat{p}(\cdot)}(Q)$, as $k \to \infty$ for every open, bounded Lipschitz domain $Q = A \times (t_1, t_2)$, provided the condition $\mathrm{diam}(A) \le R_0 = R_0(n, \omega(\cdot))$ and $t_2 - t_1 \le R_0^2$ holds. Since, we have shown the desired strong convergence in $L^{\hat{p}(\cdot)}$ on subdomains Q of Ω_T with the restriction on the diameter of these Q, our next goal is to deduce the strong convergence in $L^{\hat{p}(\cdot)}$ on Ω_T. This will be done by a covering argument. Therefore, we choose a family of dyadic cuboids $\{C_i\}_{i=1}^\infty$ such that $\bigcup_{i=1}^\infty C_i = \mathbb{R}^{n+1}$, where C_i denotes the cuboid

$$C_{\frac{R_0}{2},\left(\frac{R_0}{2}\right)^2}(z_i) := \left\{x \in \mathbb{R}^n, s \in \mathbb{R} \mid |x_i - x| < \frac{1}{2}R_0,\ |t_i - s| < \frac{1}{4}R_0^2,\ 1 \le i \le n\right\}$$

with center in $z_i = (x_i, t_i)$, side length $\frac{1}{2}R_0$ and height $\left(\frac{1}{2}R_0\right)^2$. These cuboids, we use to partition the $\mathbb{R}^{n+1}$ into dyadic cuboids C_i, where $1 \le i \le \infty$. Since, $Q \subseteq \Omega_T$ was arbitrary, we can consider $Q_i = \tilde{C}_i \cap \Omega_T \subseteq \Omega_T$, $i = 1, \ldots, M$, such that $\Omega_T = \bigcup_{i=1}^M Q_i$, where $\tilde{C}_i = C_{R_0,(R_0)^2}(z_i)$. Since, every cuboid in $\mathbb{R}^n$ has a Lipschitz boundary, C_i respectively $\tilde{C}_i$ are open, bounded domains and the intersection of two open, bounded domains yields again an open, bounded domain, it follows that Q_i are open, bounded Lipschitz domains. Therefore, we can conclude that $u_k \to u$ strongly in $L^{\hat{p}(\cdot)}(Q_i)$, as $k \to \infty$, for all $i \in \{1, \ldots, M\}$. Moreover, the covering of Ω_T yields that $u_k \to u$ strongly in $L^{\hat{p}(\cdot)}(\Omega_T)$ and thus, $W(\Omega_T) \hookrightarrow L^{\hat{p}(\cdot)}(\Omega_T)$ compact as desired. This completes the proof. ◻

4 Proof of Theorems 1.4 and 1.6

We start with the

Proof of Theorem 1.4 First of all, we construct a sequence of the Galerkin's approximations, where the limit of this sequence is equal to the solution in (1.15). Therefore, $\{\phi_i(x)\}_{i=1}^{\infty} \subset W_0^{1,\gamma_2}(\Omega)$ is an orthonormal basis in $L^2(\Omega)$. Now, we fix a positive integer m and define the approximate solution to (1.15) as follows

$$u^{(m)}(z) := \sum_{i=1}^{m} c_i^{(m)}(t)\phi_i(x),$$

where the coefficients $c_i^{(m)}(t)$ are defined via the identity

$$\int_\Omega \Big(u_t^{(m)}\phi_i(x) + \Big(a(x,t,Du^{(m)}) - |F|^{p(\cdot,t)-2}F\Big) D\phi_i(x) - f\phi_i(x)\Big)\,\mathrm{d}x = 0, \tag{4.1}$$

for $i = 0, \dots, m$ and $t \in (0,T)$ with the initial condition

$$c_i^{(m)}(0) = \int_\Omega g(\cdot,0)\phi_i \,\mathrm{d}x, \quad i = 1,\dots,m.$$

Then, the equation (4.1) together with the initial boundary condition generates a system of m ordinary differential equations:

$$\begin{cases} \left(c_i^{(m)}\right)'(t) = F_i\left(t, c_1^{(m)}(t), \dots, c_m^{(m)}(t)\right), \\ c_i^{(m)}(0) = \int_\Omega g(\cdot,0)\phi_i \,\mathrm{d}x, \quad i = 1,\dots,m, \end{cases} \tag{4.2}$$

where we abbreviated

$$F_i(t,\cdot) := -\int_\Omega (a(\cdot,t,Du^{(m)}) - |F(\cdot,t)|^{p(\cdot,t)-2}F(\cdot,t))D\phi_i(x) - f(\cdot,t)\phi_i(x)\mathrm{d}x,$$

since $\{\phi_i(x)\}$ is orthonormal in $L^2(\Omega)$. From this starting point, we will conclude the existence result to the Dirichlet problem (1.15). By [26, Theorem 1.44, p. 25]—see also [26, p. 240 ff.]—we know that, there is for every finite system (4.2) a solution $c_i^{(m)}(t)$, $i = 1,\dots,m$ on the interval $(0,T_m)$ for some $T_m > 0$. First, we multiply the equation (4.1) by the coefficients $c_i^{(m)}(t)$, $i = 1,\dots,m$. Then, we need a priori estimates that permit us to extend the solution to the whole domain $(0,T_m)$. Therefore, we integrate the equation over $(0,\tau)$ for an arbitrarily $\tau \in (0,T_m)$. Next, we sum the resulting equation over $i = 1,\dots,m$. Therefore, it follows

$$\int_{\Omega_\tau} \partial_t u^{(m)} \cdot u^{(m)} + \Big(a(z,Du^{(m)}) - |F|^{p(\cdot)-2}F\Big)\cdot Du^{(m)} - fu^{(m)}\mathrm{d}z = 0 \tag{4.3}$$

for a.e. $\tau \in (0, T_m)$. Next, we convert the first term on the left-hand side of (4.3) as follows

$$\begin{aligned}\int_{\Omega_\tau} \partial_t u^{(m)} \cdot u^{(m)} \mathrm{d}z &= \frac{1}{2} \int_{\Omega_\tau} \partial_t [u^{(m)}]^2 \mathrm{d}z \\ &= \frac{1}{2} \int_{\Omega} |u^{(m)}(\cdot, \tau)|^2 \mathrm{d}x - \frac{1}{2} \int_{\Omega} |u^{(m)}(\cdot, 0)|^2 \mathrm{d}x \\ &\geq \frac{1}{2} \int_{\Omega} |u^{(m)}(\cdot, \tau)|^2 \mathrm{d}x - \frac{1}{2} \int_{\Omega} |g(\cdot, 0)|^2 \mathrm{d}x\end{aligned}$$

for a.e. $\tau \in (0, T_m)$, since $g(\cdot, 0) \in L^2(\Omega)$, $\{\phi_i\}_{i=1}^{\infty} \subset L^2(\Omega)$ and

$$\begin{aligned}\int_{\Omega} |u^{(m)}(\cdot, 0)|^2 \mathrm{d}x &= \int_{\Omega} \Big| \sum_{i=1}^{m} c_i^{(m)}(0)\phi_i(x) \Big|^2 \mathrm{d}x \\ &= \int_{\Omega} \Big| \sum_{i=1}^{m} \int_{\Omega} g(\cdot, 0)\phi_i(x)\mathrm{d}x\phi_i(x) \Big|^2 \mathrm{d}x \\ &\leq \int_{\Omega} \Big| \sum_{i=1}^{\infty} \int_{\Omega} g(\cdot, 0)\phi_i(x)dx\phi_i(x) \Big|^2 \mathrm{d}x = \int_{\Omega} |g(\cdot, 0)|^2 \mathrm{d}x,\end{aligned}$$

cf. [19]. Therefore, it holds

$$\int_{\Omega_\tau} \partial_t u^{(m)} \cdot u^{(m)} \mathrm{d}z \geq \frac{1}{2} \int_{\Omega} |u^{(m)}(\cdot, \tau)|^2 \mathrm{d}x - \frac{1}{2} \|g(\cdot, 0)\|_{L^2(\Omega)}^2$$

for a.e. $\tau \in (0, T_m)$. Now, we have from (4.3) that

$$\begin{aligned}&\frac{1}{2} \int_{\Omega} |u^{(m)}(\cdot, \tau)|^2 \mathrm{d}x + \int_{\Omega_\tau} a(z, Du^{(m)}) \cdot Du^{(m)} \mathrm{d}z \\ &\quad \leq \frac{1}{2} \|g(\cdot, 0)\|_{L^2(\Omega)}^2 + \int_{\Omega_\tau} |F|^{p(\cdot)-2} F \cdot Du^{(m)} + f u^{(m)} \mathrm{d}z \qquad (4.4)\end{aligned}$$

for a.e. $\tau \in (0, T_m)$. Using the coercivity condition (1.7) on left-hand side of (4.4) and estimating the right-hand side of (4.4) by the absolute value and (1.14), then we get the following estimate

$$\begin{aligned}&\frac{1}{2} \int_{\Omega} |u^{(m)}(\cdot, \tau)|^2 \mathrm{d}x + \frac{\nu}{c(\gamma_1, \gamma_2)} \int_{\Omega_\tau} |Du^{(m)}|^{p(\cdot)} \mathrm{d}z - c \int_{\Omega_\tau} (1 + |v|^{p(\cdot)}) \mathrm{d}z \\ &\quad \leq \frac{1}{2} \|g(\cdot, 0)\|_{L^2(\Omega)}^2 + \int_{\Omega_\tau} |F|^{p(\cdot)-1} |Du^{(m)}| \mathrm{d}z + c \|f\|_{W^{\gamma_1'}(\Omega_\tau)} \|u^{(m)}\|_{W^{\gamma_1}(\Omega_\tau)} \qquad (4.5)\end{aligned}$$

where $c = c(\gamma_1, \gamma_2, \nu, L)$. Now, we adopt Young's inequality to the second term on the right-hand side of (4.5) with the exponents $p(\cdot)$ and $\frac{p(\cdot)}{p(\cdot)-1}$, where we use the factors $\varepsilon^{-\frac{1}{p(\cdot)}} \cdot \varepsilon^{\frac{1}{p(\cdot)}} = 1$ and $\varepsilon^{-\frac{1}{\gamma_1}} \cdot \varepsilon^{\frac{1}{\gamma_1}} = 1$, $\varepsilon \in (0,1)$ and get the following estimate

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u^{(m)}(\cdot,\tau)|^2 \mathrm{d}x + \frac{\nu}{c(\gamma_1,\gamma_2)}\int_{\Omega_\tau} |Du^{(m)}|^{p(\cdot)}\mathrm{d}z - c\int_{\Omega_\tau} 1 + |v|^{p(\cdot)}\mathrm{d}z \\
&\quad \le \frac{1}{2}\|g(\cdot,0)\|^2_{L^2(\Omega)} + \int_{\Omega_\tau} \frac{p(\cdot)-1}{p(\cdot)}\varepsilon^{\frac{1}{1-p(\cdot)}}|F|^{p(\cdot)} + \frac{1}{p(\cdot)}\varepsilon|Du^{(m)}|^{p(\cdot)}\mathrm{d}z \\
&\qquad + \varepsilon^{-\frac{1}{\gamma_1}}\|f\|_{L^{\gamma_1'}(0,\tau;W^{-1,\gamma_1'}(\Omega))}\varepsilon^{\frac{1}{\gamma_1}}\|u^{(m)}\|_{L^{\gamma_1}(0,\tau;W^{1,\gamma_1}(\Omega))} \\
&\quad \le \frac{1}{2}\|g(\cdot,0)\|^2_{L^2(\Omega)} + \frac{\gamma_2-1}{\gamma_1}\varepsilon^{\frac{1}{1-\gamma_1}}\left(\int_{\Omega_\tau}|F|^{p(\cdot)}\mathrm{d}z + \|f\|^{\gamma_1'}_{L^{\gamma_1'}(0,\tau;W^{-1,\gamma_1'}(\Omega))}\right) \\
&\qquad + \frac{1}{\gamma_1}\varepsilon\left(\int_{\Omega_\tau}|Du^{(m)}|^{p(\cdot)}\mathrm{d}z + \int_{\Omega_\tau}|u^{(m)}|^{\gamma_1} + |Du^{(m)}|^{\gamma_1}\mathrm{d}z\right) \qquad (4.6)
\end{aligned}$$

for a.e. $\tau \in (0,T_m)$ with a constant $c = c(\gamma_1,\gamma_2,\nu,L)$, where we used Young's estimate with exponents $\frac{1}{\gamma_1} + \frac{\gamma_1-1}{\gamma_1} = 1$ for the last estimate. Next, we apply the standard Poincaré inequality slicewise to get the following estimate

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |u^{(m)}(\cdot,\tau)|^2 \mathrm{d}x + \frac{\nu}{c(\gamma_1,\gamma_2)}\int_{\Omega_\tau} |Du^{(m)}|^{p(\cdot)}\mathrm{d}z - c\int_{\Omega_\tau} 1 + |v|^{p(\cdot)}\mathrm{d}z \\
&\quad \le \frac{1}{2}\|g(\cdot,0)\|^2_{L^2(\Omega)} + \frac{\gamma_2-1}{\gamma_1}\varepsilon^{\frac{1}{1-\gamma_1}}\left(\int_{\Omega_\tau}|F|^{p(\cdot)}\mathrm{d}z + \|f\|^{\gamma_1'}_{L^{\gamma_1'}(0,\tau;W^{-1,\gamma_1'}(\Omega))}\right) \\
&\qquad + \left(\frac{1}{\gamma_1} + c_*\right)\varepsilon\left(\int_{\Omega_\tau}|Du^{(m)}|^{p(\cdot)} + 1\mathrm{d}z\right)
\end{aligned}$$

with a constant $c = c(\gamma_1,\gamma_2,\nu,L)$ and the constant c_*, which depends on n,γ_1,γ_2 and $\mathrm{diam}(\Omega)$, where we used

$$\int_{\Omega_\tau}|Du^{(m)}|^{\gamma_1}\mathrm{d}z \le c(\gamma_2)\int_{\Omega_\tau}|Du^{(m)}|^{p(\cdot)} + 1\mathrm{d}z.$$

This inequality can be converted as follows

$$\begin{aligned}
&\int_\Omega |u^{(m)}(\cdot,\tau)|^2\mathrm{d}x + 2\left(\frac{\nu}{c} - \left(\frac{1}{\gamma_1}+c_*\right)\varepsilon\right)\int_{\Omega_\tau}|Du^{(m)}|^{p(\cdot)}\mathrm{d}z - c\int_{\Omega_\tau}1+|v|^{p(\cdot)}\mathrm{d}z \\
&\quad \le \|g(\cdot,0)\|^2_{L^2(\Omega)} + 2\frac{\gamma_2-1}{\gamma_1}\varepsilon^{\frac{1}{1-\gamma_1}}\left(\int_{\Omega_\tau}|F|^{p(\cdot)}\mathrm{d}z + \|f\|^{\gamma_1'}_{L^{\gamma_1'}(0,\tau;W^{-1,\gamma_1'}(\Omega))}\right) + c_*\varepsilon,
\end{aligned}$$

where $c = c(\gamma_1,\gamma_2,\nu,L)$ and $c_* = c(n,\gamma_1,\gamma_2,\mathrm{diam}(\Omega))$. Next, we choose ε, such that

$$2\left(\frac{\nu}{c(\gamma_1,\gamma_2)} - \left(\frac{1}{\gamma_1}+c_*\right)\varepsilon\right) \ge \frac{\nu}{c(\gamma_1,\gamma_2)}.$$

Hence, we have:

$$\int_{\Omega} |u^{(m)}(\cdot,\tau)|^2 \mathrm{d}x + \int_{\Omega_\tau} |Du^{(m)}|^{p(\cdot)} \mathrm{d}z$$
$$\leq c\|g(\cdot,0)\|^2_{L^2(\Omega)} + c\left(\int_{\Omega_\tau} |F|^{p(\cdot)} + |v|^{p(\cdot)} + 1 \mathrm{d}z \|f\|^{\gamma_1'}_{L^{\gamma_1'} - W^{-1,\gamma_1'}} + 1\right),$$

where $c = c(n, \gamma_1, \gamma_2, \nu, L, \mathrm{diam}(\Omega))$. This estimate holds for a.e. $\tau \in (0, T_m)$. Therefore, we have that $u^{(m)}$ is uniformly bounded in $L^\infty(0, T_m; L^2(\Omega))$ and $Du^{(m)}$ is uniformly bounded in $L^{p(\cdot)}(\Omega_{T_m})$. Now, we can estimate the right-hand side from above by the bound $\mathcal{M}_0$ introduced in (1.18). This yields

$$\sup_{0\leq t\leq T_m} \int_{\Omega} |u^{(m)}(\cdot,\tau)|^2 \mathrm{d}x + \int_{\Omega_{T_m}} |Du^{(m)}|^{p(\cdot)} \mathrm{d}z \leq c\|g(\cdot,0)\|^2_{L^2(\Omega)} + c\cdot\mathcal{M}_0, \quad (4.7)$$

where $c = c(n, \gamma_1, \gamma_2, \nu, L, \mathrm{diam}(\Omega))$. Next, we apply the Poincaré type inequality (2.4) to get an uniform $L^{p(\cdot)}$-bound for $u^{(m)}$ in the following way:

$$\|u^{(m)}\|^{\gamma_1}_{L^{p(\cdot)}(\Omega_{T_m})} \leq c\left(\|u^{(m)}\|^{\frac{4\gamma_2}{n+2}}_{L^\infty(0,T_m;L^2(\Omega))} + 1\right)\left(\int_{\Omega_{T_m}} |Du^{(m)}|^{p(\cdot)} + 1 \mathrm{d}z\right)$$

where $c = c(n, \gamma_1, \gamma_2, \mathrm{diam}(\Omega), \omega(\cdot))$. Using (4.7) then we have the following estimate

$$\|u^{(m)}\|_{L^{p(\cdot)}(\Omega_{T_m})} \leq c\left[\left(\|u^{(m)}\|^{\frac{4\gamma_2}{n+2}}_{L^\infty(0,T_m;L^2(\Omega))} + 1\right)\left(\|g(\cdot,0)\|^2_{L^2(\Omega)} + \mathcal{M}_0\right)\right]^{\frac{1}{\gamma_1}}$$

where $c = c(n, \gamma_1, \gamma_2, \nu, L, \mathrm{diam}(\Omega), \omega(\cdot))$. Applying again (4.7), we get the following $L^{p(\cdot)}$-bound for $u^{(m)}$:

$$\|u^{(m)}\|_{L^{p(\cdot)}(\Omega_{T_m})} \leq c\left(\|g(\cdot,0)\|^2_{L^2(\Omega)} + \mathcal{M}_0\right)^{\left(\frac{2\gamma_2}{n+2}+1\right)\frac{1}{\gamma_1}}$$

with a constant $c = c(n, \gamma_1, \gamma_2, \nu, L, \mathrm{diam}(\Omega), \omega(\cdot))$, where we finally used Young's inequality. Therefore, we have shown that $u^{(m)}$ is uniformly bounded in $W^{p(\cdot)}(\Omega_{T_m})$ and $L^\infty(0, T_m; L^2(\Omega))$ independently of m. Thus, the solution of system (4.2) can be continued to the maximal interval $(0, T)$.

Next, we want to derive an uniform bound for $\partial_t u^{(m)}$ in $W^{p(\cdot)}(\Omega_T)'$. Therefore, we define a subspace of the set of admissible test functions

$$\mathcal{W}_m(\Omega_T) := \left\{\eta : \eta = \sum_{i=1}^{m} d_i\phi_i,\ d_i \in C^1([0, T])\right\} \subset W_0^{p(\cdot)}(\Omega_T).$$

Then, we choose a test function

$$\varphi(z)=\sum_{i=1}^{m} d_i(t)\phi_i(x)\in \mathcal{W}_m(\Omega_T) \quad \text{with } d_i(0)=d_i(T)=0.$$

Note that $\partial_t\varphi$ exists, since the coefficients $d_i(t)$ lie in $C^1([0,T])$. Moreover, we know that $C^1([0,T],W_0^{1,\gamma_2}(\Omega_T))\subset W_0^{p(\cdot)}(\Omega_T)$ and therefore, we have also $\varphi\in W_0^{p(\cdot)}(\Omega_T)$. Thus, we can conclude by the definition of $u^{(m)}$ and (4.1) that

$$\begin{aligned}-\int_{\Omega_T} u^{(m)}\varphi_t \mathrm{d}z=\int_{\Omega_T} u_t^{(m)}\varphi \mathrm{d}z=-\int_{\Omega_T}&[a(z,Du^{(m)})\\ &-|F|^{p(\cdot)-2}F]\cdot D\varphi \mathrm{d}z+\langle\langle f,\varphi\rangle\rangle_{\Omega_T}.\end{aligned}$$

Then, we derive from the growth condition (1.16), the generalized Hölder's inequality (1.8), (1.14) and the fact that $L^{\gamma_1'}(0,T;W^{-1,\gamma_1'}(\Omega))\subset W^{p(\cdot)}(\Omega_T)'$ implies $f\in W^{p(\cdot)}(\Omega_T)'$, the following

$$\begin{aligned}\left|\int_{\Omega_T} u^{(m)}\varphi_t \mathrm{d}z\right|&\le\int_{\Omega_T}\left(|a(z,Du^{(m)})|+|F|^{p(\cdot)-1}\right)\cdot|D\varphi|\mathrm{d}z+\langle\langle f,\varphi\rangle\rangle_{\Omega_T}\\ &\le\int_{\Omega_T}\left(|a(z,Du^{(m)})|+|F|^{p(\cdot)-1}\right)\cdot(|D\varphi|+|\varphi|)\,\mathrm{d}z\\ &\quad+c(\gamma_1,\gamma_2)\|f\|_{W^{p(\cdot)}(\Omega_T)'}\|\varphi\|_{W^{p(\cdot)}(\Omega_T)}\\ &\le c\,\Big[\|(1+|Du^{(m)}|^{p(\cdot)-1}\\ &\quad+|v|^{p(\cdot)-1}+|F|^{p(\cdot)-1})\|_{L^{p'(\cdot)}(\Omega_T)}+\|f\|_{W^{p(\cdot)}(\Omega_T)'}\Big]\\ &\quad\times\left[\|\varphi\|_{W^{p(\cdot)}(\Omega_T)}\right],\end{aligned}$$

where $c=c(\gamma_1,\gamma_2,L)$. Next, we consider the term

$$\|(1+|Du^{(m)}|^{p(\cdot)-1}+|v|^{p(\cdot)-1}+|F|^{p(\cdot)-1})\|_{L^{p'(\cdot)}(\Omega_T)}$$

and use (1.9) to get the following bound

$$\begin{aligned}&\|(1+|Du^{(m)}|^{p(\cdot)-1}+|v|^{p(\cdot)-1}+|F|^{p(\cdot)-1})\|_{L^{p'(\cdot)}(\Omega_T)}\\ &\quad\le\left(\int_{\Omega_T}|(1+|Du^{(m)}|^{p(\cdot)-1}+|v|^{p(\cdot)-1}+|F|^{p(\cdot)-1})|^{p'(\cdot)}\mathrm{d}z+1\right)^{\frac{1}{\gamma_1}}\\ &\quad\le\left(\int_{\Omega_T}(1+|Du^{(m)}|^{p(\cdot)}+|v|^{p(\cdot)}+|F|^{p(\cdot)})\mathrm{d}z+1\right)^{\frac{1}{\gamma_1}}\\ &\quad\le c(\gamma_1,\gamma_2)(\mathcal{M}_0)^{\frac{1}{\gamma_1}}\le c(\gamma_1,\gamma_2,\mathcal{M}_0).\end{aligned}$$

Summarized, we have for every $\varphi \in \mathcal{W}_m(\Omega_T) \subset W_0^{p(\cdot)}(\Omega_T)$ and an arbitrary m that

$$\left|\int_{\Omega_T} u^{(m)}\varphi_t \mathrm{d}z\right| \leq c\|\varphi\|_{W^{p(\cdot)}(\Omega_T)}$$

with a constant $c = c(\gamma_1, \gamma_2, L, \|f\|_{W^{p(\cdot)}(\Omega_T)'}, \mathcal{M}_0)$, where c is independent of m. This shows that $u_t^{(m)} \in W^{p(\cdot)}(\Omega_T)'$ with the estimate

$$\|u_t^{(m)}\|_{W^{p(\cdot)}(\Omega_T)'} \leq c(\gamma_1, \gamma_2, L, \|f\|_{W^{p(\cdot)}(\Omega_T)'}, \mathcal{M}_0).$$

Therefore, we have an uniform bound of $u_t^{(m)}$ in $W^{p(\cdot)}(\Omega_T)'$ and it follows that

$$\begin{cases} u^{(m)} \in W_0^{p(\cdot)}(\Omega_T) \subseteq L^{\gamma_1}(0, T; W_0^{1,\gamma_1}(\Omega)) \\ u_t^{(m)} \in W^{p(\cdot)}(\Omega_T)' \subseteq L^{\gamma_2'}(0, T; W^{-1,\gamma_2'}(\Omega)) \end{cases}$$

are bounded. This imply the following weak convergences for the sequence $\{u^{(m)}\}$ (up to a subsequence):

$$\begin{cases} u^{(m)} \rightharpoonup^* u \text{ weakly* in } L^\infty(0, T; L^2(\Omega)), \\ Du^{(m)} \rightharpoonup Du \text{ weakly in } L^{p(\cdot)}(\Omega_T, \mathbb{R}^n), \\ u_t^{(m)} \rightharpoonup u_t \text{ weakly in } W^{p(\cdot)}(\Omega_T)'. \end{cases}$$

Moreover, by Theorem 1.3 we can conclude that the sequence $\{u^{(m)}\}$ (up to a subsequence) converges strongly in $L^{\hat{p}(\cdot)}(\Omega_T)$ with $\hat{p}(\cdot) := \max\{2, p(\cdot)\}$ to some function $u \in W(\Omega_T)$. Thus, we get the desired convergences

$$\begin{cases} u^{(m)} \to u \text{ strongly in } L^{\hat{p}(\cdot)}(\Omega_T) \\ u^{(m)} \to u \text{ a.e. in } \Omega_T \end{cases}$$

for the sequence $\{u^{(m)}\}$ (up to a subsequence). Further, the growth assumption of $a(z, \cdot)$ and the energy estimate (4.7) imply that the sequence $\{a(z, Du^{(m)})\}_{m\in\mathbb{N}}$ is bounded in $L^{p'(\cdot)}(\Omega_T, \mathbb{R}^n)$. Consequently, after passing to a subsequence once more, we can find a limit map $A_0 \in L^{p'(\cdot)}(\Omega_T, \mathbb{R}^n)$ with

$$a(z, Du^{(m)}) \to A_0 \quad \text{as } m \to \infty. \tag{4.8}$$

Our next aim is to show that $A_0 = a(z, Du)$ for almost every $z \in \Omega_T$. First of all, we should mention that each of $u^{(m)}$ satisfies the identity (4.1) with a test function $\varphi \in \mathcal{W}_m(\Omega_T)$. This follows by the method of construction, see [7]. Then, we fix an arbitrary $m \in \mathbb{N}$. Thus, we have for every $s \leq m$

$$-\int_{\Omega_T} u_t^{(m)}\varphi + [a(z, Du^{(m)}) + |F|^{p(\cdot)-2}F]D\varphi - f\varphi \mathrm{d}z = 0$$

for all test functions $\varphi \in \mathcal{W}_s(\Omega_T)$. Passing to the limit $m \to \infty$, we can conclude that for all test functions $\varphi \in \mathcal{W}_s(\Omega_T)$

$$-\int_{\Omega_T} u_t\varphi + [A_0 + |F|^{p(\cdot)-2}F]D\varphi - f\varphi \mathrm{d}z = 0 \tag{4.9}$$

with an arbitrary $s \in \mathbb{N}$, by the convergence from above. Therefore, it follows that the identity (4.9) holds for every $\varphi \in W_0^{p(\cdot)}(\Omega_T)$. According to monotonicity assumption (1.3), we know that for every $w \in \mathcal{W}_s(\Omega_T)$, $s \leq m$

$$\int_{\Omega_T} [a(z, Du^{(m)}) - a(z, Dw)]D(u^{(m)} - w)\mathrm{d}z \geq 0. \tag{4.10}$$

Moreover, it follows from (4.1), the conclusion from above and a test function $\varphi = u^{(m)} - w$ with $w \in \mathcal{W}_s(\Omega_T)$ that

$$-\int_{\Omega_T} u_t^{(m)}\varphi + [a(z, Du^{(m)}) + |F|^{p(\cdot)-2}F]D\varphi - f\varphi \mathrm{d}z = 0. \tag{4.11}$$

Adding (4.10) and (4.11), we have

$$\begin{aligned}&-\int_{\Omega_T} u_t^{(m)}\varphi + [a(z, Du^{(m)}) + |F|^{p(\cdot)-2}F]D\varphi - f\varphi \mathrm{d}z\\&\quad+\int_{\Omega_T} [a(z, Du^{(m)}) - a(z, Dw)]D\varphi \mathrm{d}z \geq 0\end{aligned}$$

with a test function $\varphi = u^{(m)} - w$. This yields

$$-\int_{\Omega_T} u_t^{(m)}\varphi + [a(z, Dw) + |F|^{p(\cdot)-2}F]D\varphi - f\varphi \mathrm{d}z \geq 0.$$

Then, we test (4.9) with $\varphi = u^{(m)} - w$ and subtract the resulting equation from the last estimate. Passing to the limit $m \to \infty$, we arrive at

$$-\int_{\Omega_T} [A_0 - a(z, Dw)]D(u - w)\mathrm{d}z \geq 0$$

for all $w \in \mathcal{W}_s(\Omega_T)$. Since, $\mathcal{W}_s(\Omega_T) \subset W_0^{p(\cdot)}(\Omega_T)$ is dense, we are allowed to choose $w \in W_0^{p(\cdot)}(\Omega_T)$. Hence, we choose $w = u \pm \varepsilon\zeta$ with an arbitrary $\zeta \in W_0^{p(\cdot)}(\Omega_T)$. This yields

$$-\varepsilon \int_{\Omega_T} [A_0 - a(z, D(u \pm \varepsilon\zeta))]D\zeta \mathrm{d}z \geq 0.$$

Finally, passing to the limit $\varepsilon \downarrow 0$, we can conclude that

$$\int_{\Omega_T} [A_0 - a(z, Du)]D\zeta \,\mathrm{d}z = 0$$

for all $\zeta \in W_0^{p(\cdot)}(\Omega_T)$. This shows that

$$A_0 = a(z, Du) \quad \text{for almost every } z \in \Omega_T.$$

Moreover we have to show, that $u(\cdot, 0) = g(\cdot, 0)$. First of all, we should mention that we get from (4.9) and integration by parts the following

$$\int_{\Omega_T} u\varphi_t - [a(z, Du) + |F|^{p(\cdot)-2}F]D\varphi + f\varphi \mathrm{d}z = \int_{\Omega} (u \cdot \varphi)(\cdot, 0)\mathrm{d}x$$

for all $\varphi \in W_0^{p(\cdot)}(\Omega_T)$ with $\varphi(\cdot, T) = 0$. Moreover, we can conclude (similar to (4.11)) that

$$\int_{\Omega_T} u^{(m)}\varphi_t - [a(z, Du^{(m)}) + |F|^{p(\cdot)-2}F]D\varphi + f\varphi \mathrm{d}z = \int_{\Omega} (u^{(m)} \cdot \varphi)(\cdot, 0)\mathrm{d}x$$

for all $\varphi \in W_0^{p(\cdot)}(\Omega_T)$ with $\varphi(\cdot, T) = 0$. Passing to the limit $m \to \infty$ and using the convergences from above we get

$$\int_{\Omega_T} u\varphi_t - [a(z, Du) + |F|^{p(\cdot)-2}F]D\varphi + f\varphi \mathrm{d}z = \int_{\Omega} g(\cdot, 0) \cdot \varphi(\cdot, 0)\mathrm{d}x,$$

where $u^{(m)}(\cdot, 0) \to g(\cdot, 0)$ as $m \to \infty$, since

$$\begin{aligned} u^{(m)}(\cdot, 0) &= \sum_{i=1}^{m} c_i^{(m)}(0)\phi_i(x) \\ &= \sum_{i=1}^{m} \int_{\Omega} g(\cdot, 0)\phi_i(x)\mathrm{d}x\phi_i(x) \to \sum_{i=1}^{\infty} \int_{\Omega} g(\cdot, 0)\phi_i(x)\mathrm{d}x\phi_i(x) = g(\cdot, 0) \end{aligned}$$

as $m \to \infty$. Furthermore, $\varphi(\cdot, 0)$ is arbitrary. Therefore, we can conclude that $u(\cdot, 0) = g(\cdot, 0)$.

Finally, we show the uniqueness of the weak solution. Therefore, we assume that there exist two weak solution u and $u_* \in C^0([0, T]; L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T)$ with $\partial_t u, \partial_t u_* \in W^{p(\cdot)}(\Omega_T)'$ of the Dirichlet Problem (1.15). Thus, we have the following weak formulations

$$\int_{\Omega_T} [u \cdot \varphi_t - a(z, Du) \cdot D\varphi] \,\mathrm{d}z = -\int_{\Omega_T} \left[f \cdot \varphi + |F|^{p(\cdot)-2}F \cdot D\varphi\right] \mathrm{d}z$$

and

$$\int_{\Omega_T} [u_* \cdot \varphi_t - a(z, Du_*) \cdot D\varphi]\,\mathrm{d}z = -\int_{\Omega_T} \left[f \cdot \varphi + |F|^{p(\cdot)-2} F \cdot D\varphi \right] \mathrm{d}z$$

with the admissible test function $\varphi = u - u_* \in W_0^{p(\cdot)}(\Omega_T)$, since $W_0^{p(\cdot)}(\Omega_T)'$ is the dual of $W_0^{p(\cdot)}(\Omega_T)$. Hence, we can conclude that

$$\int_{\Omega_T} [(u - u_*) \cdot (u - u_*)_t - (a(z, Du) - a(z, Du_*)) \cdot D(u - u_*)]\,\mathrm{d}z = 0.$$

Using the monotonicity condition (1.3), we derive at

$$0 \geq \int_{\Omega_T} (u - u_*) \cdot (u - u_*)_t \mathrm{d}z = \frac{1}{2} \int_{\Omega_T} \partial_t (u - u_*)^2 \mathrm{d}z.$$

Finally, we have that $0 \geq \frac{1}{2} \|u(t) - u_*(t)\|^2_{L^2(\Omega)} \geq 0$ for every $t \in (0, T]$, since $u(\cdot, 0) = u_*(\cdot, 0) = g(\cdot, 0)$. This shows the conclusion of the Theorem. □

The proof of Theorem 1.6 is very short and the conlusion of Theorem 1.6 derives immediately from Theorem 1.4 as follows.

Proof of Theorem 1.6 First, we define a modified vector-field $\tilde{a}(z, w) := a(z, w + Dg(z))$ for all $z \in \Omega_T$ and $w \in \mathbb{R}^n$. Moreover, we let $v \in L^\infty(0, T; L^2(\Omega)) \cap W_0^{p(\cdot)}(\Omega_T)$ be a solution to the following initial value problem

$$\begin{cases} \partial_t v - \operatorname{div}\, \tilde{a}(z, Dv) &= f - \operatorname{div}\, (|F|^{p(\cdot)-2} F) - \partial_t g \quad \text{in } \Omega_T \\ v &= 0 \quad \text{on } \partial\Omega \times (0, T) \\ v &= g(\cdot, 0) - g \quad \text{on } \Omega \times \{0\}\,. \end{cases} \tag{4.12}$$

The existence of the solution is guaranteed by Lemma 1.4, since we have $f - \partial_t g \in L^{\gamma_1'}(0, T; W^{-1,\gamma_1'}(\Omega))$. It is easy to show, that $u = v + g$ is the desired solution to the boundary value problem (1.20). From the energy estimate (1.19) with u replaced by v, we get the following energy estimate

$$\sup_{0 \leq t \leq T} \int_\Omega |v(\cdot, t)|^2 \mathrm{d}x + \int_{\Omega_T} |Dv|^{p(\cdot)} \mathrm{d}z \leq c \|v(\cdot, 0)\|_{L^2(\Omega)} + c \cdot \mathcal{M}_0,$$

where $\mathcal{M}_0$ is introduced in (1.18) with v replaced by g and f replaced by $f - \partial_t g$. Using the fact that $v = u - g$, we get the energy estimate (1.22). Therefore, we can conclude that

$$\sup_{0\le t\le T}\int_\Omega |u(\cdot,t)|^2\mathrm{d}x - 2\sup_{0\le t\le T}\int_\Omega |g(\cdot,t)|^2\mathrm{d}x + \int_{\Omega_T} |Du|^{p(\cdot)} - c(\gamma_2)|Dg|^{p(\cdot)}\mathrm{d}z$$
$$\le c(\gamma_2)\left[\sup_{0\le t\le T}\int_\Omega |v(\cdot,t)|^2\mathrm{d}x + \int_{\Omega_T}|Dv|^{p(\cdot)}\mathrm{d}z\right]$$
$$\le c\|g(\cdot,0)\|^2_{L^2(\Omega)} + c\cdot\mathcal{M}_0,$$

since $|Du|^{p(\cdot)} \le 2^{\gamma_2-1}\left[|D(u-g)|^{p(\cdot)} + |Dg|^{p(\cdot)}\right]$ and therefore, $|Du|^{p(\cdot)} - 2^{\gamma_2-1}|Dg|^{p(\cdot)} \le 2^{\gamma_2-1}|D(u-g)|^{p(\cdot)}$ and $|u|^2 - 2|g|^2 \le 2|u-g|^2$. This yields the energy estimate (1.22). Finally, we show the uniqueness of the weak solution. Therefore, we assume that there exist two weak solution u and $u_* \in C^0([0,T];L^2(\Omega)) \cap W_g^{p(\cdot)}(\Omega_T)$ with $\partial_t u, \partial_t u_* \in W^{p(\cdot)}(\Omega_T)'$ of the Cauchy–Dirichlet Problem (1.20). Thus, we consider again the difference of the weak formulations with the admissible test function $\varphi = u - u_* \in W_0^{p(\cdot)}(\Omega_T)$, since $W_0^{p(\cdot)}(\Omega_T)'$ is the dual of $W_0^{p(\cdot)}(\Omega_T)$. Hence, we can conclude that

$$\int_{\Omega_T} [(u-u_*)\cdot(u-u_*)_t - (a(z,Du) - a(z,Du_*))\cdot D(u-u_*)]\,\mathrm{d}z = 0.$$

Using the monotonicity condition (1.3), we derive at

$$0 \ge \int_{\Omega_T}(u-u_*)\cdot(u-u_*)_t\mathrm{d}z = \frac{1}{2}\int_{\Omega_T}\partial_t(u-u_*)^2\mathrm{d}z.$$

Finally, we have that $0 \ge \frac{1}{2}\|u(t)-u_*(t)\|^2_{L^2(\Omega)} \ge 0$ for every $t\in(0,T]$, since $u(\cdot,0) = u_*(\cdot,0) = g(\cdot,0)$. This completes the proof. □

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