

The Milnor fiber of the singularity

$$f(x, y) + zg(x, y) = 0$$

Baldur Sigurðsson^{1,2}

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Abstract We give a description of the Milnor fiber and the monodromy of a singularity of the form $f + zg = 0$, where f and g define germs of plane curve singularities and have no common components. In particular, this gives a description of the boundary of the Milnor fiber. The description depends only on the topological type of the two plane curve germs defined by f and g . As a corollary, we give a simple formula for the monodromy zeta function and the Euler characteristic of the fiber in terms of an embedded resolution of f and g .

Keywords Nonisolated hypersurface singularities · Milnor fiber · Monodromy zeta function

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1 Introduction

Let $\Phi : (\mathbb{C}^3, 0) \rightarrow (\mathbb{C}, 0)$, $(x, y, z) \mapsto f(x, y) + zg(x, y)$ be the germ of a function, where $f, g : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$. We require that f and g have no common factors and that both germs are singular (if either f or g is nonsingular, see Sect. 3). We determine the diffeomorphism type of the Milnor fiber F_Φ , as well as the monodromy zeta function, in terms of a simultaneous embedded resolution graph of f and g . For a precise statement, see Theorem 3.3 and its corollaries.

✉ Baldur Sigurðsson
baldur@renyi.hu

¹ Alfréd Rényi Institute of Mathematics, 13-15 Reáltánoda u., 1053 Budapest, Hungary

² Department of Mathematics, Central European University, Zrínyi u. 14, 1051 Budapest, Hungary

The result yields, in particular, a description of the boundary ∂F_Φ . This boundary is known to be a plumbed manifold, see [9] and citations therein. This result was extended in [4] for certain real analytic map germs.

Singularities of the above type play an important role in the investigations of sandwiched singularities, see [2].

In Sect. 2 we recall some topological properties of hypersurface singularities with emphasis on non-isolated singularities and the singular Milnor fiber, as well as some properties of plane curve singularities. Finally, we recall the notion of a 4 dimensional handlebody and fix some notation for surgery.

In Sect. 3 we construct a subset $T_{f,g}$ of a common embedded resolution of f and g from tubular neighbourhoods around some divisors. We obtain the space $F_{f,g}$ by performing surgery along certain embedded disks in $T_{f,g}$. This surgery does not change the homotopy type. Our main theorem states that $F_{f,g}$ has the same diffeomorphism type as the Milnor fiber F_Φ . Furthermore, $F_{f,g}$ can be decomposed into a union of sets on which the monodromy can be completely described. As a corollary, we obtain a simple formula for the monodromy zeta function and the Euler characteristic $\chi(F_\Phi)$.

Section 4 contains the proof of the main statement of the article, Theorem 3.3.

2 Hypersurface singularities

2.1 General results

In this subsection we recall some of the general properties of the Milnor fiber of a holomorphic germ $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$, the monodromy associated to such a germ, and other invariants related to these two.

Let $f : (\mathbb{C}^{n+1}, 0) \rightarrow (\mathbb{C}, 0)$ be a hypersurface singularity, denote by B_δ the closed ball with radius δ around the origin in \mathbb{C}^{n+1} , and by D_ϵ the closed disk around the origin in \mathbb{C} with radius ϵ . D will denote an arbitrary closed disk in the complex plane. Let $V_f = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$ and $S_f = \{z \in V_f : \partial f = 0\}$. The link of f is defined as $K = V_f \cap \partial B_\delta$ for $0 < \delta \ll 1$.

The Milnor fiber F_f of f is by definition the fiber $f^{-1}(\epsilon) \cap B_\delta$ for $0 < \epsilon \ll \delta \ll 1$. Then F_f is a smooth $2n$ dimensional manifold, and so has the homotopy type of a CW complex. In [8], Milnor proves that F_f is homotopy equivalent to a finite n -dimensional CW-complex. Moreover, if s is the dimension of the singular locus S_f , then F_f is $(n - s - 1)$ -connected, as proved in [6].

Let $E = f^{-1}(\partial D_\epsilon) \cap B_\delta$. The function $E \rightarrow \partial D_\epsilon, z \mapsto f(z)$ is a locally trivial fiber bundle with fiber F_f . If $T = \{z \in \partial B_\delta : |f(z)| < \epsilon\}$, we can define another fiber bundle $\partial B_\delta \setminus T \rightarrow \partial D_1, z \mapsto f(z)/|f(z)|$. These two fiber bundles are isomorphic. In fact, there is a bundle-isomorphism $E \rightarrow \partial B \setminus T$ which restricts to the identity on ∂T . In particular, we have a diffeomorphism

$$F_f \cong \{z \in \partial B_\delta \setminus T : f(z)/|f(z)| = 1\}. \tag{2.1}$$

The singular fiber of f is defined as

$$F_{f,sing} = \{z : |z| = \delta, |f(z)| > 0, f(z)/|f(z)| = 1\} \cup K.$$

Usually, $F_{f,sing}$ is not a smooth manifold. By the description of F_f in eq. (2.1), we have an inclusion $\iota : F_f \hookrightarrow F_{f,sing}$. If f is an isolated singularity, ι is a homotopy equivalence, as proved in [8]. For non-isolated singularities this does generally not hold.

Remark 2.1 In the above definition of the Milnor fiber, and in the following discussion of the Milnor fibration, one may replace the standard ball B_δ by a neighbourhood of the form $B_\delta^\rho = \{z \in \mathbb{C}^{n+1} : \rho(z) \leq \delta\}$ where ρ is any real analytic function $\rho : (\mathbb{C}^{n+1}, 0) \rightarrow [0, \infty[$ satisfying $\rho^{-1}(0) = \{0\}$, see e.g. [7]. In fact, in the case $n = 2$, one may take instead of B_δ the ball $B_\delta^{\alpha,\beta} = \{(x, y, z) \in \mathbb{C}^3 : \|(x, y)\|^\alpha \leq \delta, |z|^\beta \leq \delta\}$ for some $\alpha, \beta \in \mathbb{Z}_{>0}$. In Lemma 4.3 we will assume that the Milnor fiber is given as a subset of such a ball for some well chosen α, β . By replacing δ with δ^β , we may in fact assume that the Milnor fiber F_f is given by $f = \epsilon, \|(x, y)\| \leq \delta^{\alpha/\beta}$ and $|z| \leq \delta$.

2.2 The zeta function of the monodromy

The monodromy of the Milnor fibration is a diffeomorphism $m_f : F_f \rightarrow F_f$ with the property that this bundle is isomorphic to the bundle given by $F_f \times I / ((p, 0) \sim (m_f(p), 1)) \rightarrow I / (0 \sim 1), (p, t) \mapsto t$. The monodromy is determined by the bundle up to isotopy, and the bundle is determined up to bundle isomorphism by the monodromy. The monodromy induces linear isomorphisms $h_i : H_i(F_f; \mathbb{C}) \rightarrow H_i(F_f; \mathbb{C})$.

We call the product

$$\zeta_f(t) = \prod_{i=0}^{\infty} \det(I - th_i)^{(-1)^{i+1}}$$

the zeta function associated with the singularity f . This product is well defined because F_f is a finite CW complex, and so $\dim_{\mathbb{C}} H_*(F_f; \mathbb{C}) < \infty$. The zeta function behaves multiplicatively in the following sense.

Let C be a subset of F_f so that $\dim H_*(C; \mathbb{C}) < \infty$ and m_f restricts to a homeomorphism $m_C : C \rightarrow C$. Let us call such a subset good with respect to m . Then m_C induces a linear automorphism $h_{C,i}$ on $H_i(C; \mathbb{C})$ and we define

$$\zeta_C(t) = \prod_{i=0}^{\infty} \det(I - th_{C,i})^{(-1)^{i+1}}.$$

The following propositions are well known. For the first one, see e.g. [3], I. 4.3. The second one can be read from the results of in [1], but is easier to prove by hand.

Proposition 2.2 *Assume that $A, B \subset F_f$ so that $A, B, A \cap B$ are good subsets of F_f and the interiors of A and B cover F_f . Then we have $\zeta_f(t) = \zeta_A(t)\zeta_B(t)\zeta_{A \cap B}(t)^{-1}$.*

Proposition 2.3 *We have $\chi(F_f) = -\deg(\zeta_f)$, where we extend \deg multiplicatively to the field of rational function, i.e. $\deg(a/b) = \deg(a) - \deg(b)$ for $a, b \in \mathbb{C}[t], b \neq 0$.*

The monodromy m_f can be extended to a homeomorphism $m_{f,sing} : F_{f,sing} \rightarrow F_{f,sing}$, which is called the singular monodromy. In fact, by defining $F_{f,sing,\theta}$ in the same way as $F_{f,sing}$, only replacing the condition $f/|f| = 1$ by $f/|f| = \theta$, we get a subspace $\cup_{\theta} F_{f,sing,\theta} \times \{\theta\} \subset S^{2n+1} \times S^1$. The projection onto S^1 is a locally trivial fiber bundle with fiber $F_{f,sing}$; its monodromy is the singular monodromy.

2.3 Plane curves

In the case $n = 1$, f is a plane curve singularity. For a detailed introduction to plane curves see [10]. For a more topological treatment of general open book decompositions, see e.g. [3] I. 3 and I. 4. Note that the first reference deals mainly with the reduced case, whereas the second one allows arbitrary multiplicities. Write $f = f_1^{\alpha_1} f_2^{\alpha_2} \dots f_k^{\alpha_k}$ where f_1, \dots, f_k are the k different irreducible factors of f . In this case, K is a link in ∂B_{δ} . Let T be a tubular neighbourhood around K and \bar{T} the corresponding closed tubular neighbourhood. There exists a projection $c : \bar{T} \rightarrow K$ which is a trivial D -bundle, this is just the normal bundle of the link. Write further $K = \cup_{i=1}^k K_i$, where $K_i = \{z \in \partial B_{\delta} : f_i = 0\}$, and $T = \cup_{i=1}^k T_i$, where T_i is the component of T containing K_i . Choosing $\epsilon > 0$ small enough, we can choose $T = \{z \in \partial B_{\delta} : |f(z)| < \epsilon\}$. Then $\partial F_f \subset \partial \bar{T}$. The projection c can be chosen in such a way that the restriction $c_i = c|_{F_f \cap \partial T_i} : F_f \cap \partial T_i \rightarrow K_i$ is a covering map. This map can be described in terms of the embedded resolution graph of f ; we recall some of its properties.

Let $\Gamma_f = (\mathcal{V}, \mathcal{E})$ be the embedded resolution graph of some fixed embedded resolution of f (see [10] for definition and properties). Here \mathcal{V} is the set of vertices and \mathcal{E} the set of edges. Write $\mathcal{V} = \mathcal{W} \sqcup \mathcal{A}_f$ where \mathcal{A}_f consists of the arrowhead vertices of Γ and \mathcal{W} consists of the nonarrowhead vertices. The elements of \mathcal{A}_f correspond to the branches of f so there is a natural correspondence between the arrowhead vertices of Γ and the components of K . We will make no distinction between the indices $i = 1, \dots, k$ and the corresponding $a \in \mathcal{A}_f$.

For each $a \in \mathcal{A}_f$ there exists a unique $w_a \in \mathcal{W}$ so that $(w_a, a) \in \mathcal{E}$. The map f has multiplicity α_a on a , let m_{w_a} be its multiplicity on w_a . Then $F_f \cap T_a$ has $\gcd(\alpha_a, m_{w_a})$ components, and restricting c_a to any of these components gives a covering of degree $\alpha_a / \gcd(\alpha_a, m_{w_a})$. The singular fiber $F_{f,sing}$ of f is homeomorphic to the space F_f / \sim where the equivalence relation \sim is given by $z_1 \sim z_2$ if and only if $z_1, z_2 \in F_f \cap T_a$ for some a , and $c_a(z_1) = c_a(z_2)$.

The monodromy $m_f : F_f \rightarrow F_f$ can be chosen so that it preserves this equivalence relation, that is, $x_1 \sim x_2$ if and only if $m(x_1) \sim m(x_2)$. Therefore, we get a homeomorphism $F_{f,sing} \rightarrow F_{f,sing}$ induced by the monodromy. It is clear that under the identifications already made, this is nothing else than the singular monodromy already constructed. Note that $F_{f,sing} = F_f \cup B$ where both B and $F_f \cap B$ are homotopically equivalent to the disjoint union of copies of S^1 (here, the set B is a disjoint union of sets of the form $S^1 \times R$ where R is a union of segments in the plane with one endpoint at the origin). Moreover, these homotopy S^1 's contract to actual copies of oriented S^1 's. The singular monodromy $m_{f,sing}$ restricts to a homeomorphism $F_f \rightarrow F_f$ which coincides with the monodromy m_f . Also, $m_{f,sing}$ permutes the connected compo-

nents of B and $F_f \cap B$, respecting the orientation on the first homology. Thus, the induced maps on the homologies of B and $F_f \cap B$ are zero in degree > 1 , and can be represented by the same permutation matrix in degrees zero and one. These cancel out to give $\zeta_B(t) = \zeta_{F_f \cap B}(t) = 1$, and therefore, by Proposition 2.2,

Proposition 2.4 *If $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ defines a plane curve singularity, then $\zeta_f(t) = \zeta_{f, \text{sing}}(t)$.*

2.4 Handles and surgery

We will use handles to describe the Milnor fiber. More precisely, we will use 4 dimensional handles of index 2 in our construction. Chapter 4 of [5] gives a good presentation of the necessary theory.

Let X be a 4-manifold with boundary and $\phi : (\partial D) \times D \rightarrow \partial X$ an embedding. We obtain a new manifold $X \cup_{\phi} D \times D$ by taking the disjoint union $X \sqcup (D \times D)$ and then identifying any point $x \in (\partial D) \times D$ with $\phi(x) \in \partial X$. The map ϕ induces an isomorphism between the normal bundles of $(\partial D) \times \{0\}$ in $(\partial D) \times D$ and $\phi((\partial D) \times \{0\})$ in ∂X . Since $(\partial D) \times \{0\} \subset (\partial D) \times D$ already comes with a canonical framing, this isomorphism can be specified by a framing on $\phi((\partial D) \times \{0\})$. The diffeomorphism type of the resulting manifold is determined by the following data (see for example [5]):

- The embedding $\phi|_{(\partial D) \times \{0\}}$ of $(\partial D) \times \{0\} \cong S^1$ into ∂X .
- The framing of the normal bundle of $\phi|_{(\partial D) \times \{0\}}$.

We will now fix some notation for surgery along embedded disks. We will assume that all maps respect orientation when appropriate. Let X be an oriented 4 dimensional manifold with boundary and $\iota : \bar{D} \hookrightarrow X$ an embedding of the closed disk. We assume that the boundary $\partial \bar{D}$ is embedded into the boundary ∂X , and that $\iota(\bar{D})$ is transversal to ∂X . We can find a parametrisation $\psi : \bar{D} \times \bar{D} \rightarrow X$ of a closed tubular neighbourhood of $\iota(\bar{D})$ so that $\psi(0, z) = \iota(z)$, and $\psi|_{\bar{D} \times \partial \bar{D}}$ is a parametrisation of a tubular neighbourhood of $\iota(\partial \bar{D}) \subset \partial X$. Define $X' = X \setminus \psi(D \times \bar{D})$. For $k \in \mathbb{Z}$ let $X_{\iota, k} = X' \cup_{t_k} \bar{D} \times \bar{D}$, where the glueing map $t_k : \bar{D} \times \partial \bar{D} \rightarrow X'$ is given by $t_k(x, y) = \psi(x, x^k y)$.

Definition 2.5 We call $X_{\iota, k}$ constructed above the k th twist of X along $\iota(\bar{D})$.

Note that $X_{\iota, k}$ is obtained by thinking of $\psi(\bar{D} \times \bar{D})$ as a handle, removing it, and then attaching it again via a different glueing map. This construction is very similar to Dehn surgery. In fact, $\partial X_{\iota, k}$ is nothing else than ∂X , to which a Dehn surgery with coefficient $1/k$ has been applied along $\iota(\partial \bar{D})$.

3 Description of the fiber

In this section we start by fixing some notation for an embedded resolution of the plane curves f and g . Using this data, we define subsets of the resolution which, after

a simple surgery provide the Milnor fiber by Theorem 3.3. Let $f, g : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$ be any plane curve singularities without common factors and define

$$\Phi(x, y, z) = f(x, y) + zg(x, y).$$

Before going through these constructions, we make a remark on the case when f or g is nonsingular, as well as on the singular locus of Φ . Clearly, if f is nonsingular, then Φ is nonsingular as well. If g is nonsingular, we may assume $g(x, y) = y$. Writing $f(x, y) = f_0(x) + yf_1(x, y)$, we find $f_0 \neq 0$ since f, g have no common components. Thus, if k is the order of f_0 , then it has an k th root \hat{x} . Replacing the coordinates x, y, z with $\hat{x}, y, z + f_1(x, y)$, we find that Φ is equivalent to $x^n + yz$, i.e. is of type A_n . Now, assuming that f, g are both singular, it is clear that the singular locus of Φ contains the z axis. Furthermore, we have $\partial_z \Phi = g$, so Φ restricts to f along S_Φ . We therefore get $f = g = 0$ along S_Φ , which implies that the singular set is precisely the z axis.

3.1 An embedded resolution of f and g

Consider a fixed common embedded resolution $\phi : V \rightarrow \mathbb{C}^2$ of f and g . The resolution graph of this embedded resolution will be denoted by Γ . Denote its set of vertices by \mathcal{V} and the set of edges by \mathcal{E} . We write $\mathcal{V} = \mathcal{W} \amalg \mathcal{A}$ where \mathcal{W} is the set of non-arrowhead vertices and \mathcal{A} the set of arrowhead vertices. We decompose \mathcal{A} further as $\mathcal{A} = \mathcal{A}_f \amalg \mathcal{A}_g$, where the elements of \mathcal{A}_f and \mathcal{A}_g correspond to components of the strict transform of f and g respectively. A vertex $v \in \mathcal{V}$ corresponds to a component E_v of the exceptional divisor $\phi^{-1}(0)$, or the strict transform of f or g . In each case, we denote by m_v the multiplicity of f on E_v , and l_v the multiplicity of g on E_v . In particular, $m_v = 0$ if and only if $v \in \mathcal{A}_g$ and $l_v = 0$ if and only if $v \in \mathcal{A}_f$.

Let $f' = f \circ \phi, g' = g \circ \phi$ and $F'_f = (f')^{-1}(\epsilon) \cap \phi^{-1}(B_\delta) = \phi^{-1}(F_f)$. The map $V \setminus \phi^{-1}(0) \rightarrow \mathbb{C}^2 \setminus \{(0, 0)\}, r \rightarrow \phi(r)$ is a diffeomorphism. In particular, it restricts to a diffeomorphism $F'_f \rightarrow F_f$.

We have a map $\phi \times \text{id}_{\mathbb{C}} : V \times \mathbb{C} \rightarrow \mathbb{C}^3$ which restricts to a diffeomorphism $(V \setminus \phi^{-1}(0)) \times \mathbb{C} \rightarrow \mathbb{C}^3 \setminus \{(0, 0, z) : z \in \mathbb{C}\}$. We set $\Phi' = \Phi \circ (\phi \times \text{id}_{\mathbb{C}})$, and $F'_\Phi = (\phi \times \text{id}_{\mathbb{C}})^{-1}(F_\Phi)$. Clearly, F'_Φ is diffeomorphic to F_Φ .

3.2 Construction of the fiber

Using the resolution graph Γ defined above, we now construct a space which, as we will see in the next subsection, is diffeomorphic to the Milnor fiber. For each $w \in \mathcal{W}$, choose a small tubular neighbourhood T_w around E_w in V and a map $b_w : T_w \rightarrow E_w$ which is a smooth open disk bundle. Denote by \overline{T}_w the corresponding closed tubular neighbourhood. These can be chosen so that they satisfy the following properties:

- If $w, w' \in \mathcal{W}$ and $(w, w') \in \mathcal{E}$, then we have $b_w^{-1}(E_w \cap E_{w'}) = E_{w'} \cap T_w$ and $b_{w'}^{-1}(E_w \cap T_{w'}) = T_w \cap T_{w'}$.
- If $w \in \mathcal{W}, a \in \mathcal{A}$ and $(w, a) \in \mathcal{E}$, then $b_w^{-1}(E_w \cap E_a) = E_a \cap T_w$.

Then the set $T = \cup_{w \in \mathcal{W}} T_w$ is the plumbed 4-manifold with plumbing graph Γ .

If $w, w' \in \mathcal{W}$ and $e = (w, w') \in \mathcal{E}$, then we let $T_e = T_w \cap T_{w'}$. If $w \in \mathcal{W}$ and $a \in \mathcal{A}$ so that $e = (w, a) \in \mathcal{E}$, then we pick a small disk-shaped neighbourhood U_a in E_w around $E_w \cap E_a$ and let $T_a = T_e = b_w^{-1}(U_a)$. Then T_a is a tubular neighbourhood around E_a in T .

The Milnor fiber F_Φ can be described in terms of the embedded resolution graph Γ , with the additional arrowhead vertices, and all vertices decorated by the multiplicities of f' and g' . This description will depend on which of the two functions f' and g' has higher multiplicities on the exceptional divisors. The following definition makes this precise.

Definition 3.1 • Let $\mathcal{W}_1 = \{w \in \mathcal{W} : m_w \leq l_w\}$ and $\mathcal{W}_2 = \mathcal{W} \setminus \mathcal{W}_1$. Let Γ_i be the subgraph of Γ generated by the set \mathcal{W}_i . Define $T_i = \cup_{w \in \mathcal{W}_i} T_w$.

- Let $\mathcal{A}_{f,i} = \{a \in \mathcal{A}_f : w_a \in \mathcal{W}_i\}$ and $T_{f,i} = \cup_{a \in \mathcal{A}_{f,i}} T_a$. Repeat this with f replaced by g .
- Choose a small $\epsilon > 0$ and let T_ϵ be a small tubular neighbourhood around $f'^{-1}(\epsilon) \cap \overline{T}$.
- Let T' be a small tubular neighbourhood around the exceptional divisor inside T . This is chosen after choosing ϵ . In particular, $\overline{T'} \cap \overline{T}_\epsilon = \emptyset$.
- Let $\overline{T}_{f,g} = [\overline{T}_{f,1} \setminus T'] \cup \overline{T}_\epsilon \cup [\overline{T}_2 \setminus (T' \cup T_{g,2})]$, where $\overline{}$ denotes closure.
- Let T'_g be a tubular neighbourhood around the strict transform of g , chosen small with respect to the above.

Definition 3.2 We define $F_{f,g}$ to be a twisting of $T_{f,g}$ along the strict transform of g . More precisely, for any $a \in \mathcal{A}_g$, the set $E_a \cap T_{f,g}$ is a union of m_w disks embedded in $T_{f,g}$ as in Sect. 2.4, where $w \in \mathcal{W}$ so that $(a, w) \in \mathcal{E}$. Take the l_a th twist along each of these disks.

3.3 Main theorem and corollaries

We keep here the notation defined in the previous subsections.

Theorem 3.3 *The Milnor fiber F_Φ is diffeomorphic to the space $F_{f,g}$ constructed above. The monodromy can be chosen to satisfy the following*

- The set $\overline{T}_{f,1} \setminus T'$ is invariant under m_Φ and the restriction is homotopic to the identity.
- We have $m_\Phi|_{F_f} = m_f$
- The set $T_2 \setminus (T' \cup \overline{T}_{g,2})$ is invariant under m_Φ and the restriction is homotopic to the identity.
- For any $a \in \mathcal{A}_{g,2}$, the monodromy m_Φ permutes the m_{w_a} handles corresponding to a cyclically.

The proof of Theorem 3.3 is postponed until Sect. 4.

Corollary 3.4 *For $w \in \mathcal{W}$, let $\delta_{w,f}$ be the number of vertices in $\mathcal{W} \cup \mathcal{A}_f$ connected to w by an edge.*

(i) *The Euler characteristic of F_Φ is given by the formula*

$$\chi(F_\Phi) = \sum_{w \in \mathcal{W}_1} m_w(2 - \delta_{w,f}) + \sum_{a \in \mathcal{A}_{g,2}} m_{w_a}.$$

(ii) *The zeta function associated to Φ is given by the formula*

$$\zeta_\Phi(t) = \left(\prod_{w \in \mathcal{W}_1} (1 - t^{m_w})^{\delta_{w,f}-2} \right) \left(\prod_{a \in \mathcal{A}_{g,2}} (1 - t^{m_{w_a}})^{-1} \right) \tag{3.1}$$

Proof of corollary By Proposition 2.3, it is enough to prove Eq. (3.1).

Consider first the action of m_Φ on $\overline{T}_{f,g} \setminus T_2$. Using a similar argument as in Sect. 2.3, we see that the zeta function of the restriction is the same as that of the restriction to $F_f \cap \overline{T}_1 \setminus T_2$. An A'Campo type argument shows that this zeta function is

$$\prod_{w \in \mathcal{V}(\Gamma_1)} (1 - t^{m_w})^{\delta_{w,f}-2}. \tag{3.2}$$

Consider now the set $\overline{T}_2 \cap \overline{T}_{f,g}$. It has the homotopy type of a 3-manifold with some solid tori removed. In particular, $\chi(\overline{T}_2 \setminus T') = 0$. We can now use the same proof as that of Corollary 3.5(ii) to see that the zeta function of the restriction to $T_{f,g} \cap \overline{T}_2$ is $(\prod_{a \in \mathcal{A}_{g,2}} (1 - t^{m_{w_a}})^{-1})$.

The intersection $(\overline{T}_{f,g} \setminus T_2) \cap (\overline{T}_{f,g} \cap \overline{T}_2)$ is a disjoint union of circles which are cyclically permuted by the monodromy. The zeta function of the monodromy restricted to these circles is therefore 1.

Finally, using Proposition 2.2, we can glue these zeta functions together to get Eq. (3.1). □

Corollary 3.5 (i) *If $m_w \leq l_w$ for all $w \in \mathcal{W}$, then F_Φ and $F_{f,sing}$ have the same homotopy type and $\zeta_\Phi = \zeta_f$.*

(ii) *If $m_w > l_w$ for all $w \in \mathcal{W}$, then F_Φ has the same homotopy type as $\vee_{m-1} S^2$, where $m = \sum_{a \in \mathcal{A}_g(\Gamma_2)} m_{w_a}$. The zeta function is given by $\zeta_\Phi(t) = \prod_{a \in \mathcal{A}_g} (1 - t^{m_{w_a}})$.*

Proof In (i) we have $T_2 = \emptyset$. Twisting the handles corresponding to elements $a \in \mathcal{A}_g$ does not alter the homotopy type. Therefore, F_Φ has, by Theorem 3.3, the same homotopy type as $\overline{T}_\epsilon \cup \overline{T}_{f,1}$. Homotopically, this space is the same as F_f , where we have glued the boundary components to some circles. This can easily be seen as the same construction of $F_{f,sing}$. Furthermore, this homotopy equivalence can be seen as invariant under the actions of m_Φ and m_f , proving $\zeta_\Phi = \zeta_{f,sing}$ and thus $\zeta_\Phi = \zeta_f$ by Proposition 2.4. For a second proof of this statement, one may compare A'Campo's formula for ζ_f with Corollary 3.4.

In the case of (ii), $\mathcal{A}_{f,1} = \emptyset$. We have $A = \overline{T}_2 \setminus (T' \cup T_{g,2})$ and $\overline{T}_2 = \overline{T}$. Also, $\overline{T} \setminus \overline{T}'$ is homotopically just $\partial \overline{T} = S^3$ because the graph Γ describes a modification of the smooth germ $(\mathbb{C}^2, 0)$. In fact, $\overline{T} \setminus T'$ is a collar neighbourhood around $\partial \overline{T}$,

so $\overline{T} \setminus T' = S^3 \times I$. Furthermore, for $a \in \mathcal{A}_g$, the pair $(\overline{T} \setminus T', \overline{T}_a \cap (\overline{T} \setminus T'))$ is isomorphic to the pair $(S^3 \times I, S \times I)$ where $S \subset S^3$ is a solid torus. Therefore, A is homotopically S^3 with some solid tori removed, one for each element of \mathcal{A}_g . What's more, the attaching spheres of the handles are meridians of these tori. But removing a solid torus from a 3 manifold and adding m handles attached to meridians is equivalent to removing m spheres from the original manifold. This gives the same as $\vee_{m-1} S^2$.

The statement about ζ_Φ follows from Corollary 3.5 □

Example 3.6 Let $f(x, y) = x^d$ and $g = y^d$ where $d \geq 2$. Then we can choose the resolution V so that \mathcal{V} has a single element, say $\mathcal{V} = \{v\}$. Then $m_v = l_v = d$, so we can apply Corollary 3.5(i). The Milnor fiber F_Φ associated to Φ has the same homotopy type as $F_{f, sing}$, which is up to homotopy a bouquet of $d - 1$ two-spheres. Note that in spite of this, Φ is not isolated. The zeta function of this singularity is $\zeta(t) = t^d - 1$.

4 Proof of Theorem 3.3

In this final section we prove Theorem 3.3. To do so, we project the embedded resolution $V \times \mathbb{C} \rightarrow \mathbb{C}^3$ down to V , and study the image of the fiber F'_Φ . Denote the projection by p . Choose $r \in V$ with the property that $g'(r) \neq 0$. Assume further that there exists a $z \in \mathbb{C}$ such that $\Phi'(r, z) = \epsilon$. We can solve this equation for z , namely

$$z = \frac{\epsilon - f'(r)}{g'(r)}.$$

This means that p restricts to an injection $F'_\Phi \setminus (\text{St}_g \times \mathbb{C}) \rightarrow V$, where St_g is the strict transform of g . Define a function $Z : V \setminus \text{St}_g \rightarrow \mathbb{C}$ by $Z(r) = (\epsilon - f'(r))/g'(r)$. This way, we get a diffeomorphism $F'_\Phi \setminus \text{St}_g \times \mathbb{C} \rightarrow X$ where

$$X = \{r \in V \setminus \text{St}_g : |Z(r)| \leq \delta\}.$$

We obtain a description of $F_\Phi \cong F'_\Phi$ by considering the sets $F'_\Phi \cap p^{-1}(\overline{T} \setminus T_g)$ and $F'_\Phi \cap p^{-1}(\overline{T}_g)$, and how they glue together along their intersection. In fact, the following theorem is a reformulation of Theorem 3.3 in this language. Proving Theorem 4.1 therefore finishes the proof of Theorem 3.3.

Theorem 4.1 *The following items determine the Milnor fiber and the monodromy.*

- (i) *Let $e = (a, w) \in \mathcal{E}$ where $a \in \mathcal{A}_{f,1}$ and $w \in \mathcal{W}$. There is a diffeomorphism between $p(F'_\Phi) \cap \overline{T}_e$ and $\overline{T}_e \setminus T'$ inducing identity on $F'_f \cap \partial \overline{T}_e$ and its normal bundle in $\partial \overline{T}_e$. The set $F'_\Phi \cap p^{-1}(\overline{T}_e)$ is invariant under the monodromy, up to homotopy the monodromy action is trivial on this set.*
- (ii) *The set $p(F'_\Phi) \cap [\overline{T}_1 \setminus (T_f \cup T'_g \cup T_2)]$ is a tubular neighborhood around $F'_f \cap [\overline{T}_1 \setminus (T_f \cup T'_g \cup T_2)]$ in $T_1 \setminus (T_f \cup T'_g \cup T_2)$.*

The set $F'_\Phi \cap p^{-1}[\overline{T}_1 \setminus (T_f \cup T'_g \cup T_2)]$ is invariant under the monodromy. It can be chosen to coincide with m_f on the subset $F'_f \cap [\overline{T}_1 \setminus (T_f \cup T'_g \cup T_2)]$ which is a strong homotopy retract.

- (iii) *There is a diffeomorphism between $p(F'_\Phi) \cap \bar{T}_2 \setminus T'_g$ and $\bar{T}_2 \setminus (T' \cup T'_g)$ inducing identity on $F'_f \cap \partial(T' \cup T'_g)$ and its normal bundle in $\partial(T' \cup T'_g)$. This set is invariant under the monodromy; it's action is trivial up to homotopy.*
- (iv) *Let $e = (a, w) \in \mathcal{E}$ where $a \in \mathcal{A}_g$ and $w \in \mathcal{W}$. The set $p^{-1}(\bar{T}'_g) \cap F'_\Phi$ is a disjoint union of m_w 4 dimensional 2-handles glued to the manifold $p(F'_\Phi) \setminus T'_g$. The attaching spheres are those boundary components of $F'_f \cap (\bar{T}'_e)$ which are in \bar{T}'_e . The normal bundle of the attaching spheres has a canonical trivialisation since each component is the boundary of a disk in T'_e . The handles are attached with the $(-l_a)$ th framing. These handles are invariant under the monodromy, its action permutes them cyclically.*

Proof of (i) We can choose coordinates around the point $E_w \cap E_a$ so that $f'(u, v) = u^m v^n$ and $g(u, v) = u^l$, where $m = m_w, n = m_a$ and $l = l_w$. We can also suppose that $\bar{T}_e = \{(u, v) : |u|, |v| \leq \rho\}$ where ρ is some number so that $\epsilon \ll \rho$. By choices made, we have $m \leq l$ and $n > 0$.

Consider the space $\tilde{T}_e = \{(u, \tilde{v}) : |u|, |\tilde{v}|^{1/n} < \rho\}$ and the map $\pi_e : \bar{T}_e \rightarrow \tilde{T}_e$ given by $(u, v) \mapsto (u, \tilde{v}) = (u, v^n)$. We have then maps

$$Z_e(u, v) = \frac{u^m v^n - \epsilon}{u^l}, \quad \tilde{Z}_e(u, \tilde{v}) = \frac{u^m \tilde{v} - \epsilon}{u^l}$$

satisfying $\tilde{Z}_e \circ \pi_e = Z_e$. The function $|\tilde{Z}_e|^2$ has the divisor $u^m \tilde{v} = \epsilon$ as a nondegenerate critical manifold of index 0. This holds on \tilde{T}_e as well as $\partial \tilde{T}_e$. We will show that $|\tilde{Z}_e|^2$ has no other critical manifolds (in the interior or on the boundary) in the preimage $|\tilde{Z}_e|^2 \leq \delta^2$. This will show that the set $\tilde{F}_e = \pi_e(p(F'_\Phi) \cap T_e)$ is a tubular neighbourhood around the submanifold given by $u^m \tilde{v} = \epsilon$, that is, $\pi_e(F'_f)$. Note first that the coordinate u takes nonzero values on \tilde{F}_e , since Z_e has a pole along the exceptional divisor. We have then $\partial_{\tilde{v}} Z_e(u, \tilde{v}) = u^{m-l} \neq 0$ on \tilde{F}_e . This shows that $|\tilde{Z}_e|^2$ has no critical points in the interior \tilde{T}_e , nor on the part of the boundary given by $|u| = \rho$. For the rest of the boundary, we will show that if $|\tilde{v}| = \rho^n$, then $\partial_u Z_e \neq 0$. But we have

$$\partial_u \tilde{Z}_e = (m - l)u^{m-l-1} \tilde{v} + lu^{-l-1} \epsilon = ((m - l)u^m \tilde{v} + \epsilon l)u^{-l-1}.$$

If $m = l$, then this shows that the partial derivative does not vanish. Assuming $m < l$ we find that $\partial_u \tilde{Z}_e(u, \tilde{v}) = 0$ implies $u^m = -\epsilon l / ((m - l)\tilde{v})$. This implies

$$|\tilde{Z}_e(u, \tilde{v})| = \frac{\left| -\frac{\epsilon l}{(m-l)\tilde{v}} - \epsilon \right|}{\left| -\frac{\epsilon l}{(m-l)\tilde{v}} \right|^{l/m}} = \left| \frac{l}{(m-l)\rho^n} - 1 \right| \left| \frac{(m-l)\rho}{l} \right|^{-l/m} \epsilon^{1-l/m},$$

so that $|Z_e(u, \tilde{v})|$ is large, since ϵ is small and $l > m$. In particular, $|\tilde{Z}_e| > \delta$.

We have now showed that \tilde{F}_e is a tubular neighbourhood around the divisor $u^m \tilde{v} = \epsilon$. But the same is true about the set $\tilde{T}_e \setminus \pi_e(T')$. Thus, we have a diffeomorphism $\tilde{\psi}_e : \tilde{F}_e \rightarrow \tilde{T}_e \setminus \pi_e(T')$ and we can assume that $\tilde{\psi}_e$ equals the identity on a small neighbourhood around the divisor $u_m \tilde{v} = \epsilon$. Now, the set $\{\tilde{v} = 0\} \cap \tilde{F}_e$ is an annulus given by $|u| \geq |\epsilon/\delta|^{1/l}$. One can now easily see that the map $\tilde{\psi}_e$ can also be chosen

to map this annulus into $\pi_e(E_e) = \{\tilde{v} = 0\}$. Finally, by considering the symmetries of \tilde{F}_e and \tilde{T}_e , one can assume that $\tilde{\psi}_e$ commutes with multiplying \tilde{v} by a primitive n th root of unity. This, combined with the fact that $F_e = \pi_e^{-1}(\tilde{F}_e)$, shows that $\tilde{\psi}_e$ transfers to a diffeomorphism $\psi_e : F_e \rightarrow T_e$. \square

Lemma 4.2 *The map $F'_\Phi \cap [(\overline{T}_1 \setminus (T_f \cup T_2)) \times \mathbb{C}] \rightarrow \overline{D}_\delta, (r, z) \mapsto z$ is proper, with surjective derivative everywhere. The same holds for its restriction to the boundary.*

Proof The map is proper, since its domain is compact. The surjectivity of the derivative requires more attention:

Let $(r_0, z_0) \in F'_\Phi \cap [(\overline{T}_1 \setminus (T_f \cup T_2)) \times \mathbb{C}]$. Then, we have three cases: in the first, there is a unique $w \in \mathcal{W}_1$ so that $r_0 \in T_w$. Secondly, there might be exactly two elements $w, w' \in \mathcal{W}_1$ such that $r_0 \in T_w \cap T_{w'}$. Thirdly, we might have $r_0 \in T_e$ for some $e = (w, a) \in \mathcal{E}$ for some $w \in \mathcal{W}_1$ and $a \in \mathcal{A}_{g,1}$. In any case, we can find coordinates u, v in a neighbourhood U around r_0 in V such that $f'(u, v) = u^m v^l$ and $g'(u, v) = \alpha u^l v^k$ for $m = m_w, l = l_w$ and some non-vanishing function $\alpha : U \rightarrow \mathbb{C}$. We have $m \leq l$, and one out of three, depending on the cases above: $n = k = 0, n \leq k$ or $n = 0, k > 0$. In any case, we have $n \leq k$.

By the inverse function theorem, the map $F'_\Phi \cap U \rightarrow \mathbb{C}^2, (u, v, z) \rightarrow (v, z)$ is a coordinate chart, provided that $\partial_u \Phi' \neq 0$ on $F'_\Phi \cap U$. We have

$$\begin{aligned} \partial_u \Phi'(u, v, z) &= \partial_u(u^m v^n + z\alpha u^l v^k) = mu^{m-1} v^n + z((\partial_u \alpha)u^l v^k + \alpha lu^{l-1} v^k) \\ &= u^{m-1} v^n (m + zu^{l-m} v^{k-n} (\partial_u \alpha u + \alpha l)). \end{aligned}$$

The function $u^{l-m} v^{k-n} (\partial_u \alpha u + \alpha l)$ is continuous, and therefore bounded on U (we can assume that U is relatively compact). Since $|z| \leq \delta$, we get

$$|zu^{l-m} v^{k-n} (\partial_u \alpha u + \alpha l)| \ll m.$$

proving that $\partial_u \Phi' \neq 0$ on $F'_\Phi \cap (U \times \mathbb{C})$. Therefore, the function z is a part of a coordinate system around (r_0, z_0) . In particular, its derivative is surjective.

For the last statement, the same reasoning applies; the equation $\partial_u \Phi' \neq 0$ implies that z (as two real variables) gives part of a coordinate system on the boundary. We omit the details. \square

Proof of (ii) The argument in the proof of Lemma 4.2 can be transferred directly to the boundary components $p(F'_\Phi) \cap \partial \overline{T}_{g,1}$. We can therefore use Ehresmann’s fibration theorem to get that the restriction of Z to the set $p(F'_\Phi) \cap [\overline{T}_1 \setminus (T_f \cup T_g \cup T_2)]$ is a locally trivial fibration over D_δ . Since D_δ is contractible, this fibration is trivial. The fiber over $0 \in D_\delta$ is simply $F'_f \cap [\overline{T}_1 \setminus (T_f \cup T_g \cup T_2)]$. Therefore, the set $p(F'_\Phi) \cap [\overline{T}_1 \setminus (T_f \cup T_g \cup T_2)]$ is a product $F'_f \cap [\overline{T}_1 \setminus (T_f \cup T_g \cup T_2)] \times D$. This proves the statement. \square

Lemma 4.3 *We may assume that the inequality $|f'|/|g'| < \delta/2$ holds in $\overline{T}_2 \setminus (T_1 \cup T_g)$.*

Proof Let x, y be some generically chosen coordinates on \mathbb{C}^2 . Assume that the Milnor fibration is given inside a ball of the form $B_\delta = B_\delta^{\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{Z}_{>0}$ as in

Remark 2.1. We may then assume that F'_Φ is given by inequalities $\|(x', y')\| \leq \delta^{\beta/\alpha}$, where x', y' are the pullbacks of x, y , and $|Z| < \delta$. In $\overline{T}_2 \setminus (T_1 \cup T_g)$, the function f'/g' vanishes along E by definition of \mathcal{W}_2 . Since x' and y' vanish along E only we have $|f'/g'| \leq C\|(x', y')\|^{\alpha/\beta}$ in \overline{T}_2 for some $C > 0$ and for a suitably small α/β . Multiplying f with $2C^{-1}$, however, gives an equivalent singularity because the germ $C^{-1}f + zg = C^{-1}(f + Czg)$ is equivalent with the germ $f + zg$ via the coordinate change $(x, y, z) \leftrightarrow (x, y, Cz)$. Combining the two inequalities obtained so far yields $|f'/g'| < 2\delta$. \square

Proof of (iii) We start by investigating the intersection of $p(F'_\Phi)$ with the smaller set $\overline{T}_2 \setminus (T_1 \cup T_g)$. The remaining parts will be considered separately.

As before, we have

$$p(F'_\Phi) \cap \overline{T}_2 \setminus (T_1 \cup T_g) = \{r \in \overline{T}_2 \setminus (T_1 \cup T_g) : |Z(r)| \leq \delta\}.$$

We will start by showing that $|Z|^{-1}$ is a Morse–Bott function in $\overline{T}_2 \setminus (T_1 \cup T_g)$ which defines a small tubular neighbourhood around the exceptional divisor. More precisely, let

$$N = \{r \in \overline{T}_2 \setminus (T_1 \cup T_g) : |Z(r)|^{-1} < \delta^{-1}\}.$$

We will prove that N is a tubular neighbourhood around the exceptional divisor in $\overline{T}_2 \setminus (T_1 \cup T_g)$, and that it can be made arbitrarily small by shrinking ϵ . The restriction of g' to $\overline{T}_2 \setminus (T_1 \cup T_g)$ is a holomorphic function vanishing exactly on the exceptional divisor. Therefore, the set $\{r \in \overline{T}_2 \setminus (T_1 \cup T_g) : |g'(r)| \geq 2\epsilon/\delta\}$ is the complement of a small neighbourhood around the exceptional divisor. If $r \in \overline{T}_2 \setminus (T_1 \cup T_g)$ satisfies $|g'(r)| \geq 2\epsilon/\delta$, we get

$$|Z(r)| = \left| \frac{f'(r) - \epsilon}{g'(r)} \right| \leq \left| \frac{\epsilon}{g'(r)} \right| + \left| \frac{f'(r)}{g'(r)} \right|.$$

By the choice of r , we have $\epsilon/|g'(r)| < \delta/2$. By Lemma 4.3, we also have $|f'(r)/g'(r)| \leq \delta/2$. Therefore, we get $|Z(r)| \leq \delta$. We have proven

$$N \subset N' := \{r \in V : |g'(r)| \leq 2\epsilon/\delta\} \setminus (T_1 \cup T_g).$$

The set N' above can be made arbitrarily small, as a neighbourhood around the exceptional divisor. To show that N is a tubular neighbourhood, we will prove that the derivative of Z does not vanish in N' outside the exceptional divisor. Choose coordinates u, v around $r \in V$ such that $f'(u, v) = u^m v^l$ and $g'(u, v) = \alpha u^l v^k$ where $m = m_w, l = l_w$ for some $w \in \mathcal{W}_2$ for which $r \in \overline{T}_w$ and either there is a $w' \in \mathcal{W}_2$ so that $n = m_{w'}$ and $k = l_{w'}$, or $n = l = 0$. In any case, we have $m > l$ and $n \geq k$. We calculate:

$$\partial_u Z(r) = \frac{\partial}{\partial u} \frac{u^m v^n - \epsilon}{\alpha u^l v^k} = \frac{m u^{m-1} v^n \alpha u^l v^k - (u^m v^n - \epsilon)((\partial_u \alpha) u^l - \alpha l u^{l-1}) v^k}{(\alpha u^l v^k)^2}.$$

Simplifying, we get $\partial_u Z(r) \neq 0$ if and only if

$$mu^m v^n \alpha - (u^m v^n - \epsilon)((\partial_u \alpha)u - \alpha l) \neq 0. \tag{4.1}$$

By assumption, we have $|Z(r)| > \delta$, that is, $|u^m v^n - \epsilon| > \delta | \alpha u^l v^k |$. Thus, we prove Eq. (4.1) by showing that

$$|mu^{m-l} v^{n-k}| < \delta |(\partial_u \alpha)u - \alpha l|. \tag{4.2}$$

The number $|(\partial_u \alpha)u - \alpha l|$ is bounded from below independently of δ and ϵ , because $|u|$ is small with respect to αl , which is bounded below. The functions $|u^{m-l} v^{n-k}|$ and $g'(u, v)$ have the same zero set, thus there is a $C, \gamma \in \mathbb{R}_+$ so that $|mu^{m-l} v^{n-k}| < C |g'(u, v)|^\gamma \leq C(2\epsilon/\delta)^\gamma \ll \delta$. This proves Eq. (4.2). Hence, the set N is a tubular neighborhood around the exceptional divisor in $\overline{T}_2 \setminus (T_2 \cup T_g)$.

Now consider an edge $e = (w_1, w_2) \in \mathcal{E}$ where $w_i \in \mathcal{W}_i$. We want to prove that there is a diffeomorphism between $T_e \cap p(F'_\Phi)$ and $\overline{T}_e \setminus T'$ fixing the intersection $F'_f \cap \partial \overline{T}_e$ and its normal bundle inside $\partial \overline{T}_e$.

Consider coordinates u, v on T_e so that $f = u^m v^n$ and $g = u^l v^k$. Let $\tau_1, \tau_2 \in \mathbb{C}$ with $|\tau_1| = 1$. Then the set $\{(u, v) \in \overline{T}_e : \text{Arg}(u) = \tau_1, \text{Arg}(v) = \tau_2, |Z(u, v)| \leq \delta\}$ is a disk. In fact:

- If $\tau_1^m \tau_2^n \neq 1$, then $|Z|^2$ restricts to a Morse function on the manifold $\{(u, v) : \text{Arg}(u) = \tau_1, \text{Arg}(v) = \tau_2\}$. There are no critical points on the interior. Restricting Z to the boundary of this submanifold, we get exactly one critical point with index zero and at most one with index one.
- If $\tau_1^m \tau_2^n = 1$, then $|Z|^2$ restricts to a Morse–Bott function on the submanifold $\{(u, v) : \text{Arg}(u) = \tau_1, \text{Arg}(v) = \tau_2\}$, the critical set being the intersection with F'_f .

Proving these two statements is a simple exercise, it boils down to showing that certain partial derivatives do not vanish. The results show that each fiber of the argument map (τ_1, τ_2) is abstractly a disk. One is therefore free to choose a diffeomorphism from this disk to the set of points (u, v) where the argument of each coordinate is fixed, to the set of points with corresponding arguments in $\overline{T}_e \subset T'$. This can be done in such a way that we get a diffeomorphism with the desired properties.

The last thing we need to consider is the set $p(F'_\Phi) \cap (\overline{T}_{g,2} \setminus T'_{g,2})$. Let a be in $\mathcal{A}_{g,2}$. We have local coordinates u, v on T_a so that $f = u^m$ and $g = u^l v^k$, where $m = m_{w_a}$, $l = l_{m_w}$ and $k = m_a$. The set $p(F'_\Phi) \cap (\overline{T}_a \setminus T'_a)$ can be given by equations $|Z| \leq \delta$ and $|v| \geq \eta$ for some $\eta \ll \epsilon$, that is,

$$\left\{ (u, v) \in \overline{T}_a : \left| \frac{u^m - \epsilon}{u^l v^k} \right| \leq \delta, |v| \geq \eta \right\}$$

We proved already, that the intersection $p(F'_\Phi) \cap \{|v| = \rho_a\}$ is the complement of a tubular neighbourhood around the exceptional divisor in the set $\{|v| = \rho_a\}$. Take a point $(u_0, v_0) \in p(F'_\Phi) \cap \overline{T}_a \setminus T'_a$. From the formula $Z(u, v) = (u^m - \epsilon)/(u^l v^k)$ we see that

the segment between (u_0, v_0) and $(u_0, (\rho_a/|v_0|)v_0)$ is contained in $p(F'_\Phi) \cap (\overline{T}_a \setminus T'_a)$. From this, one quickly observes that the inclusion $p(F'_\Phi) \cap (\overline{T}_a \setminus T'_a) \rightarrow \overline{T}_a \setminus T'_a$ is isotopic to the inclusion of $(\overline{T}_a \setminus T'_a) \setminus T'$ fixing a neighborhood around both $F'_f \cap (\overline{T}_a \setminus T'_a)$ and $\{|v| = \rho_a\}$.

Finally, all these diffeomorphisms glue together to the desired map.

For the monodromy, we notice that the diffeomorphism type of the pair $(F'_{\Phi,\theta}, p^{-1}(\overline{T}_2 \setminus (T_1 \cup T_g)) \cap F'_{\Phi,\theta})$, where $\theta \in S^1$ and $F'_{\Phi,\theta} = \Phi'^{-1}(\theta\epsilon)$, is independent of θ , that is, $p^{-1}(\overline{T}_2 \setminus (T_1 \cup T_g)) \cap F'_{\Phi,\theta}$ is a subbundle of the Milnor fibration. The description of this fiber above is independent of θ however, and therefore gives a trivialisation of the bundle. Therefore, the monodromy acts trivially, up to homotopy, on this subset. □

Proof of (iv) Let $a \in \mathcal{A}_g$. As before, we consider coordinates u, v on \overline{T}_a so that $f' = u^m$ and $g' = u^l v^k$. Then $H_a := F'_\Phi \cap p^{-1}(\overline{T}'_a)$ is the set of points (u, v, z) satisfying $|z| \leq \delta, |v| \leq \eta$ for some $\eta \ll \epsilon$ and the equality $\Phi' = \epsilon$. We show first that abstractly, this set is a disjoint union of bidisks. Clearly, the map $\pi_a = (v, z) : H_a \rightarrow D_\eta \times D_\delta$ is a proper surjection which maps boundary points to boundary points. Also, the preimage of $(0, 0)$ is the set $\{(u, 0, 0) : u^m = \epsilon\}$, and so contains exactly m points. By the implicit function theorem, if $\partial_u \Phi' \neq 0$ on H , the map π_a is a local diffeomorphism, and so a covering map. Furthermore, since the bidisk is contractible, such a covering map must be a product. We get

$$\partial \Phi'(u, v, z) = \partial_u(u^m + zu^l v^k) = mu^{m-1} + zlu^{l-1}v^k.$$

The function $|mu^{m-1}|$ is bounded from below by a positive number on H_a , since it is continuous and does not vanish. Similarly, the function $|zlu^{l-1}v^k|$ is bounded from above. Taking η small enough, we get $|mu^{m-1}| > |zlu^{l-1}v^k|$ on H_a . This gives $\partial_u \Phi' \neq 0$ as required.

We have now shown that F'_Φ is given by glueing handles (such as h) to $T_{f,g} \setminus T'_g$ in the way described in Sect. 2.4. We only have to determine the twisting coefficient. We already have a parametrization of the handle h by (u, v) . The handle already contained in $T_{f,g}$ is parametrized by $(u - \xi, v)$, where ξ is some m th root of unity. Denote this parametrization by $\psi : \overline{D} \times \overline{D} \rightarrow T_{f,g}$.

Now, for any $r \in h$ with coordinates (z, v) we have $p(r) = (U(z, v), v)$ where

$$zU(z, v)^l v^k = U(z, v)^m - \epsilon$$

and we assume that $U(z, v)$ is in a small neighbourhood around some m th root of ϵ . This shows that the twisting coefficient used to glue h is k , as stated.

To finish the proof, we must consider the action of the monodromy on the handles corresponding to $a \in \mathcal{A}_{g,2}$. But the central disks of these handles are given by F'_f . It follows that they are permuted cyclically. □

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