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Lebesgue constants for the weak greedy algorithm

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Abstract We estimate the Lebesgue constants for the weak thresholding greedy algorithm in a Banach space relative to a biorthogonal system. The estimates involve the weakness (relaxation) parameter of the algorithm, as well as properties of the basis, such as its quasi-greedy constant and democracy function.

Keywords Weak greedy algorithm · Biorthogonal system · Markushevitch basis · Lebesgue constant

Mathematics Subject Classification Primary 46B15; Secondary 41A30 · 41A65 · 46A35

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1 Introduction

In this short note, we calculate the Lebesgue constants associated with the t-greedy, and the Chebyshev t-greedy, algorithms in Banach spaces (thus measuring the *efficiency* of these approximation methods, in the *worst case*).

Throughout this paper, X is a separable infinite dimensional Banach space. A family $(e_i, e_i^*)_{\in\in\mathbb{N}} \subset X \times X^*$ is called a *bounded biorthogonal system* if:

1. $X = \mathrm{span}\,[e_i : i \in \mathbb{N}]$.
2. $e_i^*(e_j) = 1$ if $i = j$, $e_i^*(e_j) = 0$ otherwise.
3. $0 < \inf_i \min\{\|e_i\|, \|e_i^*\|\} \le \sup_i \max\{\|e_i\|, \|e_i^*\|\} < \infty$.

For brevity, we refer to (e_i) as a *basis*. Note that Condition (3) is referred to as (e_i) being *seminormalized*. In this note, only seminormalized bases are considered.

It is easy to see that, for any $x \in X$, $\lim_i e_i^*(x) = 0$, and $\sup_i |e_i^*(x)| > 0$, unless $x = 0$.

Bases as above are quite common. It is known [7, Theorem 1.27] that, for any $c > 1$ any separable Banach space has a bounded biorthogonal system (a *Markushevitch basis*) with $1 \le \|e_i\|, \|e_i^*\| \le c$, and $X^* = \overline{\mathrm{span}}^{w^*}[e_i^* : i \in \mathbb{N}]$.

To consider the problem of approximating $x \in X$ by finite linear combinations of e_i's, introduce some notation. For $x \in X$ set $\mathrm{supp}\, x = \{i \in \mathbb{N} : e_i^*(x) \ne 0\}$. For finite $A \subset \mathbb{N}$, set $P_A x = \sum_{i\in A} e_i^*(x)e_i$. If $A^c = \mathbb{N}\backslash A$ is finite, write $P_A x = x - P_{A^c}x$.

The *best n-term approximation* for $x \in X$ is defined as

$$\sigma_n(x) = \inf_{|\,\mathrm{supp}\, y| \le n} \|x - y\|,$$

while the *best n-term coordinate approximation* is

$$\tilde{\sigma}_n(x) = \inf_{|B| \le n} \|x - P_B x\|.$$

It is easy to see that $\lim_n \sigma_n(x) = 0$, and

$$\sigma_n(x) = \inf_{|\,\mathrm{supp}\, y| = n} \|x - y\| \quad \text{and} \quad \tilde{\sigma}_n(x) = \inf_{|B| = n} \|x - P_B x\|$$

(the second equality is due to the fact that $\lim_i e_i^*(x) = 0$).

We also consider the *n term residual approximation*

$$\hat{\sigma}_n(x) = \|x - P_{[1,n]}x\|.$$

We say that (e_i) is a *Schauder basis* if $\lim_n \hat{\sigma}_n(x) = 0$ for every $x \in X$ (in this case, also $\lim_n \tilde{\sigma}_n(x) = 0$). Many commonly used bases (such as the Haar basis or the trigonometric basis in L_p, for $1 < p < \infty$) are, in fact, Schauder bases.

Note that calculating $\sigma_n(x)$ and $\tilde{\sigma}_n(x)$ is next to impossible, since all coordinates of x are in play. Therefore, one can naively look for a good n-term approximant of x by considering the n largest (or "nearly largest") coefficients. This is done using the *weak greedy algorithm*. To define this algorithm, fix the *relaxation parameter*

$t \in (0, 1]$. Consider a non-zero $x \in X$. A set $A \subset \mathbb{N}$ is called *t-greedy for x* if $\inf_{i \in A} |e_i^*(x)| \geq t \sup_{i \notin A} |e_i^*(x)|$ (by the above, A is finite). When there is no confusion about x, we shorten this term to *t-greedy set*. Suppose $\rho = \rho_x : \mathbb{N} \to \mathbb{N}$ is a *t-greedy ordering*—that is, $\{\rho(1), \dots, \rho(n)\}$ is t-greedy for every n. In general, a t-greedy ordering is not unique. Note that $\{\rho(n) : n \in \mathbb{N}\} = \mathfrak{S}_x := \{n \in \mathbb{N} : e_n^*(x) \neq 0\}$ if the set $\mathfrak{S}_x$ is infinite. On the other hand, if $|\mathfrak{S}_x| = m < \infty$, then $\{\rho(1), \dots, \rho(m)\} = \mathfrak{S}_x$ while $\rho(i) \notin \mathfrak{S}_x$ for $i > m$.

An *n-term t-greedy approximant of x* is defined as $\mathbf{G}_n^t(X) = P_{A_n} x$, where $A_n = \{\rho(1), \dots, \rho(n)\}$, and ρ is a t-greedy ordering for x. We define an *n-term Chebyshev t-greedy approximant* $\mathbf{CG}_n^t(x)$ as $y \in \operatorname{span}[e_i : i \in A_n]$ so that $\|x - y\|$ is minimal. We stress that these approximants are not unique, and *a fortiori*, the operators $x \mapsto \mathbf{G}_n^t(x)$ and $x \mapsto \mathbf{CG}_n^t(x)$ are not linear.

For more information on greedy approximation algorithms, we refer the reader to the survey papers [13,18], as well as to the recent monograph [14].

When $t = 1$, we omit it, and use the terms "greedy set", ("Chebyshev") "greedy approximant", as well as notation $\mathbf{G}_n(x)$ and $\mathbf{CG}_n(x)$. A basis (e_i) is called *quasi-greedy* if its *quasi-greedy constant* is finite:

$$\mathfrak{K} = \sup_{\|x\|=1} \sup_{n \in \mathbb{N}} \sup \|\mathbf{G}_n(x)\| < \infty,$$

with the inner sup taken over all realizations of $\mathbf{G}_n(x)$. In [17] it was shown that a basis is quasi-greedy if and only if $\lim_n \mathbf{G}_n(x) = x$ for any $x \in X$, and any (equivalently, some) choice of the sequence $\mathbf{G}_n(x)$. By [9], for a quasi-greedy basis we also have $\lim_n \mathbf{G}_n^t(x) = x$ for any $x \in X$, and any choice of the sequence $\mathbf{G}_n^t(x)$.

The goal of this paper is to estimate the *efficiency* of the t-greedy and t-Chebyshev greedy methods (in the worst case), by comparing $\|x - \mathbf{G}_n^t(x)\|$ and $\|x - \mathbf{CG}_n^t(x)\|$ with the best n-term approximation $\sigma_n(x)$, and similar quantities. This is done through estimating the Lebesgue constants and its relatives:

$$\textit{The Lebesgue constant } \mathbf{L}(n, t) = \sup_{x \in X, \sigma_n(x) \neq 0} \frac{\|x - \mathbf{G}_n^t(x)\|}{\sigma_n(x)}.$$

$$\textit{The Chebyshevian Lebesgue constant } \mathbf{L}_{\mathrm{ch}}(n, t) = \sup_{x \in X, \sigma_n(x) \neq 0} \frac{\|x - \mathbf{CG}_n^t(x)\|}{\sigma_n(x)}.$$

$$\textit{The residual Lebesgue constant } \mathbf{L}_{\mathrm{re}}(n, t) = \sup_{x \in X, \hat{\sigma}_n(x) \neq 0} \frac{\|x - \mathbf{G}_n^t(x)\|}{\hat{\sigma}_n(x)}.$$

We stress that the suprema in the above inequalities are taken over all $x \in X$, and all possible realizations of the (Chebyshev) weakly greedy algorithm. A basis is called *greedy* if $\sup_n \mathbf{L}(n, 1) < \infty$, and *partially greedy* if $\sup_n \mathbf{L}_{\mathrm{re}}(n, 1) < \infty$.

To estimate the Lebesgue constants, we quantify some properties of (e_i). We use the *left* and *right democracy functions* $\phi_l(k) = \inf_{|A|=k} \|\sum_{i \in A} a_i\|$ and $\phi_r(k) = \sup_{|A|=k} \|\sum_{i \in A} a_i\|$ (sometimes, ϕ_r is also referred to as the *fundamental function*). We define the *democracy parameter*

$$\boldsymbol{\mu}(n) = \max_{k \le n} \frac{\phi_r(k)}{\phi_l(k)} = \sup_{|A|=|B|\le n} \frac{\| \sum_{i\in A} e_i \|}{\| \sum_{i\in B} e_i \|}.$$

Following [12], define the *disjoint democracy parameter*

$$\boldsymbol{\mu_d}(n) = \sup_{|A|=|B|\le n, A\cap B=\emptyset} \frac{\| \sum_{i\in A} e_i \|}{\| \sum_{i\in B} e_i \|}.$$

Clearly, $\boldsymbol{\mu_d}(n) \le \boldsymbol{\mu}(n)$. By [10, Lemma 13], $\boldsymbol{\mu}(n) \le 2\mathfrak{K}\boldsymbol{\mu_d}(n)$. Related to the democracy parameter of a basis (e_i) is its *conservative parameter*:

$$\mathbf{c}(n) = \sup \left\{ \frac{\| \sum_{i\in A} e_i \|}{\| \sum_{i\in B} e_i \|} : \max A \le n < \min B, |A| = |B| \right\}.$$

Clearly $\mathbf{c}(n) \le \boldsymbol{\mu_d}(n)$. The norms of coordinate projections in a basis (e_i) are quantified by the *unconditionality parameter* and *complemented unconditionality parameter*: $\mathbf{k}(n) = \sup_{|A|\le n} \|P_A\|$, resp. $\mathbf{k}_c(n) = \sup_{|A|\le n} \|I - P_A\|$ (clearly $|\mathbf{k}(n) - \mathbf{k}_c(n)| \le 1$).

The investigation of Lebesgue constants for greedy algorithms dates back to the earliest works on greedy algorithms, with some relevant ideas appearing already in [8]. In [12], the Lebesgue constants of the Haar basis in the BMO, and the dyadic BMO, were computed. More recently, in [15,16], the Lebesgue constants for tensor product bases in L_p-spaces (in particular, for the multi-Haar basis) were calculated. The Lebesgue constants for the trigonometric basis L_p (which is not quasi-greedy) are also known, see e.g. [13, Section 1.7]. The recent paper [4] estimates the Lebesgue constants for bases in L_p spaces with specific properties (such as being uniformly bounded). Lebesgue constants for redundant dictionaries are studied in [14, Section 2.6].

This paper is structured as follows: in Sect. 2, we gather some preliminary facts about quasi-greedy bases. In Sect. 3, we estimate $\mathbf{L}(n, t)$ in terms of $\mathfrak{K}$, $\boldsymbol{\mu_d}(n)$, $\mathbf{k}(n)$, and t. For $t = 1$, related results were obtained in [5]. However, the Lebesgue constant was not explicitly calculated there. Retracing the computations, one obtains worse constants than those given by Theorem 3.1. Corollary 3.5 gives an upper estimate for the Lebesgue constant of quasi-greedy bases in Hilbert spaces, by combining Theorem 3.1 with the recent results of Garrigos and Wojtaszczyk [6]. Further, we estimate the Lebesgue constant for general (not necessarily quasi-greedy) systems in Proposition 3.6.

In Sect. 4, we estimate $\mathbf{L}_{\text{ch}}(n, t)$. The estimates involve only t, $\mathfrak{K}$, and $\boldsymbol{\mu_d}(n)$. Finally, in Sect. 5, we provide upper and lower bounds for $\mathbf{L}_{\text{re}}(n, t)$, involving t, $\mathfrak{K}$, and $\mathbf{c}(n)$. The main results are given in Theorems 4.1 and 5.1, respectively.

Most of the work in this paper is done in the real case. In Sect. 5, we indicate that the complex versions of the results of this paper also hold, albeit perhaps with different numerical constants.

Remark 1.1 After the first version of this article was circulated, the referee brought the recent paper [10] to the attention of the authors. There, order-of-magnitude estimates for the Lebesgue constant, and the Chebyshevian Lebesgue constant (similar to our Theorems 3.1, 4.1) are given. Our results have the advantage of establishing the dependence of the Lebesgue constants not only of $\boldsymbol{\mu}_d(n)$ and $\boldsymbol{k}(n)$, but also of $\mathfrak{K}$ and t.

2 Preliminary results

In this section we prove two lemmas, which will be needed throughout the paper, and may be of interest in their own right. First we sharpen some results from [9, Section 2].

Lemma 2.1 *Suppose $(e_i) \subset X$ is a basis with a quasi-greedy constant $\mathfrak{K}$. Consider $x \in X$, and let A be a t-greedy set for x. Then $\|P_A x\| \le (1 + 4t^{-1}\mathfrak{K})\mathfrak{K}\|x\|$.*

Proof For the sake of brevity, set $a_i = e_i^*(x)$. Let $M = \min_{i \in A} |a_i|$, then $|a_i| \le t^{-1}M$ for $i \notin A$. Define $B = \{i : |a_i| > t^{-1}M\}$ and $C = \{i : |a_i| \ge M\}$. Then $B \subset A \subset C$, and $P_A x = P_B x + P_{A \setminus B} x$. By the definition of $\mathfrak{K}$, $\|P_B x\| \le \mathfrak{K}\|x\|$, and $\|P_C x\| \le \mathfrak{K}\|x\|$. Write $P_C x = \sum_{i \in C} a_i e_i$.

Now define the basis (e_i') by setting

$$e_i' = \begin{cases} \operatorname{sign}(a_i) e_i & i \in C \\ e_i & \text{otherwise.} \end{cases}$$

As this basis has the same quasi-greedy constant as (e_i), Lemma 6.1(2) shows that $M\|\sum_{i \in C} e_i'\| \le 2\mathfrak{K}\|x\|$. For $i \in C$, set

$$b_i = \begin{cases} |a_i| & i \in A \setminus B \\ 0 & \text{otherwise} \end{cases}.$$

For any i, $|b_i| \le t^{-1}M$, hence, by Lemma 6.1(1)

$$\left\| \sum_{i \in A \setminus B} a_i e_i \right\| = \left\| \sum_{i \in C} b_i e_i' \right\| \le 2t^{-1}M\mathfrak{K} \left\| \sum_{i \in C} e_i' \right\| \le 4t^{-1}\mathfrak{K}^2\|x\|.$$

By the triangle inequality, $\|P_A x\| \le \|P_B x\| + \|P_{A \setminus B} x\|$. □

Lemma 2.2 *Suppose (e_i) is a $\mathfrak{K}$-quasi-greedy basis in X. Consider $x \in X$, and let $a_i = e_i^*(x)$, for $i \in \mathbb{N}$. Suppose a finite set $A \subset \mathbb{N}$ satisfies $\min_{i \in A} |a_i| \ge M$. Then $M\|\sum_{i \in A} \operatorname{sign}(a_i) e_i\| \le 2\mathfrak{K}^2\|x\|$. Furthermore, $M\|\sum_{i \in A} e_i\| \le 4\mathfrak{K}^2\|x\|$.*

Proof Consider the set $B = \{i : |a_i| \ge M\}$ (clearly $A \subset B$). By [5, Lemma 10.1], $\|\sum_{i \in A} \operatorname{sign}(a_i) e_i\| \le \mathfrak{K}\|\sum_{i \in B} \operatorname{sign}(a_i) e_i\|$. By Lemma 6.1(2), $\|\sum_{i \in B} \operatorname{sign}(a_i) e_i\| \le 2\mathfrak{K}\|x\|/M$. To establish the "moreover" part, let $A_+ = \{i \in A : \operatorname{sign}(a_i) = 1\}$, and $A_- = \{i \in A : \operatorname{sign}(a_i) = -1\}$. By the above, $M\|\sum_{i \in A_+} \operatorname{sign}(a_i) e_i\| \le 2\mathfrak{K}^2\|x\|$, and the same holds for A_-. Complete the proof using the triangle inequality. □

We close this section with a brief discussion about the values of $\boldsymbol{\mu_d}(n)$, $\mathbf{k}(n)$, and $\mathbf{c}(n)$. It was shown in [2,5] that, for a $\mathfrak{K}$-quasi-greedy basis, $\mathbf{k}(n) \leq C \log(en)$, where the constant C depends on the particular basis. For bases in L_p spaces, sharper estimates were obtained in [6]. It is easy to see that $\mathbf{c}(n) \leq \boldsymbol{\mu_d}(n) \leq Cn$, where C depends on a basis. These estimates are optimal: indeed, an appropriate enumeration of the canonical (normalized and 1-unconditional) basis in $c_0 \oplus_2 \ell_1$ gives $\mathbf{c}(n) \geq cn$.

3 The Lebesgue constant

In this section, we use some of the techniques of [5] to estimate the Lebesgue constants $\mathbf{L}(n, t)$.

Theorem 3.1 *For any $\mathfrak{K}$-quasi-greedy basis,*

$$\max\left\{\mathbf{k}_c(n), t^{-1}\boldsymbol{\mu_d}(n)\right\} \leq \mathbf{L}(n,t) \leq 1 + 2\mathbf{k}(n) + 8t^{-1}\mathfrak{K}^3\boldsymbol{\mu_d}(n).$$

The proof of the theorem relies on several lemmas, whose proofs closely resemble those given in [5] (Lemma 3.4 yields better upper estimates).

Lemma 3.2 *For any $\mathfrak{K}$-quasi-greedy basis,* $\mathbf{L}(n, t) \geq t^{-1}\boldsymbol{\mu_d}(n)$.

Proof Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. Find $A, B \subset \mathbb{N}$, so that $A \cap B = \emptyset$, $|A| = |B| = k \leq n$, and

$$\left\|\sum_{i\in A} e_i\right\| \geq (\boldsymbol{\mu_d}(n) - \varepsilon)\left\|\sum_{i\in B} e_i\right\|.$$

Pick a set C, disjoint from A and B, so that $|C| = n - k$. Consider

$$x = (t+\varepsilon)\sum_{i\in B\cup C} e_i + \sum_{i\in A} e_i.$$

Then $(t+\varepsilon)\sum_{i\in B\cup C} e_i$ is a t-greedy approximant of x, for which $\|x - \mathbf{G}_n^t(x)\| = \|\sum_{i\in A} e_i\|$. However, $|A \cup C| = n$, hence

$$\sigma_n(x) \leq \tilde{\sigma}_n(x) \leq \|x - P_{A\cup C}x\| = (t+\varepsilon)\left\|\sum_{i\in B} e_i\right\|.$$

Thus,

$$\mathbf{L}(n,t) \geq \frac{\|x - \mathbf{G}_n^t(x)\|}{\sigma_n(x)} = (t+\varepsilon)^{-1}\frac{\|\sum_{i\in A} e_i\|}{\|\sum_{i\in B} e_i\|} \geq \frac{\boldsymbol{\mu_d}(n) - \varepsilon}{t+\varepsilon}.$$

As ε can be arbitrarily small, the desired estimate follows. □

Lemma 3.3 *For any basis,* $\mathbf{L}(n, t) \geq \mathbf{k}_c(n)$.

Proof Clearly $\mathbf{L}(n, t) \geq \mathbf{L}(n, 1)$. By [5, Proposition 3.3], $\mathbf{L}(n, 1) \geq \mathbf{k}_c(n)$. □

Lemma 3.4 *For any $\mathfrak{K}$-quasi-greedy basis, $\mathbf{L}(n, t) \leq \mathbf{k}(n) + \mathbf{k}_c(n) + 8t^{-1}\mathfrak{K}^3\boldsymbol{\mu}_{\boldsymbol{d}}(n)$.*

Proof For $x \in X$, let $a_i = e_i^*(x)$, and fix $\varepsilon > 0$. Suppose $A \subset \mathbb{N}$ is a t-greedy set for x, of cardinality n. Find $z \in X$, supported on a set B of cardinality n, so that $\|x - z\| < \sigma_n(x) + \varepsilon$. Let $M = \sup_{i \notin A} |a_i|$, then $|a_i| \geq tM$ whenever $i \in A$. By the triangle inequality,

$$\|x - P_A x\| \leq \|x - P_B x\| + \|P_{A \backslash B} x\| + \|P_{B \backslash A} x\|.$$

We have

$$\|P_{A \backslash B} x\| = \|P_{A \backslash B}(x - z)\| \leq \mathbf{k}(n)\|x - z\|,$$

and

$$\|x - P_B x\| = \|x - P_B x + z - P_B z\| = \|(1 - P_B)(x - z)\| \leq \mathbf{k}_c(n)\|x - z\|.$$

It remains to estimate the third summand, in the non-trivial case of $|B \backslash A| = k > 0$. For $i \in B \backslash A$, $|a_i| \leq M$, hence by Lemma 6.1(1) (see also [3, Lemma 2.1]),

$$\|P_{B \backslash A} x\| = \Big\| \sum_{i \in B \backslash A} a_i e_i \Big\| \leq 2M\mathfrak{K}\| \sum_{i \in B \backslash A} e_i \|.$$

By Lemma 2.2, $M \leq 4t^{-1}\mathfrak{K}^2 \| \sum_{i \in A \backslash B} e_i \|^{-1} \|x - z\|$. Thus,

$$\begin{aligned}\|P_{B \backslash A} x\| &\leq 2M\mathfrak{K}\Big\| \sum_{i \in B \backslash A} e_i \Big\| \leq 8t^{-1}\mathfrak{K}^3 \frac{\| \sum_{i \in B \backslash A} e_i \|}{\| \sum_{i \in A \backslash B} e_i \|} \|x - z\| \\ &\leq 8t^{-1}\mathfrak{K}^3 \boldsymbol{\mu}_{\boldsymbol{d}}(n) \|x - z\|.\end{aligned}$$

As $\|x - z\|$ can be arbitrarily close to $\sigma_n(x)$, we are done. □

We use Theorem 3.1 to estimate the Lebesgue constant for quasi-greedy bases in a Hilbert spaces. Recall that a basis (e_i) is called *hilbertian* (*besselian*) if there exists a constant c so that, for every finite sequence of scalars (α_i). we have $\sum_i |\alpha_i|^2 \geq c\| \sum_i \alpha_i e_i \|^2$ (resp. $\sum_i |\alpha_i|^2 \leq c\| \sum_i \alpha_i e_i \|^2$).

Corollary 3.5 *For any quasi-greedy basis in a Hilbert space, there exists $\alpha \in (0, 1)$ and $C > 0$ so that, for any $n \in \mathbb{N}$ and $t \in (0, 1)$, $\mathbf{L}(n, t) \leq C(t^{-1} + (\log(en))^\alpha)$. If, moreover, the basis is either besselian or hilbertian, then there exists $\alpha \in (0, 1/2)$ with the above property.*

Proof By [6], there exists $c_1 > 0$, and α as above, so that $\mathbf{k}(n) \leq c_1(\log(en))^\alpha$. By [17, Theorem 3], $\boldsymbol{\mu}(n) \leq c_2$, for some constant c_2. To finish the proof, apply Theorem 3.1. □

We conclude this section with an estimate for $\mathbf{L}(n,t)$ for bounded Markushevitch bases which are not necessarily quasi-greedy. Let $1 \le p \le q \le \infty$. We say that (e_i) satisfies weak upper p- and lower q-estimates if there exists $K > 0$ such that for all $x \in X$,

$$\frac{1}{K}\|(e_i^*(x))\|_{q,\infty} \le \|x\| \le K\|(e_i^*(x))\|_{p,1},$$

where, letting (a_n^*) denote the decreasing rearrangement of the sequence $(|a_n|)$,

$$\|(a_n)\|_{q,\infty} := \sup_{n\ge 1} n^{1/q} a_n^*$$

and

$$\|(a_n)\|_{p,1} := \sum_{n\ge 1} n^{1/p-1} a_n^*$$

are the usual Lorentz sequence norms. Note that $p = 1$ and $q = \infty$ are just the ℓ_1 and c_0 norms, respectively.

The following result slightly extends [17, Theorem 5] by incorporating the weakness parameter t and replacing upper ℓ_p-and lower ℓ_q-estimates by weaker Lorentz sequence space estimates.

Proposition 3.6 *Suppose (e_i) satisfies weak upper p- and lower q-estimates. Then there exists $D := D(p,q,K)$ such that*

$$\mathbf{L}(n,t) \le \begin{cases} Dn^{1/p-1/q}/t, & p \ne q \\ D\log n/t, & p = q. \end{cases}$$

Proof First suppose $q > p$. Let $x \in X$ and set $a_i := e_i^*(x)$. Let A be a t-greedy set for x, with $|A| = n$, and let $\mathbf{G}_n^t(x) := \sum_{i\in A} a_i e_i$. Given $\varepsilon > 0$, choose $B \subset \mathbb{N}$, with $|B| = n$, such that $\|x - \sum_{i\in B} b_i e_i\| \le \sigma_n(x) + \varepsilon$. For convenience, set $b_i = 0$ if $i \notin B$. By the triangle inequality,

$$\begin{aligned}\|x - \mathbf{G}_n^t(x)\| &\le \|x - \sum_{i\in B} b_i e_i\| + \|\sum_{i\in B} b_i e_i - \sum_{i\in A} a_i e_i\| \\ &\le \sigma_n(x) + \varepsilon + \|\sum_{i\in B} b_i e_i - \sum_{i\in A} a_i e_i\|. \qquad (3.1)\end{aligned}$$

Setting $C = C(p,q) := (1/p - 1/q)^{1/q-1/p}$, we obtain:

$$\begin{aligned}\Big\|\sum_{i\in A}(b_i - a_i)e_i\Big\| &\le K\|(b_i-a_i)_{i\in A}\|_{p,1}\\ &\le KCn^{1/p-1/q}\|(b_i-a_i)_{i\in A}\|_{q,\infty}\\ &\le K^2Cn^{1/p-1/q}\|x-\textstyle\sum_{i\in B}b_ie_i\|\\ &\le K^2Cn^{1/p-1/q}(\sigma_n(x)+\varepsilon).\end{aligned} \tag{3.2}$$

Similarly,

$$\begin{aligned}\Big\|\sum_{i\in B\setminus A}b_ie_i\Big\| &\le \Big\|\sum_{i\in B\setminus A}(b_i-a_i)e_i\Big\| + \Big\|\sum_{i\in B\setminus A}a_ie_i\Big\|\\ &\le K^2Cn^{1/p-1/q}(\sigma_n(x)+\varepsilon) + \Big\|\sum_{i\in B\setminus A}a_ie_i\Big\|.\end{aligned} \tag{3.3}$$

We clearly have $|A\setminus B| = |B\setminus A|$. As A is t-greedy set for x, we have $\min_{A\setminus B}|a_i| \ge t\max_{B\setminus A}|a_i|$. Therefore,

$$\begin{aligned}\Big\|\sum_{i\in B\setminus A}a_ie_i\Big\| &\le KCn^{1/p-1/q}\|(a_i)_{i\in B\setminus A}\|_{q,\infty}\\ &\le \frac{KCn^{1/p-1/q}}{t}\|(a_i)_{i\in A\setminus B}\|_{q,\infty}\\ &\le \frac{K^2Cn^{1/p-1/q}}{t}\|x-\sum_{i\in B}b_ie_i\|\\ &\le \frac{K^2Cn^{1/p-1/q}}{t}(\sigma_n(x)+\varepsilon).\end{aligned} \tag{3.4}$$

Since $\varepsilon > 0$ is arbitrary, combining (3.1)–(3.4) gives

$$\|x - \mathbf{G}_n^t(x)\| \le \Big(1 + 2K^2C + \frac{K^2C}{t}\Big)n^{1/p-1/q}\sigma_n(x),$$

and hence $\mathbf{L}(n,t) \le \Big(1 + 2K^2C + \frac{K^2C}{t}\Big)n^{1/p-1/q}$. The case $p = q$ is similar except $Cn^{1/p-1/q}$ is replaced by $1 + \log n$ throughout. □

Corollary 3.7 *Let $1 \le p < \infty$ and let (e_i) be a bounded Markushevitch basis such that $\phi_r(k) \le Ck^{1/p}$ for some $C > 0$. Then $\mathbf{L}(n,t) \le C'n^{1/p}/t$, for some constant C'.*

Proof Any basis satisfies the lower ∞-estimate. In order to apply Proposition 3.6, we need to show that (e_i) has a weak upper p-estimate.

By the triangle inequality $\|\sum_{i\in A}\pm e_i\| \le 2Cn^{1/p}$ for all $A \subset \mathbb{N}$ with $|A| = n$. Suppose, for $x \in X$, the sequence $a_n = e_n^*(x)$ satisfies $\sum_n n^{1/p-1}a_n^* = \gamma$. Let (n_i) be a non-decreasing enumeration of this sequence – that is, $|a_{n_i}| = a_i^*$ for every i.

Set $\varepsilon_i = \text{sign}(a_{n_i})$, $c_i = a_i^* - a_{i+1}^*$, and $y_i = \sum_{j=1}^i \varepsilon_j e_{n_j}$. Note that, for every i, $i^{1/p} - (i-1)^{1/p} \le i^{1/p-1}$, hence

$$(2C)^{-1} \sum_i |c_i| \|y_i\| \le \sum_i (a_i^* - a_{i+1}^*) i^{1/p} = \sum_i a_i^* (i^{1/p} - (i-1)^{1/p}) \le \gamma.$$

Consequently, $\sum_i c_i y_i$ converges in X. For every i, we have $e_i^*(\sum_i c_i y_i) = e_i^*(x)$, hence $\sum_i c_i y_i = x$. By the above, $\|x\| \le 2C\gamma$. □

Remark 3.8 The estimates of Proposition 3.6 and Corollary 3.7 are sharp, even for unconditional (hence quasi-greedy) bases. For $q > p$, consider the canonical basis of $\ell_q \oplus_q \ell_p$ ($c_0 \oplus_\infty \ell_p$ if $q = \infty$). This basis clearly possesses the lower q- and upper p-estimates, with constant 1. Denote the bases of ℓ_q and ℓ_p by (e_i) and (f_i) respectively. Fix $c > 1$, and let $x = \sum_{i=1}^n \big(cte_i + f_i\big)$. One possible realization of the t-greedy algorithm gives $\mathbf{G}_m^t(x) = ct \sum_{i=1}^n e_i$, hence $\|x - \mathbf{G}_m^t(x)\| = n^{1/p}$. On the other hand, $\sigma_n(x) \le \tilde{\sigma}_n(x) \le \|ct \sum_{i=1}^n e_i\| = ctn^{1/q}$. As c can be arbitrarily close to 1, we obtain $\mathbf{L}(n,t) \ge n^{1/p-1/q}/t$, showing the optimality of Proposition 3.6. Note that $\phi_r(k) = k^{1/p}$, hence, for $q = \infty$, we witness the optimality of Corollary 3.7.

We can also show the optimality of Proposition 3.6 for $p = q = 2$, once more for quasi-greedy basis. By [6, Theorem 3.1 and Corollary 3.11], there exists a quasi-greedy democratic basis in $c_0 \oplus \ell_1 \oplus \ell_2$, so that $\phi_r(n) \sim \phi_l(n) \sim \sqrt{n}$. The weak upper 2-estimate follows from the proof of Corollary 3.7, whereas the weak lower 2-estimate follows from Lemma 6.1(2). Furthermore, [6, Corollary 3.11] gives $\mathbf{k}(n) \ge c \log n$ for this basis (c is a constant). By Theorem 3.1, $\mathbf{L}(n,t) \ge \mathbf{k}(n) - 1$.

Remark 3.9 We also present two examples of sharpness of Proposition 3.6 for bases which are not quasi-greedy. Throughout, we use some well-known facts about Lorentz spaces, see e.g. the survey [1].

First pick $p \in (1,2)$. Set $q = p/(p-1)$ and $\gamma = 2/p - 1$ (so $1/p = (1+\gamma)/2$, and $1/q = 1 - 1/p = (1-\gamma)/2$). Consider the measures μ and ν on $[-\pi, \pi]$, by setting $d\mu = |t|^{-\gamma}\, dt$ and $d\nu = |t|^{\gamma}\, dt$. The trigonometric system forms a non-quasi-greedy Schauder basis in both $L_2(\mu)$ and $L_2(\nu)$, see e.g. [11]. Denote by $e_1, e_2, \ldots$ ($f_1, f_2, \ldots$) the trigonometric basis in $L_2(\mu)$ (resp. $L_2(\nu)$), enumerated as $1, e^{it}, e^{-it}, e^{2it}, e^{-2it}, \ldots$.

First concentrate on the basis (e_i) in $L_2(\mu)$. Clearly this basis satisfies the lower 2-estimate:

$$\Big\| \sum_i \alpha_i e_i \Big\|_{L_2(\mu)} \ge \pi^{-\gamma} \left(\int_{-\pi}^{\pi} \Big| \sum_i \alpha_i e_i \Big|^2 dt \right)^{1/2} = \sqrt{2}\pi^{1/2-\gamma} \left(\sum_i |\alpha_i|^2 \right)^{1/2}.$$

Next show that $\phi_r(n) \sim n^{1/p}$ (once this is established, the weak upper p-estimate will follow, as in the proof of Corollary 3.7). The lower estimate on ϕ_r is proved in [6, Lemma 3.7]. For the upper estimate, recall the well-known fact that $\int |\phi\psi| \le \int \phi^* \psi^*$ (ϕ^* and ψ^* are decreasing rearrangments of ϕ and ψ respectively). Consequently, if f is a function of $[0, \pi]$ with $0 \le f \le n^2$, and $\int f(t)\, dt = n$, then $\int f(t) t^{-\gamma}\, dt \le n^{1+\gamma}/(1-\gamma)$ (the equality is attained when $f(t) = n^2 \mathbf{1}_{[0,1/n]}$). Now suppose $A \subset \mathbb{N}$ has cardinality n. Applying our observation to $f = |\sum_{j\in A} e_j|^2$, we obtain $\|\sum_{j\in A} e_j\|_{L_2(\mu)} \prec n^{(1+\gamma)/2} = n^{1/p}$.

Use [6, Lemma 3.7] to find $\varepsilon_1, \ldots, \varepsilon_{2n+1} \in \{-1, 1\}$ so that $\|\sum_i \varepsilon_i e_i\|_{L_2(\mu)} \sim \sqrt{n}$, while $\|\sum_i e_i\|_{L_2(\mu)} \sim n^{1/p}$. Let $B = \{i : \varepsilon_i = 1\}$ and $C = \{i : \varepsilon_i = -1\}$. For $\varepsilon > 0$ set $x = (1+\varepsilon)\sum_{i\in B} e_i - \sum_{i\in C} e_i$. For $\varepsilon < 1/n$ we have $\|x\| \sim \sqrt{n}$, yet $\|x - \mathbf{G}_{|B|}(x)\| = \|\sum_{i\in C} e_i\| \sim n^{1/p}$. Consequently, $\mathbf{L}(|B|, 1) \succ |B|^{1/p-1/2}$. By the above, $|B| \sim n$. Thus, the estimates on $\mathbf{L}(n, t)$ obtained in Proposition 3.6 are optimal for this basis.

In the second example the optimality of these estimates is shown for a basis with a weak upper p-estimate, and a weak lower q-estimate. Following [6, Section 3], define the Schauder basis (g_j) in $L_2(\mu) \oplus_2 L_2(\nu)$ by setting, for $k \in \mathbb{N}$, $g_{2k-1} = (e_k + f_k)/\sqrt{2}$ and $g_{2k} = (e_k - f_k)/\sqrt{2}$. By the proof of [6, Proposition 3.10], for any odd n we can have $\|\sum_{k=1}^{2n} g_k\| \sim n^{1/q}$, yet $\|\sum_{k=1}^{n} g_{2k-1}\| \sim n^{1/p}$. As in the previous paragraph, we conclude that $\mathbf{L}(n, 1) \succ n^{1/p-1/q}$.

Next show that (g_j) satisfies the weak upper p-estimate, and the weak lower q-estimate. Consider

$$x = \sum_k \left(\alpha_k g_{2k-1} + \beta_k g_{2k}\right) = \frac{1}{\sqrt{2}}\left(\sum_k (\alpha_k + \beta_k) e_k\right) \oplus_2 \left(\sum_k (\alpha_k - \beta_k) f_k\right). \tag{3.5}$$

We have to show that

$$\|(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots)\|_{q,\infty} \prec \|x\| \prec \|(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots)\|_{p,1}. \tag{3.6}$$

Start by recalling that, for any sequence (γ_i),

$$\|(\gamma_i)\|_{q,\infty} \prec \left(\sum_i |\gamma_i|^2\right)^{1/2} = \|(\gamma_i)\|_2 \prec \|(\gamma_i)\|_{p,1}. \tag{3.7}$$

The basis (f_k) satisfies the upper 2-estimate:

$$\Big\|\sum_i \alpha_i f_i\Big\|_{L_2(\nu)} \le \pi^{\gamma}\left(\int_{-\pi}^{\pi} \Big|\sum_i \alpha_i f_i\Big|^2\, dt\right)^{1/2} = \sqrt{2}\pi^{1/2+\gamma}\left(\sum_i |\alpha_i|^2\right)^{1/2}.$$

Thus, by (3.7), (3.5), and the triangle inequality for $\|\cdot\|_{p,1}$,

$$\begin{aligned}\|x\| &\prec \|(\alpha_k + \beta_k)\|_{p,1} + \|(\alpha_k - \beta_k)\|_2 \prec \|(\alpha_k + \beta_k)\|_{p,1} + \|(\alpha_k - \beta_k)\|_{p,1} \\ &\sim \|(\alpha_k)\|_{p,1} + \|(\beta_k)\|_{p,1} \prec \|(\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots)\|_{p,1},\end{aligned}$$

yielding the right hand side of (3.6).

Next note that (f_i) satisfies the weak lower q-estimate. Indeed, the functions $f_i'(t) = e_i(t)|t|^{\gamma}$ are biorthogonal to (e_i) in $L_2(\mu)$. By duality, the sequence (f_i') satisfies the weak lower q-estimate. Now observe that $U : L_2(\mu) \to L_2(\nu) : f_i' \to f_i$

is an isometry. Moreover, (e_i) satisfies the lower 2-estimate, hence the weak lower q-estimate as well. As $\| \cdot \|_{q,\infty}$ is a quasi-norm, we obtain

$$\|x\| \succ \|(\alpha_k + \beta_k)\|_2 + \|(\alpha_k - \beta_k)\|_{q,\infty} \succ \|(\alpha_k + \beta_k)\|_{q,\infty} + \|(\alpha_k - \beta_k)\|_{q,\infty} \sim \|(\alpha_k)\|_{q,\infty} + \|(\beta_k)\|_{q,\infty} \succ \|(\alpha_1, \beta_1, \alpha_2, \beta_2, \dots)\|_{q,\infty}.$$

This yields the left hand side of (3.6).

4 The Chebyshevian Lebesgue constant

Theorem 4.1 *For any $\mathfrak{K}$-quasi-greedy basis,*

$$\frac{\boldsymbol{\mu_d}(n)}{2t\mathfrak{K}} \le \mathbf{L}_{\mathrm{ch}}(n,t) \le \frac{20\mathfrak{K}^3 \boldsymbol{\mu_d}(n)}{t}.$$

As a corollary, we recover a result from [2].

Corollary 4.2 *Any almost greedy basis is semi-greedy.*

Recall that (e_i) is *almost greedy* if there exists a constant C so that $\|x - \mathbf{G}_n(x)\| \le C\tilde{\sigma}_n(x)$ for any $n \in \mathbb{N}$ and $x \in X$, and *semi-greedy* if there exists a constant C so that $\|x - \mathbf{CG}_n(x)\| \le C\sigma_n(x)$, for any n and x.

Proof By [2], a basis is almost greedy if and only if it is quasi-greedy and democratic (that is, $\sup_n \boldsymbol{\mu}(n) < \infty$). In this case $\sup_n \mathbf{L}_{\mathrm{ch}}(n, 1) < \infty$, hence the basis is semi-greedy. □

Below, we shall use the "truncation function"

$$\mathbf{F}_M : \mathbb{R} \to \mathbb{R} : t \mapsto \begin{cases} -M & t < -M \\ t & -M \le t \le M \\ M & t > M \end{cases}.$$

Abusing the notation slightly, we shall write

$$\mathbf{F}_M(x) = x - \sum_i \Big(e_i^*(x) - \mathbf{F}_M\big(e_i^*(x)\big)\Big) e_i.$$

The sum above converges, since the set $\{i \in \mathbb{N} : |e_i^*(x)| > M\}$ is finite. Moreover, $\mathbf{F}_M(x)$ is the only element $y \in X$ with the property that, for every i, $e_i^*(y) = \mathbf{F}_M(e_i^*(x))$. By [2, Proposition 3.1], $\|\mathbf{F}_M(x)\| \le (1 + 3\mathfrak{K})\|x\|$.

Proof (The upper estimate in Theorem 4.1) For $x \in X$ let $a_i = e_i^*(x)$, and fix $\varepsilon > 0$. Suppose a set $A \subset \mathbb{N}$ of cardinality n is t-greedy for x. Let $M = \max_{i \notin A} |a_i|$, then $\min_{i \in A} |a_i| \ge tM$. We have to show that there exists $w \in X$ so that $\mathrm{supp}(x - w) \subset A$, and $\|w\| \prec 20t^{-1}\mathfrak{K}^3 \boldsymbol{\mu_d}(n)(\sigma_n(x) + \varepsilon)$.

Pick $z = \sum_{i \in B} b_i e_i$, where $|B| \leq n$, and $\|x - z\| < \sigma_n(x) + \varepsilon$. Set $y = x - z$ and

$$y_i = e_i^*(y) = \begin{cases} a_i - b_i & i \in B \\ a_i & i \notin B \end{cases}.$$

We claim that $w = P_A \mathbf{F}_M(y) + P_{A^c} x$ has the desired properties. Indeed, $x - w$ is supported on A. To estimate $\|w\|$, note that, for $i \notin B$, $y_i = a_i$. For $i \notin A$, $\mathbf{F}_M(a_i) = a_i$, hence, for $i \notin A \cup B$, $a_i = \mathbf{F}_M(y_i)$. Thus,

$$w = \mathbf{F}_M(y) + \sum_{i \in B \backslash A} (a_i - \mathbf{F}_M(y_i)) e_i. \tag{4.1}$$

We use [2, Proposition 3.1] to estimate on the first summand:

$$\|\mathbf{F}_M(y)\| \leq (1 + 3\mathfrak{K}) \|y\| = (1 + 3\mathfrak{K}) \|x - z\|. \tag{4.2}$$

To handle the second summand, set $k = |B \backslash A|$. For $i \in B \backslash A$, $|a_i| \leq M$, hence $|a_i - \mathbf{F}_M(y_i)| \leq 2M$. By Lemma 6.1(1),

$$\left\| \sum_{i \in B \backslash A} (a_i - \mathbf{F}_M(y_i)) e_i \right\| \leq 4M\mathfrak{K} \left\| \sum_{i \in B \backslash A} e_i \right\|. \tag{4.3}$$

On the other hand, for $i \in A \backslash B$, $a_i = y_i$, and $|a_i| \geq tM$, hence by Lemma 2.2,

$$M \leq t^{-1} \frac{4\mathfrak{K}^2 \|x - z\|}{\| \sum_{i \in A \backslash B} e_i \|}.$$

Plugging this into (4.3), we get:

$$\left\| \sum_{i \in B \backslash A} (a_i - \mathbf{F}_M(y_i)) e_i \right\| \leq \frac{16}{t} \frac{\| \sum_{i \in B \backslash A} e_i \|}{\| \sum_{i \in A \backslash B} e_i \|} \mathfrak{K}^3 \|x - z\| \leq \frac{16}{t} \boldsymbol{\mu_d}(n) \mathfrak{K}^3 \|x - z\|.$$

Together with (4.2), we obtain:

$$\|w\| \leq \left(\frac{16}{t} \boldsymbol{\mu_d}(n) \mathfrak{K}^3 + 1 + 3\mathfrak{K} \right) \|x - z\| \leq \frac{20\mathfrak{K}^3 \boldsymbol{\mu_d}(n)}{t} (\sigma_n(x) + \varepsilon).$$

As ε can be arbitrarily close to 0, we are done. □

Proof (The lower estimate in Theorem 4.1) Fix $n \in \mathbb{N}$ and $\varepsilon > 0$. Find $A, B \subset \mathbb{N}$, so that $A \cap B = \emptyset$, $|A| = |B| = k \leq n$, and

$$\left\| \sum_{i \in A} e_i \right\| \geq (\boldsymbol{\mu_d}(n) - \varepsilon) \left\| \sum_{i \in B} e_i \right\|.$$

Pick a set C, disjoint from A and B, so that $|C| = n - k$. Consider

$$x = (t+\varepsilon) \sum_{i \in B \cup C} e_i + \sum_{i \in A} e_i .$$

We can find a Chebyshev t-greedy approximant $\mathbf{CG}_n^t(x)$ supported on $B \cup C$, and then $y = x - \mathbf{CG}_n^t(x) = \sum_{i \in A} e_i + \sum_{i \in B \cup C} y_i e_i$. Let $D = \{i \in B \cup C : |y_i| \ge 1\}$. Both $\sum_{i \in A} e_i + \sum_{i \in D} y_i e_i$ and $\sum_{i \in D} y_i e_i$ are greedy approximants of y, hence

$$\max \left\{ \left\| \sum_{i \in A} e_i + \sum_{i \in D} y_i e_i \right\|, \left\| \sum_{i \in D} y_i e_i \right\| \right\} \le \mathfrak{K} \|y\|.$$

By the triangle inequality, $\| \sum_{i \in A} e_i \| \le 2\mathfrak{K} \|y\|$. Thus,

$$\begin{aligned} \|x - \mathbf{CG}_n^t(x)\| &\ge \frac{1}{2\mathfrak{K}} \left\| \sum_{i \in A} e_i \right\| \ge \frac{\boldsymbol{\mu_d}(n) - \varepsilon}{2(t+\varepsilon)\mathfrak{K}} \left\| (t+\varepsilon) \sum_{i \in B} e_i \right\| \\ &= \frac{\boldsymbol{\mu_d}(n) - \varepsilon}{2(t+\varepsilon)\mathfrak{K}} \|x - P_{A \cup C} x\| \ge \frac{\boldsymbol{\mu_d}(n) - \varepsilon}{2(t+\varepsilon)\mathfrak{K}} \tilde{\sigma}_n(x) \ge \frac{\boldsymbol{\mu_d}(n) - \varepsilon}{2(t+\varepsilon)\mathfrak{K}} \sigma_n(x) \end{aligned}$$

(since $|A \cup C| = n$). As ε can be arbitrarily small, we are done. □

5 The residual Lebesgue constant

Theorem 5.1 *For any $\mathfrak{K}$-quasi-greedy basis,*

$$t^{-1} \mathbf{c}(n) \le \mathbf{L}_{\mathrm{re}}(n, t) \le 1 + 4\mathfrak{K}^2 + 8t^{-1} \mathfrak{K}^3 \mathbf{c}(n).$$

Proof (The upper estimate in Theorem 5.1) For $x \in X$ set $a_i = e_i^*(x)$. Suppose A is a t-greedy subset of $\mathbb{N}$, of cardinality n, and set $B = [1, n]$. Let $M = \min_{i \in A} |a_i|$, then $|a_i| \le t^{-1} M$ for $i \notin A$. By the triangle inequality,

$$\|x - \mathbf{G}_n^t(x)\| = \|P_{A^c} x\| \le \|x - P_B x\| + \|P_{A \setminus B} x\| + \|P_{B \setminus A} x\|. \tag{5.1}$$

Let $y = P_{B^c} x$, then $\|y\| = \hat{\sigma}_n(x)$. For $i \in A \setminus B$, we have $|e_i^*(y)| \ge M$, hence by Lemma 2.2, $M \| \sum_{i \in A \setminus B} e_i \| \le 4\mathfrak{K}^2 \|y\|$. By Lemmas 2.2 and 6.1(1),

$$\|P_{B \setminus A} x\| \le 2t^{-1} M \mathfrak{K} \left\| \sum_{i \in B \setminus A} e_i \right\| \le 2t^{-1} M \mathfrak{K} \mathbf{c}(n) \left\| \sum_{i \in A \setminus B} e_i \right\| \le 8t^{-1} \mathfrak{K}^3 \mathbf{c}(n) \|y\|.$$

Plug the above results into (5.1) to obtain the upper estimate for $\mathbf{L}_{\mathrm{re}}(n, t)$. □

Proof (The lower estimate in Theorem 5.1) Fix $\varepsilon > 0$, and find sets $A \subset [1, n]$ and $B \subset [n+1, \infty)$ so that $|A| = k = |B|$, and

$$\mathbf{c}(n) - \varepsilon < \frac{\| \sum_{i \in A} e_i \|}{\| \sum_{i \in B} e_i \|}.$$

Consider $x = \sum_{i=1}^{n} e_i + (t + \varepsilon) \sum_{i \in B} e_i$. Then $B \cup ([1, n] \backslash A)$ is a t-greedy set for x, hence one can run the t-greedy algorithm in such a way that $\|x - \mathbf{G}_n^t(x)\| = \| \sum_{i \in A} e_i \|$. On the other hand, $\hat{\sigma}_n(x) = \|P_{[n+1,\infty)} x\| = (t + \varepsilon) \| \sum_{i \in B} e_i \|$. The lower estimate follows from comparing these two quantities. □

Appendix: The complex case

The results above are stated for the real case. The complex case is similar, but the constants are different. As customary, we set

$$\operatorname{sign} z = \begin{cases} z/|z| & z \neq 0 \\ 0 & z = 0 \end{cases}.$$

The following result is present (implicitly or explicitly) in [5, Appendix] (the better-known real case is in [3, Lemmas 2.1 and 2.2]):

Lemma 6.1 *Suppose (e_i) is a $\mathfrak{K}$-quasi-greedy basis in a Banach space X.*

1. *If A is a finite set, then $\| \sum_{i \in A} a_i e_i \| \leq 4\sqrt{2}\mathfrak{K} \max_i |a_i| \| \sum_{i \in A} e_i \|$. Moreover, if the a_i's are real, then $\| \sum_{i \in A} a_i e_i \| \leq 2\mathfrak{K} \max_i |a_i| \| \sum_{i \in A} e_i \|$.*
2. *Suppose A is a greedy set for $x \in X$. Let $M = \min_{i \in A} |e_i^*(x)|$. Then*

$$\frac{M}{8\sqrt{2}\mathfrak{K}^2} \| \sum_{i \in A} e_i \| \leq \frac{M}{2\mathfrak{K}} \| \sum_{i \in A} \operatorname{sign}\big(e_i^*(x)\big) e_i \| \leq \|x\|.$$

For $M > 0$, define

$$\mathbf{F}_M : \mathbb{C} \to \mathbb{C} : z \mapsto \begin{cases} \operatorname{sign}(z) M & |z| > M \\ z & |z| \leq M \end{cases}.$$

For $x \in X$, we set $\mathbf{F}_M(x) = x - \sum_i \big(e_i^*(x) - \mathbf{F}_M(e_i^*(x))\big) e_i$ (the sum converges, and $e_i^*(\mathbf{F}_M(x)) = \mathbf{F}_M(e_i^*(x))$ for every i). As in [2, Proposition 3.1], one can prove:

Lemma 6.2 *In the above notation, $\|\mathbf{F}_M(x)\| \leq (1 + 3\mathfrak{K})\|x\|$.*

As in Sect. 2, we obtain:

Lemma 6.3 *Suppose $(e_i) \subset X$ is a basis with a quasi-greedy constant $\mathfrak{K}$, and a set A is t-greedy for $x \in X$. Then $\|P_A x\| \leq (1 + 8\sqrt{2}t^{-1}\mathfrak{K})\mathfrak{K}\|x\|$.*

Lemma 6.4 *Suppose (e_i) is a $\mathfrak{K}$-quasi-greedy basis in X. Consider $x \in X$, and let $a_i = e_i^*(x)$, for $i \in \mathbb{N}$. Suppose a finite set $A \subset \mathbb{N}$ satisfies $\min_{i\in A}|a_i| \geq M$. Then $M\|\sum_{i\in A} \mathrm{sign}\,(a_i)e_i\| \leq 2\mathfrak{K}^2\|x\|$. Furthermore, $M\|\sum_{i\in A} e_i\| \leq 8\mathfrak{K}^2\|x\|$.*

Proof Consider $C = \{i : |a_i| \geq M\}$ (note that $A \subset C$). For the brevity of notation, let $e_i' = \mathrm{sign}\,(a_i)e_i$ (if $a_i = 0$, let $e_i' = e_i$). Clearly the basis (e_i') is $\mathfrak{K}$-quasi-greedy. Set $y = \sum_{i\in C} e_i'$. By Lemma 6.1(2), $M\|y\| \leq 2\mathfrak{K}\|x\|$. For $\varepsilon > 0$, let

$$y_\varepsilon = \sum_{i\in A} e_i' + (1+\varepsilon)\sum_{i\in C\setminus A} e_i' = \sum_{i\in C} e_i' + \varepsilon \sum_{i\in C\setminus A} e_i'.$$

By the triangle inequality, $\|y_\varepsilon\| \leq \|y\| + \varepsilon\sum_{i\in C\setminus A}\|e_i\|$. Furthermore, $\|\sum_{i\in A} e_i'\| \leq \mathfrak{K}\|y_\varepsilon\|$. As ε is arbitrary, we establish the first statement of the lemma.

The reasoning above also shows that $M\|\sum_{i\in B} e_i'\| \leq 2\mathfrak{K}^2\|x\|$ for any $B \subset A$. Let S be the absolute convex hull of the elements $\sum_{i\in B} e_i'$—that is,

$$S = \left\{ \sum_{B\subset A} t_B \sum_{i\in B} e_i' : \sum_{B\subset A} |t_B| \leq 1 \right\}.$$

We claim that $\sum_{i\in A} e_i = \sum_{i\in A} \omega_i e_i' \in 4S$ here $|\omega_i| = 1$. Otherwise, by Hahn-Banach Separation Theorem, there exists a sequence $(b_i)_{i\in A} \in \mathbb{C}^{|A|}$ so that $|\sum_{i\in B} b_i| < 1$ whenever $B \subset A$, yet $|\sum_{i\in A}\omega_i b_i| > 4$. Let $B_+ = \{i \in A : \Re b_i \geq 0\}$ and $B_- = \{i \in A : \Re b_i < 0\}$.

$$\sum_{i\in B_+} \Re b_i \leq \left|\sum_{i\in B_+} b_i\right| \leq 1,$$

and similarly, $\sum_{i\in B_-}(-\Re b_i) \leq 1$. Therefore,

$$\sum_{i\in A} |\Re b_i| = \sum_{i\in B_+} |\Re b_i| + \sum_{i\in B_-} |\Re b_i| \leq 2.$$

The same way, we show that $\sum_{i\in A} |\Im b_i| \leq 2$. Consequently,

$$\left|\sum_{i\in A} \omega_i b_i\right| \leq \sum_{i\in A} |b_i| \leq \sum_{i\in A} \left(|\Re|b_i| + |\Im b_i|\right) \leq 4,$$

yielding a contradiction. This establishes the second statement of our lemma. □

These results allow us to emulate the proofs of previous sections, and to estimate the Lebesgue constants:

Theorem 6.5 *Suppose (e_i) is a $\mathfrak{K}$-quasi-greedy basis in a complex Banach space X. Then:*

1.

$$\max\left\{\mathbf{k}_c(n), t^{-1}\boldsymbol{\mu}_{\boldsymbol{d}}(n)\right\} \leq \mathbf{L}(n,t) \leq 1 + 2\mathbf{k}(n) + 32\sqrt{2}t^{-1}\mathfrak{K}^3\boldsymbol{\mu}_{\boldsymbol{d}}(n).$$

2.

$$\frac{\boldsymbol{\mu}_{\boldsymbol{d}}(n)}{2t\mathfrak{K}} \leq \mathbf{L}_{\mathrm{ch}}(n,t) \leq \frac{100\mathfrak{K}^3\boldsymbol{\mu}_{\boldsymbol{d}}(n)}{t}.$$

3.

$$t^{-1}\mathbf{c}(n) \leq \mathbf{L}_{\mathrm{re}}(n,t) \leq 1 + 8\mathfrak{K}^2 + 32\sqrt{2}t^{-1}\mathfrak{K}^3\mathbf{c}(n).$$

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