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Dicriticals of pencils and Dedekind's Gauss lemma

Shreeram S. Abhyankar

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Abstract The dicritical divisors of a pencil at a simple point of a surface constitute an important tool in affine algebraic geometry, i.e., in the study of polynomial rings. These dicriticals may be viewed as certain nodes of the singularity tree of a generic member of the pencil. Dedekind's generalization of Gauss Lemma plays a significant role.

Keywords Dicritical divisor · Pencil · Singularity tree · Generic member

Mathematics Subject Classification Primary 14A05

1 Introduction

In this paper we shall connect the dicritical divisors of a pencil at a simple point of a surface with certain nodes in the singularity tree of the generic member of the pencil. For the definition of dicritical divisors and other terminology to be used in this paper see Sect. 2 and Remark (4.0) of [11] as well as Sects. 2 to 4 of [13] which may be viewed as a preamble to the present paper. For singularity trees see Fig. 5 on page 426 of [4] and Fig. 18.2 on page 132 of [6]. The original analytical theory of dicritical divisors was developed by many authors such as Artal-Bartolo [17], Lê-Weber [22] and Mattei-Moussu [23], and then it was algebracized in [9–12, 14–16]. For basic background material see my papers and books [1–3, 5–8] and the books of Nagata [25] and Zariski-Samuel [27].

In Sect. 2 we discuss the ideal theory as we pass from a ring S to the polynomial ring $S[t]$ to the localization $S^t = S[t]_N$ where N is the multiplicative set of all polynomials whose coefficients generate the unit ideal in S. In Lemma 2.2 we

S.S. Abhyankar (✉)
Mathematics Department, Purdue University, West Lafayette, IN 47907, USA
e-mail: ram@cs.purdue.edu

cite the McCoy Lemma and in Lemma 2.4 we cite Dedekind's version of Gauss Lemma. In Lemma 2.7, for a regular local ring S of any dimension n, we relate the QDTs = Quadratic Transforms of S to the QDTs of S^t. In Theorem 2.9 we specialize to the $n = 2$ case. In Sect. 3 we use this together with the material of [13] to give a bijection of the dicriticals of a pencils to the maximal pines of the singularity tree of its generic member. It may be noted that Theorem 3.2 of Sect. 3 is one of the main results of this paper. In Sects. 4 and 5, after discussing completions and classification of valuations, we give a bijection from maximal pines to analytic branches.

Thanks are due to Heinzer, Luengo, Sathaye, and Shannon for valuable discussions. The possibility of relating dicriticals to branches was first suggested by Luengo.

2 Extension of the quadratic tree

Given any two-dimensional regular local domain R, consider the rational function field $L^t = L(t)$ where t is an indeterminate over the quotient field L of R. In the beginning of Sect. 2 of [13] we introduced the following eight subsets of $\rho(L)$ = the set of all subrings of L and $\rho(L^t)$ = the set of all subrings of L^t:

$D(R)^\Delta$ = set of all prime divisors of R.
$Q(R)$ = the **quadratic tree** of R.
$Q(R)^\Delta = Q(R) \coprod D(R)^\Delta$ = the **full quadratic tree** of R.
$Q^t(R) = \{S^t : S \in Q(R)\}$ = t-extension of the quadratic tree of R.
$\mathfrak{Q}(R, I)$ = the **ideal tree** of a nonzero ideal I in R.
$\mathfrak{P}(R, I)$ = the **singularity tree** of a nonzero ideal I in R.
$\mathfrak{Q}(R, z) = \mathfrak{Q}(R, J_R(z))$ = the **ideal tree** of $0 \neq z \in L$ in R.
$\mathfrak{P}(R, F) = \mathfrak{P}(R, FR)$ = the **singularity tree** of $0 \neq F \in R$ in R.

Here $D(R)^\Delta$ is the set of all DVRs S with quotient field L such that S dominates R and is residually transcendental over R, while $Q(R)$ is the set of all two dimensional regular local domains S with quotient field L such that S dominates R. Moreover, $\mathfrak{Q}(R, I)$ is the set of all $T \in Q(R)$ at which the transform $(R, T)(I)$ is a nonprincipal ideal, while $\mathfrak{P}(R, I)$ is the set of all $T \in Q(R)$ at which the transform $(R, T)(I)$ is a nonunit ideal. We also introduced the j-th levels $D_j(R)^\Delta, \dots, \mathfrak{P}_j(R, F)$ of these sets. Recall that the map $\rho^t : \rho(L) \to \rho(L^t)$ is given by $S \mapsto S^t$ = the localization of the polynomial ring $S[t]$ at the multiplicative set of all those members of $S[t]$ whose coefficients generate the unit ideal in S. In Nagata's notation (page 18 of [25]), $S^t = S(t)$.

As a first step in describing $Q^t(R)$, we prove the following Lemma 2.1 which was stated as item (4.1) in Sect. 4 of [13] without a detailed proof.

Lemma 2.1 *Let $S \in \rho(L)$ be a local domain. Then S^t is a local domain which dominates S, and we have* $\dim(S^t) = \dim(S)$ *with $M(S^t) = M(S)S^t$ and $S^t \cap L = S$. Moreover, if S is a regular local domain then so is S^t.*

Proof By (T30) on page 233 of [8] we see that $M(S)S[t]$ is a prime ideal in $S[t]$ whose height equals the height of $M(S)$ in S. Also S^t is the localization of $S[t]$

at $M(S)S[t]$. Therefore we get everything except the inclusion $S^t \cap L \subset S$. Now L contains the quotient field E of S and we have $S^t \subset E(t)$ with $E(t) \cap L = E$, and hence $S^t \cap L \subset S^t \cap E$. Therefore it suffices to show that $S^t \cap E \subset S$. Now any nonzero element w of E can be expressed as $w = u/v$ with $u \neq 0 \neq v$ in S; moreover, $w \in S^t \Leftrightarrow uS^t \subset vS^t$, whereas $w \in S \Leftrightarrow uS \subset vS$. Consequently it suffices to show that for any $u \neq 0 \neq v$ in S with $uS^t \subset vS^t$ we have $uS \subset vS$. Therefore it is enough to prove that for any ideal I in S we have $(IS^t) \cap S = I$, because we can apply this with $I = uS$ and $I = vS$. Hence it suffices to show that for any ideal I in S, upon letting $J = IS[t]$, we have $(IS[t]) \cap S = I$ and $(JS^t) \cap S[t] = J$. Out of this the first equation directly follows from the above (T30). In the following Lemma 2.2 Supplement to (T30), we shall show that if $I = \cap_{1 \le i \le h} Q_i$ is any irredundant primary decomposition of I in S then $IS[t] = \cap_{1 \le i \le h}(Q_i S[t])$ is an irredundant primary decomposition of $IS[t]$ in $S[t]$. In our case it follows that all the associated (minimal or not) primes of J are contained in the prime ideal $P = M(S)S[t]$ in $D = S[t]$ and hence the second equation $(JD_P) \cap D = J$ follows from the following fact proved in Sect. §7.1 of Lecture L4 of [8]: if a prime ideal P in a noetherian domain D contains all the associated primes of an ideal J in D then $(JD_P) \cap D = J$. □

Lemma 2.2 (Supplement to (T30)) [*The statement and proof of Lemma* 2.2 *are taken from Lemma* 14 *on pages* 85–86 *of Cohen's famous paper* [18] *on which I was weaned*; *I am reproducing it here for the benefit of the reader*; *I include an alternative proof communicated* [26] *by Sathaye*.] *For the univariate polynomial ring* $D = S[t]$ *over any* (*not necessarily noetherian*) *ring* S *we have the following*.

(1) *For any ideal* I *in* S *we have* $ID = I[t]$ *where* $I[t]$ *denotes the set of all polynomials* $f = f(t)$ *all whose coefficients belong to* I.
(2) *Any finite ideal intersection* $I = \cap_{1 \le i \le h} Q_i$ *in* S *gives the finite ideal intersection* $ID = \cap_{1 \le i \le h}(Q_i D)$ *in* D. *Moreover, if no* Q_i *can be deleted from the first intersection then no* $Q_i D$ *can be deleted from the second intersection.*
(3) *If for some* i, Q_i *is a primary ideal in* S *with* $P_i = \mathrm{rad}_S Q_i$ *then* $Q_i D$ *is a primary ideal in* D *with* $P_i D = \mathrm{rad}_D(Q_i D)$.
(4) *If* $fg = 0$ *with* $0 \neq f \in D$ *and* $g \in D$ *then* $cg = 0$ *for some* $0 \neq c \in S$. *This is called MCCOY'S Lemma* [24] *with implicit stipulation that* c *be a monomial in the coefficients of* f *and* g. *The stipulation is explicit in Sathaye's proof which is similar to Forsythe's* [20].

Proof (1) Was noted in (C12) on page 235 of [8], and noetherianness was not used in its proof. (2) Follows from (1). To prove (3) assume $\mathfrak{q} = Q_i$ is primary with $\mathrm{rad}_S \mathfrak{q} = \mathfrak{p} = P_i$. Then clearly $\mathfrak{q}D \subset \mathfrak{p}D$ and every element of $\mathfrak{p}D$ has some power belonging to $\mathfrak{q}D$. It remains to show that $fg \in \mathfrak{q}D$ with $f \notin \mathfrak{p}D \Rightarrow g \in \mathfrak{q}D$. Let $f = \sum f_i t^i$ and $g = \sum g_j t^j$ with f_i, g_j in S, and suppose that $f_0, \ldots, f_{m-1}$ belong to $\mathfrak{p}$ but f_m does not. Let $\mathfrak{a}$ be the ideal in S generated by $f_0, \ldots, f_{m-1}$. Then (i) $\mathfrak{a}^k \subset \mathfrak{q}$ for some $k \in \mathbb{N}_+$; we fix the smallest such k. Let $\mathfrak{q}_i = (\mathfrak{q} : \mathfrak{a}^{k-i})_S$ for $0 \le i \le k$. Then (ii) $fg \in \mathfrak{q}D \subset \mathfrak{q}_{i+1}D$ for $0 \le i < k$. Moreover we have (iii) $\mathfrak{a}\mathfrak{q}_i \subset \mathfrak{q}_{i+1}$ for $0 \le i < k$. Because k is the smallest, by (O8)($1^\bullet$) on page 110 of [8] we see that (iv) for $0 \le i < k$, the ideal $\mathfrak{q}_{i+1}$ is $\mathfrak{p}$-primary. (Cohen forgot to fix k to be the smallest.) Next we claim that (v) for $0 \le i < k$ we have $g \in \mathfrak{q}_i D \Rightarrow g \in \mathfrak{q}_{i+1} D$. So let $0 \le i < k$

be such that $g \in \mathfrak{q}_i D$; suppose if possible that $g \notin \mathfrak{q}_{i+1} D$; let g_n be the first g_j not in $\mathfrak{q}_{i+1}$; now the coefficient of t^{m+n} in fg equals

$$\cdots + f_{m-1}g_{n+1} + f_m g_n + f_{m+1}g_{n-1} + \cdots$$

and by (ii) this belongs to $\mathfrak{q}_{i+1}$; all terms following $f_m g_n$ are in $\mathfrak{q}_{i+1}$; so are those preceding it since $g \in \mathfrak{q}_i D \Rightarrow g_j \in \mathfrak{q}_i$ and by (iii) $\mathfrak{a}\mathfrak{q}_i \subset \mathfrak{q}_{i+1}$; thus $f_m g_n \in \mathfrak{q}_{i+1}$ which is impossible by (iv) since $f_m \notin \mathfrak{p}$ and $g_n \notin \mathfrak{q}_{i+1}$. Clearly $\mathfrak{q}_0 = S$ with $\mathfrak{q}_k = \mathfrak{q}$, and hence by (v) and induction on i we get $g \in \mathfrak{q}$.

To prove (4) let f, g in D be such that $fg = 0 \neq f$. We want to show $cf = 0$ for some $0 \neq c \in S$. Let S_1 be the smallest subring of S containing the coefficients of f and g. Then S_1 noetherian and we can try to find c in S_1. So replacing S by S_1 we may assume S to be noetherian. Let $(0) = \cap_{1 \le i \le h} Q_i$ be an irredundant primary decomposition in S with $P_i = \mathrm{rad}_S Q_i$. By (2) and (3), $P_i D$ are the associated primes of 0 in D and hence, by (3) on page 217 of [8], $g \in P_i D$ for some i and so all the coefficients of g belong to P_i and therefore, by (1) on page 216 of [8], they are annihilated by a single nonzero c in S.

Sathaye's Proof of (3). If g_j belongs to $\mathfrak{q}$ for all j then we have nothing to show. So assume the contrary and let g_n be the earliest g_j not belonging to $\mathfrak{q}$. Now

$$f = u - v \quad \text{with } u = f_m t^m + f_{m+1}t^{m+1} + \cdots \quad \text{and} \quad v = -f_0 - \cdots - f_{m-1}t^{m-1},$$

where $f_0, \dots, f_{m-1}$ belong to $\mathfrak{p}$ but f_m does not. By (i) we get $v^k \in \mathfrak{q}D$; here k need not be the smallest. Also

$$\left(u^k - v^k\right)g = fg\left(u^{k-1} + u^{k-2}v + \cdots + v^{k-1}\right) \in \mathfrak{q}D$$

and hence $u^k g \in \mathfrak{q}D$. Moreover

$$u^k g = f_m^k g_n t^{mk+n} + \text{higher degree terms} + \text{an element of } \mathfrak{q}D$$

but $f_m^k g_n \notin \mathfrak{q}$ which is a contradiction. Therefore $g \in \mathfrak{q}D$.

Sathaye's Proof of (4). We induct on $\deg(f) - \mathrm{ord}(f)$. Obvious when $\deg(f) - \mathrm{ord}(f) = 0$ because then we can take c to be the unique nonzero coefficient of f. If $fg_j = 0$ for all j then $f_i g_j = 0$ for all i, j, and we can take $c = f_i$ for any nonzero f_i. Henceforth suppose $fg_j \neq 0$ for some j; fix the smallest such j and let $f' = fg_j$; now $f'g = 0 \neq f'$ with $\deg(f') \le \deg(f)$ and $\mathrm{ord}(f') \ge \mathrm{ord}(f)$; if $\mathrm{ord}(f') = \mathrm{ord}(f) = \mu$ then, because $fg_{j'} = 0$ for all $j' < j$, we would get that the coefficient of $t^{\mu+j}$ in fg is $f_\mu g_j$ and hence nonzero. This would contradict the equation $fg = 0$; therefore $\mathrm{ord}(f') > \mathrm{ord}(f)$ and hence $\deg(f') - \mathrm{ord}(f') < \deg(f) - \mathrm{ord}(f)$. Consequently by the induction hypothesis there is a nonzero monomial c in the coefficients of f' and g such that $cg = 0$. Since $f' = fg_j$, c is a monomial in the coefficients of f and g. □

Lemma 2.3 (First Supplement to Lemma 2.1) *Lemma 2.1 remains true for any field L without assuming it to be the quotient field of a two dimensional regular local domain.*

Proof This is obvious because in the proof of Lemma 2.1 we never used the assumption that L is the quotient field of a two dimensional regular local domain.

In Lemma 2.7 we shall prove the equation $S^t \cap L = S$ of Lemma 2.1 in a more general context. For this purpose, in Lemmas 2.4 and 2.5 we shall visit a beautiful paper of Dedekind. □

Dedekind's Gauss Lemma 2.4 [*The statement and proof of Lemmas* 2.4 *and* 2.5 *are taken from pages* 36–38 *of Dedekind's paper* [19]; *also see Heinzer–Huneke* [21].] *For any polynomial $f = f(t)$ in an indeterminate t with coefficients in a ring S let $C(f)$ denote the ideal in S generated by all the coefficients of f. Let a, b in $S[t]$ and m, n in $\mathbb{N}$ be such that* $\deg(a) \le m$ *and* $\deg(b) \le n$. *Then $C(a)^n C(ab) = C(a)^{n+1} C(b)$.*

Proof Writing

$$a = \sum_{0 \le \mu \le m} a_\mu t^{m-\mu} \quad \text{and} \quad b = \sum_{0 \le \nu \le n} b_\mu t^{n-\nu} \quad \text{and} \quad ab = \sum_{0 \le \lambda \le m+n} c_\lambda t^{m+n-\lambda}$$

with a_μ, b_ν, c_λ in S, and putting $a_\mu = 0$ for all $\mu \in \mathbb{Z} \setminus \{0, \ldots, m\}$, we get

$$(1) \qquad \sum_{0 \le \nu \le n} a_{\lambda-\nu} b_\nu = c_\lambda \quad \text{for } 0 \le \lambda \le m+n.$$

Applying Cramer's Rule (E4.1) on page 164 of [8] to the above equation for $\lambda = r_i$ with $1 \le i \le n$, where $r = (r_0, \ldots, r_n) \in \{0, \ldots, m+n\}^{n+1}$, we get

$$(2) \qquad \alpha_r b_\nu = \beta_{r\nu} \quad \text{with } \alpha_r = \det(A) \quad \text{and} \quad \beta_{r\nu} = \det\big(B^{(\nu)}\big) \quad \text{for } 0 \le \nu \le n$$

where the $(n+1) \times (n+1)$ matrices $A = (A_{ij})$ and $B^{(\nu)} = (B_{ij}^{(\nu)})$ are given by

$$(3) \qquad A_{ij} = a_{r_i - j} \quad \text{and} \quad B_{ij}^{(\nu)} = \begin{cases} A_{ij} & \text{if } j \ne \nu \\ c_{r_i} & \text{if } j = \nu. \end{cases}$$

(1) shows $C(ab) \subset C(a)C(b)$ and hence $C(a)^n C(ab) \subset C(a)^{n+1} C(b)$. (3) tells us that $B^{(\nu)} \in C(a)^n C(ab)$ and hence (2) gives $(\{\alpha_r : r \in \{0, \ldots, m+n\}^{n+1}\}S)C(b) \subset C(a)^n C(ab)$. Therefore it suffices to show that $C(a)^{n+1} = \{\alpha_r : r \in \{0, \ldots, m+n\}^{n+1}\}S$. But this follows from Lemma 2.5 below. □

Dedekind's Basis Lemma 2.5 *Let m, n in $\mathbb{N}$. Let H_{mn} be the $\mathbb{Z}$-module consisting of all homogeneous polynomials of degree $n+1$ in indeterminates $X_0, \ldots, X_m$ with coefficients in $\mathbb{Z}$. For every $r = (r_0, \ldots, r_n) \in I_{mn} = \{0, \ldots, m+n\}^{n+1}$ let $\alpha_r = \det(A^{(r)})$ where $A^{(r)} = (A_{ij}^{(r)})$ is the $(n+1) \times (n+1)$ matrix with $A_{ij}^{(r)} = X_{r_i - j}$ where $X_\mu = 0$ for all $\mu \in \mathbb{Z} \setminus \{0, \ldots, m\}$. For every $r \in I_{mn}$ let $\sigma(r) = (\sigma_0(r), \ldots, \sigma_n(r)) \in \mathbb{Z}^{n+1}$ be given by putting $\sigma_i(r) = r_i - i$ for $0 \le i \le n$. Let $J_{mn} = \{r \in I_{mn} : r_0 < r_1 < \cdots < r_n\}$. Let $K_{mn} = \{r \in J_{mn} : \sigma(r) \in \{0, \ldots, m\}^{n+1}\}$. Then $|K_{mn}| = \binom{m+n+1}{n+1}$ and $\{\alpha_r : r \in K_{mn}\}$ is a free $\mathbb{Z}$-module basis of H_{mn}.*

Proof For r, s in K_{mn} let $r > s$ in K_{mn} mean that for some $l \in \{0, \ldots, n\}$ we have $r_0 = s_0, \ldots, r_{l-1} = s_{l-1}, r_l > s_l$. This converts K_{mn} into a linearly ordered set. Let M_{mn} be the set of all monomials of degree $n+1$ in $X_0, \ldots, X_m$. Now $|K_{mn}| = \binom{m+n+1}{n+1} = |M_{mn}|$ and hence we get a bijection $\tau : K_{mn} \to M_{mn}$ by putting $\tau(r) = \prod_{0 \le i \le n} X_{r_i - i}$. This converts M_{mn} into a linearly ordered set. Now for any $r \in K_{mn}$, the principal diagonal of α_r is $\tau(r)$ and every other term of α_r is smaller than $\tau(r)$. It follows that for every $r \in K_{mn}$ we have $\tau(r) = \sum_{s\in K_{mn}:s<r} p_{rs}\alpha_s$ with $p_{rs} \in \mathbb{Z}$. Therefore $\{\alpha_r : r \in K_{mn}\}$ is a free $\mathbb{Z}$-module basis of H_{mn}. □

Before reading Definition 2.6 needed for parts (4) and (5) of Lemma 2.7, the reader may like to review blowing up and quadratic transformations by browsing in pages 134–161 and 536–577 of [8]. Special attention may be given to the material on pages 559–566 of [8] which may be used tacitly. As a precursor to this material see (1.4) on pages 16–21 of [3] and Sect. 14 on pages 72–74 of [2].

Definition 2.6 For any regular local domain S we put $\overline{Q}_1(S) = \mathfrak{W}(S, M(S))^{\Delta}$ and $\overline{Q}_1(S)_i = \mathfrak{W}(S, M(S))^{\Delta}_i$ and we call these the first layer of the **total quadratic tree** of S and the i—the component of the first layer of the **total quadratic tree** of S respectively. Note that $\overline{Q}_1(S)_i$ is a set of i-dimensional regular local domains which dominate S and have the same quotient field as S, and we have the disjoint union $\overline{Q}_1(S) = \coprod_{i\in\mathbb{N}} \overline{Q}_1(S)_i$. Moreover, upon letting $n = \dim(S)$ we have: $\overline{Q}_1(S)_0 = \emptyset$ or $\{\mathrm{QF}(S)\}$ according as $n > 0$ or $n = 0$, $\overline{Q}_1(S)_1 = \{o(S)\}$ if $n > 0$, $\overline{Q}_1(S)_i =$ an infinite set if $2 \le i \le n$, and the set $\overline{Q}_1(S)_i$ is in a natural bijective correspondence with the set $(\mathbb{P}^{n-1}_{H(S)})^{\delta}_{n-i}$ of all $(n-i)$-dimensional irreducible subvarieties of the modelic projective space $(\mathbb{P}^{n-1}_{H(S)})^{\delta}$ as defined on page 158 [8]. Recall that for any ideal J in a domain A, the set of all prime ideals P in A with $J \subset P$ is denoted by $\mathrm{vspec}_A J$, and let us put $\mathfrak{V}(A, J) = \{T \in \mathfrak{V}(A) : JT \ne T\} = \{A_P : P \in \mathrm{vspec}_A J\}$, where $\mathfrak{V}(A)$ is the set of all localizations A_P at the various prime ideals P in A, and where $\mathrm{vspec}_A J$ is the set of all prime ideals P in A for which $J \subset P$.

Lemma 2.7 (Second Supplement to Lemma 2.1) *For any subring S of any field L, without assuming S to be a local domain, we have the following.*

(1) $S^t \cap L = S$.
(2) *The map $\rho^t : \rho(L) \to \rho(L^t)$ given by $S \mapsto S^t$ is an inclusion preserving map of the set $\rho(L)$ of all subrings of L into the set $\rho(L^t)$ of all subrings of $L^t = L(t)$.*
(3) *Assume that S is quasilocal. Then S^t is quasilocal with $M(S^t) = M(S)S^t$ such that S^t dominates S with $H(S^t) = K(\tau)$ where $K = H_{S^t}(S) =$ a field which is naturally isomorphic to $H(S)$ and $\tau = H_{S^t}(t) =$ a transcendental element over K. If $T \in \rho(L)$ is a quasilocal domain dominating S then T^t is a quasilocal domain dominating S^t.*
(4) *Assume that S is a regular local domain whose dimension is a positive integer n. Then for $1 \le i \le n$ we have $\rho^t(\overline{Q}_1(S)_i) \subset \overline{Q}_1(S^t)_i$ and for every $T' \in \overline{Q}_1(S^t)_i$ with $T' \notin \rho^t(\overline{Q}_1(S)_i)$, upon letting $T = T' \cap L$, we have $T \in \overline{Q}_1(S)_{i-1}$ and $T' \subsetneqq \rho^t(T) \in \mathfrak{V}(T')$. Observe that the containment $\rho^t(\overline{Q}_1(S)_i) \subset \overline{Q}_1(S^t)_i$ is induced by a natural bijection of these sets onto the sets $(\mathbb{P}^{n-1}_K)^{\delta}_{n-i} \subset (\mathbb{P}^{n-1}_{K(\tau)})^{\delta}_{n-i}$ where $K(\tau)$ is as in* (3) *above. In particular, for $2 \le i \le n$ we have $|\overline{Q}_1(S^t)_i \setminus \rho^t(\overline{Q}_1(S)_i)| = \infty$.*
(5) *Assume that S is a regular local domain whose dimension is a positive integer n. Let $x_1, \dots, x_n$ be generators of $M(S)$. Let*

$$A = S[x_2/x_1, \dots, x_n/x_1] \quad \textit{and} \quad A' = S^t[x_2/x_1, \dots, x_n/x_1].$$

Then we have the following.

(i) *For any* $P \in \mathrm{vspec}_A(x_1A)$, *upon letting* $P' = PA'$, *we have* $P' \in \mathrm{vspec}_{A'}(x_1A')$ *with* $\mathrm{ht}_A P = \mathrm{ht}_{A'} P'$ *and* $P' \cap A = P$ *with* $(A_P)^t = A'_{P'}$. *In particular* $x_1A \in \mathrm{vspec}_A(x_1A)$ *and* $x_1A' \in \mathrm{vspec}_{A'}(x_1A')$ *with* $\mathrm{ht}_A(x_1A) = 1 = \mathrm{ht}_{A'}(x_1A')$ *and* $(x_1A') \cap A = x_1A$.

(ii) *Let* $Q^\sharp \in \mathrm{vspec}_{A'}(x_1A')$ *be such that for every* $Q \in \mathrm{vspec}_A(x_1A)$ *we have* $(A_Q)^t \neq A'_{Q^\sharp}$. *Then upon letting* $Q^\flat = Q^\sharp \cap A$, *we have* $Q^\flat \in \mathrm{vspec}_A(x_1A)$ *with* $\mathrm{ht}_A Q^\flat = (\mathrm{ht}_{A'} Q^\sharp) - 1$ *and we have* $A_{Q^\flat} = A'_{Q^\sharp} \cap L$ *with* $A'_{Q^\sharp} \subsetneqq (A_{Q^\flat})^t \in \mathfrak{V}(A'_{Q^\sharp})$.

(iii) *For* $2 \le i \le n$ *we have* $|U'_i \setminus U''_i| = \infty$ *where* $U'_i = \{P' \in \mathrm{vspec}_{A'}(x_1A') : \mathrm{ht}_{A'} P' = i\}$ *and* $U''_i = \{PA' : P \in U_i\}$ *with* $U_i = \{P \in \mathrm{vspec}_A(x_1A) : \mathrm{ht}_A P = i\}$.

Note (6) *In the proof of* (5)(ii), *instead of using the assumption that* $(A_Q)^t \neq A'_{Q^\sharp}$ *for every* $Q \in \mathrm{vspec}_A(x_1A)$, *we shall only use the weaker assumption that* $(A_{Q^\flat})^t \neq A'_{Q^\sharp}$ *where* $Q^\flat = Q^\sharp \cap A$.

Note (7) *As a familiar example of* t*-extension, if* S *is the valuation ring* R_v *of a valuation* v *of a subfield* K *of* L *and* $Y = t$, *then* S^t *is the valuation ring* R_w *of the valuation* w *of* $K(Y)$ *as defined in* (*J*9) *on page* 80 *of* [8]; w *is sometimes called the* ***Gauss extension*** *of* v.

Note (8) *Some of the proofs, especially the following proof of* (5), *inspire a relative and hence more general definition of* t*-extension. Namely, given any subrings* $S \subset A$ *of the field* L, *we may introduce the subring* $(S, A)^t$ *of* $L(t)$ *as the localization of* $A[t]$ *at the multiplicative set* N *of all* $a = \sum a_i t^i \in S[t]$ *such that the ideal in* S *generated by all the coefficients* a_i *is the unit ideal* S. *As an example, if* A *is the multivariate polynomial ring* $K[X_1, \dots, X_m]$ *over a subfield* $S = K$ *of* L *then* $(S, A)^t$ *is the polynomial ring* $K(t)[X_1, \dots, X_m]$ *over the field* $K(t)$.

Proof of (1) As in the above proof of Lemma 2.1, it suffices to show that for any ideal I in S, upon letting $J = IS[t]$, we have $(IS[t]) \cap S = I$ and $(JS^t) \cap S[t] = J$. The first equation follows from (T30) on page 233 of [8] by observing that the noetherianness of the rings R and S was not used in much of the proof. The noetherianness was not used in the related comment (C12) on page 235 of [8] either, and hence $J = I[t]$ where $I[t]$ denotes the set of all polynomials $f(t)$ all whose coefficients belong to I. Now $S^t = S[t]_N$ where N is the multiplicative set of all $a \in S[t]$ whose coefficients generate the unit ideal in S. Therefore by Sect. §7 of Lecture L4 of [8] we see that $(JS^t) \cap S[t] = J$ iff: $b \in S[t]$ with $ab \in J$ for some $a \in N \Rightarrow b \in J$. Hence we are done by taking $C(a) = S$ in Lemma 2.4. □

Proof of (2) Follows from (1). □

Proof of (3) Follows by noting that, as in the proofs of Lemmas 2.1 and 2.2, upon letting $P = M(S)S[t]$, we have that P is a prime ideal in $S[t]$ with $P \cap S = M(S)$ and $S^t = S[t]_P$. □

Proof of (4) Let $x_1, \dots, x_n$ be generators of $M(S)$. Then

$$\mathfrak{W}\big(S, M(S)\big)^{\Delta} = \bigcup_{1\le j\le n} \mathfrak{V}(A_j, x_jA_j) \quad \text{and}$$
$$\mathfrak{W}\big(S^t, M(S^t)\big)^{\Delta} = \bigcup_{1\le j\le n} \mathfrak{V}\big(A'_j, x_jA'_j\big)$$

where

$$A_j = S[x_1/x_j, \ldots, x_{j-1}/x_j, \ldots, x_n/x_j] \quad \text{and}$$
$$A'_j = S^t[x_1/x_j, \ldots, x_{j-1}/x_j, \ldots, x_n/x_j]$$

and hence we are reduced to (5). □

Proof of (5) Let

$$A^* = A[t] \quad \text{and} \quad P^* = PA^*.$$

By (T30) and (C12) on pages 233–235 of [8] we get

(1*) $P^* = P[t] \in \text{vspec}_{A^*}\big(x_1A^*\big)$ with $\text{ht}_A P = \text{ht}_{A^*} P^*$ and $P^* \cap A = P$.

Consider polynomials

$$f = \sum f_i t^i \in L[t] \quad \text{with } f_i \in L \quad \text{and} \quad g = \sum g_i t^i \in L[t] \quad \text{with } g_i \in L.$$

By definition $(A_P)^t = \{f/g : f_i, g_i \text{ in} A_P \text{ for all } i \text{ but } g_j \notin PA_P \text{ for some } j\}$; for any such f, g we can find $r \in A \setminus P$ such that the elements rf_i, rg_i belong to A for all i; now $f/g = (rf)/(rg)$ with $rf \in A^*$ and $rg \in A^* \setminus P^*$ because $rg_j \notin P$ for some j; consequently $f/g \in A^*_{P*}$; thus $(A_P)^t \subset A^*_{P*}$. Conversely, any element of A^*_{P*} can be written as f/g with f, g in A^* with $g \notin P^* = P[t]$; it follows that f_i, g_i are in A for all i but $g_j \notin P$ for some j; therefore $f/g \in (A_P)^t$; thus $A^*_{P*} \subset (A_P)^t$. This proves that

(2*) $(A_P)^t = A^*_{P*}$.

By (T157) on page 560 of [8] we see that $x_1A \in \text{spec}(A)$ with $(x_1A) \cap S = M(S)$ and hence

(3*) $x_1A \in \text{vspec}_A(x_1A)$ and $Q \cap S = M(S)$ for all $Q \in \text{vspec}_A(x_1A)$.

Concerning the rings $A^* \subset A'$ we claim that:

(4*) for any $Q \in \text{vspec}_A(x_1A)$ upon letting $Q^* = QA^*$ we have $Q^* = Q[t] \in \text{vspec}_{A^*}(x_1A^*)$ with $\text{ht}_A Q = \text{ht}_{A^*} Q^*$ and $Q^* \cap A = Q$;
(5*) and upon letting $Q' = Q^*A'$ we have $Q' \in \text{vspec}_{A'}(x_1A')$ with $\text{ht}_{A^*} Q^* = \text{ht}_{A'} Q'$ and $Q' \cap A^* = Q^*$ with $A^*_{Q^*} = A'_{Q'}$;
(6*) moreover $\mathfrak{q}^* \mapsto \mathfrak{q}' = \mathfrak{q}^*A'$ gives a bijection ϕ of $W^* = \{\mathfrak{p}^* \in \text{vspec}_{A^*}(Q^*) : \mathfrak{p}^* \cap A = Q\}$ onto $W' = \{\mathfrak{p}' \in \text{vspec}_{A'}(Q') : \mathfrak{p}' \cap A = Q\}$ such that $\mathfrak{q}' \cap A^* = \mathfrak{q}^*$;
(7*) and if $\mathfrak{q}' \in W'$ is such that $(A_Q)^t \neq A'_{\mathfrak{q}'}$ then we have $\text{ht}_A Q = (\text{ht}_{A'}\mathfrak{q}') - 1$ and $A_Q = A'_{\mathfrak{q}'} \cap L$ with $A'_{\mathfrak{q}'} \subsetneqq (A_Q)^t \in \mathfrak{V}(A'_{\mathfrak{q}'})$.

Now (5)(i) follows from (1*) to (5*) where in (4*) and (5*) we take $Q = P$. Likewise (5)(ii) follows from (6*) and (7*) by taking $(Q, \mathfrak{q}') = (Q^\flat, Q^\sharp)$.

So we proceed to prove (4*) to (7*). (4*) is (1*) with $Q = P$. Let N be the multiplicative set of all g such that $g_i \in S$ for all i but $g_j \notin M(S)$ for some j. By definition $S^t = S[t]_N$ and hence we get the equation $A' = A^*_N$ which we shall use tacitly. By (3*) and (4*) we have $N \subset A^* \setminus Q^*$ and hence by (T10) and (T12) on page 139 of [8] we get (5*). By (3*) we see that for every $\mathfrak{q}* \in W^*$ we have $N \subset A^* \setminus \mathfrak{q}^*$ and hence, in view of (5*), by (T12) on page 139 of [8] we get (6*). To prove (7*) let $\mathfrak{q}' \in W'$ be such that $(A_Q)^t \neq A'_{\mathfrak{q}'}$. By (2*) and (5*) we have $(A_Q)^t = A'_{Q'}$ and hence, in view of (T15) on page 144 of [8], we get $A'_{\mathfrak{q}'} \subsetneqq (A_Q)^t \in \mathfrak{V}(A'_{\mathfrak{q}'})$. Now (6*) and the inclusion $A'_{\mathfrak{q}'} \subsetneqq A'_{Q'}$ give the inclusion $Q^* \subsetneqq \mathfrak{q}^*$ and hence, in view of (4*) and (5*) together with the equation $Q^* \cap A = Q = \mathfrak{q}^* \cap A$, by (C13) on page 235 of [8] we get $\mathrm{ht}_A Q = (\mathrm{ht}_{A'}\mathfrak{q}') - 1$. By (1*) we get $(A_Q)^t \cap L = A_Q$; we also have the inclusions $A_Q \subset A'_{\mathfrak{q}'} \subset (A_Q)^t$ where the first one comes out of the equation $\mathfrak{q}' \cap A = Q$; therefore $A_Q = A'_{\mathfrak{q}'} \cap L$.

To prove (5)(iii) consider the polynomial rings

$$B = K[Z_2, \ldots, Z_n] \subset K(\tau)[Z_2, \ldots, Z_n] = B'$$

in indeterminates $Z_2, \ldots, Z_n$ over the fields $K \subset K(\tau)$ mentioned in the above item (3). By (Q35.5) on pages 559–566 of [8], there exist unique epimorphisms $\Theta : A \to B$ and $\Theta' : A' \to B'$ with $\ker(\Theta) = x_1 A$ and $\ker(\Theta') = x_1 A'$ such that $\Theta'(x_2/x_1, \ldots, x_n/x_1) = (Z_2, \ldots, Z_n)$ with $\Theta'(r) = \Theta(r)$ for all $r \in A$ and $\Theta'(s) = H_{S^t}(s)$ for all $s \in S^t$. Clearly $\Theta(U_i)$ and $\Theta'(U'_i)$ are the sets of all prime ideals of height $i - 1$ in B and B' respectively. The rest is obvious. □

Definition 2.8 Getting back to the quotient field L of a two dimensional regular local domain R, and referring to Sect. 3 of [13] for definitions concerning subsets of $Q(R)$, we add some more. Recall that a **thicket** at R is a bush B at R such that for every $T \in B$ we have $Q_1(T) \cap B \neq \emptyset$. By analogy, we define an **antithicket** at R to be a nonempty bush B at R such that for every $T \in B$ we have $|Q_1(T) \setminus B| = \infty$.

Given any nonempty bush B at R and any S in $Q(R)$, upon letting $(R_j)_{0 \le j \le \nu}$ to be the unique finite QDT sequence of R with $S = R_\nu$, we define the **antecedent** of S in B (relative to R) to be R_μ where μ is the largest nonnegative integer $\le \nu$ such that $R_\mu \in B$; we denote this μ by $\chi_R(S, B)$ or $\chi(S, B)$.

By the **halo** of any positive dimensional regular local domain S we mean the DVR $o(S)$.

Theorem 2.9 *Given any two dimensional regular local domain R, considering the rational function field $L(t)$ in an indeterminate t over the quotient field L of R, the map $\rho^t : \rho(L) \to \rho(L^t)$ given by $S \mapsto S^t$ relates the sets $Q(R)^\Delta$ and $Q(R^t)^\Delta$ thus.*

(2.9.1) *For any $S \in Q(R^t)$ we have*:

$$S \in Q^t(R) \quad \Leftrightarrow \quad S \cap L \in Q(R) \quad \Leftrightarrow \quad (S \cap L)^t = S \quad \Leftrightarrow \quad \dim(S \cap L) = 2.$$

(2.9.2) *For any $S \in Q_j(R^t)$ with $j \in \mathbb{N}$ we have*:

$$S \in Q^t(R) \quad \Leftrightarrow \quad S \cap L \in Q_j(R) \quad \Leftrightarrow \quad S \cap L \notin D_j(R)^\Delta \quad \Leftrightarrow \quad \dim(S \cap L) \neq 1.$$

(2.9.3) *$Q^t(R)$ is an antithicket at R^t and the map $Q(R) \to Q^t(R)$ given by $S \mapsto S^t$ is a domination preserving and inclusion preserving bijection. For any $S \in Q(R^t) \setminus Q^t(R)$, letting $T = \chi(S, Q^t(R))$, we have $S \cap L = o(T \cap L)$.*

(2.9.4) *For any finite QDT sequence $(S_j)_{0 \le j \le \nu}$ of R^t let $\mu = \chi(S_\nu, Q^t(R))$ with $R_j = S_j \cap L$ and $V_j = o(S_j) \cap L$. Then $(R_j)_{0 \le j \le \nu}$ is a finite QDT sequence of R such that for $0 \le j \le \mu$ we have $V_j = o(R_j)$ with $(V_j)^t = o(S_j)$ and $(R_j)^t = S_j$, and for $\mu < j \le \nu$ we have $R_j = V_j = V_\mu$.*

Proof By a simple induction on the layer index j in $Q_j(R)$ and $Q_j(R^t)$, everything follows from Lemma 2.7 by using the following observations. Let V be a DVR with quotient field L, let T be a quasilocal domain such that T dominates R^t and $V = T \cap L$, and let S be a quasilocal domain which dominates T. Then S dominates R and hence $L \not\subset S$. Also $V \subset S$ and hence $V = S \cap L$ because V is a maximal subring of L. □

3 Dicriticals and shoots

Again let R be a two dimensional regular local domain with quotient field L, let t be an indeterminate over L, and let $\rho^t : \rho(L) \to \rho(L^t)$ be the map given by $S \mapsto S^t$ where $\rho(L)$ is the set of all subrings of L and $\rho(L^t)$ is the set of all subrings of $L^t = L(t)$. For describing how ρ^t maps the ideal tree of a pencil into the singularity tree of its generic member, let us continue the project of adding definitions started in Definition 2.8.

Definition 3.1 Let B be a bush at R. Note that every pine $S = (S_j)_{0 \le j < \infty}$ of B is the subpine of a unique maximal pine S^* of B; we denote S^* by $B(S)$ and call it the **pine-closure** of S in B. By $\mathfrak{U}(B)$ we denote the set of all **maximal pines** of B.

Let $B \subset \overline{B}$ be bushes at R. By a **shoot** of $(B, \overline{B})$ we mean a pine $\overline{S} = (\overline{S}_j)_{0 \le j < \infty}$ of $\overline{B}$ such that $\overline{S}_0 \in B$ with $\overline{S}_1 \notin B$ and $(Q_1(\overline{S}_0) \cap \overline{B}) \setminus \{\overline{S}_1\} \subset B$; by $\mathfrak{S}(B, \overline{B})$ we denote the set of all shoots of $(B, \overline{B})$. By a **graft** of $(B, \overline{B})$ we mean $T \in B$ such that T is the base of some (obviously unique) shoot $\overline{S}$ of $(B, \overline{B})$; we denote $\overline{S}$ by $(B, \overline{B})(T)$ and call it the **sprout** of T in $(B, \overline{B})$, and we denote $\overline{B}(\overline{S})$ by $(B, \overline{B})(T)^*$ and call it the **maximal sprout** of T in $(B, \overline{B})$; by $\mathfrak{T}(B, \overline{B})$ we denote the set of all grafts of $(B, \overline{B})$.

If $B \subset \overline{B}$ are bushes at R, we say that B is a subbush of $\overline{B}$ or $\overline{B}$ is an overbush of B. If moreover B is a nonempty finite antithicket at R and $\overline{B}$ is a thicket at R such that every maximal pine $\overline{S}^*$ of $\overline{B}$ is the pine-closure of a unique shoot $\overline{S}$ of $(B, \overline{B})$ then we say that B is a **wellpined** subbush of $\overline{B}$ at R, and $\overline{B}$ is a **wellpined** overbush of B at R; we denote $\overline{S}$ by $(B, \overline{B})(\overline{S}^*)'$ and call it the **truncation** of $\overline{S}^*$ in $(B, \overline{B})$. If B is a wellpined subbush of $\overline{B}$ at R and I is a nonzero ideal in R such that for every shoot $\overline{S} = (\overline{S}_j)_{0 \le j < \infty}$ of $(B, \overline{B})$ we have $M(\overline{S}_1) = (z, x)\overline{S}_1$ with $(R, \overline{S}_1)(I) = z\overline{S}_1$ and $M(\overline{S}_0)\overline{S}_1 = x\overline{S}_1$ then we say that B is a **wellshot** subbush of $\overline{B}$ at (R, I), and $\overline{B}$ is a **wellshot** overbush of B at (R, I). Note that if B is a wellpined subbush of $\overline{B}$ then $\overline{S} \mapsto \overline{B}(\overline{S})$ gives a bijection $\mathfrak{S}(B, \overline{B}) \to \mathfrak{U}(\overline{B})$ whose inverse is given by $\overline{S}^* \mapsto (B, \overline{B})(\overline{S}^*)'$.

Descriptively speaking, as our main result we shall show that, for a two-generated primary pencil at R, letting Q be the t-extension of its ideal tree and letting P be the singularity tree of its generic member Φ, we have that Q is a wellshot subbush of P at $(R^t, \Phi R^t)$. Moreover the t-transforms of the big stars (see definition below) of the pencil are the grafts of (Q, P) and the t-transforms of the dicritical divisors of the pencil are the halos of these grafts. In particular, the dicritical divisors of the pencil are in a bijective correspondence with the shoots of the singularity tree of the generic member, which themselves are in a bijective correspondence with the maximal pines of the singularity tree.

Symbolically speaking we have the following Theorem where for the definitions of the set $\mathfrak{B}(R, J)$ of **big stars** of the pencil J and the set $\mathfrak{D}(R, J)$ of its **dicritical divisors** we refer to Sect. 2 of [11], for the definitions of a **strongly square free stable thicket** and the **breadth** of a thicket we refer to Sect. 3 of [13], and for the concept of **pine-closure** and the definitions of the set $\mathfrak{U}$ of all **maximal pines** and the set $\mathfrak{S}$ of all **shoots** we refer to the above four paragraphs.

Theorem 3.2 *Let $F \neq 0 \neq G$ in R be such that $J = (F, G)R$ is $M(R)$-primary and let $Q = \rho^t(\mathfrak{Q}(R, J))$ and $P = \mathfrak{P}(R^t, \Phi)$ where $\Phi = F + tG$. Then Q is a wellshot subbush of P at $(R^t, \Phi R^t)$ and we have $\rho^t(\mathfrak{B}(R, J)) = \mathfrak{T}(Q, P)$ and hence $S \mapsto S^t$ gives a bijection $\mathfrak{B}(R, J) \to \mathfrak{T}(Q, P)$. Moreover the singularity tree P of the generic member Φ of the pencil J is a strongly square free stable thicket at $(R^t, \Phi R^t)$. Furthermore, the above bijection $\mathfrak{B}(R, J) \to \mathfrak{T}(Q, P)$, when composed with the inverses of the bijections $\mathfrak{B}(R, J) \to \mathfrak{D}(R, J)$ and $\mathfrak{S}(Q, P) \to \mathfrak{T}(Q, P)$ given by $S \mapsto o(S)$ and pine $\mapsto$ its base respectively, gives a bijection $\mathfrak{D}(R, J) \to \mathfrak{S}(Q, P)$ which, when composed with the bijection $\mathfrak{S}(Q, P) \to \mathfrak{U}(P)$ given by pine $\mapsto$ its pine-closure, gives a bijection $\mathfrak{D}(R, J) \to \mathfrak{U}(P)$.*

Proof Follows from (4.1)(ii) of [11], (3.1) and (4.6) of [13], and (2.9) above. In greater details, (4.6.3) of [13] tells us that Q is a wellshot subbush of P, whereas (4.1)(ii) of [11] together with (4.6) of [13] imply that $\rho^t(\mathfrak{B}(R, J) \cap Q_j(R)) = \mathfrak{T}(Q, P) \cap Q_j(R^t)$ for every $j \in \mathbb{N}$, and so on. □

4 Completions and quadratic transforms

To continue the discussion started in Definition 2.6, referring to pages 7–11 of [3] and pages 248–270 of [27] for definitions and basic properties of completions of local rings and complete local rings, note that the completion of a local ring R is a complete local ring $\widehat{R}$ which is an overring of R such that, every $z \in \widehat{R}$ can be written as $\lim z_n = z$ with $z_n \in R$, i.e., $z - z_n \in M(\widehat{R})^{u(n)}$ with positive integers $u(n) \to \infty$ as $n \to \infty$, and

$$(*) \qquad M(R)^i \widehat{R} = M(\widehat{R})^i \quad \text{and} \quad M(\widehat{R})^i \cap R = M(R)^i \quad \text{for all } i \in \mathbb{N}.$$

This gives a natural isomorphism $g(R) \to g(\widehat{R})$, where $g(R) = \text{grad}(R) =$ **graded ring** of R. Also nonzerodivisors in R stay nonzerodivisors in $\widehat{R}$ and hence we get the subset monomorphism $\text{QR}(R) \to \text{QR}(\widehat{R})$ of total quotient rings, i.e., we may regard

$\mathrm{QR}(R)$ as a subring of $\mathrm{QR}(\widehat{R})$ or, equivalently, $\mathrm{QR}(\widehat{R})$ as an overring of $\mathrm{QR}(R)$; for sets $A \subset B$ the **subset map** $A \to B$ is given by $z \mapsto z$; note that $\mathrm{QR}(R) \cap \widehat{R} = R$. R is **analytically irreducible** means $\widehat{R}$ is a domain, and then the quotient field $\mathrm{QF}(\widehat{R})$ may be regarded as an overfield of the quotient field $\mathrm{QF}(R)$.

By a **local homomorphism** we mean a ring homomorphism $f : R \to R^\dagger$ where R and $R^\dagger$ are local rings with $f(M(R)) \subset M(R^\dagger)$; f uniquely extends to its **completion** $\widehat{f} : \widehat{R} \to \widehat{R^\dagger}$ by which we mean a local homomorphism such that $\widehat{f}(z) = f(z)$ for all $z \in R$. We call f a **subcompletion** of R if f is a subset map and $\widehat{f}$ is an identity map; in other words, if $R^\dagger$ dominates R and $\widehat{R} = \widehat{R^\dagger}$; note that then (*) holds with $\widehat{R}$ replaced by $R^\dagger$ and hence we get an isomorphism $g(R) \to g(R^\dagger)$ which we denote by $g(f)$. Note that if $f : R \to R^\dagger$ is a subcompletion of R then f extends to a subset monomorphism $f^\dagger : \mathrm{QR}(R) \to \mathrm{QR}(R^\dagger)$ with $\mathrm{QR}(R) \cap R^\dagger = R$, and we have $\dim(R) = \dim(R^\dagger)$ with $\mathrm{emdim}(R) = \mathrm{emdim}(R^\dagger)$, and hence R is regular iff $R^\dagger$ is regular. For all this see pages 9 and 10 of [3].

Assumption 4.1 Let $p = \mathrm{proj}$ denote the set of all nonmaximal homogeneous prime ideals. Assuming R to be a local domain of dimension $n \in \mathbb{N}_+$, as on pages 534–577 of [8], there is a natural bijection $\delta_R : p(g(R)) \to (\mathfrak{W}(R, M(R)))^\Delta$, which is a generalization of the bijection $\delta : \kappa[X, Y]^{hmi} \to Q_1(R)$ given in (2B)(3) of [13]. For any subcompletion $f : R \to R^\dagger$ of R with local domain $R^\dagger$, by putting together the three bijections $\delta_R, g(f), \delta_{R^\dagger}$ we obtain a bijection $f_1' : (\mathfrak{W}(R, M(R)))^\Delta \to (\mathfrak{W}(R^\dagger, M(R^\dagger)))^\Delta$ called the **first quadratic derivative** of f. Assuming R to be regular, for every $S \in (\mathfrak{W}(R, M(R)))^\Delta$ and $S^\dagger = f_1'(S) \in (\mathfrak{W}(R^\dagger, M(R^\dagger)))^\Delta$, we have that S and $S^\dagger$ are regular local domains of equal positive dimension $m \le n$ such that $S^\dagger$ dominates S, $M(S)S^\dagger = M(S^\dagger)$, and $H_{S^\dagger}(S) = H(S^\dagger)$; therefore by (10.1) on page 238 of [3] we see that the subset map $S \to S^\dagger$ is a subcompletion of S.

Assumption 4.2 Assume that R is a two dimensional regular local domain and let $f : R \to R^\dagger$ be any subcompletion of R. In Assumption 4.1 we defined the first quadratic derivative of f as a bijection $f_1' : Q_1(R)^\Delta \to Q_1(R^\dagger)^\Delta$. Iterating the procedure we define the j**-th quadratic derivative** of f as a bijection $f_j' : Q_j(R)^\Delta \to Q_j(R^\dagger)^\Delta$ for all $j \in \mathbb{N}$. Putting these together we define the **quadratic derivative** of f as a bijection $f' : Q(R)^\Delta \to Q(R^\dagger)^\Delta$ such that, for any $T \in Q(R)^\Delta$ and $T^\dagger = f'(T) \in Q(R^\dagger)^\Delta$, $T^\dagger$ dominates T and the subset map $f_T : T \to T^\dagger$ is a subcompletion of T. Note that if $T \in Q_j(R)^\Delta$ for some $j \in \mathbb{N}$ then $T^\dagger = f_j(T) \in Q_j(R^\dagger)^\Delta$; moreover, if $T \in Q_j(R)$ then for any $S \in Q_i(T)^\Delta$ with $i \in \mathbb{N}$ we have $(f_T)_i'(S) = f_{j+i}'(S)$, and in particular $f_j'(o(T)) = o(T^\dagger)$. Also note that the common quotient field L of all the members of $Q(R)^\Delta$ is a subfield of the common quotient field $L^\dagger$ of all the members of $Q(R^\dagger)^\Delta$, and for all $T \in Q(R)^\Delta$ we have $L \cap f'(T) = T$. Now let (x, y) be any generators of $M(R)$ and let κ be any coefficient set of R. Then, upon letting the bijection $\delta : \kappa[X, Y]^{hmi} \to Q_1(R)$ be as in (2B)(3) of [13] and $\delta^\dagger : \kappa[X, Y]^{hmi} \to Q_1(R^\dagger)$ be the corresponding bijection for $R^\dagger$, for every $\lambda \in \kappa[X, Y]^{hmi}$ we have $f_1'(\delta(\lambda)) = \delta^\dagger(\lambda)$. Moreover by slightly modifying the proof of the Tangent Lemma given on pages 140–141 of [6] we get the:

Generalized Tangent Lemma 4.3 *In the situation of Assumptions 4.1 and 4.2 let* $K = H(R)$. *Given any* $0 \neq F \in M(R)$ *let* $\overline{F} = \overline{F}(X, Y) \in K[X, Y] = g(R)$ *be the initial form of* F *relative to* R, x, y, *and let* $\overline{F}(X, Y) = \widehat{F} \prod_{1 \le i \le h} H_R(\overline{F}_i)(X, Y)^{e_i}$ *be the factorization described in the beginning of* (2C) *of* [13]. *If* R *is complete then there exist nonzero elements* $F_1, \ldots, F_h$ *in* $M(R)$ *such that info*$(F_i) = \widehat{F} H_R(\overline{F}_i)(X, Y)^{e_i}$ *or info*$(F_i) = H_R(\overline{F}_i)(X, Y)^{e_i}$ *according as* $i = 1$ *or* $2 \le i \le h$. *In other words, if* R *is complete and* F *is irreducible then* $h = 1$.

By Cohen Structure Theorem, given on pages 106–112 of [25], every one-dimensional complete local domain of embedding dimension ≤ 2 can be written as a homomorphic image of a two-dimensional complete regular local domain modulo a nonzero principal prime ideal, and hence Lemma 4.3 is equivalent to saying that for any one-dimensional complete local domain S of embedding dimension ≤ 2 we have $|p(g(R))| = 1$, i.e., $|\mathfrak{W}(R, M(R))^{\Delta}| = 1$. Here is a:

More General Tangent Lemma 4.4 *Let* S *be a one-dimensional complete local domain of any embedding dimension, and let* S' *be the integral closure of* S *in its quotient field* L_S. *Then* S' *is a complete DVR which dominates* S *and is a finite* S*-module. Moreover* $\mathfrak{W}(S, M(S)) = \{S', L_S\}$ *and* $|p(g(S))| = 1$. *Also there exists a unique nonnegative integer* μ *together with an infinite sequence* $S = S_0 \subset S_1 \subset \cdots \subset S_\mu = S' = S_{\mu+1} = S_{\mu+2} = \ldots$ *of one-dimensional complete local domains with quotient field* L_S *such that for all* $j \in \mathbb{N}$ *we have that* S_{j+1} *dominates* S_j *and* $\mathfrak{W}(S_j, M(S_j)) = \{S_{j+1}, L_S\}$ *with* $|p(g(S_j))| = 1$, *and for* $0 \le j < \mu$ *we have that* S_j *is nonregular with* $S_j \neq S_{j+1}$.

Proof By items 1.19 to 1.24 of [2] and items (17.8), (30.3), (30.5), (32.1) of [25] we see that S is a pseudogeometric henselian ring and hence S' is a complete DVR which dominates S and is a finite S-module. In view of Assumption 4.1 and the above references, the rest is now clear. □

Lifting Lemma 4.5 *In the situation of Assumption 4.2 let* $q : R \to S$ *be a local epimorphism where* S *is a one-dimensional local domain with quotient field* L_S *such that* $|p(g(S))| = 1$. *Let* $\ker(q) = I$. *Then* $\mathfrak{P}_1(R, I)$ *and* $\mathfrak{W}(S, M(S))^{\Delta}$ *consist of singletons* R_1 *and* S_1 *and, upon letting* $I_1 = (R, R_1)(I)$, *there exists a unique local epimorphism* $q_1 : R_1 \to S_1$ *with* $\ker(q_1) = I_1$ *such that* $q_1(z) = q(z)$ *for all* $z \in R$.

Proof Although the proof is "straightforward," for understanding this Sect. 4, the reader may profit by studying §4 on pages 108–148 of [3]. □

Iterated Tangent Lemma 4.6 *In the situation of Assumption 4.2, assume that* R *is complete and let* $q : R \to S$ *be a local epimorphism where* S *is a one-dimensional local domain with quotient field* L_S. *Let* $\ker(q) = I$. *Then there exists a unique infinite sequence* $(S_j)_{0 \le j < \infty}$ *of one-dimensional complete local domains with quotient field* L_S *such that* $S_0 = S$ *and, for all* $j \in \mathbb{N}$, S_{j+1} *dominates* S_j *and* $\mathfrak{W}(S_j, M(S_j)) = \{S_{j+1}, L_S\}$ *with* $|p(g(S_j))| = 1$. *Moreover there exists a unique nonnegative integer* μ *such that for* $0 \le j < \mu$ *we have that* S_j *is nonregular with* $S_j \neq S_{j+1}$, *and for* *all $j \geq \mu$ we have that S_j is regular with $S_j = S_\mu$. Also there exists a unique infinite QDT sequence $(R_j)_{0\leq j<\infty}$ of R such that $\mathfrak{P}_j(R, I) = \{R_j\}$ for all $j \in \mathbb{N}$. Finally, upon letting $I_j = (R, R_j)(I)$, for every $j \in \mathbb{N}$ there exists a unique local epimorphism $q_j : R_j \to S_j$ with $\ker(q_j) = I_j$ and $q_0 = q$ such that $q_{j+1}(z) = q_j(z)$ for all $z \in R_j$.*

Proof S is complete by (17.8) of [25]. Hence everything follows from Assumption 4.2 to Lemma 4.5. □

Pine Lemma 4.7 *In the situation of Assumption 4.2, assume that $R^\dagger = \widehat{R}$. For $0 \neq \phi \in M(R)$ let us write $\phi = \widetilde{\phi}\phi_1^{\epsilon_1} \ldots \phi_\beta^{\epsilon_\beta}$ where $\beta, \epsilon_1, \ldots, \epsilon_\beta$ are positive integers, $\widetilde{\phi}$ is a unit in $R^\dagger$, and $\phi_1, \ldots, \phi_\epsilon$ are pairwise coprime irreducible elements in $R^\dagger$. Let $I = \phi R$ and $I_i = \phi_i R^\dagger$. Then, for $1 \leq i \leq \beta$, there exist unique infinite QDT sequences $(R_{ij})_{0\leq j<\infty}$ and $(R^\dagger_{ij})_{0\leq j<\infty}$ of R and $R^\dagger$ respectively such that, for all $j \in \mathbb{N}$, we have: $\mathfrak{P}_j(R^\dagger, I_j) = \{R^\dagger_{ij}\}$, $R^\dagger_{ij}$ dominates R_{ij}, $L \cap R^\dagger_{ij} = R_{ij}$, and the subset monomorphism $R_{ij} \to R^\dagger_{ij}$ is a subcompletion of R_{ij}. Moreover, for all $j \in \mathbb{N}$, we have $\mathfrak{P}_j(R^\dagger, IR^\dagger) = \{R^\dagger_{ij} : 1 \leq i \leq \beta\}$ with $\mathfrak{P}_j(R, IR) = \{R_{ij} : 1 \leq i \leq \beta\}$ and, for $1 \leq i \leq \beta$, we have $(R^\dagger, R^\dagger_{ij})(IR^\dagger) = ((R, R_{ij})(I))R^\dagger_{ij}$ with $((R^\dagger, R^\dagger_{ij})(IR^\dagger)) \cap R_{ij} = (R, R_{ij})(I)$. Also there exists $j_0 \in \mathbb{N}$ such that for every integer $j \geq j_0$ we have $(R^\dagger, R^\dagger_{ij})(I) = (R^\dagger, R^\dagger_{ij})(I_i^{\epsilon_i})$ with $\mathrm{ord}_{R^\dagger_{ij}}(R^\dagger, R^\dagger_{ij})(I_i) = 1$ for $1 \leq i \leq \beta$, and $R^\dagger_{ij} \neq R^\dagger_{i'j}$ with $R_{ij} \neq R_{i'j}$ for all $i \neq i'$. Finally, for $1 \leq i \leq \beta$, $(R^\dagger_{ij})_{j_0\leq j<\infty}$ and $(R_{ij})_{j_0\leq j<\infty}$ are pines of the bushes $P^\dagger = \mathfrak{P}(R^\dagger, IR^\dagger)$ and $P = \mathfrak{P}(R, I)$ and their pine-closures in these bushes are exactly all the distinct members of $\mathfrak{U}(P^\dagger)$ and $\mathfrak{U}(P)$ respectively.*

Proof Follows from Assumption 4.2 to Lemma 4.6. □

Note 4.8 In Lemma 4.7, $\phi_1, \ldots \phi_\beta$ are called the **analytic branches** of ϕ. By taking ϕ to be a bivariate polynomial $\phi(X, Y)$ over a field and R to be the local ring of the origin in the (X, Y)-plane, this reduces to the idea of analytic branches of the plane curve $\phi(X, Y) = 0$ at the origin $(0, 0)$. In the situation of Theorem 3.2, by taking $\phi = \Phi$ and $R = R^t$, it follows that the set $\mathfrak{U}(P)$ of maximal pines of P is in a natural bijective correspondence with the set of analytic branches of Φ and hence in particular β is the breadth of its singularity tree P; the part of (3.2) asserting that P is strongly square free says that $\epsilon_1 = \cdots = \epsilon_\beta = 1$.

5 Analytic branches and maximal pines

Let R be a two dimensional regular local domain with quotient field L, coefficient set κ, and residue field $K = H(R)$. Let (x, y) be generators of $M(R)$. We now refine the concepts of infinite pillar and pillar from the end of Sect. 2 of [11]. We use the refinement to decompose a QDT sequence into packets called flips, nonflips, and so on. This enables us to codify the classification of valuations dominating R as in my Princeton Book [2]. In particular it characterizes analytic branches via nonreal valuations which correspond to what we call infinite pseudopillars.

Let $S = (S_j, x_j, y_j, \kappa_j)_{0\le j<\infty}$ be any infinite QDT sequence of (R, x, y, κ); we define its **height** $h(S)$ by putting $h(S) = \infty$. As suggestive abbreviation, a positive integer j is called an X-**operation** or Y-**operation** or **translation** of S according as $(x_j, y_j) = (x_{j-1}, y_{j-1}/x_{j-1})$ or $(x_j, y_j) = (x_{j-1}/y_{j-1}, y_{j-1})$ or $(x_{j-1}, y_{j-1}/x_{j-1}) \neq (x_j, y_j) \neq (x_{j-1}/y_{j-1}, y_{j-1})$; it is called a **nontranslation** of S if it is not a translation of S, i.e., if it is either an X-operation or a Y-operation of S; in these terms the reference to S may be omitted when it is clear from the context. We define the **translation index** $t(S) \in \mathbb{N}$ of S by letting $t(S)$ be the largest positive integer which is a translation with the understanding that if there is no positive integer which is a translation then $t(S) = 0$ and if there are infinitely many positive integers which are translations then $t(S) = \infty$. We call S an **infinite** X-**pillar** at (R, x, y, κ) if every positive integer is an X-operation. We call S an **infinite** Y-**pillar** at (R, x, y, κ) if every positive integer is a Y-operation. We call S an **infinite protopillar** at (R, x, y, κ) if S is either an infinite X-pillar at (R, x, y, κ) or an infinite Y-pillar at (R, x, y, κ). We call S an **infinite retropillar** at (R, x, y, κ) if every positive integer is a nontranslation, infinitely many positive integers are X-operations, and infinitely many positive integers are Y-operations. We call S an **infinite pillar** at (R, x, y, κ) if $(S_j)_{0\le j<\infty}$ is an infinite pillar at (R, x) (as defined at the end of Sect. 2 of [11]), i.e., if every positive integer j is either an X-operation or a translation. We define the **pillar number** $n(S) \in \mathbb{N} \cup \{\infty\}$ of S by letting $n(S)$ be the smallest nonnegative integer π such that $(S_{\pi+j}, x_{\pi+j}, y_{\pi+j}, \kappa_{\pi+j})_{0\le j<\infty}$ is either an infinite pillar at $(S_\pi, x_\pi, y_\pi, \kappa_\pi)$ or an infinite Y-pillar at $(S_\pi, x_\pi, y_\pi, \kappa_\pi)$ with the understanding that if there is no such π then $n(S) = \infty$. If $n(S) \neq \infty$ then clearly

$$(S_{n(S)+j}, x_{n(S)+j}, y_{n(S)+j}, \kappa_{n(S)+j})_{0\le j<\infty}$$

is an **infinite pillar or infinite** Y-**pillar** at $(S_{n(S)}, x_{n(S)}, y_{n(S)}, \kappa_{n(S)})$ which we call the **top** of S; in this case we call $(S_j, x_j, y_j, \kappa_j)_{0\le j\le n(S)}$ the **bottom** of S; if $n(S) = \infty$ then S is its own bottom and it has no top. If $n(S) = \infty \neq t(S)$ then we call

$$(S_{t(S)+j}, x_{t(S)+j}, y_{t(S)+j}, \kappa_{t(S)+j})_{0\le j<\infty} \quad \text{and} \quad (S_j, x_j, y_j, \kappa_j)_{0\le j\le t(S)}$$

the **head** and **tail** of S respectively, and we note that the head of S is an **infinite retropillar** at $(S_{t(S)}, x_{t(S)}, y_{t(S)}, \kappa_{t(S)})$; in other cases we leave the head and tail undefined. We call S an **infinite pseudopillar** at (R, x, y, κ) if $n(S) \in \mathbb{N}$. We call S an **infinite antipillar** at (R, x, y, κ) if $n(S) = \infty$. It will turn out that infinite pseudopillars in $\mathfrak{P}(R, F)$ correspond to analytic branches of $F \in R^\times$.

Let $S = (S_j, x_j, y_j, \kappa_j)_{0\le j\le\nu}$ be a finite QDT sequence of (R, x, y, κ); we define its **height** $h(S)$ by putting $h(S) = \nu$; note that the above four terms, X-operation, ..., nontranslation, are now defined for every positive integer $j \le \nu$. We define the **translation index** $t(S) \in \mathbb{N}$ of S by letting $t(S)$ be the largest positive integer $\le \nu$ which is a translation with the understanding that if there is no such positive integer which is a translation then $t(S) = 0$. We call S a **preflip** at (R, x, y, κ) if $\nu > 0$ and every positive integer $\le \nu$ is a nontranslation. We call S an X-**preflip** at (R, x, y, κ) if S is a preflip at (R, x, y, κ) and ν is an X-operation. We call S a Y-**preflip** at (R, x, y, κ) if S is a preflip at (R, x, y, κ) and ν is a Y-operation. We call S an X-**flip** at (R, x, y, κ)

if S is a preflip at (R, x, y, κ) and every positive integer $\le \nu$ is an X-operation. We call S a Y-**flip** at (R, x, y, κ) if S is a preflip at (R, x, y, κ) and every positive integer $\le \nu$ is a Y-operation. We call S a **protoflip** at (R, x, y, κ) if S is either an X-flip at (R, x, y, κ) or a Y-flip at (R, x, y, κ). We call S a **retroflip** at (R, x, y, κ) if S is preflip at (R, x, y, κ) but S is neither an X-flip at (R, x, y, κ) nor a Y-flip at (R, x, y, κ). We call S a **pillar** at (R, x, y, κ) if $(R_j)_{0\le j\le \nu}$ is a pillar at (R, x), i.e., if every positive integer $j \le \nu$ is either an X-operation or a translation. We call S a **nonflip** at (R, x, y, κ) if either $\nu = 0$, or S is a finite pillar at (R, x, y, κ) with $\nu > 0$ and ν is a translation. We call S a **flip** at (R, x, y, κ) if: $\nu \ge 2$, every positive integer $j < \nu$ is a nontranslation, some positive integer $j < \nu$ is a Y-operation, and ν is a translation. We define the **pillar number** $n(S) \in \mathbb{N}$ of S by letting $n(S)$ be the smallest nonnegative integer π such that $(S_{\pi+j}, x_{\pi+j}, y_{\pi+j}, \kappa_{\pi+j})_{0\le j\le \nu-\pi}$ is a pillar or a Y-flip at $(S_\pi, x_\pi, y_\pi, \kappa_\pi)$; we call $(S_{n(S)+j}, x_{n(S)+j}, y_{n(S)+j}, \kappa_{n(S)+j})_{0\le j\le \nu-n(S)}$ the **top** of S, and we call $(S_j, x_j, y_j, \kappa_j)_{0\le j\le n(S)}$ the **bottom** of S. We call

$$(S_{t(S)+j}, x_{t(S)+j}, y_{t(S)+j}, \kappa_{t(S)+j})_{0\le j\le \nu-t(S)} \quad \text{and} \quad (S_j, x_j, y_j, \kappa_j)_{0\le j\le t(S)}$$

the **head** of S and **tail** of S respectively; note that the head of S either has **height zero**, or is a **protoflip**, or is a **retroflip**. We define the **pillar length** of S to be a positive integer $l(S)$, its **height sequence** $q(S) = q_i(S)_{0\le i\le l(S)}$ where $q_i(S) \in \mathbb{N}$ with $0 = q_0(S) \le \cdots \le q_{l(S)}(S)$ and $q_{l(S)-1}(S) = t(S) \le \nu = q_{l(S)}(S)$ with $l(S) > 2 \Rightarrow q_{l(S)-2}(S) < q_{l(S)-1}(S)$, and its **flip sequence** $(S^{(i)})_{1\le i\le l(S)}$ where

$$S^{(i)} = (S_{q_{i-1}(S)+j}, x_{q_{i-1}(S)+j}, y_{q_{i-1}(S)+j}, \kappa_{q_{i-1}(S)+j})_{0\le j\le q_i(S)-q_{i-1}(S)}$$
$$\text{for } 1 \le i < l(S)$$

is a nonflip or flip at $(S_{q_{i-1}(S)}, x_{q_{i-1}(S)}, y_{q_{i-1}(S)}, \kappa_{q_{i-1}(S)})$ according as i is odd or even and

$$S^{(l(S))} = \text{the head of } S.$$

Again let $S = (S_j, x_j, y_j, \kappa_j)_{0\le j<\infty}$ be any infinite QDT sequence of (R, x, y, κ). Assuming either $n(S) \ne \infty$ or $t(S) \ne \infty$, we define the **pillar length** of S to be a positive integer $l(S)$, its **height sequence** $q(S) = q_i(S)_{0\le i\le l(S)}$ where $q_0(S) = 0$ and $q_{l(S)}(S) = \infty$ and $q_i(S) \in \mathbb{N}$ with $q_i(S) \le q_{i+1}(S)$ for $0 \le i < l(S)$ and $q_{l(S)-1}(S) = t(S)$ with $l(S) > 2 \Rightarrow q_{l(S)-2} < q_{l(S)-1}$, or $q_{l(S)-1}(S) = n(S)$, according as $t(S) \ne \infty$ or $t(S) = \infty$, and its **flip sequence** $(S^{(i)})_{1\le i\le l(S)}$ where

$$S^{(i)} = (S_{q_{i-1}(S)+j}, x_{q_{i-1}(S)+j}, y_{q_{i-1}(S)+j}, \kappa_{q_{i-1}(S)+j})_{0\le j\le q_i(S)-q_{i-1}(S)}$$
$$\text{for } 1 \le i < l(S)$$

is a nonflip or flip at $(S_{q_{i-1}(S)}, x_{q_{i-1}(S)}, y_{q_{i-1}(S)}, \kappa_{q_{i-1}(S)})$ according as i is odd or even and

$$S^{(l(S))} = \text{the head or top of } S \text{ according as } t(S) \ne \infty \text{ or } t(S) = \infty.$$

Finally assuming $n(S) = \infty = t(S)$ we define the **pillar length** $l(S) = \infty$ of S, its **height sequence** $q(S) = q_i(S)_{0\le i<\infty}$ where $q_i(S) \in \mathbb{N}$ with $0 = q_0(S) \le q_1(S) \le q_2(S) \le \ldots$, and its **flip sequence** $(S^{(i)})_{0\le i<\infty}$ where

$$S^{(i)} = (S_{q_{i-1}(S)+j}, x_{q_{i-1}(S)+j}, y_{q_{i-1}(S)+j}, \kappa_{q_{i-1}(S)+j})_{0\le j\le q_i(S)-q_{i-1}(S)} \quad \text{for all } i \in \mathbb{N}_+$$

is a nonflip or flip at $(S_{q_{i-1}(S)}, x_{q_{i-1}(S)}, y_{q_{i-1}(S)}, \kappa_{q_{i-1}(S)})$ according as i is odd or even.

The unique existence of the quantities $l(S), q(S), S^{(i)}$ is easily established by looking at the three or four cases separately.

Real valuations 5.1 In the situation of Assumption 4.2, assume that $R^\dagger = \widehat{R}$. Let $W(L/R)$ be the set of all valuation rings V with quotient field L such that V dominates R and $H(V)$ is algebraic over $H_V(R)$. For basic information on valuations see [1] and [2], as well as (0.1) and (10.5) of [3]; the said (0.1) says that the union of an infinite QDT sequence of two dimensional regular local domains is a valuation ring, and the said (10.5) says that a valuation dominating an analytically irreducible local domain can be extended to its completion. By (0.1) of [3] we see that every $V \in W(L/R)$ can be uniquely expressed as $V = \cup_{j\in\mathbb{N}} S(V)_j$ where $(S(V)_j)_{0\le j<\infty}$ is an infinite QDT sequence of R; let us denote the corresponding infinite QDT sequence $(S(V)_j, x(V)_j, y(V)_j, \kappa(V)_j)_{0\le j<\infty}$ of (R, x, y, κ) by $S(V)$. By (0.1) of [3] we see that $V \mapsto S(V)$ gives a bijection of $W(L/R)$ onto the set $\overline{W}$ of all infinite QDT sequences of (R, x, y, κ). Upon letting $S^\dagger(V)_j = f'(S(V)_j)$, by Assumption 4.2 we see that $(S^\dagger(V)_j, x(V)_j, y(V)_j, \kappa(V)_j)_{0\le j<\infty}$ is an infinite QDT sequence of $(R^\dagger, x, y, \kappa)$; we denote this sequence by $S^\dagger(V)$. Upon letting $V^\dagger = \cup_{j\in\mathbb{N}} S^\dagger(V)_j$ by (0.1) of [3] we see that $V^\dagger \in W(L^\dagger/R^\dagger)$, and $V \mapsto V^\dagger$ gives a bijection $W(L/R) \to W(L^\dagger/R^\dagger)$; this provides an alternative proof of a special case of (10.5) of [3]. Now the value group G_v, of the valuation v whose valuation ring is a given member V of $W(L/R)$, is either irrational (i.e., real but not rational), or rational nondiscrete (i.e., rational but not isomorphic to $\mathbb{Z}$), or discrete (i.e., isomorphic to $\mathbb{Z}$), or nonreal (i.e., lexicographically ordered pairs of integers); we attach these adjectives of G_v to V. We observe that: (1) V is irrational $\Leftrightarrow n(S(V)) = \infty \neq t(S(V)) \Leftrightarrow n(S^\dagger(V)) = \infty \neq t(S^\dagger(V)) \Leftrightarrow V^\dagger$ is irrational, (2) V is rational nondiscrete $\Leftrightarrow n(S(V)) = \infty = t(S(V)) \Leftrightarrow n(S^\dagger(V)) = \infty = t(S^\dagger(V)) \Leftrightarrow V^\dagger$ is rational nondiscrete,

Nonreal valuations 5.2 In the situation of (5.1) we observe that:

(3) V is discrete or nonreal $\Leftrightarrow n(S(V)) \neq \infty \Leftrightarrow n(S^\dagger(V)) \neq \infty \Leftrightarrow V^\dagger$ is nonreal $\Leftrightarrow S^\dagger(V) = \mathfrak{P}(R^\dagger, F^\epsilon)$ for some irreducible $F \in M(R^\dagger)^\times$ and some $\epsilon \in \mathbb{N}_+$.

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