

Endpoint boundedness of Riesz transforms on Hardy spaces associated with operators

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Abstract Let L_1 be a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates and L_2 a second order divergence form elliptic operator with complex bounded measurable coefficients. A typical example of L_1 is the Schrödinger operator $-\Delta + V$, where Δ is the Laplace operator on \mathbb{R}^n and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. Let $H^p_{L_i}(\mathbb{R}^n)$ be the Hardy space associated to L_i for $i \in \{1, 2\}$. In this paper, the authors prove that the Riesz transform $D(L_i^{-1/2})$ is bounded from $H^p_{L_i}(\mathbb{R}^n)$ to the classical weak Hardy space $WH^p(\mathbb{R}^n)$ in the critical case that $p = n/(n+1)$. Recall that it is known that $D(L_i^{-1/2})$ is bounded from $H^p_{L_i}(\mathbb{R}^n)$ to the classical Hardy space $H^p(\mathbb{R}^n)$ when $p \in (n/(n+1), 1]$.

Keywords Riesz transform · Davies-Gaffney estimate · Schrödinger operator · Second order elliptic operator · Hardy space · Weak Hardy space

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1 Introduction

The Hardy spaces, as a suitable substitute of Lebesgue spaces $L^p(\mathbb{R}^n)$ when $p \in (0, 1]$, play an important role in various fields of analysis and partial differen-

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tial equations. For example, when $p \in (0, 1]$, the Riesz transform $\nabla(-\Delta)^{-1/2}$ is not bounded on $L^p(\mathbb{R}^n)$, but bounded on the Hardy space $H^p(\mathbb{R}^n)$, where Δ is the Laplacian operator $\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ and ∇ is the gradient operator $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ on \mathbb{R}^n . It is well known that the classical Hardy spaces $H^p(\mathbb{R}^n)$ are essentially related to Δ , which has been intensively studied in, for example, [7, 15, 30, 32, 33] and their references.

In recent years, the study of Hardy spaces associated to differential operators inspires great interests; see, for example, [2–4, 10–13, 17–20] and their references. In particular, Auscher, Duong and McIntosh [3] first introduced the Hardy space $H_L^1(\mathbb{R}^n)$ associated to L , where the heat kernel generated by L satisfies a pointwise Poisson type upper bound. Later, Duong and Yan [9, 10] introduced its dual space $\text{BMO}_L(\mathbb{R}^n)$ and established the dual relation between $H_L^1(\mathbb{R}^n)$ and $\text{BMO}_{L^*}(\mathbb{R}^n)$, where L^* denotes the adjoint operator of L in $L^2(\mathbb{R}^n)$. Yan [35] further introduced the Hardy space $H_L^p(\mathbb{R}^n)$ for some $p \in (0, 1]$ but near to 1 and generalized these results to $H_L^p(\mathbb{R}^n)$ and their dual spaces. A theory of the Orlicz-Hardy space and its dual space associated to a such L was developed in [24, 25].

Moreover, for the Schrödinger operator $-\Delta + V$, Dziubański and Zienkiewicz [12, 13] first introduced the Hardy spaces $H_{-\Delta+V}^p(\mathbb{R}^n)$ with the nonnegative potential V belonging to the reverse Hölder class $B_q(\mathbb{R}^n)$ for certain $q \in (1, \infty)$. As a special case, the Hardy space $H_{-\Delta+V}^p(\mathbb{R}^n)$ associated with $-\Delta + V$ with $0 \leq V \in L_{\text{loc}}^1(\mathbb{R}^n)$ and $p \in (0, 1]$ but near to 1 was also studied in, for example, [8, 10, 19, 23, 25, 35–37]. More generally, for nonnegative self-adjoint operators L satisfying the Davies-Gaffney estimates, Hofmann et al. [19] introduced a new Hardy space $H_L^1(\mathbb{R}^n)$. In particular, when $L \equiv -\Delta + V$ with $0 \leq V \in L_{\text{loc}}^1(\mathbb{R}^n)$, Hofmann et al. originally showed that the Riesz transform $\nabla(L^{-1/2})$ is bounded from $H_L^1(\mathbb{R}^n)$ to the classical Hardy space $H^1(\mathbb{R}^n)$. These results in [19] were further extended to the Orlicz-Hardy space and its dual space in [23]. In particular, as a special case of [23, Theorem 6.3], it was proved that $\nabla(-\Delta + V)^{-1/2}$ with $0 \leq V \in L_{\text{loc}}^1(\mathbb{R}^n)$ is bounded from the Hardy space $H_{-\Delta+V}^p(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ if $p \in (\frac{n}{n+1}, 1]$.

Also, Auscher and Russ [2] studied the Hardy space H_L^1 on strongly Lipschitz domains associated with a second order divergence form elliptic operator L whose heat kernels have the Gaussian upper bounds and certain regularity. Hofmann and Mayboroda [17, 18] and Hofmann et al. [20] introduced the Hardy and Sobolev spaces associated to a second order divergence form elliptic operator L on \mathbb{R}^n with complex bounded measurable coefficients. Notice that, for the second order divergence form elliptic operator L , the kernel of the heat semigroup may fail to satisfy the Gaussian upper bound estimate and, moreover, L may not be nonnegative self-adjoint in $L^2(\mathbb{R}^n)$. Hofmann et al. [20] also proved that the associated Riesz transform $\nabla L^{-1/2}$ is bounded from $H_L^p(\mathbb{R}^n)$ to the classical Hardy space $H^p(\mathbb{R}^n)$ with $p \in (\frac{n}{n+1}, 1]$, which was also independently obtained by Jiang and Yang in [21, Theorem 7.4]. Moreover, a theory of the Orlicz-Hardy space and its dual space associated to L was developed in [21, 22].

Recently, the Hardy space $H_{(-\Delta)^2+V^2}^1(\mathbb{R}^n)$ associated to the Schrödinger-type operators $(-\Delta)^2 + V^2$ with $0 \leq V$ satisfying the reverse Hölder inequality was also studied in [6]. Moreover, the Hardy space $H_L^p(\mathbb{R}^n)$ associated to a one-to-one oper-

ator of type ω satisfying the k -Davies-Gaffney estimate and having a bounded H_∞ functional calculus was introduced in [5], where $k \in \mathbb{N}$. Notice that when $k = 1$, the k -Davies-Gaffney estimate is just the Davies-Gaffney estimate. Typical examples of such operators include the $2k$ -order divergence form homogeneous elliptic operator T_1 with complex bounded measurable coefficients and the $2k$ -order Schrödinger-type operator $T_2 \equiv (-\Delta)^k + V^k$, where $0 \leq V \in L^k_{\text{loc}}(\mathbb{R}^n)$. It was further proved that the associated Riesz transform $\nabla^k T_i^{-1/2}$ for $i \in \{1, 2\}$ is bounded from $H^p_{T_i}(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ with $p \in (\frac{n}{n+k}, 1]$ in [5].

On the other hand, the weak Hardy space $WH^1(\mathbb{R}^n)$ was first introduced by Fefferman and Soria in [14]. Then, Liu [26] studied the weak $WH^p(\mathbb{R}^n)$ space for $p \in (0, \infty)$ and established a weak atomic decomposition for $p \in (0, 1]$. Liu in [26] also showed that the δ -Calderón-Zygmund operator is bounded from $H^p(\mathbb{R}^n)$ to $WH^p(\mathbb{R}^n)$ with $p = n/(n + \delta)$, which was extended to the weighted weak Hardy spaces in [29].

Let L_1 be a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ satisfying the Davies-Gaffney estimates and L_2 a second order divergence form elliptic operator with complex bounded measurable coefficients. A typical example of L_1 is the Schrödinger operator $-\Delta + V$, where $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. Let $H^p_{L_i}(\mathbb{R}^n)$ be the Hardy space associated to L_i for $i \in \{1, 2\}$. In this paper, we prove that the Riesz transform $D(L_i^{-1/2})$ is bounded from $H^p_{L_i}(\mathbb{R}^n)$ to the weak Hardy space $WH^p(\mathbb{R}^n)$ in the critical case that $p = n/(n + 1)$. To be precise, we have the following general result.

Theorem 1.1 *Let $p \equiv n/(n + 1)$, L_1 be a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ satisfying the assumptions (A₁) and (A₂) as in Sect. 2 and D the operator satisfying the assumptions (B₁), (B₂) and (B₃) as in Sect. 2. Then the operator $D(L_1^{-1/2})$ is bounded from $H^p_{L_1}(\mathbb{R}^n)$ to the classical weak Hardy space $WH^p(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $f \in H^p_{L_1}(\mathbb{R}^n)$,*

$$\|D(L_1^{-1/2})f\|_{WH^p(\mathbb{R}^n)} \leq C\|f\|_{H^p_{L_1}(\mathbb{R}^n)}.$$

As an application of Theorem 1.1, we obtain the boundedness of $\nabla(-\Delta + V)^{-1/2}$ with $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$ from $H^p_{-\Delta+V}(\mathbb{R}^n)$ to the classical weak Hardy space $WH^p(\mathbb{R}^n)$ in the critical case that $p = n/(n + 1)$ as follows.

Corollary 1.1 *Let $p \equiv n/(n + 1)$ and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then the Riesz transform $\nabla(-\Delta + V)^{-1/2}$ is bounded from $H^p_{-\Delta+V}(\mathbb{R}^n)$ to $WH^p(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $f \in H^p_{-\Delta+V}(\mathbb{R}^n)$,*

$$\|\nabla(-\Delta + V)^{-1/2}f\|_{WH^p(\mathbb{R}^n)} \leq C\|f\|_{H^p_{-\Delta+V}(\mathbb{R}^n)}.$$

On the Riesz transform defined by the second order divergence form elliptic operator with complex bounded measurable coefficients, we also have the following endpoint boundedness in the critical case that $p \equiv n/(n + 1)$.

Theorem 1.2 *Let $p \equiv n/(n+1)$ and L_2 be the second order divergence form elliptic operator with complex bounded measurable coefficients. Then the Riesz transform $\nabla(L_2^{-1/2})$ is bounded from $H_{L_2}^p(\mathbb{R}^n)$ to $WH^p(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that for all $f \in H_{L_2}^p(\mathbb{R}^n)$,*

$$\|\nabla(L_2^{-1/2})f\|_{WH^p(\mathbb{R}^n)} \leq C\|f\|_{H_{L_2}^p(\mathbb{R}^n)}.$$

Recall that the second order divergence form elliptic operator with complex bounded measurable coefficients may not be nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$. Thus, we cannot deduce the conclusion of Theorem 1.2 from Theorem 1.1. However, if L is a second order divergence form elliptic operator with real symmetric bounded measurable coefficients, then L satisfies the assumptions of both Theorems 1.1 and 1.2.

We prove Theorems 1.1 and 1.2 by using the characterization of $WH^p(\mathbb{R}^n)$ in terms of the radial maximal function, namely, we need to estimate the weak $L^p(\mathbb{R}^n)$ quasi-norm of the radial maximal function of the Riesz transform acting on the atoms or molecules of the Hardy spaces $H_{L_i}^p(\mathbb{R}^n)$. Unlike the proof of the endpoint boundedness of the classical Riesz transform $\nabla(-\Delta)^{-1/2}$, whose kernel has the pointwise size estimate and regularity, the strategy to show Theorems 1.1 and 1.2 is to divide the radial maximal function into two parts by the time t based on the radius of the associated balls of atoms or molecules and then estimate each part via using L^2 off-diagonal estimates (see [16, 20] or Lemma 2.1 below).

This paper is organized as follows. In Sect. 2, we describe some assumptions on the operator L_1 ; then we recall some notion and properties concerning the Hardy space associated to L_1 and the second order divergence form elliptic operator L_2 with complex bounded measurable coefficients. We also recall the definition of weak Hardy spaces and present some technical lemmas which are used later in the next section. Section 3 is devoted to the proof Theorem 1.1, Corollary 1.1, and Theorem 1.2. In Sect. 4, a similar result on the Riesz transforms defined by *higher order divergence form homogeneous elliptic operators with complex bounded measurable coefficients* or *Schrödinger-type operators* is also presented.

Finally, we make some conventions on the notation. Throughout the whole paper, we always let $\mathbb{N} \equiv \{1, 2, \dots\}$ and $\mathbb{Z}_+ \equiv \mathbb{N} \cup \{0\}$. We use C to denote a *positive constant*, that is independent of the main parameters involved but whose value may differ from line to line. *Constants with subscripts*, such as C_0 , do not change in different occurrences. If $f \leq Cg$, we then write $f \lesssim g$; and if $f \lesssim g \lesssim f$, we then write $f \sim g$. For all $x \in \mathbb{R}^n$ and $r \in (0, \infty)$, let $B(x, r) \equiv \{y \in \mathbb{R}^n : |x - y| < r\}$ and $\alpha B(x, r) \equiv B(x, \alpha r)$ for any $\alpha > 0$. Also, for any set $E \in \mathbb{R}^n$, we use E^c to denote the set $\mathbb{R}^n \setminus E$ and χ_E the *characteristic function* of E .

2 Preliminaries

We begin with recalling some known results on the Hardy spaces associated to operators and the weak Hardy spaces.

Let L_1 be a linear operator initially defined in $L^2(\mathbb{R}^n)$ satisfying the following assumptions:

- (A₁) L_1 is nonnegative self-adjoint;
- (A₂) The semigroup $\{e^{-tL_1}\}_{t>0}$ generated by L_1 is analytic on $L^2(\mathbb{R}^n)$ and satisfies the *Davies-Gaffney estimates*, namely, there exist positive constants C_1 and C_2 such that for all closed sets $E, F \subset \mathbb{R}^n$, $t \in (0, \infty)$ and $f \in L^2(\mathbb{R}^n)$ supported in E ,

$$\|e^{-tL_1} f\|_{L^2(F)} \leq C_1 \exp\left\{-\frac{[\text{dist}(E, F)]^2}{C_2 t}\right\} \|f\|_{L^2(E)}, \tag{2.1}$$

where and in what follows, $\text{dist}(E, F) \equiv \inf_{x \in E, y \in F} |x - y|$ is the distance between E and F .

Typical examples of operators satisfying assumptions (A₁) and (A₂) include the second order divergence form elliptic operator with real symmetric bounded measurable coefficients and the Schrödinger operator $-\Delta + V$ with $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Let $\Gamma(x) \equiv \{(y, t) \in \mathbb{R}^n \times (0, \infty) : |x - y| < t\}$ be the cone with the vertex $x \in \mathbb{R}^n$. For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, the L_1 -adapted square function $S_{L_1} f(x)$ is defined by

$$S_{L_1} f(x) \equiv \left\{ \iint_{\Gamma(x)} |t^2 L_1 e^{-t^2 L_1} f(y)|^2 \frac{dy dt}{t^{n+1}} \right\}^{1/2}.$$

As in [19, 23], we define the Hardy space $H^p_{L_1}(\mathbb{R}^n)$ associated to the operator L_1 as follows.

Definition 2.1 [19, 23] Let $p \in (0, 1]$ and L_1 be an operator defined in $L^2(\mathbb{R}^n)$ satisfying the assumptions (A₁) and (A₂). A function $f \in L^2(\mathbb{R}^n)$ is said to be in $\mathbb{H}^p_{L_1}(\mathbb{R}^n)$ if $S_{L_1} f \in L^p(\mathbb{R}^n)$; moreover, define $\|f\|_{H^p_{L_1}(\mathbb{R}^n)} \equiv \|S_{L_1} f\|_{L^p(\mathbb{R}^n)}$. The Hardy space $H^p_{L_1}(\mathbb{R}^n)$ is then defined to be the completion of $\mathbb{H}^p_{L_1}(\mathbb{R}^n)$ with respect to the quasinorm $\|\cdot\|_{H^p_{L_1}(\mathbb{R}^n)}$.

For all $p \in (0, 1]$ and $M \in \mathbb{N}$, a function $a \in L^2(\mathbb{R}^n)$ is called a $(p, 2, M)_{L_1}$ -atom if there exists a function $b \in D(L^M_1)$ and a ball $B \equiv B(x_B, r_B) \subset \mathbb{R}^n$ such that

- (i) $a = L^M_1 b$;
- (ii) for each $\ell \in \{0, 1, \dots, M\}$, $\text{supp } L^\ell_1 b \subset B$;
- (iii) for all $\ell \in \{0, 1, \dots, M\}$,

$$\|(r^2_B L_1)^\ell b\|_{L^2(\mathbb{R}^n)} \leq r_B^{2M+n(\frac{1}{2}-\frac{1}{p})}. \tag{2.2}$$

We then have the following atomic decomposition of $H^p_{L_1}(\mathbb{R}^n)$.

Theorem 2.1 [19, 23] Let $p \in (0, 1]$. Suppose that $M \in \mathbb{N}$ and $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$. Then for all $f \in L^2(\mathbb{R}^n) \cap H^p_{L_1}(\mathbb{R}^n)$, there exist a sequence $\{a_j\}_{j=0}^\infty$ of $(p, 2, M)_{L_1}$ -atoms

and a sequence $\{\lambda_j\}_{j=0}^\infty$ of numbers such that $f = \sum_{j=0}^\infty \lambda_j a_j$ in both $H_{L_1}^p(\mathbb{R}^n)$ and $L^2(\mathbb{R}^n)$, and $\|f\|_{H_{L_1}^p(\mathbb{R}^n)} \sim \{\sum_{j=0}^\infty |\lambda_j|^p\}^{1/p}$.

For the second order divergence form operator, the associated Hardy space was studied in [17, 18, 20, 21]. More precisely, let $L_2 \equiv -\operatorname{div}(A\nabla)$ be a *second order divergence form elliptic operator with complex bounded measurable coefficients*. We say that L_2 is *elliptic* if the matrix $A \equiv \{a_{i,j}\}_{i,j=1}^n$ satisfies the *elliptic condition*, namely, there exist positive constants $0 < \lambda \leq \Lambda < \infty$ such that $\lambda|\xi|^2 \leq \Re(A\xi \cdot \bar{\xi})$ and $|A\xi \cdot \bar{\xi}| \leq \Lambda|\xi|^2$, where for any $z \in \mathbb{C}$, $\Re z$ denotes the *real part* of z .

Definition 2.2 [17, 20, 21] Let $p \in (0, 1]$ and L_2 be the second order divergence form elliptic operator with complex bounded measurable coefficients. A function $f \in L^2(\mathbb{R}^n)$ is said to be in $\mathbb{H}_{L_2}^p(\mathbb{R}^n)$ if $S_{L_2} f \in L^p(\mathbb{R}^n)$; moreover, define $\|f\|_{\mathbb{H}_{L_2}^p(\mathbb{R}^n)} \equiv \|S_{L_2} f\|_{L^p(\mathbb{R}^n)}$. The *Hardy space* $H_{L_2}^p(\mathbb{R}^n)$ is then defined to be the completion of $\mathbb{H}_{L_2}^p(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H_{L_2}^p(\mathbb{R}^n)}$.

Recall that in [20, 21], for all $p \in (0, 1]$, $\epsilon \in (0, \infty)$ and $M \in \mathbb{N}$, a function $A \in L^2(\mathbb{R}^n)$ is called an $(H_{L_2}^p, \epsilon, M)$ -*molecule* if there exists a ball $B \equiv B(x_B, r_B) \subset \mathbb{R}^n$ such that

- (i) for each $\ell \in \{1, \dots, M\}$, A belongs to the range of L_2^ℓ in $L^2(\mathbb{R}^n)$;
- (ii) for all $i \in \mathbb{Z}_+$ and $\ell \in \{0, 1, \dots, M\}$,

$$\|(r_B^{2\ell} L_2)^\ell A\|_{L^2(S_i(B))} \leq (2^i r_B)^{n(\frac{1}{2} - \frac{1}{p})} 2^{-i\epsilon}, \tag{2.3}$$

where $S_0(B) \equiv B$ and $S_i(B) \equiv 2^i B \setminus 2^{i-1} B$ for all $i \in \mathbb{N}$.

Assume that $\{m_j\}_j$ is a sequence of $(H_{L_2}^p, \epsilon, M)$ -molecules and $\{\lambda_j\}_j$ a sequence of numbers satisfying $\sum_j |\lambda_j|^p < \infty$. For any $f \in L^2(\mathbb{R}^n)$, if $f = \sum_j \lambda_j m_j$ in $L^2(\mathbb{R}^n)$, then $\sum_j \lambda_j m_j$ is called a *molecular $(H_{L_2}^p, 2, \epsilon, M)$ -representation* of f . The *molecular Hardy space* $H_{L_2, \text{mol}, M}^p(\mathbb{R}^n)$ is then defined to be the completion of the space

$$\mathbb{H}_{L_2, \text{mol}, M}^p(\mathbb{R}^n) \equiv \{f : f \text{ has a molecular } (H_{L_2}^p, 2, \epsilon, M)\text{-representation}\}$$

with respect to the quasi-norm

$$\|f\|_{H_{L_2, \text{mol}, M}^p(\mathbb{R}^n)} \equiv \inf \left\{ \left(\sum_{j=0}^\infty |\lambda_j|^p \right)^{1/p} : f = \sum_{j=0}^\infty \lambda_j A_j \text{ is a molecular } (H_{L_2}^p, 2, \epsilon, M)\text{-representation} \right\},$$

where the infimum is taken over all the molecular $(H_{L_2}^p, 2, \epsilon, M)$ -representations of f as above.

We have the following molecular characterization of $H_{L_2}^p(\mathbb{R}^n)$.

Theorem 2.2 [20, 21] *Let $p \in (0, 1]$. Suppose that $M > \frac{n}{2}(\frac{1}{p} - \frac{1}{2})$ and $\epsilon > 0$. Then $H^p_{L^2}(\mathbb{R}^n) = H^p_{L^2, \text{mol}, M}(\mathbb{R}^n)$. Moreover, $\|f\|_{H^p_{L^2}(\mathbb{R}^n)} \sim \|f\|_{H^p_{L^2, \text{mol}, M}(\mathbb{R}^n)}$, where the implicit constants depend only on M, n, p, ϵ and the constants appearing in the ellipticity.*

We now recall the definition of the weak Hardy space (see, for example, [14, 26, 27]). Let $p \in (0, 1]$ and $\varphi \in \mathcal{S}(\mathbb{R}^n)$ with support in the unit ball $B(0, 1)$. The weak Hardy space $WH^p(\mathbb{R}^n)$ is defined to be the space

$$\left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{WH^p(\mathbb{R}^n)} \equiv \sup_{\alpha > 0} \left(\alpha^p \left| \left\{ x \in \mathbb{R}^n : \sup_{t > 0} |\varphi_t * f(x)| > \alpha \right\} \right| \right)^{1/p} < \infty \right\}.$$

Let L_1 be a nonnegative self-adjoint operator in $L^2(\mathbb{R}^n)$ satisfying the assumptions (A₁) and (A₂). Following [1], let the operator D be a linear operator defined densely in $L^2(\mathbb{R}^n)$ and satisfy the following assumptions:

- (B₁) $DL^{-1/2}_1$ is bounded on $L^2(\mathbb{R}^n)$;
- (B₂) the family of operators, $\{\sqrt{t}De^{-tL_1}\}_{t>0}$, satisfy the Davies-Gaffney estimates as in (2.1);
- (B₃) for all $(p, 2, M)_{L_1}$ -atoms a , $\int_{\mathbb{R}^n} DL^{-1/2}_1 a(x) dx = 0$.

Typical examples of D and L_1 satisfying the assumptions (B₁), (B₂) and (B₃) include that D is the gradient operator ∇ on \mathbb{R}^n , and L_1 is the second order divergence form elliptic operator with real symmetric bounded measurable coefficients or the Schrödinger operator $-\Delta + V$ with $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$ as proved below.

Lemma 2.1 *Let $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then the Schrödinger operator $T \equiv -\Delta + V$ satisfies the assumptions (A₁) and (A₂), and both T and the gradient operator ∇ satisfy the assumptions (B₁), (B₂) and (B₃).*

Proof It is easy to see that T is nonnegative self-adjoint.

Let $e^{-tT}(\cdot, \cdot)$ be the integral kernel of the semigroup e^{-tT} . By Trotter’s formula (see, for example, [34]), we know that for all $t \in (0, \infty)$ and $x, y \in \mathbb{R}^n$,

$$0 \leq e^{-tT}(x, y) \leq e^{t\Delta}(x, y) \sim t^{-\frac{n}{2}} \exp\left\{-\frac{|x - y|^2}{t}\right\},$$

which implies that the semigroup $\{e^{-tT}\}_{t>0}$ satisfies (2.1). Thus, T satisfies the assumptions (A₁) and (A₂).

Moreover, by [19, Lemma 8.5], we conclude that there exists a positive constant C_2 such that for all closed sets $E, F \subset \mathbb{R}^n$, $t \in (0, \infty)$ and $f \in L^2(\mathbb{R}^n)$ supported in E ,

$$\|t\nabla e^{-t^2T} f\|_{L^2(F)} \lesssim \exp\left\{-\frac{[\text{dist}(E, F)]^2}{C_2 t^2}\right\} \|f\|_{L^2(E)},$$

which, combining the $L^2(\mathbb{R}^n)$ -boundedness of the Riesz transform $\nabla(T^{-1/2})$ (see [19, (8.20)]) and the fact that $\int_{\mathbb{R}^n} \nabla(T^{-1/2})a(y) dy = 0$ (see, for example [19, 23]),

implies that both T and the gradient operator satisfy the assumptions (B₁), (B₂) and (B₃). This finishes the proof of Lemma 2.1. \square

We also need the following technical lemmas.

Lemma 2.2 [27, 31] *Let $p \in (0, 1)$ and $\{f_j\}_j$ be a sequence of measurable functions. If $\sum_j |\lambda_j|^p < \infty$ and there exists a positive constant \tilde{C} such that for all $\{f_j\}_j$ and $\alpha \in (0, \infty)$, $|\{x \in \mathbb{R}^n : |f_j| > \alpha\}| \leq \tilde{C}\alpha^{-p}$. Then, for all $\alpha \in (0, \infty)$,*

$$\left| \left\{ x \in \mathbb{R}^n : \left| \sum_j \lambda_j f_j(x) \right| > \alpha \right\} \right| \leq \tilde{C} \frac{2-p}{1-p} \alpha^{-p} \sum_j |\lambda_j|^p.$$

Lemma 2.3 [1, 16] *Let L_1 be a nonnegative self-adjoint operator satisfying the assumptions (A₁) and (A₂) and D the operator satisfying the assumptions (B₁), (B₂) and (B₃). Let $M \in \mathbb{N}$. Then there exists a positive constant C , depending on M , such that for all closed sets E, F in \mathbb{R}^n with $\text{dist}(E, F) > 0$, $f \in L^2(\mathbb{R}^n)$ supported in E and $t \in (0, \infty)$,*

$$\|DL_1^{-1/2}(I - e^{-tL_1})^M f\|_{L^2(F)} \leq C \left(\frac{t}{[\text{dist}(E, F)]^2} \right)^M \|f\|_{L^2(E)} \tag{2.4}$$

and

$$\|DL_1^{-1/2}(tL_1 e^{-tL_1})^M f\|_{L^2(F)} \leq C \left(\frac{t}{[\text{dist}(E, F)]^2} \right)^M \|f\|_{L^2(E)}. \tag{2.5}$$

Moreover, if L_2 is a second order divergence form elliptic operator with complex bounded measurable coefficients, then (2.4) and (2.5) still hold when D and L_1 are replaced, respectively, by the gradient operator ∇ and L_2 .

3 Proofs of main results

In this section, we show Theorem 1.1, Corollary 1.1 and Theorem 1.2.

Proof of Theorem 1.1 Let $p \equiv \frac{n}{n+1}$. By the density of $H_{L_1}^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ in $H_{L_1}^p(\mathbb{R}^n)$, we only need to consider $f \in H_{L_1}^p(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Let $M \in \mathbb{N}$ and $M > \max\{\frac{1}{2} + \frac{n}{4}, 1\}$. By Theorem 2.1, we know that there exist a sequence $\{a_j\}_j$ of $(p, 2, M)_{L_1}$ -atoms and a sequence $\{\lambda_j\}_j$ of numbers such that

$$f = \sum_j \lambda_j a_j \tag{3.1}$$

in $L^2(\mathbb{R}^n)$ and $\|f\|_{H_{L_1}^p(\mathbb{R}^n)} \sim \{\sum_j |\lambda_j|^p\}^{1/p}$. To show Theorem 1.1, by (3.1) and the definition of $WH^p(\mathbb{R}^n)$, we see that it suffices to prove that for all $\alpha \in (0, \infty)$,

$$\left| \left\{ x \in \mathbb{R}^n : \sup_{0 < t < \infty} \left| \varphi_t * \left(\sum_j \lambda_j DL_1^{-1/2} a_j \right) (x) \right| > \alpha \right\} \right| \lesssim \frac{1}{\alpha^p} \sum_j |\lambda_j|^p, \tag{3.2}$$

where $\varphi \in C_c^\infty(\mathbb{R}^n)$ satisfies $\text{supp } \varphi \subset B(0, 1)$ and, for all $x \in \mathbb{R}^n$ and $t \in (0, \infty)$, $\varphi_t(x) \equiv \frac{1}{t^n} \varphi(\frac{x}{t})$. In order to prove (3.2), by Lemma 2.2, it suffices to show that for any $(p, 2, M)_{L_1}$ -atom a associated with the ball $B \equiv B(x_B, r_B)$ and $\alpha \in (0, \infty)$,

$$\left| \left\{ x \in \mathbb{R}^n : \sup_{0 < t < \infty} \left| \varphi_t * (DL_1^{-1/2} a)(x) \right| > \alpha \right\} \right| \lesssim \frac{1}{\alpha^p}.$$

Let \mathcal{M} be the Hardy-Littlewood maximal function. It is easy to see that

$$\sup_{0 < t < \infty} \left| \varphi_t * (DL_1^{-1/2} a) \right| \lesssim \mathcal{M}(DL_1^{-1/2} a).$$

Then by Chebyshev’s inequality, Hölder’s inequality, the $L^2(\mathbb{R}^n)$ -boundedness of \mathcal{M} , the $L^2(\mathbb{R}^n)$ -boundedness of $DL_1^{-1/2}$ via (B₁), and (2.2), we know that

$$\begin{aligned} & \left| \left\{ x \in 16B : \sup_{0 < t < \infty} \left| \varphi_t * (DL_1^{-1/2} a)(x) \right| > \alpha \right\} \right| \\ & \lesssim \frac{1}{\alpha^p} \left\| \sup_{0 < t < \infty} \left| \varphi_t * (DL_1^{-1/2} a) \right| \right\|_{L^p(16B)}^p \lesssim \frac{1}{\alpha^p} \left\| \mathcal{M}(DL_1^{-1/2} a) \right\|_{L^p(16B)}^p \\ & \lesssim \frac{1}{\alpha^p} \left\| \mathcal{M}(DL_1^{-1/2} a) \right\|_{L^2(\mathbb{R}^n)}^p |B|^{1-\frac{p}{2}} \lesssim \frac{1}{\alpha^p} \|a\|_{L^2(\mathbb{R}^n)}^p |B|^{1-\frac{p}{2}} \lesssim \frac{1}{\alpha^p}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \left\{ x \in (16B)^c : \sup_{0 < t < \infty} \left| \varphi_t * (DL_1^{-1/2} a)(x) \right| > \alpha \right\} \\ & \subset \left\{ x \in (16B)^c : \sup_{0 < t < r_B} \left| \varphi_t * (DL_1^{-1/2} a)(x) \right| > \alpha/2 \right\} \\ & \cup \left\{ x \in (16B)^c : \sup_{r_B < t < \infty} |\dots| > \alpha/2 \right\} \equiv \text{I} \cup \text{J}. \end{aligned}$$

To estimate I, let $S_i(B) \equiv 2^i B \setminus 2^{i-1} B$ and $\tilde{S}_i(B) \equiv 2^{i+1} B \setminus 2^{i-2} B$ with $i \in \mathbb{N}$. For all $i \geq 5$, $x \in S_i(B)$ and $y \in B(x, r_B)$, from $\text{supp } \varphi \subset B(0, 1)$, it follows that $y \in \tilde{S}_i(B)$. For $i \geq 5$, let

$$\text{I}_i \equiv \left\{ x \in S_i(B) : \sup_{0 < t < r_B} \left| \varphi_t * (DL_1^{-1/2} a)(x) \right| > \alpha/2 \right\}.$$

By Chebyshev’s inequality, Hölder’s inequality, the $L^2(\mathbb{R}^n)$ -boundedness of \mathcal{M} , Lemma 2.3 and (2.2), we conclude that

$$\begin{aligned} |\text{I}_i| & \lesssim \alpha^{-p} \int_{S_i(B)} \left[\sup_{0 < t < r_B} \left| \int_{\tilde{S}_i(B)} t^{-n} \varphi\left(\frac{x-y}{t}\right) [\chi_{\tilde{S}_i(B)}(y) DL_1^{-1/2} a(y)] dy \right| \right]^p dx \\ & \lesssim \alpha^{-p} \int_{S_i(B)} [\mathcal{M}(\chi_{\tilde{S}_i(B)} DL_1^{-1/2} a)(x)]^p dx \\ & \lesssim \alpha^{-p} |S_i(B)|^{1-p/2} \|DL_1^{-1/2} a\|_{L^2(\tilde{S}_i(B))}^p \end{aligned}$$

$$\begin{aligned} &\lesssim \alpha^{-p} |S_i(B)|^{1-p/2} \left[\|DL_1^{-1/2} (I - e^{-r_B^2 L_1})^M a\|_{L^2(\tilde{S}_i(B))}^p \right. \\ &\quad \left. + \sum_{k=1}^M \|DL_1^{-1/2} (r_B^2 L_1 e^{-\frac{k}{M} r_B^2 L_1})^M r_B^{-2M} b\|_{L^2(\tilde{S}_i(B))}^p \right] \\ &\lesssim \alpha^{-p} |S_i(B)|^{1-p/2} \left[\frac{r_B^2}{(2^i r_B)^2} \right]^{Mp} |B|^{p/2-1} \sim 2^{-i[2Mp-n(1-p/2)]} \alpha^{-p}. \end{aligned}$$

From this, the definition of I_i , $p = \frac{n}{n+1}$ and $M > \frac{1}{2} + \frac{n}{4}$, we deduce that $|I| \lesssim \sum_{i=1}^\infty |I_i| \lesssim \frac{1}{\alpha^p}$, which is a desired estimate for I.

To estimate J, by the assumption that $\int_{\mathbb{R}^n} DL_1^{-\frac{1}{2}} a(y) dy = 0$ via (B₃), we know that

$$|J| \lesssim \left| \left\{ x \in (16B)^{\mathbb{G}} : \sum_{i=0}^\infty \sup_{r_B < t < \infty} \left| \int_{S_i(B)} \frac{1}{t^n} \left[\varphi\left(\frac{x-y}{t}\right) - \varphi\left(\frac{x-x_B}{t}\right) \right] DL_1^{-\frac{1}{2}} a(y) dy \right| > \alpha/2 \right\} \right|.$$

Let $F_i(x) \equiv \sup_{r_B < t < \infty} \left| \int_{S_i(B)} \frac{1}{t^n} [\varphi(\frac{x-y}{t}) - \varphi(\frac{x-x_B}{t})] DL_1^{-\frac{1}{2}} a(y) dy \right|$ and

$$J_i \equiv \{x \in (16B)^{\mathbb{G}} : F_i(x) > \alpha/2\}.$$

To obtain a desired estimate for J, by Lemma 2.2, it suffices to show that there exists a positive constant C_0 such that

$$|J_i| \lesssim \frac{2^{-C_0 i}}{\alpha^p}. \tag{3.3}$$

From the mean value theorem, Hölder’s inequality, $\text{supp } \varphi \subset B(0, 1)$, Lemma 2.3 and (2.2), we infer that

$$\begin{aligned} F_i(x) &\leq \sup_{j \in \mathbb{Z}_+} \sup_{2^j r_B \leq t < 2^{j+1} r_B} \chi_{(2^{i+1}+2^{j+1})B}(x) \\ &\quad \times \int_{S_i(B)} \frac{1}{t^n} \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \left| \frac{y-x_B}{t} \right| \left\| DL_1^{-\frac{1}{2}} a(y) \right\| dy \\ &\lesssim \sup_{j \in \mathbb{Z}_+} \chi_{(2^{i+1}+2^{j+1})B}(x) \sup_{2^j r_B \leq t < 2^{j+1} r_B} 2^{-j(n+1)} |B|^{-1} 2^i |S_i(B)|^{1/2} \\ &\quad \times \|DL_1^{-\frac{1}{2}} a\|_{L^2(S_i(B))} \\ &\lesssim \sup_{j \in \mathbb{Z}_+} \chi_{(2^{i+1}+2^{j+1})B}(x) \sup_{2^j r_B \leq t < 2^{j+1} r_B} 2^{-j(n+1)} 2^{i(n/2+1)} \left[\frac{r_B^2}{(2^i r_B)^2} \right]^M |B|^{-1/p} \end{aligned}$$

$$\equiv C_3 \sup_{j \in \mathbb{Z}_+} \chi_{(2^{i+1}+2^{j+1})B}(x) \sup_{2^j r_B \leq t < 2^{j+1} r_B} 2^{-j(n+1)} 2^{-i(2M-n/2-1)} |B|^{-1/p}.$$

Let

$$j_0 \equiv \max \{ j \in \mathbb{Z}_+ : C_3 2^{-j(n+1)} 2^{-i(2M-n/2-1)} |B|^{-1/p} > \alpha/2 \}.$$

For all $x \in [(2^{i+1} + 2^{j_0+1})B]^c$, we see that

$$\begin{aligned} F_i(x) &\leq C_3 \sup_{j \geq j_0} \chi_{(2^{i+1}+2^{j+1})B}(x) \\ &\quad \times \sup_{2^j r_B \leq t < 2^{j+1} r_B} 2^{-j(n+1)} 2^{-i(2M-n/2-1)} |B|^{-1/p} \leq \alpha/2, \end{aligned}$$

which implies that $x \in J_i^c$. Thus, $J_i \subset (2^{i+1} + 2^{j_0+1})B$. From this and Chebyshev’s inequality, we then deduce that

$$\begin{aligned} |J_i| &\lesssim \alpha^{-p} \int_{(2^{i+1}+2^{j_0+1})B} 2^{-pj_0(n+1)} 2^{-ip(2M-1+n)} |B|^{-1} dx \\ &\lesssim 2^{-i[(2M-1)p-n(1-p)]} \alpha^{-p}, \end{aligned}$$

which implies that (3.3) holds with $C_0 \equiv (2M - 1)p - n(1 - p)$. Observe that $C_0 > 0$, since $M > 1$ and $p = \frac{n}{n+1}$. Thus, combining the estimate of I and J, we then complete the proof of Theorem 1.1. \square

Proof of Corollary 1.1 From Lemma 2.1, we deduce that the Schrödinger operator $-\Delta + V$ with $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies the assumptions (A_1) and (A_2) as in Sect. 2, and both $-\Delta + V$ and the gradient operator ∇ satisfy the assumptions (B_1) , (B_2) and (B_3) as in Sect. 2. Thus, from Theorem 1.1, we deduce that the Riesz transform $\nabla(-\Delta + V)^{-1/2}$ is bounded from $H^p_{-\Delta+V}(\mathbb{R}^n)$ to the classical weak Hardy space $WH^p(\mathbb{R}^n)$ in the critical case that $p = n/(n + 1)$, which completes the proof of Corollary 1.1. \square

Proof of Theorem 1.2 Let $p \equiv \frac{n}{n+1}$ and $M \in \mathbb{N}$ satisfy $M > \frac{n}{4} + \frac{1}{2}$. To prove Theorem 1.2, similar to the proof of Theorem 1.1, by Theorem 2.2 and Lemma 2.2, for each (H^p_L, ϵ, M) -molecule A associated to the ball $B(x_B, r_B)$, $m \in \mathbb{Z}_+$ and $\alpha \in (0, \infty)$, we only need estimate the measure of the following sets:

$$\tilde{I} \equiv \left\{ x \in (16B)^c : \sup_{0 < t < r_B} |\varphi_t * (\nabla L_2^{-1/2} A)(x)| > \alpha/2 \right\}$$

and

$$\tilde{J} \equiv \left\{ x \in (16B)^c : \sup_{r_B \leq t < \infty} |\varphi_t * (\nabla L_2^{-1/2} A)(x)| > \alpha/2 \right\}.$$

The estimate of \widetilde{I} is similar to that of I in the proof of Theorem 1.1. We omit the details. Now we estimate \widetilde{J} . Since

$$\begin{aligned} |\widetilde{J}| \lesssim & \left\{ \left| x \in (16B)^{\mathbb{G}} : \sum_{i=0}^{\infty} \sup_{r_B \leq t < \infty} \left| \int_{S_i(B)} \frac{1}{t^n} \left[\varphi\left(\frac{x-y}{t}\right) - \varphi\left(\frac{x-x_B}{t}\right) \right] \right. \right. \right. \\ & \left. \left. \times \nabla L_2^{-\frac{1}{2}} (I - e^{-r_B^2 L_2})^M A(y) dy \right| > \alpha/2 \right\} \\ & + \left\{ \left| x \in (16B)^{\mathbb{G}} : \sum_{i=0}^{\infty} \sum_{k=1}^M \sup_{r_B \leq t < \infty} \left| \int_{S_i(B)} \frac{1}{t^n} \left[\varphi\left(\frac{x-y}{t}\right) - \varphi\left(\frac{x-x_B}{t}\right) \right] \right. \right. \right. \\ & \left. \left. \times \nabla L_2^{-\frac{1}{2}} (r_B^2 L_2 e^{-\frac{k}{M} r_B^2 L_2})^M (r_B^2 L_2)^{-M} A(y) dy \right| > \alpha/2 \right\}. \end{aligned}$$

Let $\widetilde{F}_{1,i}(x) \equiv \sup_{r_B \leq t < \infty} \left| \int_{S_i(B)} \frac{1}{t^n} [\varphi(\frac{x-y}{t}) - \varphi(\frac{x-x_B}{t})] \nabla L_2^{-\frac{1}{2}} (I - e^{-r_B^2 L_2})^M \times A(y) dy \right|$,

$$\begin{aligned} \widetilde{F}_{2,i}(x) \equiv & \sum_{k=1}^M \sup_{r_B \leq t < \infty} \left| \int_{S_i(B)} \frac{1}{t^n} \left[\varphi\left(\frac{x-y}{t}\right) - \varphi\left(\frac{x-x_B}{t}\right) \right] \right. \\ & \left. \times \nabla L_2^{-\frac{1}{2}} (r_B^2 L_2 e^{-\frac{k}{M} r_B^2 L_2})^M (r_B^2 L_2)^{-M} A(y) dy \right|, \end{aligned}$$

$\widetilde{J}_{1,k} \equiv \{x \in (16B)^{\mathbb{G}} : \widetilde{F}_{1,i}(x) > \alpha/2\}$ and $\widetilde{J}_{2,k} \equiv \{x \in (16B)^{\mathbb{G}} : \widetilde{F}_{2,i}(x) > \alpha/2\}$. By Lemma 2.2, it suffices to show that there exist positive constants C_4 and C_5 such that for all $\alpha \in (0, \infty)$, $|\widetilde{J}_{1,k}| \lesssim \frac{2^{-C_4 i}}{\alpha^p}$ and $|\widetilde{J}_{2,k}| \lesssim \frac{2^{-C_5 i}}{\alpha^p}$. We only prove the first inequality, the proof of the second inequality is similar. Take $\epsilon \in (n + 1 - 1/(n + 1), \infty)$. By the mean value theorem, Hölder’s inequality, Lemma 2.3, (2.3) and $\text{supp } \varphi \subset B(0, 1)$, we conclude that

$$\begin{aligned} \widetilde{F}_{1,i}(x) & \lesssim \sup_{j \in \mathbb{Z}_+} \chi_{(2^{i+1}+2^{j+1})B}(x) \sup_{2^j r_B \leq t < 2^{j+1} r_B} \int_{S_i(B)} \frac{1}{t^n} \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \left| \frac{y-x_B}{t} \right| \\ & \quad \times \left| \nabla L_2^{-\frac{1}{2}} (I - e^{-r_B^2 L_2})^M A(y) \right| dy \\ & \lesssim \sup_{j \in \mathbb{Z}_+} \chi_{(2^{i+1}+2^{j+1})B}(x) \sup_{2^j r_B \leq t < 2^{j+1} r_B} \int_{S_i(B)} \frac{1}{t^n} \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \left| \frac{y-x_B}{t} \right| \\ & \quad \times \left| \nabla L_2^{-\frac{1}{2}} (I - e^{-r_B^2 L_2})^M (\chi_{\widetilde{S}_i(B)} A)(y) \right| dy \\ & + \sup_{j \in \mathbb{Z}_+} \chi_{(2^{i+1}+2^{j+1})B}(x) \sup_{2^j r_B \leq t < 2^{j+1} r_B} \int_{S_i(B)} \frac{1}{t^n} \|\nabla \varphi\|_{L^\infty(\mathbb{R}^n)} \left| \frac{y-x_B}{t} \right| \\ & \quad \times \left| \nabla L_2^{-\frac{1}{2}} (I - e^{-r_B^2 L_2})^M (\chi_{\mathbb{R}^n \setminus \widetilde{S}_i(B)} A)(y) \right| dy \end{aligned}$$

$$\begin{aligned} &\lesssim \sup_{j \in \mathbb{Z}_+} \chi_{(2^{j+1}+2^{j+1})B}(x) \\ &\quad \times \sup_{2^j r_B \leq t < 2^{j+1} r_B} 2^{-j(n+1)} [2^{-i(\epsilon+n/p-n-1)} + 2^{-i(2M-n/2-1)}] |B|^{-1/p}, \end{aligned}$$

where $S_i(B)$ and $\tilde{S}_i(B)$ are as in the proof of Theorem 1.1. The rest of the proof is similar to that of Theorem 1.1; we omit the details. This finishes the proof of Theorem 1.2. \square

4 Further remarks

In this section, we establish a variant of Theorems 1.1 and 1.2 for the higher order divergence form elliptic operators with complex bounded measurable coefficients and the higher order Schrödinger-type operators.

To this end, we first recall some notions and notation. For $\theta \in [0, \pi)$, the *closed sector*, S_θ , of angle θ in the complex plane \mathbb{C} is defined by $S_\theta \equiv \{z \in \mathbb{C} \setminus \{0\} : |\arg z| \leq \theta\} \cup \{0\}$. Let $\omega \in [0, \pi)$. A closed operator T in $L^2(\mathbb{R}^n)$ is called of *type* ω (see, for example, [28]), if its spectrum, $\sigma(T)$, is contained in S_ω , and for each $\theta \in (\omega, \pi)$, there exists a nonnegative constant C such that for all $z \in \mathbb{C} \setminus S_\theta$, $\|(T - zI)^{-1}\|_{\mathcal{L}(L^2(\mathbb{R}^n))} \leq C|z|^{-1}$, where and in what follows, $\|S\|_{\mathcal{L}(\mathcal{H})}$ denotes the *operator norm* of the linear operator S on the normed linear space \mathcal{H} . Let T be a one-to-one operator of type ω , with $\omega \in [0, \pi)$ and $\mu \in (\omega, \pi)$, and $f \in H_\infty(S_\mu^0) \equiv \{f \text{ is holomorphic on } S_\mu^0 : \|f\|_{L^\infty(S_\mu^0)} < \infty\}$, where S_μ^0 denotes the *interior* of S_μ . By the H_∞ functional calculus, the function of the operator T , $f(T)$ is well defined. The operator T is said to have a *bounded H_∞ functional calculus* in the Hilbert space \mathcal{H} , if there exist $\mu \in (0, \pi)$ and positive constant C such that for all $\psi \in H_\infty(S_\mu^0)$, $\|f(T)\|_{\mathcal{L}(\mathcal{H})} \leq C\|\psi\|_{L^\infty(S_\mu^0)}$.

As in [5], let T be an operator defined in $L^2(\mathbb{R}^n)$ which satisfies the following *assumptions*:

- (E₁) The operator T is a one-to-one operator of type ω in $L^2(\mathbb{R}^n)$ with $\omega \in [0, \pi/2)$;
- (E₂) The operator T has a bounded H_∞ functional calculus in $L^2(\mathbb{R}^n)$;
- (E₃) Let $k \in \mathbb{N}$. The operator T generates a holomorphic semigroup $\{e^{-tT}\}_{t>0}$ which satisfies the *k-Davies-Gaffney estimate*, namely, there exist positive constants C_6 and C_7 such that for all closed sets E and F in \mathbb{R}^n , $t \in (0, \infty)$ and $f \in L^2(\mathbb{R}^n)$ supported in E ,

$$\|e^{-tT} f\|_{L^2(F)} \leq C_6 \exp\left\{-\frac{[\text{dist}(E, F)]^{2k/(2k-1)}}{C_7 t^{1/(2k-1)}}\right\} \|f\|_{L^2(E)}.$$

When $k = 1$, the *k-Davies-Gaffney estimate* is just (2.1).

Let $k \in \mathbb{N}$. Typical examples of operators, satisfying the above assumptions (E₁), (E₂) and (E₃), include the following *2k-order divergence form homogeneous elliptic operator*

$$T_1 \equiv (-1)^k \sum_{|\alpha|=|\beta|=k} \partial^\alpha (a_{\alpha,\beta} \partial^\beta) \tag{4.1}$$

with complex bounded measurable coefficients $\{a_{\alpha, \beta}\}_{|\alpha|=|\beta|=k}$, and the following $2k$ -order Schrödinger-type operator

$$T_2 \equiv (-\Delta)^k + V^k \tag{4.2}$$

with $0 \leq V \in L^k_{\text{loc}}(\mathbb{R}^n)$.

For all $f \in L^2(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, define the T -adapted square function $S_T f(x)$ by

$$S_T f(x) \equiv \left\{ \iint_{\Gamma(x)} |t^{2k} T e^{-t^{2k} T} f(y)|^2 \frac{dy dt}{t^{n+1}} \right\}^{1/2}.$$

Using the T -adapted square function $S_T f$, Cao and Yang [5] introduced the following Hardy space $H^p_T(\mathbb{R}^n)$ associated to T .

Definition 4.1 [5] Let $p \in (0, 1]$ and T satisfy the assumptions (E₁), (E₂) and (E₃). A function $f \in L^2(\mathbb{R}^n)$ is said to be in $\mathbb{H}^p_T(\mathbb{R}^n)$ if $S_T f \in L^p(\mathbb{R}^n)$; moreover, define $\|f\|_{H^p_T(\mathbb{R}^n)} \equiv \|S_T f\|_{L^p(\mathbb{R}^n)}$. The Hardy space $H^p_T(\mathbb{R}^n)$ is then defined to be the completion of $\mathbb{H}^p_T(\mathbb{R}^n)$ with respect to the quasi-norm $\|\cdot\|_{H^p_T(\mathbb{R}^n)}$.

Let $i \in \{1, 2\}$. By first establishing the molecular characterization of $H^p_{T_i}(\mathbb{R}^n)$, Cao and Yang [5] then obtained the following boundedness of the Riesz transform $\nabla^k(T_i^{-1/2})$ from $H^p_{T_i}(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$ when $p \in (n/(n+k), 1]$.

Theorem 4.1 [5] Let $k \in \mathbb{N}$, $p \in (n/(n+k), 1]$, T_1 be the $2k$ -order divergence form homogeneous elliptic operator with complex bounded measurable coefficients as in (4.1), and T_2 the $2k$ -order Schrödinger-type operator as in (4.2). Then, for $i \in \{1, 2\}$, the Riesz transform $\nabla^k(T_i^{-1/2})$ is bounded from $H^p_{T_i}(\mathbb{R}^n)$ to $H^p(\mathbb{R}^n)$.

Again, for $i \in \{1, 2\}$, applying the molecular characterization of $H^p_{T_i}(\mathbb{R}^n)$ from [5], by an argument similar to that used in the proof of Theorem 1.2, we obtain the endpoint boundedness of $\nabla^k(T_i^{-1/2})$ in the critical case that $p = n/(n+k)$. We omit the details by similarity.

Theorem 4.2 Let $k \in \mathbb{N}$, $p \equiv n/(n+k)$, T_1 be the $2k$ -order divergence form homogeneous elliptic operator with complex bounded measurable coefficients as in (4.1), and T_2 the $2k$ -order Schrödinger-type operator as in (4.2). Then, for $i \in \{1, 2\}$, the Riesz transform $\nabla^k(T_i^{-1/2})$ is bounded from $H^p_{T_i}(\mathbb{R}^n)$ to $WH^p(\mathbb{R}^n)$.

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