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# On the metric projection onto $\varphi$-convex subsets of Hadamard manifolds

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**Abstract** We introduce the concept of a $\varphi$-convex subset of a Hadamard manifold. Then we prove that for a $\varphi$-convex subset $S$ of a Hadamard manifold $M$ there exists an open set $U$ containing $S$ such that the metric projection is a single valued locally Lipschitz mapping on $U$.

**Keywords** Convex sets · $\varphi$-convex sets · Limiting subdifferential · Hadamard manifolds

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## 1 Introduction

The concepts of convexity and generalized convexity for sets and functions play a central role in many areas of mathematics. An important generalization of convexity is $\varphi$-convexity. A $\varphi$-convex set satisfies an external sphere condition with locally uniform radius. Such sets under the name of "sets with positive reach" were studied in finite dimensional linear spaces by Federer in [9], where apparently this concept was started for the first time. The notion of $\varphi$-convexity in linear spaces (as titled $p$-convexity) was introduced in a slightly different manner in [8] and framed in the

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concept of $\varphi$-convex functions. Canino in [4] introduced several important properties of p-convex sets in infinite dimensional Hilbert spaces and then in [5] the existence of closed geodesics on p-convex sets in linear spaces was studied. In a Hilbert space $\varphi$-convexity implies that all normal cones coincide (see [6, 7]). It turns out that certain properties which hold globally for convex sets are still valid locally for $\varphi$-convex sets. For example it is well known that a closed subset of a Hilbert space is convex if and only if its corresponding metric projection is globally nonempty and single valued. On the other hand, it was proved in [4] that the metric projection into a $\varphi$-convex subset $S$ of a Hilbert space $H$ is locally nonempty unique and Lipschitz continuous. Moreover, as a consequence it was shown that $d_S^2$ is of class $C^{1+}$ in a neighborhood of $S$.

Unlike a Hilbert space, a manifold in general does not have a linear structure and therefore new techniques are needed for dealing with the concepts of the metric projection and distance function from sets in manifolds. Moreover, these notions are not of local type and can not be studied by local techniques. A number of results regarding the metric projection and distance function corresponding to the convex sets in Riemannian manifolds have been obtained. In [17] differentiability of the metric projection for a closed locally convex subset $S$ of a finite dimensional Riemannian manifold $M$ was shown. Moreover, the author proved that the distance function from $S$, near and outside of $S$ is of class $C^1$.

In 1981, Greene and Shiohama proved that for a closed totally convex subset $S$ of a finite dimensional Riemannian manifold $M$, there exists an open set $W$ containing $S$ such that the metric projection is locally Lipschitz on $W$; see [11]. It was shown in [12] for a closed convex subset $S$ of a finite dimensional Hadamard manifold $M$, the metric projection is single valued and Lipschitz on $M$. We do not know if there is any result regarding the metric projection onto convex subsets of infinite dimensional Riemannian manifolds.

Our aim is to extend the results regarding the metric projection of $\varphi$-convex subsets of Hilbert spaces to infinite dimensional Hadamard manifolds.

The paper is organized as follows. Section 2 is concerned with the concept of limiting subdifferential and some results related to nonsmooth analysis on Riemannian manifolds. In Section 3, the notion of $\varphi$-convex subsets of Hadamard manifolds is introduced. Then some properties of metric projection onto these subsets are studied.

## 2 Limiting subdifferential and limiting normal cone

In this paper, we use the standard notations and known results of Riemannian manifolds; see [14]. In what follows $M$ is a $C^\infty$ smooth manifold modelled on a Hilbert space $H$, either finite dimensional or infinite dimensional, endowed with a Riemannian metric $\langle ., .\rangle_x$ on the tangent space $T_xM \cong H$. In the case when $\gamma$ is a minimizing geodesic and $\gamma(t_0) = x, \gamma(t_1) = y$, the parallel transport from $T_xM$ to $T_yM$ along the curve $\gamma$ is denoted by $L_{xy}$. Recall that the set $S$ in a Riemannian manifold $M$ is called to be convex if every two points $x_1,\ x_2 \in S$ can be joined by a unique geodesic whose image belongs to $S$. Also, $f$ defined on a Riemannian manifold $M$ is called to be convex provided $f \circ \gamma : I \subset \mathbb{R} \to \mathbb{R}$ is convex for every geodesic $\gamma : I \to M$ (parameterized by arc length).

For a nonempty set $S$ of a Riemannian manifold $M$, the set of metric projection of the point $q \in M$ to the set $S$ denoted by $P_S(q)$ is defined as follows,

$$P_S(q) = \left\{ p \in S : d(q,p) = \inf_{z \in S} d(q,z) = d_S(q) \right\}.$$

Let us start with some definitions of nonsmooth analysis on Riemannian manifolds; for more details see [1, 2].

Let $M$ be a Riemannian manifold, and let $f : M \to (-\infty, +\infty]$ be a lower semicontinous function. The proximal subdifferential of $f$ at a point $x \in \operatorname{dom}(f) = \{x \in M : f(x) < \infty\}$ denoted by $\partial_P f(x)$ consists of all $\zeta \in T_x M$ such that

$$f(y) \geq f(x) + \left\langle \zeta, \exp_x^{-1}(y) \right\rangle_x - \sigma d(x,y)^2, \tag{2.1}$$

for each $y$ in a neighborhood of $x$.

The Fréchet subdifferential of $f$ at a point $x \in \operatorname{dom}(f)$ denoted by $\partial_F f(x)$ is defined as follows,

$$\partial_F f(x) = \left\{ d\varphi(x) \mid \varphi \in C^1(M),\ f - \varphi \text{ attains a local minimum at } x \right\}.$$

Now using the Fréchet subdifferential, we can define the limiting subdifferential of a lower semicontinous function defined on a Riemannian manifold; see [15, 16].

**Definition 2.1** Let $M$ be a Riemannian manifold, and let $f : M \to (-\infty, +\infty]$ be a lower semicontinous function. We define the limiting subdifferential of $f$ at a point $x \in M$ denoted by $\partial f(x)$ as

$$\partial f(x) := \left\{ w - \lim \zeta_i : \zeta_i \in \partial_F f(x_i),\ \left(x_i, f(x_i)\right) \to \left(x, f(x)\right) \right\},$$

where $w$-lim signifies weak limit.

Note that a sequence $(\zeta_i)$ with $\zeta_i \in T_{x_i} M$ is said to converge weakly to $\zeta \in T_x M$ provided $x_i$ converges to $x$ in $M$ and for every vector field $V$ defined on an open neighborhood of $x$;

$$\left\langle \zeta_i, V(x_i) \right\rangle_{x_i} \to \left\langle \zeta, V(x) \right\rangle_x.$$

It is worthwhile to mention that

$$\partial_P f(x) \subseteq \partial_F f(x) \subseteq \partial f(x). \tag{2.2}$$

Let $S$ be a closed subset of a Riemannian manifold $M$. The Fréchet and proximal normal cones of $S$ at a point $x \in S$ are defined, respectively, by

$$N_F(x,S) := \partial_F \delta_S(x),$$

and

$$N_P(x,S) := \partial_P \delta_S(x),$$

where $\delta_S$ is the indicator function of $S$ defined by $\delta_S = 0$ if $x \in S$ and $\delta_S = +\infty$ if $x \notin S$. It is easy to verify that $\xi \in N_P(x, S)$ if and only if there is $\sigma > 0$ such that

$$\left\langle \xi, \exp_x^{-1}(y) \right\rangle_x \leq \sigma d(y, x)^2, \tag{2.3}$$

for every $y$ in a neighborhood of $x$; see [13].

Similarly, we can define the limiting normal cone of $S$ at $x \in S$ as follows,

$$N(x, S) := \partial \delta_S(x).$$

**Proposition 2.1** *Let $M$ be a Riemannian manifold and $f : M \to (-\infty, +\infty]$ be a lower semicontinous function.*

(i) *If $f$ is locally Lipschitz, then $\partial f(x) \neq \phi$ for every $x \in \mathrm{dom}(f)$.*
(ii) *For every $x \in \mathrm{dom}(f)$,*

$$\partial f(x) = \left\{ w - \lim \zeta_i : \zeta_i \in \partial_P f(x_i), \left(x_i, f(x_i)\right) \to \left(x, f(x)\right) \right\}. \tag{2.4}$$

*Proof* (i) Fix $x \in \mathrm{dom}(f)$. By [2, Theorem 3.2], there exist sequences $\{x_n\}$ and $\{\zeta_n\}$ with $\zeta_n \in \partial_P f(x_n)$ such that $(x_n, f(x_n)) \to (x, f(x))$. Suppose that $f$ is Lipschitz of rank $L$ in a neighborhood of $x$. Then [2, Proposition 3.1] implies $\|\zeta_n\|_{x_n} \leq L$ for $n$ large enough. We extract a subsequence of $\{L_{x_n x}(\zeta_n)\}$ which converges to an element $\zeta \in T_x M$ in weak topology. Hence, (without relabeling) for every $C^\infty$-vector field $V$ on a neighborhood of $x \in M$ we obtain

$$\left\langle \zeta_n, V(x_n) \right\rangle_{x_n} = \left\langle L_{x_n x}(\zeta_n), L_{x_n x}\left(V(x_n)\right) \right\rangle_x \to \left\langle \zeta, V(x) \right\rangle_x,$$

which means $\zeta_n$ converges to $\zeta$.

(ii) Set $A = \{w - \lim \zeta_i : \zeta_i \in \partial_P f(x_i), (x_i, f(x_i)) \to (x, f(x))\}$. Let $\zeta \in \partial f(x)$. Hence, there exist sequences $\{x_n\}$ and $\{\zeta_n\}$ with $\zeta_n \in \partial_F f(x_n)$ such that $(x_n, f(x_n)) \to (x, f(x))$ and $\zeta_n \to \zeta$. By [2, Proposition 3.10] there exist $y_n \in B(x_n, \frac{1}{n})$ and $\psi_n \in \partial_P f(y_n)$ such that $|f(y_n) - f(x_n)| < \frac{1}{n}$ and $\|\zeta_n - L_{y_n x_n}(\psi_n)\|_{x_n} < \frac{1}{n}$. It is easy to see that $(y_n, f(y_n)) \to (x, f(x))$ and $\psi_n \to \zeta$. Therefore, $\zeta \in A$ and (2.2) completes the proof. □

We recall that a simply connected complete Riemannian manifold of nonpositive sectional curvature is called a Hadamard manifold.

**Proposition 2.2** *Let $M$ be a Hadamard manifold and $S$ be a closed convex subset of $M$, then $\zeta \in N_F(s, S)$ if and only if $\langle \zeta, \exp_s^{-1}(s') \rangle_s \leq 0$ for every $s' \in S$.*

*Proof* Let $\zeta \in N_F(s, S)$. By [1, Theorem 4.3] there exists $\varphi \in C^1(M)$ such that $\zeta = d\varphi(s)$ and $\delta_S - \varphi$ attains a global minimum at $s$. Hence $\varphi(s') \leq \varphi(s)$ for every $s' \in S$. Fix an arbitrary $s' \in S$. Since $S$ is convex, the unique geodesic $\gamma : [0, 1] \to M$ joining the points $s, s'$, defined by $\gamma(t) := \exp_s(t \exp_s^{-1}(s'))$, belongs entirely to $S$. Therefore, $\varphi \circ \gamma(t) \leq \varphi \circ \gamma(0)$ for $t \in [0, 1]$. So we can deduce that

$$0 \geq (\varphi \circ \gamma)'(0) = \left\langle \zeta, \exp_s^{-1}\left(s'\right) \right\rangle_s.$$

By (2.2) and (2.3) one can obtain the reverse implication. □

The following lemma has an essential role in the next section.

**Lemma 2.1** *Let $M$ be a Hadamard manifold, $S$ be a closed subset of $M$ and $y_0 \notin S$. Suppose that there exists $x_0 \in S$ such that $x_0 \in P_S(y_0)$. Then* $\exp_{x_0}^{-1}(y_0) \in N_F(x_0, S)$.

*Proof* Since the function $g(s) = d(s, y_0)^2$ attains a minimum at $x_0$ on $S$ and is $C^2$ on $M$, it follows from [2, Proposition 3.1] that

$$0 \in 2d(x_0, y_0)\frac{\partial d}{\partial x}(x_0, y_0) + \partial_P \delta_S(x_0).$$

Hence, [1, Lemma 6.5] implies

$$2\exp_{x_0}^{-1}(y_0) = -2d(x_0, y_0)\frac{\partial d}{\partial y}(x_0, y_0) \in N_P(x_0, S).$$

Now using (2.2) for the indicator function corresponding to $S$ completes the proof. □

## 3 Metric projection onto $\varphi$-convex subsets

In this section we establish our main results. First we introduce the notion of $\varphi$-convexity for subsets of Hadamard manifolds. This class of sets is of particular importance, since it includes convex sets. Moreover, some results regarding the existence and properties of the metric projection of convex sets are extended to $\varphi$-convex sets of Hadamard manifolds.

**Definition 3.1** Let $M$ be a Hadamard manifold, and $S$ be a closed subset of $M$. We say that $S$ is $\varphi$-convex if there exists a continuous function $\varphi : S \to (0, +\infty)$ such that

$$\left\langle \zeta, \exp_x^{-1}(y)\right\rangle_x \le \varphi(x)\|\zeta\|_x d(x, y)^2, \quad \text{for every } x, y \in S \text{ and } \zeta \in N_F(x, S). \quad (3.1)$$

Note that in Definition 3.1 when the function $\varphi$ is a constant $\varphi_0$ then we say that $S$ is $\varphi_0$-convex.

*Example 3.1* If $S$ is a closed convex subset of a Hadamard manifold $M$, then by Proposition 2.2, $S$ is 0-convex.

In the following example, it is demonstrated that a $\varphi_0$-convex subset of a Hadamard manifold is not necessarily convex.

*Example 3.2* Let $M$ be a Hadamard manifold modelled on a separable Hilbert space $H$. Fix $x \in M$ and suppose that $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $T_x M$. Set $x_n := (1 + 2^{-n})e_n$, $n \in \mathbb{N}$ and define $y_n := \exp_x(x_n)$, $S := \{y_n\}_{n\in\mathbb{N}}$. It is obvious that $\|x_m - x_n\|_x^2 = \|x_m\|_x^2 + \|x_n\|_x^2 > 2$. It follows from [14, Corollary 3.10, p. 252] that

$$d(y_m, y_n) = d\big(\exp_x(x_m), \exp_x(x_n)\big) \ge \|x_m - x_n\|_x.$$

Therefore

$$\frac{1}{\sqrt{2}}\left\|\exp_{y_m}^{-1}(y_n)\right\|_{y_m} \geq \frac{1}{\sqrt{2}}\|x_m - x_n\|_x > 1. \tag{3.2}$$

Now (3.2) implies that for $\zeta \in T_{y_m}M$,

$$\begin{aligned}\left\langle\zeta, \exp_{y_m}^{-1}(y_n)\right\rangle_{y_m} &\leq \|\zeta\|_{y_m}\left\|\exp_{y_m}^{-1}(y_n)\right\|_{y_m} \leq \frac{1}{\sqrt{2}}\|\zeta\|_{y_m}\left\|\exp_{y_m}^{-1}(y_n)\right\|_{y_m}^2 \\ &= \frac{1}{\sqrt{2}}\|\zeta\|_{y_m} d(y_m, y_n)^2,\end{aligned}$$

which means $S$ is $\frac{1}{\sqrt{2}}$-convex. Note that since $S$ is countable, it is not convex.

Using [1, Lemma 6.5], Proposition 2.1 in [4] can be extended to Hadamard manifolds.

**Proposition 3.1** *Let $S$ be a closed subset of a Hadamard manifold $M$ and $\varphi : S \to (0, +\infty)$ be a continuous function. Then $S$ is $\varphi$-convex if and only if for every $x, y \in S$ and $\zeta \in \partial_F \delta_S(x)$, $\psi \in \partial_F \delta_S(y)$*

$$\left\langle\zeta - L_{yx}(\psi), \exp_x^{-1}(y)\right\rangle_x \leq \left[\varphi(x)\|\zeta\|_x + \varphi(y)\|\psi\|_y\right] d(x, y)^2. \tag{3.3}$$

The following theorem provides a sufficient condition for the uniqueness of the closest point from a $\varphi$-convex subset of a Hadamard manifold.

**Theorem 3.1** *Let $S$ be a $\varphi$-convex subset of a Hadamard manifold $M$ and $x \in S$. Suppose that $y \in M$ is such that $\exp_x^{-1}(y) \in N_F(x, S)$ and $2\varphi(x)d(x, y) < 1$. Then $P_S(y) = \{x\}$.*

*Proof* Fix an arbitrary $z \in S$. It follows from [14, p. 261] that

$$d(y, z)^2 \geq d(y, x)^2 + d(z, x)^2 - 2\left\langle\exp_x^{-1}(y), \exp_x^{-1}(z)\right\rangle_x. \tag{3.4}$$

Since $S$ is $\varphi$-convex it follows that

$$\begin{aligned}d(y, z)^2 &\geq d(y, x)^2 + d(z, x)^2 - 2\varphi(x)d(x, y)d(x, z)^2 \\ &= d(y, x)^2 + \left(1 - 2\varphi(x)d(x, y)\right)d(x, z)^2 \\ &> d(y, x)^2.\end{aligned}$$

Hence, $d(y, z) > d(y, x)$. Since $z$ is arbitrary, $x$ is the unique element of $S$ satisfying $d(y, x) = d_S(y)$. □

Let $S$ be a $\varphi$-convex subset of a Hadamard manifold $M$ and $y \in M$. We define a new distance between $y$ and $S$ as follows:

$$\delta_\varphi(y, S) := \limsup_{d(x,y) \to d_S(y),\ x \in S} 2\varphi(x)d(x, y).$$

Moreover, we define a tabular set containing $S$ denoted by $\hat{S}$ which consists of all $y \in M$ with the following properties:

(i) $\delta_\varphi(y, S) < 1$.

(ii) There exists a real number $r \geq 0$ such that $S \cap \{x \in M : d(x, y) \leq r\}$ is nonempty.

In the following theorem employing Shapiro's variational principle, we prove that $P_S : \hat{S} \to S$ is single valued.

Let us recall Shapiro's variational principle, see [18]. Consider the optimization problems

$$\min_{x \in S} f(x), \tag{3.5}$$

$$\min_{x \in T} g(x), \tag{3.6}$$

where $f, g : H \to \mathbb{R}$ and $S$, $T$ are subsets of a Hilbert space $H$. Let $x_0$ be an optimal solution of the problem (3.5) and let $\bar{x}$ be an $\epsilon$-optimal of (3.6), i.e., $\bar{x} \in T$ and

$$g(\bar{x}) \leq \inf_{x \in T} g(x) + \epsilon.$$

Suppose that there exist a positive constant $\alpha$ and a neighborhood $W$ of $x_0$ such that for all $x \in S \cap W$,

$$f(x) \geq f(x_0) + \alpha \|x - x_0\|^2.$$

Suppose further that $f(x)$ and $g(x)$ are Lipschitz continuous on $W$ with Lipschitz constants $k_1$ and $k_2$, respectively, and that $\bar{x} \in W$. Then

$$\|\bar{x} - x_0\| \leq \alpha^{-1}\lambda + \alpha^{-1/2}\epsilon^{1/2} + 2\delta_1 + \alpha^{-1/2}(k_1\delta_1 + k_2\delta_2)^{1/2},$$

where $\delta_1 = \sup_{x \in T \cap W} d_{S \cap W}(x)$ and $\delta_2 = d_{T \cap W}(x_0)$, $\lambda$ is a Lipschitz constant of the function $h(x) = g(x) - f(x)$ on $W$. Note that if $S = T$, then $\delta_1 = \delta_2 = 0$.

**Theorem 3.2** *Let $S$ be a $\varphi$-convex subset of a Hadamard manifold $M$. Then $P_S : \hat{S} \to S$ is single valued.*

*Proof* Let $x \in \hat{S}$ and $r > 0$ be such that $S \cap \{y \in M : d(x, y) \leq r\}$ is nonempty.

If $d_S(x) = r$, then there exists $y \in S$ such that $d(x, y) \leq r = d_S(x) \leq d(x, y)$, which implies the existence of a point $y \in S$ satisfying $d(x, y) = d_S(x)$. Now, from Theorem 3.1 and Lemma 2.1 it follows that $y$ is unique.

If $d_S(x) = 0$, then $P_S(x) = \{x\}$ and the proof is complete.

If $0 < d_S(x) < r$, then by Corollary 3.6 in [2] there exist sequences $x_i$ in $M$ and $y_i \in S$ such that $\lim_{i\to\infty} x_i = x$ and $d(x_i, y_i) = d_S(x_i)$. Therefore

$$\lim_{i\to\infty} d(x, y_i) = \lim_{i\to\infty} d(x_i, y_i) = d_S(x),$$

and

$$\limsup_{i\to\infty} 2\varphi(y_i) d(x, y_i) < 1. \tag{3.7}$$

Now, we claim that $\{y_i\}$ is convergent to a point $y \neq x$, hence

$$\lim_{i\to\infty} y_i = y, \qquad 2\varphi(y)d(x,y) < 1 \quad \text{and} \quad d(x,y) = d_S(x),$$

and Lemma 2.1 implies that $\exp_y^{-1}(x) \in N_F(y,S)$. Hence, by Theorem 3.1 the proof is complete.

To prove the claim, by (3.7) there exists a subsequence, still denoted by $\{y_i\}$, such that $\limsup_{i\to\infty} \varphi(y_i) < +\infty$, hence $\limsup_{i\to\infty} 2\varphi(y_i)d(x_i,y_i) < 1$.

Now since $d(x_i,y_i) = d_S(x_i)$, by Lemma 2.1 we get $\exp_{y_i}^{-1}(x_i) \in N_F(y_i,S)$ and from $\varphi$-convexity property of $S$, for arbitrary $z \in S$ and $i \in \mathbb{N}$,

$$\big\langle \exp_{y_i}^{-1}(x_i), \exp_{y_i}^{-1}(z)\big\rangle_{y_i} \leq \varphi(y_i)d(y_i,x_i)d(y_i,z)^2. \tag{3.8}$$

On the other hand, we consider the following optimization problems for $n,\ m \in \mathbb{N}$,

$$\min_{z\in S} d(x_n,z)^2 = \min_{v\in\exp_x^{-1}(S)} d\big(x_n, \exp_x(v)\big)^2, \tag{3.9}$$

$$\min_{z\in S} d(x_m,z)^2 = \min_{v\in\exp_x^{-1}(S)} d\big(x_m, \exp_x(v)\big)^2. \tag{3.10}$$

By [14, p. 261] for every $z \in S$

$$d(x_n,z)^2 - d(x_n,y_n)^2 \geq -2\big\langle \exp_{y_n}^{-1}(x_n), \exp_{y_n}^{-1}(z)\big\rangle_{y_n} + d(y_n,z)^2. \tag{3.11}$$

Hence it follows from (3.11) and (3.8) that for every $z \in S$

$$d(x_n,z)^2 - d(x_n,y_n)^2 \geq \big(1 - 2\varphi(y_n)d(y_n,x_n)\big)d(y_n,z)^2.$$

Now we define a new sequence in $T_xM$ as follows, $\{w_i\} = \{\exp_x^{-1}(y_i)\}$. Therefore, if $\exp_x(v) = z$, [14, Corollary 3.10, p. 252] implies that

$$d(x_n,z)^2 - d(x_n,y_n)^2 \geq \big(1 - 2\varphi(y_n)d(y_n,x_n)\big)d(w_n,v)^2.$$

By Shapiro's variational principle,

$$d(w_n,w_m) \leq \frac{2}{1 - 2\varphi(y_n)d(y_n,x_n)} d(x_n,x_m),$$

which means $\{w_i\}$ is Cauchy in complete space $T_xM$. Therefore, there exists $w \in T_xM$ such that $w_i \to w$. Since $S$ is closed in $M$ and $\exp_x$ is continuous, $y_i = \exp_x(w_i) \to \exp_x(w) = y \in S$, as required. □

Now we prove that $P_S$ is locally Lipschitz on a neighborhood of a $\varphi$-convex subset $S$ of a Hadamard manifold $M$. Hence it follows that $P_S$ is almost everywhere differentiable on a neighborhood of $S$.

**Theorem 3.3** *Let $S$ be a $\varphi$-convex subset of a Hadamard manifold $M$. Then there exists neighborhood $U$ containing $S$ such that $P_S$ is locally Lipschitz on $U$.*

*Proof* Let $x$ be an arbitrary point of $S$ and $\varphi(x) < M_x$. Since $\varphi$ is continuous, it follows that there exists $B(x, r_x)$ such that $\varphi(y) < M_x + 1$ for every $y$ in $B(x, r_x)$. Set $\alpha_x < \min\{\frac{r_x}{2}, \frac{1}{4(M_x+1)}\}$ and $4\alpha_x(1 + M_x) = C_x$. Using [1, Theorem 2.3] there exists $\beta_x$ such that $\exp_x^{-1}$ is Lipschitz of constant $\frac{1}{1-C_x}$ on $B(x, \beta_x)$. Set $\rho_x < \min\{\beta_x, \alpha_x\}$. One can deduce that $d_S(y) \le d(x, y) < \rho_x$, for every $y \in B(x, \rho_x)$. Hence

$$\delta_\varphi(y, S) \le 4\rho_x(1 + M_x) < C_x < 1.$$

For each $x \in S$ we choose number $\rho_x$ and set $U = \bigcup_{x \in S} B(x, \rho_x)$. The open set $U$ contains $S$ and by Theorem 3.2 for every $y \in U$ there exists one and only one $x \in S$ such that $d(x, y) = d_S(y)$. We claim that $P_S$ is Lipschitz on $B(x, \rho_x)$ for every $x \in S$. Fix an arbitrary $x \in S$ and let $x_1,\ x_2$ be two arbitrary points in $B(x, \rho_x)$. To prove the theorem, we shall use Shapiro's variational principle. Consider the optimization problems

$$\min_{z \in S} d(x_1, z)^2 = \min_{v \in \exp_x^{-1}(S)} d\big(x_1, \exp_x(v)\big)^2, \tag{3.12}$$

and

$$\min_{z \in S} d(x_2, z)^2 = \min_{v \in \exp_x^{-1}(S)} d\big(x_2, \exp_x(v)\big)^2. \tag{3.13}$$

Let $P_S(x_i) = x_i',\ i = 1, 2$. By [14, p. 261] for every $z \in S$

$$d(x_1, z)^2 - d\big(x_1, x_1'\big)^2 \ge -2\big\langle \exp_{x_1'}^{-1}(x_1), \exp_{x_1'}^{-1}(z)\big\rangle_{x_1'} + d\big(x_1', z\big)^2. \tag{3.14}$$

From $\varphi$-convexity property of $S$,

$$\big\langle \exp_{x_1'}^{-1}(x_1), \exp_{x_1'}^{-1}(z)\big\rangle_{x_1'} \le \varphi\big(x_1'\big) d\big(x_1', x_1\big) d\big(x_1', z\big)^2 \le \frac{1}{2} C_x d\big(x_1', z\big)^2.$$

Hence it follows from (3.14) that

$$d(x_1, z)^2 - d\big(x_1, x_1'\big)^2 \ge (1 - C_x) d\big(x_1', z\big)^2.$$

Therefore, if $\exp_x(w_i) = x_i',\ i = 1,\ 2$ and $\exp_x(v) = z$, then [14, Corollary 3.10, p. 252] implies that

$$d(x_1, z)^2 - d\big(x_1, x_1'\big)^2 \ge (1 - C_x) d(w_1, v)^2.$$

Due to the choice of $\rho_x$ and employing Shapiro's variational principle,

$$d\big(P_S(x_1), P_S(x_2)\big) = d\big(x_1', x_2'\big) \le \frac{1}{1 - C_x} d(w_1, w_2) \le \frac{2}{(1 - C_x)^2} d(x_1, x_2). \qquad \square$$

**Definition 3.2** Let $M$ be a Riemannian manifold. A mapping $X : M \to TM$ satisfying $X_y \in T_y M$ for all $y \in M$ is said to be Lipschitz vector field of rank $k$ near a given point $x \in M$, if for some $\varepsilon > 0$, we have

$$\|L_{yz} X(y) - X(z)\|_z \le k d(y,z) \quad \text{for all } z, y \in B(x;\varepsilon),$$

where $B(x;\varepsilon)$ is convex, and $L_{yz}$ is parallel transport along the unique geodesic connecting $z$ and $y$.

Note that if we consider the Riemannian metrics on $M$ and $TM$, then above definition is equivalent to the usual definition of locally Lipschitz functions on metric spaces, see [3, p. 241]. Any two Riemannian metrics being each bounded locally by a constant multiple of the other, give equivalent concepts of Lipschitz continuity though not the same local Lipschitz constant.

**Definition 3.3** The function $f : M \to \mathbb{R}$ defined on a Riemannian manifold $M$ is said to be $C^{1+}$ if $f$ is differentiable with the locally Lipschitz gradient vector field $\operatorname{grad}(f) : M \to TM$.

**Theorem 3.4** *Let $S$ be a $\varphi$-convex subset of a Hadamard manifold $M$. Then there exists a neighborhood $U$ containing $S$ such that $d_S^2$ is $C^{1+}$ on $U \setminus S$.*

*Proof* We consider the neighborhood $U$ as in Theorem 3.3. We claim that for every $x \in S$, there exists a number $\lambda_x$ such that $d_S^2 + \lambda_x d(x,.)^2$ is convex on $B(x,\rho_x)$. Let $x_1,\ x_2$ be two arbitrary points in $B(x,\rho_x)$ and $P_S(x_i) = x_i',\ i = 1,\ 2$. We may assume that $\partial_P d_S^2(x_i),\ i = 1,\ 2$ is nonempty; see [2, Theorem 3.2]. Then [2, Theorem 3.3] implies $d_S^2$ is differentiable at $x_i,\ i = 1, 2$. Since $d(.,x)^2 : M \to \mathbb{R}$ is strongly monotone; see [10],

$$\begin{aligned}&\big\langle \exp_{x_1}^{-1}(x), \exp_{x_1}^{-1}(x_2)\big\rangle_{x_1} + \big\langle \exp_{x_2}^{-1}(x), \exp_{x_2}^{-1}(x_1)\big\rangle_{x_2}\\ &\quad \ge d(x_1,x_2)^2. \end{aligned} \tag{3.15}$$

Theorem 3.3 implies

$$\left(\frac{4}{(1-C_x)^2} - 2\right) d(x_1,x_2)^2 \ge -2d(x_1,x_2)^2 + 2d(x_1,x_2) d\big(x_1', x_2'\big).$$

On the other hand it follows from [14, p. 261] that

$$\begin{aligned}&-d\big(x_1,x_1'\big)^2 - d\big(x_2,x_2'\big)^2 - d(x_1,x_2)^2 - d(x_1,x_2)^2 + d\big(x_1',x_2\big)^2 + d\big(x_2',x_1\big)^2\\ &\quad \ge -2\big\langle \exp_{x_1}^{-1}\big(x_1'\big), \exp_{x_1}^{-1}(x_2)\big\rangle_{x_1} - 2\big\langle \exp_{x_2}^{-1}\big(x_2'\big), \exp_{x_2}^{-1}(x_1)\big\rangle_{x_2}. \end{aligned} \tag{3.16}$$

Therefore by [10, Proposition 3.4], (3.15) and (3.16) it can be deduced that $d_S^2 + \frac{2-(1-C_x)^2}{(1-C_x)^2} d(.,x)^2$ is convex on $B(x,\rho_x)$. This shows our goal, and it also in turn implies that $\partial_P d_S^2$ is nonempty valued on $B(x,\rho_x)$ which proves differentiability of $d_S$ on $B(x,\rho_x)$.

It remains to show that vector field $\operatorname{grad}(d_S^2) : U \setminus S \to TM$ is locally Lipschitz. Assume that $X(z) = \frac{\partial d}{\partial x}(z, q_z)$ is the unite tangent to the unique minimizing geodesic segment from $z$ to $P_S(z)$, where $q_z$ is on the unique geodesic connecting $z$ and $P_S(z)$ and closed enough to $z$. Along the same lines as [11, Proposition 4.1] one can prove that the vector field $X$ is $k_x$ Lipschitz on $B(x, \rho_x) \setminus S$. Also [2, Theorem 3.3] implies that for every $x_1 \in B(x, \rho_x)$ with $P_S(x_1) = x_1'$, $\operatorname{grad}(d_S^2)(x_1) = 2d(x_1, x_1')\frac{\partial d}{\partial x}(x_1, q_1)$ where $q_1$ is on the unique geodesic connecting $x_1$ and $x_1'$ and closed enough to $x_1$.

Now, let $x_1,\ x_2$ be two arbitrary points in $B(x, \rho_x)$ and $P_S(x_i) = x_i',\ i = 1,\ 2$. Without loss of generality suppose that $d(x_2, x_2') \le d(x_1, x_1')$. So that

$$
\begin{aligned}
&\left\| L_{x_1x_2}\left(2d\big(x_1, x_1'\big)\frac{\partial d}{\partial x}(x_1, q_1)\right) - 2d\big(x_2, x_2'\big)\frac{\partial d}{\partial x}(x_2, q_2)\right\|_{x_2} \\
&\quad \le 2d\big(x_1, x_1'\big)\left\| L_{x_1x_2}\left(\frac{\partial d}{\partial x}(x_1, q_1)\right) - \frac{\partial d}{\partial x}(x_2, q_2)\right\|_{x_2} + 2\big\| d\big(x_1, x_1'\big) - d\big(x_2, x_2'\big)\big\| \\
&\quad \le 2k_x d\big(x_1, x_1'\big) d(x_1, x_2) + 2\big(d\big(x_1, x_2'\big) - d\big(x_2, x_2'\big)\big) \\
&\quad \le 2k_x d\big(x_1, x_1'\big) d(x_1, x_2) + 2d(x_1, x_2) \le (2\rho_x k_x + 2) d(x_1, x_2),
\end{aligned}
$$

which completes the proof. □

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