Some remarks on the planar Kouchnirenko's theorem

Gert-Martin Greuel · Hong Duc Nguyen

Abstract We consider different notions of non-degeneracy, as introduced by Kouchnirenko (NND), Wall (INND) and Beelen-Pellikaan (WNND) for plane curve singularities $\{f(x, y) = 0\}$ and introduce the new notion of weighted homogeneous Newton non-degeneracy (WHNND). It is known that the Milnor number μ resp. the delta-invariant δ can be computed by explicit formulas μ_N resp. δ_N from the Newton diagram of f if f is NND resp. WNND. It was however unknown whether the equalities $\mu = \mu_N$ resp. $\delta = \delta_N$ can be characterized by a certain non-degeneracy condition on f and, if so, by which one. We show that $\mu = \mu_N$ resp. $\delta = \delta_N$ is equivalent to INND resp. WHNND and give some applications and interesting examples related to the existence of "wild vanishing cycles". Although the results are new in any characteristic, the main difficulties arise in positive characteristic.

Keywords Milnor number · Delta-invariant · Newton non-degenerate · Inner Newton non-degenerate · Weak Newton non-degenerate

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1 Introduction

Let K be an algebraically closed field, $K[[x]] = K[[x_1, \ldots, x_n]]$ the formal power series ring and $\mathfrak{m}$ its maximal ideal. Let us recall the definition of the Newton diagram and Wall's notion of a C-polytope (see [13]). To each power series $f = \sum_\alpha c_\alpha x^\alpha \in K[[x]]$ we can associate its *Newton polyhedron* $\Gamma_+(f)$ as the convex hull of the set

$$\bigcup_{\alpha \in \mathrm{supp}(f)} (\alpha + \mathbb{R}^n_{\geq 0}),$$

where $\mathrm{supp}(f) = \{\alpha | c_\alpha \neq 0\}$ denotes the support of f. This is an unbounded polytope in $\mathbb{R}^n$. We call the union $\Gamma(f)$ of its compact faces the *Newton diagram* of f. By $\Gamma_-(f)$ we denote the union of all line segments joining the origin to a point on $\Gamma(f)$. We always assume that $f \in \mathfrak{m}$ if not explicitly stated otherwise.

If the Newton diagram of a singularity f meets all coordinate axes we call f *convenient*. However, not every isolated singularity is convenient, and one then has to enlarge the Newton diagram. A compact rational polytope P of dimension $n - 1$ in the positive orthant $\mathbb{R}^n_{\geq 0}$ is called a C-*polytope* if the region above P is convex and if every ray in the positive orthant emanating from the origin meets P in exactly one point. The Newton diagram of f is a C-polytope iff f is convenient.

We first introduce the different notions of non-degeneracy. For this let $f = \sum_\alpha c_\alpha x^\alpha \in \mathfrak{m}$ be a power series, let P be a C-polytope and let Δ be a face of P. By $f_\Delta := \mathrm{in}_\Delta(f) := \sum_{\alpha \in \Delta} c_\alpha x^\alpha$ we denote the initial form or principal part of f along Δ. Following Kouchnirenko we call f *non-degenerate* ND along Δ if the Jacobian ideal[1] $j(f_\Delta)$ has no zero in the torus $(K^*)^n$. f is then said to be *Newton non-degenerate* NND if f is non-degenerate along each face (of any dimension) of the Newton diagram $\Gamma(f)$. We do not require f to be convenient.

To define inner non-degeneracy we need to fix two more notions. The face Δ is an *inner face* of P if it is not contained in any coordinate hyperplane. Each point $q \in K^n$ determines a coordinate hyperspace $H_q = \bigcap_{q_i=0} \{x_i = 0\} \subset \mathbb{R}^n$ in $\mathbb{R}^n$. We call f *inner non-degenerate* IND *along* Δ if for each zero q of the Jacobian ideal $j(\mathrm{in}_\Delta(f))$ the polytope Δ contains no point on H_q. f is called *inner Newton non-degenerate* INND *w.r.t. a C-polytope P* if no point of $\mathrm{supp}(f)$ lies below P and f is IND along each inner face of P. We call f simply *inner Newton non-degenerate* INND if it is INND w.r.t. some C-polytope.

Finally, we call f *weakly non-degenerate* WND *along* Δ if the Tjurina ideal[2] $tj(\mathrm{in}_\Delta(f))$ has no zero in the torus $(K^*)^n$, and f is called *weakly Newton non-degenerate* WNND if f is weakly non-degenerate along each top-dimensional face of $\Gamma(f)$. Note that NND implies WNND while NND does not imply INND and vice versa. See [3, Remark 3.1] for facts on and relations between the different types of non-degeneracy.

For any compact polytope Q in $\mathbb{R}^n_{\geq 0}$ we denote by $V_k(Q)$ the sum of the k-dimensional Euclidean volumes of the intersections of Q with the k-dimensional

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[1]The Jacobian ideal $j(f)$ denotes the ideal generated by all partials of $f \in K[[x]]$.

[2]For $f \in K[[x]]$ we call $tj(f) = \langle f \rangle + j(f)$ the Tjurina ideal of f.

coordinate subspaces of $\mathbb{R}^n$ and, following Kouchnirenko, we then call

$$\mu_N(Q) = \sum_{k=0}^{n} (-1)^{n-k} k! V_k(Q)$$

the Newton number of Q. For a power series $f \in K[[x]]$ we define the *Newton number* of f to be

$$\mu_N(f) = \sup\{\mu_N(\Gamma_-(f_m)) | f_m := f + x_1^m + \cdots + x_n^m, m \geq 1\}.$$

If f is convenient then

$$\mu_N(f) = \mu_N(\Gamma_-(f)).$$

The following theorem was proved by Kouchnirenko in arbitrary characteristic. We recall that $\mu(f) := \dim K[[x, y]]/j(f)$ is the *Milnor number* of f.

Theorem 1.1 [10] *For $f \in K[[x]]$ we have $\mu_N(f) \leq \mu(f)$, and if f is NND and convenient then $\mu_N(f) = \mu(f) < \infty$.*

Since Theorem 1.1 does not cover all semi-quasihomogeneous singularities, Wall introduced the condition INND (denoted by NPND* in [13]). Using Theorem 1.1, Wall proved the following theorem for $K = \mathbb{C}$ which was extended to arbitrary K in [3].

Theorem 1.2 [3, 13] *If $f \in K[[x]]$ is INND, then*

$$\mu(f) = \mu_N(f) = \mu_N(\Gamma_-(f)) < \infty.$$

Kouchnirenko proved that the condition "convenient" is not necessary in Theorem 1.1 if char$(K) = 0$. The authors in [3] show that in the planar case Kouchnirenko's result holds in arbitrary characteristic without the assumption that f is convenient (allowing $\mu(f) = \infty$):

Proposition 1.3 [3, Proposition 4.5] *Suppose that $f \in K[[x, y]]$ is NND, then $\mu_N(f) = \mu(f)$.*

2 Milnor number

In the following we consider only the case of plane curve singularities. The main result of this section says that for $f \in K[[x, y]]$, the condition $\mu(f) = \mu_N(f) < \infty$ is equivalent to f being INND (Theorem 2.13). In characteristic zero this is also equivalent to f being NND and $\mu_N(f) < \infty$ (Corollary 2.17). However, in positive characteristic, this is in general not true as the following example shows.

Example 2.1 $f = x^3 + xy + y^3$ in characteristic 3 satisfies $\mu(f) = \mu_N(f) = 1$ but f is not NND.

Fig. 1
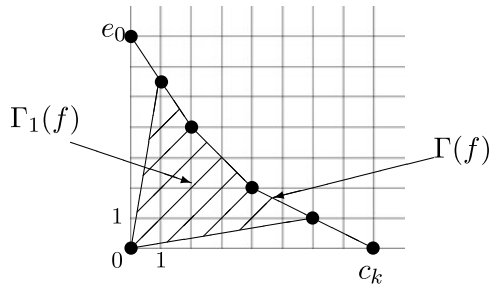
Remark 2.2 Let $f \in K[[x, y]]$ be convenient and $A_i = (c_i, e_i), i = 0, \ldots, k$ the vertices of $\Gamma(f)$ with $c_0 = e_k = 0, c_i < c_{i+1}$ and $e_i > e_{i+1}$. Then

$$\mu_N(f) = 2V_2(\Gamma_-(f)) - c_k - e_0 + 1.$$

Lemma 2.3 *Let $f, g \in K[[x, y]]$ be convenient such that $\Gamma_-(f) \subset \Gamma_-(g)$. Then*

(a) $\mu_N(f) \leq \mu_N(g)$.
(b) *The equality holds if and only if $\Gamma_-(f) \cap \mathbb{R}^2_{\geq 1} = \Gamma_-(g) \cap \mathbb{R}^2_{\geq 1}$, where*

$$\mathbb{R}^2_{\geq 1} = \{(x, y) \in \mathbb{R}^2 | x \geq 1, y \geq 1\}.$$

Part (a) of the lemma was also shown in [2, Corollary 5.6]. Let us denote by $\Gamma_1(f)$ the cone joining the origin with $\Gamma(f) \cap \mathbb{R}^2_{\geq 1}$ (cf. Fig. 1).

Proof First, we prove that

$$\mu_N(f) = V_2(\Gamma_1(f)) + 1.$$

It is easy to see that $\Gamma_1(f)$ divides $\Gamma_-(f)$ into three parts whose volumes are $c_k/2, V_2(\Gamma_1(f))$ and $e_0/2$. Therefore

$$\mu_N(f) = 2V_2(\Gamma_-(f)) - c_k - e_0 + 1 = 2V_2(\Gamma_1(f)) + 1.$$

(a) Clearly, if $\Gamma_-(f) \subset \Gamma_-(g)$ then $\Gamma_1(f) \subset \Gamma_1(g)$ and hence

$$\mu_N(f) = V_2(\Gamma_1(f)) + 1 \leq 2V_2(\Gamma_1(g)) + 1 = \mu_N(g).$$

(b) follows easily from the formula $\mu_N(f) = 2V_2(\Gamma_1(f)) + 1$. □

We recall some classical notions. Let $f \in K[[x, y]]$ be irreducible. A couple $(x(t), y(t)) \in K[[t]]^2$ is called a *(primitive) parametrization* of f, if $f(x(t), y(t)) = 0$ and if the following universal factorization property holds: for each $(u(t), v(t)) \in K[[t]]^2$ with $f(u(t), v(t)) = 0$, there exists a unique series $h(t) \in K[[t]]$ such that $u(t) = x(h(t))$ and $v(t) = y(h(t))$.

If $g \in K[[x, y]]$ is irreducible and $(x(t), y(t))$ its parametrization, then the *intersection multiplicity* of any $f \in K[[x, y]]$ with g is given by $i(f, g) = \text{ord} f(x(t), y(t))$, and if u is a unit then $i(f, u) = 0$. The *intersection multiplicity* of f with a reducible power series $g = g_1 \cdots g_s$ is defined to be the sum $i(f, g) = i(f, g_1) + \cdots + i(f, g_s)$.

Proposition 2.4 [7, Proposition 3.12] *Let* $f, g \in K[[x, y]]$. *Then*

$$i(f, g) = i(g, f) = \dim K[[x, y]]/\langle f, g \rangle.$$

The proof in [7] was given for $K = \mathbb{C}$ but works in any characteristic.

Let $f = \sum_{i,j} c_{ij} x^i y^j \in K[[x, y]]$ and $\Gamma(f)$ be its Newton diagram. We call

$$f_{in} := \sum_{(i,j) \in \Gamma(f)} c_{ij} x^i y^j$$

the *initial part* of f.

Proposition 2.5 *Let* $f \in \mathfrak{m} \subset K[[x, y]]$ *be irreducible, x-general of order m and y-general of order n. Let $(x(t), y(t))$ be parametrization of f. Then*

(a) $\operatorname{ord}(x(t)) = n$ *and* $\operatorname{ord}(y(t)) = m$.
(b) *The Newton diagram of f is the straight line segment.*
(c) *There exist* $\xi, \lambda \in K^*$ *such that*

$$f_{in}(x, y) = \xi \cdot (x^{m/q} - \lambda y^{n/q})^q,$$

where $q = (m, n)$.

Proof cf. [5, Lemma 3.4.3, 3.4.4, 3.4.5]. □

Proposition 2.6 [4, Lemma 3] *Let* $f \in K[[x, y]]$ *and let* $E_i, i = 1, \ldots, k$ *be the edges of its Newton diagram. Then there is a factorization of f:*

$$f = monomial \cdot \bar{f}_1 \cdots \bar{f}_k$$

such that $\bar{f}_i$ *is convenient,* $f_{E_i} = monomial \times (\bar{f}_i)_{in}$. *In particular, if f is convenient then* $f = \bar{f}_1 \cdots \bar{f}_k$.

A polynomial $f = \sum_{i,j} c_{ij} x^i y^j \in K[x, y]$ is called *weighted homogeneous* or *quasihomogeneous* of type $(n, m; d)$ if m, n, d are positive integers satisfying $ni + mj = d$, for each $(i, j) \in \operatorname{supp}(f)$.

Let $f \in K[[x, y]]$ be a formal power series and n, m positive integers. We can decompose f into a sum

$$f = f_d^w + f_{d+1}^w + \cdots,$$

where $f_d^w \neq 0$ and f_l^w is weighted homogeneous of type $(n, m; l)$ for $l \geq d$. We call f_d^w the first term of the decomposition.

For each series $\varphi(t) = c_1 t^{\alpha_1} + c_2 t^{\alpha_2} + \cdots$ with $c_1 \neq 0, \alpha_1 < \alpha_2 < \cdots$, we set

$$\operatorname{LT}(\varphi(t)) := c_1 t^{\alpha_1} \quad \text{and} \quad \operatorname{LC}(\varphi(t)) := c_1.$$

Lemma 2.7 *Let* m, n *be two positive integers. Let* $x(t), y(t) \in K[[t]]$ *with* $\operatorname{LT}(x(t)) = at^\alpha$ *and* $\operatorname{LT}(y(t)) = bt^\beta$ *such that* $\alpha : \beta = n : m$. *Let* $f = f_d^w + f_{d+1}^w + \cdots$, *be a* (n, m)-*weighted homogeneous decomposition of* $f \in K[[x, y]]$. *Then* $\operatorname{ord} f(x(t), y(t)) \geq \frac{d\alpha}{n}$. *Equality holds if and only if* $f_d(a, b) \neq 0$.

Proof We can write $x(t) = t^\alpha (a + u(t))$ and $y(t) = t^\beta (b + v(t))$, where $\mathrm{ord}\, u(t) > 0$ and $\mathrm{ord}\, v(t) > 0$. Then

$$f_l^{\mathrm{w}}(x(t), y(t)) = \sum_{ni+mj=l} c_{ij} (t^\alpha (a + u(t)))^i (t^\beta (b + v(t)))^j$$

$$= t^{\frac{l\alpha}{n}} f_l^{\mathrm{w}}(a + u(t), b + v(t)).$$

Thus $\mathrm{ord}\, f_l^{\mathrm{w}}(x(t), y(t)) \geq \frac{l\alpha}{n}$ and hence $\mathrm{ord}\, f(x(t), y(t)) \geq \frac{d\alpha}{n}$.

Since $f_d^{\mathrm{w}}(a + u(t), b + v(t)) = f_d^{\mathrm{w}}(a, b) + t h(t)$ for some power series h,

$$\mathrm{ord}\, f(x(t), y(t)) = \mathrm{ord}\, f_d^{\mathrm{w}}(x(t), y(t)) = \frac{d\alpha}{n}$$

iff $f_d^{\mathrm{w}}(a, b) \neq 0$.   □

Lemma 2.8 *Let $f \in K[[x, y]]$ be convenient such that $\Gamma(f)$ has only one edge. Let $m = \mathrm{ord}\, f(x, 0), n = \mathrm{ord}\, f(0, y)$ and $f = f_1 \cdot \cdots \cdot f_r$ a factorization of f into its branches (irreducible factors).*

(a) *Let $(x_j(t), y_j(t))$ be a parametrization of $f_j, j = 1, \ldots, r$ with $\mathrm{LT}(x_j(t)) = a_j t^{\alpha_j}$ and $\mathrm{LT}(y_j(t)) = b_j t^{\beta_j}$. Then $f_{\mathrm{in}}(a_j, b_j) = 0$, $\alpha_j : \beta_j = n : m$ and $\alpha_1 + \cdots + \alpha_r = n$.*

(b) *Let $a, b \in K^*$ such that $f_{\mathrm{in}}(a, b) = 0$. Then there is a parametrization $(x(t), y(t))$ of a branch of f satisfying $\mathrm{LC}(x(t)) = a$ and $\mathrm{LC}(y(t)) = b$.*

Proof Let $f = f_d^{\mathrm{w}} + f_{d+1}^{\mathrm{w}} + \cdots$ with $f_d^{\mathrm{w}} \neq 0$ be the (n, m)-weighted homogeneous decomposition of f. Then $f_d^{\mathrm{w}} = f_{\mathrm{in}}$.

(a) It is easily verified that $f_{\mathrm{in}} = \prod (f_j)_{\mathrm{in}}$ then $(f_j)_{\mathrm{in}}$ is also a (n, m)-weighted homogeneous polynomial of order some d_j. By Proposition 2.5, $\mathrm{ord}\, f_j(x, 0) = \beta_j$ and $\mathrm{ord}\, f_j(0, y) = \alpha_j$, i.e. x^{β_j} and y^{α_j} are monomials of $(f_j)_{\mathrm{in}}$. Thus $n\beta_j = d_j = m\alpha_j$ and hence $\alpha_j : \beta_j = n : m$.

Since $f(x_j(t), y_j(t)) = 0$, i.e. $\mathrm{ord}\, f(x_j(t), y_j(t)) = +\infty > \frac{d\alpha_j}{n}$, Lemma 2.7 yields that $f_d^{\mathrm{w}}(a_j, b_j) = 0$, i.e. $f_{\mathrm{in}}(a_j, b_j) = 0$.

Now, by the definition of intersection multiplicity we have

$$n = i(f, x) = \sum_{j=1}^{r} \mathrm{ord}\, x_j(t) = \sum_{j=1}^{r} \alpha_j.$$

(b) Since $f = g \cdot h$ implies $f_{\mathrm{in}} = g_{\mathrm{in}} \cdot h_{\mathrm{in}}$, it suffices to prove part (b) for the irreducible case. Then by Proposition 2.5(c), there exist $\xi, \lambda \in K^*$ such that

$$f_{\mathrm{in}}(x, y) = \rho \cdot (x^{m'} - \lambda y^{n'})^q,$$

where $q = (m, n)$; $m' = m/q$ and $n' = n/q$. It is impossible that the characteristic p of K divides both m' and n' since $(n', m') = 1$. We may assume that p does not divide n'.

Let $(\bar{x}(t), \bar{y}(t))$ be a parametrization of f. It follows from Proposition 2.5 that $\mathrm{LT}(\bar{x}(t)) = \bar{a}t^n$ and $\mathrm{LT}(\bar{y}(t)) = \bar{b}t^m$ for some $\bar{a}, \bar{b} \in K^*$ satisfying $f_{\mathrm{in}}(\bar{a}, \bar{b}) = 0$. Set

$$g(y) := a^{m'} - \lambda y^{n'} \quad \text{and} \quad \sqrt[n']{a/\bar{a}} := \{\xi_i \,|\, i = 1, \ldots, n'\}.$$

Then

$$g(\bar{b}\xi_i^{m'}) = f_{\mathrm{in}}(a, \bar{b}\xi_i^{m'}) = f_{\mathrm{in}}(\bar{a}\xi_i^n, \bar{b}\xi_i^{m'}) = \xi_i^d f_{\mathrm{in}}(\bar{a}, \bar{b}) = 0.$$

Since $(m', n') = 1$, it is easy to see that $\xi_i^{m'} \neq \xi_j^{m'}$ for all $i \neq j$. Thus the set

$$\{\bar{b}\xi_i^{m'} \,|\, i = 1, \ldots, n'\}$$

contains all of roots of g. Since $0 = f_{\mathrm{in}}(a, b) = \rho \cdot g(b)^q$, $g(b) = 0$. Then there is an index i_0 such that $b = \bar{b}\xi_{i_0}^{m'}$. Choose a ϵ in $\sqrt[q]{\xi_{i_0}}$ and put

$$x(t) = \bar{x}(\epsilon t) \quad \text{and} \quad y(t) = \bar{y}(\epsilon t),$$

we get $\mathrm{LC}(x(t)) = a$ and $\mathrm{LC}(y(t)) = b$. $\qquad\square$

Definition 2.9 Let $f = \sum c_{ij} x^i y^j \in K[[x, y]]$ be such that $(0, n)$ is the vertex on the y-axis of $\Gamma(f)$. Let $(1, j_1)$ be the intersection point of $\Gamma(f)$ and the line $x = 1$. We define f to be *ND1 along* $(0, n)$ if either $\mathrm{char}(K) = p = 0$ or if $p \neq 0$ then $p \nmid n$ or $j_1 \in \mathbb{N}$ and the coefficient c_{1j_1} of xy^{j_1} in f is different from zero. ND1 along $(m, 0)$, with $(m, 0)$ the vertex on the x-axis of $\Gamma(f)$, is defined analogously.

f is called NND1 if f is convenient, ND along each inner face and ND1 along each vertex on the axes of $\Gamma(f)$.

Proposition 2.10 *Let $f = \sum c_{ij} x^i y^j \in K[[x, y]]$ be convenient and let $(0, n)$ (resp. $(m, 0)$) be the vertex on the y-axis (resp. on the x-axis) of $\Gamma(f)$. Assume that f is not ND1 along the point $(0, n)$ or $(m, 0)$ then $\mu(f) > \mu_N(f)$.*

Proof We consider only $(0, n)$ since $(m, 0)$ is analogous. Let $(1, j_1)$ be the intersection point of $\Gamma(f)$ and the line $x = 1$. The assumption that f is not ND1 along the point $(0, n)$ implies that $p|n$ and $c_{1j_1} = 0$. Putting $g(x, y) = f(x, y) - c_{0n}y^n$ one then has $\mu(f) = \mu(g)$ and $\Gamma_-(f) \subset \Gamma_-(g)$. On the other hand, it is easy to see that $(1, j_1) \in \Gamma_+(f) \setminus \Gamma_+(g)$. This means $\Gamma_-(f) \cap \mathbb{R}^2_{\geq 1} \subsetneq \Gamma_-(g) \cap \mathbb{R}^2_{\geq 1}$. It hence follows from Lemma 2.3 that $\mu_N(g) > \mu_N(f)$. Thus

$$\mu(f) = \mu(g) \geq \mu_N(g) > \mu_N(f). \qquad\square$$

Proposition 2.11 *Let $f \in K[[x, y]]$ be convenient. If f is degenerate along some inner vertex of $\Gamma(f)$ then $\mu(f) > \mu_N(f)$.*

Proof Assume that f is degenerate along some vertex (i_0, j_0) of $\Gamma(f)$ with $i_0 > 0$ and $j_0 > 0$. Then $p \neq 0$ and i_0 and j_0 are divisible by p. Put $g(x, y) = f(x, y) -$

$c_{i_0 j_0} x^{i_0} y^{j_0}$, then $j(f) = j(g)$ and hence $\mu(f) = \mu(g)$. Clearly, $\Gamma_+(g)$ does not contain the point (i_0, j_0). Thus

$$\Gamma_-(f) \cap \mathbb{R}^2_{\geq 1} \subsetneq \Gamma_-(g) \cap \mathbb{R}^2_{\geq 1}.$$

Lemma 2.3 hence implies that $\mu_N(g) > \mu_N(f)$. We then have

$$\mu(f) = \mu(g) \geq \mu_N(g) > \mu_N(f). \qquad \square$$

Proposition 2.12 *Let $f \in K[[x, y]]$ be convenient. If f is degenerate along some edge of $\Gamma(f)$ then $\mu(f) > \mu_N(f)$.*

Proof Let $f(x, y) = \sum c_{\alpha\beta} x^\alpha y^\beta$. Let f_x, f_y be the partials of f and put $h(x, y) := x f_x(x, y) + \lambda y f_y(x, y)$, where $\lambda \in K$ is generic. Then

$$h(x, y) = \sum (\alpha + \lambda\beta) c_{\alpha\beta} x^\alpha y^\beta.$$

Thus $\operatorname{supp}(h) = \operatorname{supp}(f) \setminus (p\mathbb{N})^2$ and if $p = 0$ then $\operatorname{supp}(h) = \operatorname{supp}(f)$. Hence $\Gamma_+(h) \subset \Gamma_+(f)$.

Case 1: f is ND along each vertex of $\Gamma(f)$.

Assume now that (i, j) is a vertex of $\Gamma(f)$. Since f is ND along (i, j), $p = 0$ or $p \neq 0$ and one of i, j is not divisible by p. Therefore $(i, j) \in \operatorname{supp}(f) \setminus (p\mathbb{N})^2 = \operatorname{supp}(h)$ and then $\Gamma_+(f) \subset \Gamma_+(h)$. Hence $\Gamma(h) = \Gamma(f)$.

Let $E_i, i = 1, \ldots, k$ be edges of $\Gamma(h)$. By Proposition 2.6, we can write $h = \bar{h}_1 \ldots \bar{h}_k$, where $\bar{h}_i$ are convenient and $h_{E_i}(x, y) = monomial \times (\bar{h}_i)_{\text{in}}$. We denote by m_i and n_i the lengths of the projections of E_i on the horizontal and vertical axes.

Let $h = h^{\text{w}}_{d_i} + h^{\text{w}}_{d_i+1} + \cdots$ with $h^{\text{w}}_{d_i} \neq 0$ be the (n_i, m_i)-weighted homogeneous decomposition of h. Then $h^{\text{w}}_{d_i} = h_{E_i}$. Since E_i is also an edge of $\Gamma(f)$, $f = f^{\text{w}}_{d_i} + f^{\text{w}}_{d_i+1} + \cdots$ is the (n_i, m_i)-weighted homogeneous decomposition of f with $f^{\text{w}}_{d_i} = f_{E_i}$ and then

$$h^{\text{w}}_{d_i} = \sum_{n_i \alpha + m_i \beta = d_i} (\alpha + \lambda\beta) c_{\alpha\beta} x^\alpha y^\beta = x \frac{\partial f^{\text{w}}_{d_i}}{\partial x} + \lambda y \frac{\partial f^{\text{w}}_{d_i}}{\partial y}.$$

Let $y f_y = g^{\text{w}}_{d_i'} + g^{\text{w}}_{d_i'+1} + \cdots$ be the (n_i, m_i)-weighted homogeneous decomposition of $y f_y$. It is easy to see that $d_i' \geq d_i$ and $d_i' = d_i$ iff $y \frac{\partial f^{\text{w}}_{d_i}}{\partial y} \neq 0$.

Claim 1 *Let A_{i-1}, A_i be the vertices of the edge E_i and let $V_2(O A_{i-1} A_i)$ be the volume of triangle $O A_{i-1} A_i$. Then $d_i = 2 V_2(O A_{i-1} A_i)$.*
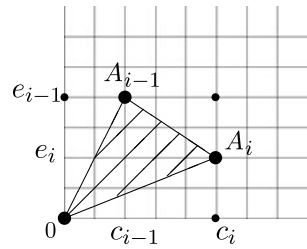
Proof Let (c_i, e_i) be the coordinates of $A_i, i = 0, \ldots, k$. Then $m_i = c_i - c_{i-1}$ and $n_i = e_{i-1} - e_i$. (cf. Fig. 2).

Considering the rectangle $(0, 0); (c_i, 0); (c_i, e_{i-1}); (0, e_{i-1})$ we have

$$2 V_2(O A_{i-1} A_i) = 2 c_i e_{i-1} - c_i e_i - c_{i-1} e_{i-1} - m_i n_i$$

$$= (c_{i-1} + m_i) e_{i-1} + c_i (e_i + n_i) - c_i e_i - c_{i-1} e_{i-1} - m_i n_i$$

Fig. 2

$$= m_i e_{i-1} + c_i n_i - m_i n_i = m_i(e_i + n_i) + c_i n_i - m_i n_i$$

$$= m_i e_i + n_i c_i = d_i.$$

This proves Claim 1.

Claim 2 $i(\bar{h}_i, yf_y) \geq d_i$, and if f is degenerate along E_i then $i(\bar{h}_i, yf_y) > d_i$.

Proof Let $(x_j(t), y_j(t))$, $j = 1, \ldots, r$ be parametrizations of the branches $\bar{h}_{i,j}$ of $\bar{h}_i$. Then by Lemma 2.8, we have $LT(x_j(t)) = a_j t^{\alpha_j}$ and $LT(y_j(t)) = b_j t^{\beta_j}$, where $a_j, b_j \in K^*, \bar{h}_i(a_j, b_j) = 0, \alpha_j : \beta_j = n_i : m_i$ for all $j = 1, \ldots, r$ and $\alpha_1 + \cdots + \alpha_r = n_i$. It follows from Lemma 2.7 that $\mathrm{ord}(yf_y)(x_j(t), y_j(t)) \geq \frac{d_i' \alpha_j}{n_i}$ for all $j = 1, \ldots, r$. Thus

$$i(\bar{h}_i, yf_y) = \sum_{j=1}^{r} \mathrm{ord}(yf_y)(x_j(t), y_j(t)) \geq \sum_{j=1}^{r} \frac{d_i' \alpha_j}{n_i} = d_i' \geq d_i.$$

Assume that f is degenerate along E_i then there exist $a, b \neq 0$ such that

$$x \frac{\partial f_{d_i}^{\mathrm{w}}}{\partial x}(a, b) = y \frac{\partial f_{d_i}^{\mathrm{w}}}{\partial y}(a, b) = 0.$$

Therefore $h_{d_i}(a, b) = 0$. Lemma 2.8 implies that there is a parametrization of a branch of $\bar{h}_i$ such that $LT(\bar{x}(t)) = at^\alpha$ and $LT(\bar{y}(t)) = bt^\beta$. We may assume that $(\bar{x}(t), \bar{y}(t))$ is a parametrization of the branch $\bar{h}_{i,1}$. Then $\alpha = \alpha_1$ and $\beta = \beta_1$.

To show $i(\bar{h}_i, yf_y) > d_i$, we may restrict to the case that $d_i' = d_i$, because of the inequality $i(\bar{h}_i, yf_y) \geq d_i' \geq d_i$. As $d_i' = d_i$ then $g_{d_i}^{\mathrm{w}}(a, b) = y \frac{\partial f_{d_i}^{\mathrm{w}}}{\partial y}(a, b) = 0$. Lemma 2.7 yields

$$\mathrm{ord}(yf_y)(\bar{x}(t), \bar{y}(t)) > \frac{d_i \alpha_1}{n_i}.$$

Thus

$$i(\bar{h}_i, yf_y) = \mathrm{ord}(yf_y)(\bar{x}(t), \bar{y}(t)) + \sum_{j=2}^{r} \mathrm{ord}(yf_y)(x_j(t), y_j(t))$$

$$> \frac{d_i \alpha_1}{n_i} + \sum_{j=2}^{r} \frac{d_i' \alpha_j}{n_i} = d_i.$$

This proves Claim 2.

It now follows from Claim 1 and Claim 2 that

$$i(h, yf_y) \geq \sum_{i=1}^{k} 2V_2(O A_{i-1} A_i) = 2V_2(\Gamma_-(f)).$$

Hence

$$\mu(f) = i(f_x, f_y) = i(h, yf_y) - i(x, f_y) - i(f_x, y) - 1$$
$$\geq 2V_2(\Gamma_-(f)) - (e_0 - 1) - (c_k - 1) - 1$$
$$= \mu_N(f).$$

Moreover, if f is degenerate along some edge of $\Gamma(f)$ then $\mu(f) > \mu_N(f)$ by Claim 1 and 2. This proves of Case 1.

Case 2: In the general case, by Propositions 2.10, 2.11 we may assume that f is ND along each inner vertex and ND1 along the two vertice on the axes of $\Gamma(f)$. For m sufficiently large and $p \nmid m$, we put

$$\bar{f}_m(x, y) = \sum_{(\alpha, \beta) \notin (p\mathbb{N})^2} c_{\alpha\beta} x^\alpha y^\beta + x^m + y^m.$$

Then

$$\mu(\bar{f}_m) = \mu(f_m) = \mu(f) \quad \text{and} \quad \mu_N(\bar{f}_m) \geq \mu_N(f_m) = \mu_N(f),$$

where the inequality follows from Lemma 2.3.

Claim 3 $\bar{f}_m$ *is degenerate along some edge of* $\Gamma(\bar{f})$.

Proof By the assumption f is degenerate along some edge E of $\Gamma(f)$. If E is also an edge of $\Gamma(\bar{f}_m)$ then $j(\text{in}_E(\bar{f}_m)) = j(\text{in}_E(f))$ and hence $\bar{f}_m$ is degenerate along E. If E is not an edge of $\Gamma(\bar{f}_m)$, then E must meet the axes since f is ND along each inner vertex of $\Gamma(f)$. We may assume that $(0, n)$ is a vertex of E. We will show that

$$\sharp(\text{supp}(\bar{f}_m) \cap E) \geq 2.$$

Let $(1, j_1)$ be the intersection point of E and the line $x = 1$. Since f is ND1 along $(0, n)$, either $(0, n) \in \text{supp}(\bar{f}_m) \cap E$ or $(1, j_1) \in \text{supp}(\bar{f}_m) \cap E$, i.e. $\text{supp}(\bar{f}_m) \cap E \neq \emptyset$ On the other hand, it is easy to see that $\sharp(\text{supp}(\bar{f}_m) \cap E) \neq 1$ since f is degenerate along the edge E. Hence $\sharp(\text{supp}(\bar{f}_m) \cap E) \geq 2$. Let us denote by $\bar{E}$ the convex hull of the set $\text{supp}(\bar{f}_m) \cap E$. Then $\bar{E}$ is an edge of $\Gamma(\bar{f}_m)$ and $j(\text{in}_{\bar{E}}(\bar{f}_m)) = j(\text{in}_E(f))$. Thus $\bar{f}_m$ is degenerate along $\bar{E}$ since f is degenerate along the edge E, which proves Claim 3.

Now, by definition, $\bar{f}_m$ is ND along each vertex of $\Gamma(\bar{f}_m)$. Since $\bar{f}_m$ is degenerate along some edge of $\Gamma(\bar{f}_m)$, applying the first case to $\bar{f}_m$, we get $\mu(\bar{f}_m) > \mu_N(\bar{f}_m)$.

Hence

$$\mu(f) = \mu(\bar{f}_m) > \mu_N(\bar{f}_m) \geq \mu_N(f).$$

This proves Proposition 2.12. □

Theorem 2.13 *Let $f \in \mathfrak{m} \subset K[[x, y]]$ and let $f_m = f + x^m + y^m$. Then the following are equivalent*

(i) $\mu(f) = \mu_N(f) < \infty$.
(ii) $\mu(f) < \infty$ and f_m is NND1 for some large integer number m.
(iii) f is INND.

Proof (i) $\Rightarrow$ (ii): Since $\mu(f) = \mu_N(f) < \infty$ we have by definition of $\mu_N(f)$

$$\mu(f_m) = \mu(f) = \mu_N(f) = \mu_N(f_m) < \infty.$$

Combining Propositions 2.10, 2.11 and 2.12 we get the claim.

(ii) $\Rightarrow$ (iii): Assume that $\mu(f) < \infty$ and f_m is NND1. Firstly, it is easy to see that there is an $M \in \mathbb{N}$ such that $\Gamma(f) \subset \Gamma(f_M)$. It suffices to show f is INND w.r.t. $\Gamma(f_m)$ for all $m > M$. We argue by contradiction. Suppose that it is not true. Then f is not IND along some edge Δ of $\Gamma(f_m)$ which meets the axes, since f_m is NND1. We may assume that Δ meets the axes at $(0, n)$. Let (k, l) be the second vertex of Δ. We consider two cases:

• If $l = 0$, i.e. $\Gamma(f_m)$ has only one edge Δ. Then Δ is also a unique edge of $\Gamma(f)$ and $\mathrm{in}_\Delta(f) = \mathrm{in}_\Delta(f_m)$. Since f is not IND along Δ, there exists $(a, b) \in K \setminus \{(0, 0)\}$ which is a zero point of $j(\mathrm{in}_\Delta(f))$. Beside, since f_m is ND along Δ, either $a = 0$ or $b = 0$. Assume that $a = 0$ and $b \neq 0$. We will show that f_m is not ND1 along $(0, n)$. Firstly, we write $\mathrm{in}_\Delta(f_m) = c_{0n} y^n + x \cdot g(x, y)$, then $\frac{\partial \mathrm{in}_\Delta(f_m)}{\partial y} = n y^{n-1} + x \cdot \frac{\partial g}{\partial y}$. Thus

$$\frac{\partial \mathrm{in}_\Delta(f_m)}{\partial y}(0, b) = n y^{n-1} = 0 \Rightarrow p \neq 0 \text{ and } p | n.$$

We now write $\mathrm{in}_\Delta(f_m) = c_{0n} y^n + c_{1j} x y^j + x^2 \cdot h(x, y)$, then

$$\frac{\partial \mathrm{in}_\Delta(f_m)}{\partial x} = c_{1j} y^j + 2x \cdot h(x, y) + x^2 \cdot \frac{\partial h}{\partial x}.$$

Since $\frac{\partial \mathrm{in}_\Delta(f_m)}{\partial x}(0, b) = 0$, $c_{1j} = 0$. Hence f_m is not ND1 along $(0, n)$, a contradiction.

• Assume that $l > 0$. If Δ is also an edge of $\Gamma(f)$ then $\mathrm{in}_\Delta(f) = \mathrm{in}_\Delta(f_m)$. Since f is not IND along Δ, there exists $(a, b) \in K \times K^*$ being a zero of $j(\mathrm{in}_\Delta(f))$. Since f_m is ND along Δ, $a = 0$. Analogously as above f_m is not ND1 along $(0, n)$ and we get a contradiction. Assume now that Δ is not an edge of $\Gamma(f)$, i.e. $m = n$ and $x | f(x, y)$. Let P be the end point of $\Gamma(f)$ closest to y-axis. It follows from $\Gamma(f) \subset \Gamma(f_M)$ and $m > M$ that P must be a vertex of Δ, i.e. $P = (k, l)$. This implies $f = x^k \cdot h(x, y)$. Since $\mu(f) < \infty$, $k = 1$. Then $\mathrm{in}_\Delta(f) = c_{0n} y^n + c_{1l} x y^l$ and clearly f is always IND along Δ, a contradiction. Hence f is INND w.r.t. $\Gamma(f_m)$ and then it is INND.

(iii) $\Rightarrow$ (i): See Theorem 1.2. □

Corollary 2.14 *Let $f \in K[[x, y]]$ and let $M \in \mathbb{N}$ such that $\Gamma(f) \subset \Gamma(f_M)$. Then f is INND if and only if it is INND w.r.t. $\Gamma(f_m)$ for some (equivalently for all) $m > M$.*

Proof One direction is obvious, it remains to show f is INND $\Rightarrow$ f is INND w.r.t. $\Gamma(f_m)$ for all $m > M$. We take $m_1 > M$ satisfying Theorem 2.13 and then

$$f \text{ is INND} \Rightarrow \mu(f) < \infty \quad \text{and} \quad f_{m_1} \text{ is NND1} \Rightarrow f \text{ is INND w.r.t. } \Gamma(f_{m_1}).$$

For each inner face Δ_m of $\Gamma(f_m)$, since $m, m_1 > M$, there is an inner face Δ_{m_1} of $\Gamma(f_{m_1})$ such that $\text{in}_{\Delta_m}(f) = \text{in}_{\Delta_{m_1}}(f)$. Thus f is IND along Δ_m since it is IND along Δ_{m_1}. Hence f is INND w.r.t. $\Gamma(f_m)$. $\qquad\square$

Corollary 2.15 *Let $M \in \mathbb{N}$ be such that $\Gamma(f) \subset \Gamma(f_M)$. Then Theorem 2.13 holds for each $m > M$.*

Remark 2.16 Let $\mu(f) < \infty$. Then M can be chosen as the maximum of n_1 and m_1, where $n_1 = n$ if $\Gamma(f) \cap \{x = 0\} = \{(0, n)\}$ and $n_1 = 2i_1$ if $\Gamma(f) \cap \{x = 0\} = \emptyset$ and $\Gamma(f) \cap \{x = 1\} = \{(1, i_1)\}$. Similarly we define m_1 with x replaced by y. This remark and the previous corollaries are important for concrete computation.

Proof of Corollary 2.15 Clearly, the equivalence (i) $\Leftrightarrow$ (iii) does not depend on m and as in the proof of Theorem 2.13 the implication (ii) $\Rightarrow$ (iii) holds for all $m > M$. It remains to show that f is INND $\Rightarrow$ f_m is NND1. By Corollary 2.14, it suffices to show that f is INND w.r.t. $\Gamma(f_m) \Rightarrow f_m$ is NND1. By contradiction, suppose that f is INND w.r.t. $\Gamma(f_m)$ and f_m is not NND1. Then f is not ND1 along some vertex of $\Gamma(f_m)$ in the axes. Assume that f is not ND1 along $(0, n) \in \Gamma(f_m)$. Then

$$p \neq 0, \; p|n \quad \text{and} \quad \Gamma(f_m) \cap \{x = 1\} \cap \text{supp}(f_m) = \emptyset,$$

i.e. $(f_m)_{in} = c_{0n} y^n + x^2 \cdot h(x, y)$. This implies $\mu((f_m)_{in}) = \infty$. By Theorem 1.2, $(f_m)_{in}$ is not INND and then f_m is also not INND, a contradiction. $\qquad\square$

Corollary 2.17 *Let K is a field of characteristic zero and $f \in \mathfrak{m} \subset K[[x, y]]$. Then the following are equivalent*

(i) $\mu(f) = \mu_N(f) < \infty$.
(ii) f *is INND.*
(iii) f *is NND and $\mu_N(f) < \infty$.*

In particular, if f is convenient then (i)–(iii) *are equivalent to*

(iv) f *is NND.*

Proof The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (i) follow from Theorem 2.13 and Proposition 1.3. It remains to prove (ii) $\Rightarrow$ (iii).

Assume that f is INND. Then by Theorem 2.13, $\mu_N(f) < \infty$. We will show that f is ND along each vertex and each edge of $\Gamma(f)$. Since $\text{char}(K) = 0$, f is ND along each vertex of $\Gamma(f)$. Let Δ be an edge of $\Gamma(f)$. Clearly, it is an inner edge of $\Gamma(f_m)$, where m sufficiently large. Since f is INND, by Corollary 2.14 f is INND w.r.t.

$\Gamma(f_m)$. Then f is IND along Δ, and hence it is also ND along Δ. This implies f is NND. $\qquad\square$

Corollary 2.18 *If f is NND and $\mu_N(f) < \infty$ then f is INND.*

Proof This follows from Proposition 1.3 and Theorem 2.13. $\qquad\square$

Note that $\mathrm{char}(K) = 0$ is only used to assure that f is ND along each vertex of $\Gamma(f) \cap (\{0\} \times \mathbb{N} \cup \mathbb{N} \times \{0\})$. Hence, the last corollary holds also if $p > 0$ and $p \nmid n$ if $(0, n) = \Gamma(f) \cap \{0\} \times \mathbb{N}$ and $p \nmid m$ if $(m, 0) = \Gamma(f) \cap \mathbb{N} \times \{0\}$. Example 2.1 shows that this condition is necessary.

3 δ-Invariant

We consider now another important invariant of plane curve singularities, the invariant δ and its combinatorial counterpart, the Newton invariant δ_N. We show that both coincide iff f is weighted homogeneous Newton non-degenerate (WHNND), a new non-degenerate condition introduced below.

Let $f \in \mathfrak{m} \subset K[[x, y]]$ be a power series. We define *the multiplicity* of f, denoted by $\mathrm{mt}(f)$, to be the minimal degree of the homogeneous part of f. So

$$f = \sum_{k \geq m := \mathrm{mt}(f)} f_k(x, y),$$

where f_k is homogeneous of degree k and $f_m \neq 0$. Then f_m decomposes into linear factors,

$$f_m = \prod_{i=1}^{s} (\alpha_i x - \beta_i y)^{r_i},$$

with $(\beta_i : \alpha_i) \in \mathbb{P}^1$ pairwise distinct. We call f_m *the tangent cone* and the points $(\beta_i : \alpha_i), i = 1, \ldots, s$, *the tangent directions* of f.

We fix a minimal resolution of the singularity computed via successively blowing up points, denote by $Q \to 0$ that Q is an infinitely near point of the origin on f. If Q is an infinitely near point in the n-th neighbourhood of 0, we denote by m_Q the multiplicity of the n-th strict transform of f at Q. If P is an infinitely near point in the l-th neighbourhood of 0, we denote by $Q \to P$ that Q is also an infinitely near point of P on the l-th strict transform $\tilde{f}_l$ of f at P. Note that if $Q \to P$ then $n \geq l$ and we set $n(\tilde{f}_l, Q) := n - l$. In particular, we have $n(f, Q) = n$.

Let $E_1, \ldots, E_k$ be the edges of the Newton diagram of f. We denote by $l(E_i)$ the lattice length of E_i, i.e. the number of lattice points on E_i minus one and by $s(f_{E_i})$ the number of non-monomial irreducible (reduced) factors of f_{E_i}. We set

(a) $\delta(f) := \sum_{Q \to 0} \frac{m_Q(m_Q - 1)}{2}$ the *delta invariant* of f. The delta invariant $\delta(f)$ equals also $\dim_K(\bar{R}/R)$ where $R = K[[x, y]]/\langle f \rangle$ and $\bar{R}$ is the integral closure of R in its total ring of fractions.

(b) $\nu(f) := \sum_{Q \text{ special}} \frac{m_Q(m_Q - 1)}{2}$, where an infinitely near point Q is *special* if it is the origin or the origin of the corresponding chart of the blowing up.

(c) $r(f)$ the *number of branches* of f counted with multiplicity.

(d) If f is convenient, we define

$$\delta_N(f) := V_2(\Gamma_-(f)) - \frac{V_1(\Gamma_-(f))}{2} + \frac{\sum_{i=1}^k l(E_i)}{2},$$

and otherwise we set $\delta_N(f) := \sup\{\delta_N(f^{(m)}) | f^{(m)} := f + x^m + y^m, m \in \mathbb{N}\}$ and call it the *Newton δ-invariant* of f.

(e) $r_N(f) := \sum_{i=1}^k l(E_i) + \max\{j | x^j \text{ divides } f\} + \max\{l | y^l \text{ divides } f\}$.

(f) $s_N(f) := \sum_{i=1}^k s(f_{E_i}) + \max\{j | x^j \text{ divides } f\} + \max\{l | y^l \text{ divides } f\}$.

Note that $\delta(f)$ and $r(f)$ are coordinate-independent while all the other ones depend (only) on the Newton diagram of f and hence are coordinate-dependent (for $\nu(f)$ see Proposition 3.9).

Proposition 3.1 *For $0 \neq f \in \langle x, y \rangle$ we have $r(f) \leq r_N(f)$, and if f is WNND then $r(f) = r_N(f)$.*

Proof cf. [3, Lemma 4.10]   □

Let E be an edge of the Newton diagram of f. Then we can write f_E as follows,

$$f_E = monomial \times \prod_{i=1}^s (a_i x^{m_0} - b_i y^{n_0})^{r_i},$$

where $a_i, b_i \in K^*$, $(a_i : b_i)$ pairwise distinct; $m_0, n_0, r_i \in \mathbb{N}_{>0}$, $\gcd(m_0, n_0) = 1$. It easy to see that

$$s = s(f_E) \quad \text{and} \quad l(E) = \sum_{i=1}^s r_i.$$

This implies $s(f_E) \leq l(E)$ and hence $s_N(f) \leq r_N(f)$.

Let $f = f_d^w + f_{d+1}^w + \cdots$ with $f_d^w \neq 0$ be the (n_0, m_0)-weighted homogeneous decomposition of f.

Definition 3.2 We say that f is *weighted homogeneous non-degenerate* (WHND) *along E* if either $r_i = 1$ for all $i = 1, \ldots, s$ or $(a_i x^{m_0} - b_i y^{n_0})$ does not divide f_{d+1}^w for each $r_i > 1$.

f is called *weighted homogeneous Newton non-degenerate* (WHNND) if its Newton diagram has no edge or if it is WHND along each edge of its Newton diagram.

Lemma 3.3 *Let $f \in K[[x, y]]$. Then f is not WHNND if and only if there exist $a, b \in K^*, m, n \in \mathbb{N}_{>0}$ with $(m, n) = 1$ such that f_d^w is divisible by $(ax^m - by^n)^2$ and f_{d+1}^w is divisible by $(ax^m - by^n)$, where f_d^w (resp. f_{d+1}^w) is the first (resp. the second) term of the (n, m)-weighted decomposition of f.*

Proof Straightforward from the above definition. □

Remark 3.4 (a) In [11] the author introduced superisolated singularities to study the μ-constant stratum. We recall that $f \in K[[x, y]]$ is *superisolated* if it becomes regular after only one blowing up. By ([11, Lemma 1]), this is equivalent to: $f_{m+1}(\beta_i, \alpha_i) \neq 0$ for all tangent directions $(\beta_i : \alpha_i)$ of f with $r_i > 1$, where $f = f_m + f_{m+1} + \cdots$ is the homogeneous decomposition of f and

$$f_m = \prod_{i=1}^{s} (\alpha_i x - \beta_i y)^{r_i}.$$

Note that this condition concerns all factors of f_m including monomials. For WHNND singularities we require a similar condition, but for "all weights" and without any condition on the monomial factors of the first term of the weighted homogeneous decomposition of f.

(b) Since a plane curve singularity is superisolated iff it becomes regular after only one blowing up, we have $\delta(f) = \nu(f) = m(m-1)/2$ and hence $\delta(f) = \delta_N(f) = m(m-1)/2$, by Proposition 3.9. It follows from Theorem 3.12 that

(c) A superisolated plane curve singularity is WHNND.

(d) The plane curve singularity $x^2 + y^5$ is WHNND but not superisolated.

Proposition 3.5 *With notations as above, f is WND along E if and only if $s(f_E) = l(E)$ or, equivalently, iff $r_i = 1$ for all $i = 1, \ldots, s$. In particular, WNND implies WHNND.*

Proof Firstly we can see that the equation $s(f_E) = l(E)$ is equivalent to $r_i = 1$ for all $i = 1, \ldots, s$ since $s(f_E) = s$ and $l(E) = r_1 + \cdots + r_s$. It remains to prove that f is WND along E iff $r_i = 1$ for all $i = 1, \ldots, s$. Assume that there is an i_0 s.t. $r_{i_0} > 1$. It is easy to see that $f_E, \frac{\partial f_E}{\partial x}, \frac{\partial f_E}{\partial y}$ are divisible by $(a_{i_0} x_0^{m_0} - b_{i_0} y_0^{n_0})$. Hence f is weakly degenerate (WD) along E.

We now assume that f is weakly degenerate (WD) along E. Then there exist $x_0, y_0 \in K^*$ such that

$$f_E(x_0, y_0) = \frac{\partial f_E}{\partial x}(x_0, y_0) = \frac{f_E}{\partial y}(x_0, y_0) = 0,$$

and hence there exists an index i_0 such that $a_{i_0} x_0^{m_0} - b_{i_0} y_0^{n_0} = 0$. We will show that $r_{i_0} > 1$. In fact, if this is not true then $f_E(x, y) = (a_{i_0} x^{m_0} - b_{i_0} y^{n_0}) \cdot h(x, y)$ with $h(x_0, y_0) \neq 0$. Since

$$\frac{\partial f_E}{\partial x}(x_0, y_0) = \frac{f_E}{\partial y}(x_0, y_0) = 0,$$

this is impossible if $p = 0$ and implies that p divides m_0 and n_0 if $p > 0$. This contradicts the assumption $\gcd(m_0, n_0) = 1$. □

Let $f \in K[[x, y]]$ and let $E_i, i = 1, \ldots, k$ be the edges of its Newton diagram. Then by Proposition 2.6 there is a factorization of f,

$$f = monomial \cdot \bar{f}_1 \cdots \bar{f}_k,$$

such that $\bar{f}_i$ is convenient and $f_{E_i} = monomial \times (\bar{f}_i)_{in}$. Note that $\bar{f}_i$ is in general not irreducible. On the other hand, f can be factorized into its irreducible factors as $f = m_1 \cdots m_l \cdot f_1 \cdots f_r$, where m_j are monomials, and f_j are convenient.

Proposition 3.6

(a) *Let $g, h \in K[[x, y]]$ such that $f = g \cdot h$. If f is WHNND then g and h are also WHNND.*

(b) *With the above notations, the following are equivalent*:

    (i) *f is WHNND.*

    (ii) *$\bar{f}_1, \ldots, \bar{f}_k$ are WHNND.*

    (iii) *$f_1, \ldots, f_r$ are WHNND and $(f_i)_{in}$ are pairwise coprime.*

Proof (a) It suffice to show that if g is not WHNND then neither is f. In fact, since g is not WHNND, by Lemma 3.3, there exist $a, b \in K^*, m, n \in \mathbb{N}_{>0}$ with $(m, n) = 1$ such that g_c^w is divisible by $(ax^m - by^n)^2$ and g_{c+1}^w is divisible by $(ax^m - by^n)$, where g_c^w (resp. g_{c+1}^w) is the first (resp. the second) term of the (n, m)-weighted decomposition of g. Let $f = f_d^w + f_{d+1}^w + \cdots$ (resp. $h = h_e^w + h_{e+1}^w + \cdots$) be the (n, m)-weighted homogeneous decomposition of f (resp. h). Then

$$f_d^w = g_c^w \cdot h_e^w \quad \text{and} \quad f_{d+1}^w = g_c^w \cdot h_{e+1}^w + g_{c+1}^w \cdot h_e^w.$$

This implies that f_d^w is divisible by $(ax^m - by^n)^2$ and f_{d+1}^w is divisible by $(ax^m - by^n)$. Again by Lemma 3.3, f is not WHNND.

    (b) It is easily verified that we may restrict to the case that f is convenient. The implication (i) $\Rightarrow$ (ii) follows from part (a).

    (ii) $\Rightarrow$ (iii): Assume that $\bar{f}_1, \ldots, \bar{f}_k$ are WHNND. By part (a) we can deduce that $f_1, \ldots, f_r$ are WHNND since for each i, f_i is an irreducible factor of some $\bar{f}_j$. We now show that the $(f_i)_{in}$ are pairwise coprime. By contradiction, suppose that $(f_1)_{in}$ and $(f_2)_{in}$ are not coprime. It follows from Proposition 2.5 that there exist $a, b \in K^*, m, n \in \mathbb{N}_{>0}$ with $(m, n) = 1$ such that $(ax^m - by^n)$ is the unique irreducible factor of $(f_1)_{in}$ and $(f_2)_{in}$. Consequently, $(f_1)_{in}$ and $(f_2)_{in}$ are both (n, m)-weighted homogeneous. Assume that f_1 resp. f_2 is an irreducible factor of $\bar{f}_{j_1}$ resp. $\bar{f}_{j_2}$ for some j_1 and j_2. Since $(\bar{f}_{j_1})_{in}$ and $(\bar{f}_{j_2})_{in}$ are weighted homogeneous, $(f_1)_{in}$ resp. $(f_2)_{in}$ is a factor of $(\bar{f}_{j_1})_{in}$ resp. $(\bar{f}_{j_2})_{in}$. This implies that $(\bar{f}_{j_1})_{in}$ and $(\bar{f}_{j_2})_{in}$ and therefore $f_{E_{j_1}}$ and $f_{E_{j_2}}$ are all (n, m)-weighted homogeneous. Then the edge E_{j_1} must coincide the edge E_{j_2} and hence $\bar{f}_{j_1} = \bar{f}_{j_2}$. It yields that the product $g := f_1 \cdot f_2$ is a factor of $\bar{f}_{j_1}$. Now, we decompose g, f_1, f_2 into their (n, m)-weighted homogeneous terms as follows:

$$g = g_c^w + g_{c+1}^w + \cdots, \qquad f_1 = (f_1)_{d_1}^w + (f_1)_{d_1+1}^w + \cdots,$$

$$f_2 = (f_2)_{d_2}^w + (f_2)_{d_2+1}^w + \cdots.$$

Then $c = d_1 + d_2$, $(f_1)^{\mathrm{w}}_{d_1} = (f_1)_{in}$, $(f_2)^{\mathrm{w}}_{d_2} = (f_2)_{in}$, $g^{\mathrm{w}}_c = (f_1)^{\mathrm{w}}_{d_1} \cdot (f_2)^{\mathrm{w}}_{d_2}$ and $g^{\mathrm{w}}_{c+1} = (f_1)^{\mathrm{w}}_{d_1} \cdot (f_2)^{\mathrm{w}}_{d_2+1} + (f_1)^{\mathrm{w}}_{d_1+1} \cdot (f_2)^{\mathrm{w}}_{d_2}$. This implies that g^{w}_c is divisible by $(ax^m - by^n)^2$ and g^{w}_{c+1} is divisible by $(ax^m - by^n)$. It follows from Lemma 3.3 that g is not WHNND and hence $\bar{f}_{j_1}$ is also not WHNND by part (a) with g a factor of $\bar{f}_{j_1}$, which is a contradiction.

(iii) $\Rightarrow$ (i): Suppose that f is not WHNND and that the $(f_i)_{in}$ are pairwise coprime. We will show that f_i is not WHNND for some i. Indeed, since f is not WHNND, by Lemma 3.3, there exist $a, b \in K^*, m, n \in \mathbb{N}_{>0}$ with $(m, n) = 1$ such that f^{w}_d is divisible by $(ax^m - by^n)^2$ and f^{w}_{d+1} is divisible by $(ax^m - by^n)$, where f^{w}_d (resp. f^{w}_{d+1}) is the first (resp. the second) term of the (n, m)-weighted decomposition of f. Let $(f_i) = (f_i)^{\mathrm{w}}_{d_i} + (f_i)^{\mathrm{w}}_{d_i+1} + \cdots$ be the (n, m)-weighted homogeneous decomposition of $f_i, i = 1, \ldots, r$. Then we have

$$d = \sum_{i=1}^r d_i; \qquad (f_i)^{\mathrm{w}}_{d_i} = (f_i)_{in}; \qquad f^{\mathrm{w}}_d = \prod_{i=1}^r (f_i)^{\mathrm{w}}_{d_i};$$

$$f^{\mathrm{w}}_{d+1} = \sum_{i=1}^r \left((f_i)^{\mathrm{w}}_{d_i+1} \cdot \prod_{l \neq i} (f_l)^{\mathrm{w}}_{d_l} \right).$$

Since $f^{\mathrm{w}}_d = \prod_{i=1}^r (f_i)^{\mathrm{w}}_{d_i}$ and since the $(f_i)^{\mathrm{w}}_{d_i}$ are pairwise coprime, there exists an i_0 such that $(f_{i_0})^{\mathrm{w}}_{d_{i_0}}$ is divisible by $(ax^m - by^n)^2$ and $(f_l)^{\mathrm{w}}_{d_l}$ is not divisible by $(ax^m - by^n)$ for all $l \neq i_0$. This implies that $(f_{i_0})^{\mathrm{w}}_{d_{i_0}+1}$ is divisible by $(ax^m - by^n)$ since

$$f^{\mathrm{w}}_{d+1} = (f_{i_0})^{\mathrm{w}}_{d_{i_0}+1} \cdot \prod_{l \neq i_0} (f_l)^{\mathrm{w}}_{d_l} + \sum_{i \neq i_0} \left((f_i)^{\mathrm{w}}_{d_i+1} \cdot \prod_{l \neq i} (f_l)^{\mathrm{w}}_{d_l} \right).$$

Then f_{i_0} is not WHNND by Lemma 3.3. $\qquad \square$

Proposition 3.7 *For $0 \neq f \in \langle x, y \rangle$ we have $s_N(f) \leq r(f)$ and if f is WHNND then $s_N(f) = r(f)$.*

Proof If $f = x^j y^l \cdot g(x, y)$ with g convenient, then

$$s_N(f) = s_N(g) + j + l \quad \text{and} \quad r(f) = r(g) + j + l,$$

so we may assume that f is convenient.

Step 1. Assume first that the Newton diagram $\Gamma(f)$ has only one edge E. Then we can see that $f_{in} = \prod_{i=1}^r (f_i)_{in}$. It follows from Proposition 2.5 that for each i, $(f_i)_{in}$ has only one irreducible factor and therefore f_{in} has at most r irreducible factors. This means that $r \geq s_N(f)$.

If $r(f) > s_N(f)$, then there exist $i \neq j$ such that $(f_i)_{in}$ and $(f_j)_{in}$ have the same factor. This means that $(f_i)_{in}$ and $(f_j)_{in}$ are not coprime. Then by Proposition 3.6, f is not WHNND.

Step 2. Assume now that the Newton diagram $\Gamma(f)$ has k edges $E_1, \ldots, E_k$. By Proposition 2.6, f can be factorized as $f = \bar{f}_1 \cdot \cdots \cdot \bar{f}_k$, where $\bar{f}_j$ is convenient,

its Newton diagram has only one edge and $f_{E_j} = monomial \cdot (\bar{f}_j)_{in}$ for each $j = 1, \ldots, k$. This implies that $s_N(\bar{f}_j) = s(f_{E_j})$. Then we obtain

$$r(f) = \sum_{j=1}^{k} r(\bar{f}_j) \geq \sum_{j=1}^{k} s_N(\bar{f}_j) = \sum_{j=1}^{k} s(f_{E_j}) = s_N(f).$$

Now we assume that $r(f) > s_N(f)$. Then there exists a $j = 1, \ldots, k$ such that $r(\bar{f}_j) > s(f_{E_j}) = s_N(\bar{f}_j)$. It follows from Step 1 that $\bar{f}_j$ is not WHNND. Hence f is not WHNND by Proposition 3.6, which proves the proposition. $\square$

Proposition 3.8 *For* $0 \neq f \in \langle x, y \rangle$ *we have* $s_N(f) \leq r(f) \leq r_N(f)$, *and both equalities hold if and only if* f *is WNND.*

Proof The inequalities follow from Proposition 3.1 and Proposition 3.7. For each edge E of $\Gamma(f)$, by Proposition 3.5, f is WND along E iff $s(f_E) = l(E)$. This implies that f is WNND if and only if $s_N(f) = r_N(f)$ since $s(f_E) \leq l(E)$ and both sides are additive with respect to edges of $\Gamma(f)$. $\square$

We investigate now the relations between $\nu(f)$, $\delta_N(f)$ and $\delta(f)$, which were studied in [1] and [3].

Proposition 3.9 [3, Lemma 4.8] *If* $f \in K[[x, y]]$ *then* $\delta_N(f) = \nu(f)$.

Proposition 3.10 [3, Proposition 4.9] *For* $0 \neq f \in \langle x, y \rangle$ *we have* $\delta_N(f) \leq \delta(f)$, *and if* f *is WNND then* $\delta_N(f) = \delta(f)$.

Hence WNND is sufficient but, by the following example, not necessary for $\delta_N(f) = \delta(f)$.

Example 3.11 Let $f(x, y) = (x + y)^2 + y^3 \in K[[x, y]]$. Then f is not WNND but $\delta_N(f) = \delta(f) = 1$. This easy example shows also that WNND depends on the coordinates since $x^2 + y^3$ is WNND. Note that f is WHNND.

Now we prove that WHNND is necessary and sufficient for $\delta_N(f) = \delta(f)$.

Theorem 3.12 *Let* $f \in K[[x, y]]$ *be reduced. Then* $\delta(f) = \delta_N(f)$ *if and only if* f *is WHNND.*

We will prove the theorem after three technical lemmas.
Let E be an edge of the Newton diagram of f. We write

$$f_E = monomial \times \prod_{i=1}^{s} (a_i x^{m_0} - b_i y^{n_0})^{r_i},$$

where $a_i, b_i \in K^*$, $(a_i : b_i)$ pairwise distinct; $m_0, n_0, r_i \in \mathbb{N}_{>0}$, $(m_0, n_0) = 1$.

Lemma 3.13 *With the above notations, there exist an integer n and an infinitely near point P_n in the n-th neighbourhood of 0, such that*

$$(\tilde{f}_n)_{E_n}(u, v) = monomial \times \prod_{i=1}^{s}(a_i u - b_i v)^{r_i},$$

where $\tilde{f}_n$ is a local equation of the strict transform of $\tilde{f}$ at P_n and E_n is some edge of its Newton diagram $\Gamma(\tilde{f}_n)$. Moreover, f is WHND along E if and only if $\tilde{f}_n$ is WHND along E_n.

Proof We prove the lemma by induction on $m_0 + n_0$. If $m_0 + n_0 = 2$, i.e. $m_0 = n_0 = 1$, then the claim is trivial.

Suppose $m_0 + n_0 > 2$. Now we show the induction step. Since $m_0 + n_0 > 2$ and $\gcd(m_0, n_0) = 1$, $m_0 \neq n_0$. We may then assume that $m_0 < n_0$. Then $P_1 := (1, 0)$ is a special infinitely near point of 0 and the local equation of $\tilde{f}_1$ at P_1 in chart 2, is:

$$\tilde{f}_1(x_1, y_1) = \frac{f(x_1 y_1, y_1)}{y_1^m}, \quad \text{where } m = \text{mt}(f).$$

Let $f = f_{d_0}^{\text{w}} + f_{d_0+1}^{\text{w}} + \cdots$ with $f_{d_0}^{\text{w}} \neq 0$ be the (n_0, m_0)-weighted homogeneous decomposition of f. It is easy to see that $\tilde{f}_1 = (\tilde{f}_1)_{e_0}^{\text{w}} + (\tilde{f}_1)_{e_0+1}^{\text{w}} + \cdots$ is the $(n_0 - m_0, m_0)$-weighted homogeneous decomposition of $\tilde{f}_1$ with

$$e_0 = d_0 - m \cdot m_0 \quad \text{and} \quad (\tilde{f}_1)_{e_0+v}^{\text{w}} = \frac{f_{d_0+v}^{\text{w}}(x_1 y_1, y_1)}{y_1^m}, \quad \forall v \geq 0.$$

In particular,

$$(\tilde{f}_1)_{e_0}^{\text{w}} = monomial \times \prod_{i=1}^{s}(a_i x^{m_0} - b_i y^{n_0-m_0})^{r_i}.$$

We denote by E_1 the convex hull of the support of $(\tilde{f}_1)_{e_0}^{\text{w}}$. Clearly, E_1 is an edge of $\Gamma(\tilde{f}_1)$. Since

$$(\tilde{f}_1)_{E_1} = (\tilde{f}_1)_{e_0}^{\text{w}} = monomial \times \prod_{i=1}^{s}(a_i x^{m_0} - b_i y^{n_0-m_0})^{r_i}$$

and since $(\tilde{f}_1)_{e_0+1}^{\text{w}} = \frac{f_{d_0+1}^{\text{w}}(x_1 y_1, y_1)}{y_1^m}$, it follows that f is WHND along E iff $\tilde{f}_1$ is also WHND along E_1. Hence the induction step is proven by applying the induction hypothesis to $\tilde{f}_1$. □

The above lemma yields that $Q_{E,i} := (b_i : a_i)$, $i = 1, \ldots, s$, are determined by f_E and they are tangent directions of $\tilde{f}_n$. Then they are infinitely near points in the first neighbourhood of P_n. Consequently, they are infinitely near points in the $(n + 1)$-th neighbourhood of 0. To compute the multiplicity $m_{Q_{E,i}}$, we consider the local

equation of the strict transform $\tilde{f}_{n+1}$ of $\tilde{f}_n$ at $Q_{E,i} = (b_i : a_i)$ in chart 2:

$$\tilde{f}_{n+1}(u_1, v_1) = \frac{\tilde{f}_n((u_1 + \frac{b_i}{a_i})v_1, v_1)}{v_1^{e_0}}$$

$$= \frac{(\tilde{f}_n)_{e_0}^{\mathrm{w}}((u_1 + \frac{b_i}{a_i})v_1, v_1)}{v_1^{e_0}} + \frac{(\tilde{f}_n)_{e_0+1}^{\mathrm{w}}((u_1 + \frac{b_i}{a_i})v_1, v_1)}{v_1^{e_0}} + \cdots$$

$$= (\tilde{f}_n)_{e_0}^{\mathrm{w}}\left(u_1 + \frac{b_i}{a_i}, 1\right) + v_1 \cdot (\tilde{f}_n)_{e_0+1}^{\mathrm{w}}\left(u_1 + \frac{b_i}{a_i}, 1\right) + \cdots,$$

where $\tilde{f}_n = (\tilde{f}_n)_{e_0}^{\mathrm{w}} + (\tilde{f}_n)_{e_0+1}^{\mathrm{w}} + \cdots$ with $(\tilde{f}_n)_{e_0}^{\mathrm{w}} \neq 0$, is the $((1, 1)$-weighted) homogeneous decomposition of f. Since

$$(\tilde{f}_n)_{E_n}(u, v) = (\tilde{f}_n)_{e_0}^{\mathrm{w}}(u, v) = (a_i u - b_i v)^{r_i} \cdot g(u, v) \quad \text{with } g(b_i, a_i) \neq 0,$$

we get

$$\tilde{f}_{n+1}(u_1, v_1) = (a_i u_1)^{r_i} \cdot g\left(u_1 + \frac{b_i}{a_i}, 1\right) + v_1 \cdot (\tilde{f}_n)_{e_0+1}^{\mathrm{w}}\left(u_1 + \frac{b_i}{a_i}, 1\right) + \cdots,$$

with $g(u_1 + \frac{b_i}{a_i}, 1)$ a unit. In the following, this equality will be used to compare the multiplicity $\mathrm{mt}(\tilde{f}_{n+1})$ with 1.

Lemma 3.14 *With the above notations,*

(a) *if f is WHND along E, then $m_{Q_{E,i}} = 1$ for all i;*
(b) *if f is not WHND along E, then $m_{Q_{E,i}} > 1$ for some i.*

Proof (a) Since f is WHND along E, it follows from Lemma 3.13 that f_n is WHND along E_n, i.e. either $r_i = 1$ for all i or $(a_i u - b_i v)$ is not a factor of $(\tilde{f}_n)_{e_0+1}^{\mathrm{w}}$ for each $r_i > 1$. If $r_i = 1$, it is easy to see that $m_{Q_{E,i}} = \mathrm{mt}(\tilde{f}_{n+1}(u_1, v_1)) = 1$ for all i. If $r_i > 1$ and $(a_i u - b_i v)$ is not a factor of $(\tilde{f}_n)_{e_0+1}^{\mathrm{w}}$. Then $(\tilde{f}_n)_{e_0+1}^{\mathrm{w}}(b_i, a_i) \neq 0$. This implies that $(\tilde{f}_n)_{e_0+1}^{\mathrm{w}}(u_1 + \frac{b_i}{a_i}, 1)$ is a unit. Hence

$$m_{Q_{E,i}} = \mathrm{mt}(\tilde{f}_{n+1}(u_1, v_1)) = 1.$$

(b) Assume that f is not WHND along E. By Lemma 3.13, $\tilde{f}_n$ is not WHND along E_n, i.e. there exists an i such that $r_i > 1$ and $(a_i u - b_i v)$ is a factor of $(\tilde{f}_n)_{e_0+1}^{\mathrm{w}}$. Therefore $(\tilde{f}_n)_{e_0+1}^{\mathrm{w}}(u, v) = (a_i u - b_i v) \cdot h(u, v)$ and then

$$(\tilde{f}_n)_{e_0+1}^{\mathrm{w}}\left(u_1 + \frac{b_i}{a_i}, 1\right) = (a_i u_1) \cdot h\left(u_1 + \frac{b_i}{a_i}, 1\right).$$

Hence $m_{Q_{E,i}} = \mathrm{mt}(\tilde{f}_{n+1}(u_1, v_1)) > 1$. □

Lemma 3.15 *With the above notations, if Q is not special, then there exists an edge E of $\Gamma(f)$ such that $Q \to Q_{E,i}$ for some i.*

Proof We will prove the lemma by induction on $n(f, Q)$. First, since Q is not special, $n(f, Q) \geq 1$. If $n(f, Q) = 1$, then Q is a tangent direction of f and we can write $Q = (b : a)$, where $(ax - by)$ is a factor of the tangent cone f_m of f. Since Q is not special, f_m is not monomial. This implies that there exists an edge E of $\Gamma(f)$ such that $f_E = f_m$. We can write

$$f_E = f_m = monomial \times \prod_{i=1}^{s} (a_i x - b_i y)^{r_i}$$

with $(b : a) = (b_1 : a_1)$, consequently $Q = Q_{E,1}$.

Now we prove the induction step. Suppose that $n(f, Q) > 1$. Then $Q \to P$ for some infinitely near point P in the first neighbourhood of 0. If P is not special, then as above, $P = Q_{E,1}$ for some edge E of $\Gamma(f)$ and hence $Q \to Q_{E,1}$. If P is special, we may assume that $P = (0 : 1)$. Then the local equation of the strict transform $\tilde{f}$ of f at P in chart 2, is:

$$\tilde{f}(u, v) = \frac{f(uv, v)}{v^m}.$$

Since $n(\tilde{f}, Q) = n(f, Q) - 1$ and by induction hypothesis, there is an edge E' of $\Gamma(\tilde{f})$ such that

$$\tilde{f}_{E'} = monomial \times \prod_{i=1}^{s} (a_i u^{m'_0} - b_i v^{n'_0})^{r_i},$$

where $a_i, b_i \in K^*$, $(a_i : b_i)$ pairwise distinct; $m'_0, n'_0, r_i \in \mathbb{N}_{>0}$, $\gcd(m'_0, n'_0) = 1$ and $Q \to Q_{E',i}$ for some i. Let $m_0 = m'_0, n_0 = m'_0 + n'_0$ and let $f = f_d^w + f_{d+1}^w + \cdots$ be the (n_0, m_0)-weighted homogeneous decomposition of f. Then for each $l > d$, we have

$$\frac{f_l^w(uv, v)}{v^m} = \sum_{n_0\alpha + m_0\beta = l} c_{\alpha\beta} (uv)^\alpha v^{\beta - m}$$

$$= \sum_{n'_0\alpha + m'_0(\alpha + \beta - m) = l - mm'_0} c_{\alpha\beta} u^\alpha v^{\alpha + \beta - m}.$$

This implies that $\tilde{f} = \tilde{f}_e^w + \tilde{f}_{e+1}^w + \cdots$ is the (n'_0, m'_0)-weighted homogeneous decomposition of $\tilde{f}$, where $e = d - mm'_0$ and $\tilde{f}_{l-mm'_0}^w = \frac{f_l(uv,v)}{v^m}$. Note that $\tilde{f}_e^w = \tilde{f}_{E'}$. It is easy to see that

$$f_d^w(x, y) = y^m \tilde{f}_e^w\left(\frac{x}{y}, y\right) = monomial \times \prod_{i=1}^{s} (a_i x^{m_0} - b_i y^{n_0})^{r_i}.$$

Since E' is an edge of $\Gamma(\tilde{f}_{E'})$, $\tilde{f}_{E'}$ and then $f_d^w(x, y)$ are not monomials. By E we denote the convex hull of the support of f_d^w. Then E is an edge of $\Gamma(f)$ and $f_E = f_d^w$. Therefore $Q_{E,i} = Q_{E',i}$ and hence $Q \to Q_{E,i}$. $\qquad\square$

Proof of Theorem 3.12 ($\Longrightarrow$): Assume f is not WHNND, then f is not WHND along some edge E of $\Gamma(f)$. By Lemma 3.13, there is an infinitely near point $Q_{E,i}$ of 0, such that $m_{Q_{E,i}} > 1$. Clearly, $Q_{E,i}$ is not special. It then follows from Proposition 3.9 that

$$\delta(f) > \nu(f) = \delta_N(f).$$

($\Longleftarrow$): Assume now that f is WHNND. To show $\delta(f) = \delta_N(f)$, it suffices to show that there is no infinitely near point Q of 0 such that Q is not special and $m_Q > 1$. We argue by contradiction. Suppose that there is such an infinitely near point Q. By Lemma 3.15, there is an edge E of $\Gamma(f)$ such that $Q \to Q_{E,i}$ for some i, and then $m_Q \le m_{Q_{E,i}}$. Since f is WHND along E, it follows from Lemma 3.14 that $m_{Q_{E,i}} = 1$. Hence $m_Q \le m_{Q_{E,i}} = 1$, which is a contradiction. $\qquad\square$

If $\text{char}(K) = 0$ we have Milnor's famous formula $\mu(f) = 2\delta(f) - r(f) + 1$, where $r(f)$ is the number of branches of f. The formula is wrong in general if $\text{char}(K) > 0$ but still holds if f is NND by [3, Theorem 4.13]. Using the general inequality

$$\mu_N(f) = 2\delta_N(f) - r_N(f) + 1 \le 2\delta(f) - r(f) + 1 \le \mu(f)$$

from [3], then Theorem 2.13, Proposition 3.1 and Proposition 3.10 imply

Corollary 3.16 *Let* $f \in K[[x, y]]$ *be reduced. Then* f *is INND if and only if* f *is WNND and* $\mu(f) = 2\delta(f) - r(f) + 1$.

Remark 3.17

(1) The difference $\text{wvc}(f) := \mu(f) - 2\delta(f) + r(f) - 1$ counts the number of *wild vanishing cycles* of (the Milnor fiber) of f (cf. [3, 6, 12]), which vanishes if $\text{char}(K) = 0$ or if f is INND.
(2) $\text{wvc}(f)$ is computable for any given f. This follows since $\mu(f)$ is computable by a standard basis computation w.r.t. a local ordering (cf. [8]) and $\delta(f)$ and $r(f)$ are computable by computing a Hamburger-Noether expansion (cf. [5]). Both algorithms are implemented in SINGULAR (cf. [9]).

Example 3.18 Consider $f = x(x - y)^2 + y^7$ and $g = x(x - y)^2 + y^7 + x^6$ and $\text{char}(K) = 3$. Using SINGULAR we compute $\mu(f) = 8, \delta(f) = 5, r(f) = 3$ and $\mu(g) = 8, \delta(f) = 4, r(g) = 2$. We have $\text{wvc}(f) = 0, \text{wvc}(g) = 1, \Gamma(f) = \Gamma(g)$ and f is not INND. This shows

- INND is sufficient but not necessary for the absence of wild vanishing cycles,
- the Newton diagram can not distinguish between singularities which have wild vanishing cycles and those which have not.

Although we can compute the number of wild vanishing cycles, it seems hard to understand them. We like to pose the following

Problem Is there any "geometric" way to understand the wild vanishing cycles, distinguishing them from the ordinary vanishing cycles counted by $2\delta - r + 1$? Is there at least a "reasonable" characterization of those singularities without wild vanishing cycles?

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