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On geometric properties of the spaces $L^{p(x)}$

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Abstract We study the reflexivity, the uniform convexity, the Daugavet property and the Radon-Nikodym property of the generalized Lebesgue spaces $L^{p(x)}$.

Keywords Reflexivity · Uniform convexity · Daugavet property · Radon-Nikodym property · Generalized Lebesgue spaces · Variable exponent

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1 Introduction

In what follows, $(\Omega, \mathcal{S}, \mu)$ will be a measure space with a σ-finite complete measure μ. Let $\mathcal{M}(\Omega)$ be the set of all $\mathcal{S}$-measurable functions on Ω. Let $\mathcal{P}(\Omega)$ denote the family of all $p \in \mathcal{M}(\Omega)$ for which

$$1 \le p(x) \le \infty, \quad x \in \Omega.$$

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For a function $p \in \mathcal{P}(\Omega)$, we denote

$$p_{\max} := \operatorname*{esssup}_{x \in \Omega} p(x) \quad \text{and} \quad p_{\min} := \operatorname*{essinf}_{x \in \Omega} p(x).$$

For $x \in \Omega$, we denote the *conjugate function* p' by

$$p'(x) := \begin{cases} \frac{p(x)}{p(x)-1}, & \text{if } 1 < p(x) < \infty, \\ 1 & \text{if } p(x) = \infty, \\ \infty & \text{if } p(x) = 1. \end{cases}$$

The *generalized Lebesgue space* $L^{p(x)}(\Omega, \mathcal{S}, \mu)$ is the collection of all $\mathcal{S}$-measurable functions f on Ω for which there exists a $\lambda > 0$ such that

$$\int_{\Omega_0} \left(\frac{|f(x)|}{\lambda} \right)^{p(x)} d\mu(x) + \operatorname*{esssup}_{x \in \Omega_\infty} |f(x)| < \infty,$$

where

$$\Omega_0 := \{x \in \Omega : p(x) < \infty\} \quad \text{and} \quad \Omega_\infty := \{x \in \Omega : p(x) = \infty\}.$$

We also denote $\Omega_1 := \{x \in \Omega : p(x) = 1\}$.

The functional

$$\rho_p : f \mapsto \int_{\Omega_0} |f(x)|^{p(x)} \, d\mu(x) + \operatorname*{esssup}_{x \in \Omega_\infty} |f(x)|$$

is a *convex modular* on $\mathcal{M}(\Omega)$ (cf. H. Nakano [17] for details), hence the set $L^{p(x)}$ can be endowed with the *Luxemburg norm*

$$\|f\|_p := \inf\{\lambda > 0 : \rho_p(f/\lambda) \le 1\}.$$

(Where no confusion can occur, $L^{p(x)}(\Omega, \mathcal{S}, \mu)$ will be abbreviated to $L^{p(x)}$, $\|f\|_p$ by short to $\|f\|$, and the prefix "$\mathcal{S}$-" in the notion of measurability will be dropped.)

The space $L^{p(x)}$ with the Luxemburg norm is an example of a so-called *Banach function space* (for the precise definition see below, cf. also C. Bennett and R. Sharpley [2]).

The space $L^{p(x)}$ can be equipped also with another equivalent norm. If we denote

$$|||f|||_p := \sup_{\rho_{p'}(g) \le 1} \int_\Omega f g \, d\mu, \qquad f \in L^{p(x)},$$

then $|||.|||_p$ is an equivalent norm on $L^{p(x)}$. This was shown by O. Kováčik and J. Rákosník in their paper [13, Theorem 2.3] under the assumption that Ω is a measurable subset of the Euclidean space $\mathbb{R}^m$ and μ is Lebesgue measure. Since their proof works also in a more general setting for any σ-finite measure space, we state the next theorem without proof.

Theorem 1.1 (Kováčik–Rákosník) *We have*

$$L^{p(x)} = \{f : |||f|||_p < \infty\}$$

and

$$c_p^{-1} \|f\|_p \le |||f|||_p \le r_p \|f\|_p$$

for any $f \in L^{p(x)}$. *Here,*

$$c_p = \|\chi_{\Omega_0 \setminus \Omega_1}\|_\infty + \|\chi_{\Omega_1}\|_\infty + \|\chi_{\Omega_\infty}\|_\infty, \qquad r_p = c_p + \frac{1}{p_{\min}} - \frac{1}{p_{\max}},$$

and $\|\cdot\|_\infty$ *stands for the* L^∞*-norm.*

The spaces $L^{p(x)}$ appeared first in the literature as early as in 1931 in the article [20] by W. Orlicz. Their first systematic investigation was carried out in the 1950's by H. Nakano in [18], and continued later by J. Musiełak [17]. The theory of variable exponent spaces of functions defined on the real line was also developed by I. Tsenov [25], I. Sharapudinov [23] and V.V. Zhikov [28] and [29].

In the late 1980's these spaces were investigated in connection with certain specific applications in the continuum mechanics (see, for example, V.V. Zhikov [28]). Some of their fundamental properties were established in [13] by O. Kováčik and J. Rákosník. A deeper study of the norm in $L^{p(x)}$ can be found in [8] by D.E. Edmunds, J. Lang and A. Nekvinda. Yet other authors considered inequalities of Sobolev type and related questions.

Recently, these spaces have seen a true renaissance, caused by the discovery of M. Růžička ([22]) that they constitute a natural functional setting for the mathematical model of electrorheological fluids which involves a nonlinear system of partial differential equations with coefficients of variable rate of growth. As a natural consequence, attention of various authors has been attracted to this area of functional analysis, and many new deep results were obtained. On the other hand, plenty of important problems remain still open. Let us recall that one of the key differences between $L^{p(x)}$ and the classical Lebesgue spaces is that $L^{p(x)}$ is not, in general, invariant under translation (see Example 2.9 in [13] by O. Kováčik and J. Rákosník). This fact causes serious difficulties in the study of various aspects of the theory, in particular of the action of operators such as convolutions or the Hardy–Littlewood maximal operator on $L^{p(x)}$, of the density of smooth functions in Sobolev spaces built upon them, and so on.

Our paper is a contribution to the investigation of the spaces $L^{p(x)}$. We present a characterization of four of their basic geometric properties, namely reflexivity, uniform convexity, the Daugavet property and the Radon–Nikodym property. The corresponding criteria are in each case formulated in terms of necessary and sufficient conditions on the function $p(x)$.

The structure of the paper is as follows. We fix notation and collect some indispensable background material in Sect. 2. In subsequent sections we respectively characterize reflexivity and uniform convexity, the Daugavet property, and the Radon–Nikodym property of spaces $L^{p(x)}$.

2 Preliminaries

A set $A \in \mathcal{S}$ is called an *atom* of μ if $\mu(A) > 0$ and $B \in \mathcal{S}$, $B \subset A$ implies either $\mu(B) = 0$ or $\mu(A \setminus B) = 0$. We say that μ is *nonatomic* if there are no atoms for μ and that μ is (*purely*) *atomic* if every set $M \in \mathcal{S}$ with $\mu(M) > 0$ contains an atom. Since we suppose that μ is σ-finite, there exist unique measures μ_a and μ_c such that $\mu = \mu_a + \mu_c$ and such that μ_a is atomic and μ_c is nonatomic (cf., for example, R.A. Johnson [11, Corollary 2.6]).

In most of the theorems below we assume that the underlying measure is nonatomic. This gives us a possibility to obtain equivalent conditions in rather simple formulations.

Now, we fix some notation. Let X be a Banach space. We denote by B_X its closed unit ball $\{x \in X : \|x\| \leq 1\}$ and by S_X its unit sphere $\{x \in X : \|x\| = 1\}$. We denote by χ_A the characteristic function of a set A.

Let $X \subset \mathcal{M}(\Omega)$ be a vector space and $\|\cdot\|$ be a real function on $\mathcal{M}(\Omega)$ having the norm property. We say that X is a *Banach function space* if the following axioms are satisfied:

(i) $f \in X$ if and only if $\|f\| < \infty$,
(ii) $\|f\| = \||f|\|$ for any $f \in \mathcal{M}(\Omega)$,
(iii) $0 \leq f_n \nearrow f$ μ-a.e. implies $\|f_n\| \nearrow \|f\|$,
(iv) $\|\chi_E\| < \infty$ for any $E \in \mathcal{S}$ such that $\mu(E) < \infty$,
(v) for every $E \subset \Omega$ with $\mu(E) < \infty$, there exists a constant C_E such that

$$\int_E f(x)\, d\mu(x) \leq C_E \|f\| \quad \text{for any } f \in X.$$

Let X be a Banach function space and let $\|\cdot\|_X$ denote the norm $\|\cdot\|$ on X.

We say that $f \in X$ has an *absolutely continuous norm* if for every decreasing sequence $\{G_n\}$ of subsets of Ω satisfying $\mu(G_n) \to 0$ we have $\|f\chi_{G_n}\| \to 0$. Let X_a denote the family of all functions in X having absolutely continuous norm.

The set

$$X' = \left\{ f : \Omega \to \mathbb{R} : \int_\Omega |f(x)g(x)|\, d\mu(x) < \infty \text{ for all } g \in X \right\},$$

endowed with the norm

$$\|f\|_{X'} := \sup_{g \neq 0} \frac{\int_\Omega |f(x)g(x)|\, d\mu(x)}{\|g\|_X},$$

is called the *associate space* of X. Recall that X' is again a Banach function space and $(X')' = X$ (see C. Bennett and R. Sharpley [2, Theorem 2.2, p. 8] and [Theorem 2.7, p. 10]). Moreover, the Hölder inequality

$$\int_\Omega |f(x)g(x)|\, d\mu(x) \leq \|f\|_X\ \|g\|_{X'}$$

holds.

The main properties of spaces $L^{p(x)}$ are collected in the next theorem. The proof, for the case of a bounded measurable set Ω in the Euclidean space $\mathbb{R}^m$ and of the Lebesgue measure, can be found in several papers and forms nowadays a mathematical folklore. We present the proof that $L^{p(x)}$ is a Banach function space for the sake of completeness.

Theorem 2.1 *Every space $L^{p(x)}$ is a Banach function space. Its associate space is isomorphic to $L^{p'(x)}$. If $p_{\max} < \infty$, then the space $L^{p'(x)}$ is isomorphic to the (topological) dual of $L^{p(x)}$.*

Proof Obviously, the space $L^{p(x)}$ satisfies the properties (i), (ii) and (iv).

To show (iii), choose a sequence $0 \le f_n \nearrow f$ μ-a.e. Obviously, the sequence $\{\|f_n\|\}$ is nondecreasing. Suppose that $\|f\| < \infty$ and that, for some $\lambda > 0$, $\|f_n\| \nearrow \lambda < \|f\|$. Then

$$\begin{aligned}1 &\ge \int_{\Omega_0} \left|\frac{f_n}{\|f_n\|}\right|^p d\mu + \operatorname*{esssup}_{\Omega_\infty} \left|\frac{f_n}{\|f_n\|}\right| \\ &\ge \int_{\Omega_0} \left|\frac{f_n}{\lambda}\right|^p d\mu + \operatorname*{esssup}_{\Omega_\infty} \left|\frac{f_n}{\lambda}\right| \nearrow \int_{\Omega_0} \left|\frac{f}{\lambda}\right|^p d\mu + \operatorname*{esssup}_{\Omega_\infty} \left|\frac{f}{\lambda}\right| > 1,\end{aligned}$$

which is a contradiction. The case $\|f\| = \infty$ can be treated analogously.

To verify (v), fix $E \in \mathcal{S}$ with $\mu(E) < \infty$. Denote $M := \{x \in E \cap \Omega_0 : |f(x)| < 1\}$ and $N := \{x \in E \cap \Omega_0 : |f(x)| \ge 1\}$. It suffices to write

$$\begin{aligned}\frac{1}{\|f\|}\left(\int_{E\cap\Omega_0} |f| d\mu + \operatorname*{esssup}_{E\cap\Omega_\infty} |f|\right) &= \int_{E\cap\Omega_0} \frac{|f|}{\|f\|} d\mu + \operatorname*{esssup}_{E\cap\Omega_\infty} \frac{|f|}{\|f\|} \\ &\le \int_M \frac{|f|}{\|f\|} d\mu + \int_N \frac{|f|}{\|f\|} d\mu + \operatorname*{esssup}_{E\cap\Omega_\infty} \frac{|f|}{\|f\|} \\ &\le \mu(M) + \int_N \left|\frac{f}{\|f\|}\right|^p d\mu + \operatorname*{esssup}_{E\cap\Omega_\infty} \frac{|f|}{\|f\|} \\ &\le \mu(E) + 1.\end{aligned}$$

Now, in terms of Banach function spaces, Theorem 1.1 says that the space $L^{p'(x)}$ is isomorphic to the associated space of $L^{p(x)}$.

Let now $p_{\max} < \infty$. In the general case of a σ-finite measure, we can follow the proof of Theorem 2.6 by O. Kováčik and J. Rákosník [13] to obtain that $L^{p'(x)}$ is isomorphic to the (topological) dual of $L^{p(x)}$. □

The following Lemma can be found in O. Kováčik and J. Rákosník [13], p. 594, under more restrictive assumptions. We add a proof for the sake of completeness.

Lemma 2.2 *Let $f \in L^{p(x)}$, $f \not\equiv 0$. Assume that*

$$T = \operatorname*{esssup}_{\{x: f(x)\neq 0\}} p(x) < \infty.$$

Then

$$\int_{\Omega_0}\left(\frac{|f|}{\|f\|}\right)^p d\mu + \operatorname*{esssup}_{\Omega_\infty}\frac{|f|}{\|f\|} = 1.$$

Proof First, suppose that

$$\int_{\Omega_0}\left(\frac{|f|}{\|f\|}\right)^p d\mu + \operatorname*{esssup}_{\Omega_\infty}\frac{|f|}{\|f\|} =: K < 1.$$

Choose $0 < \lambda < \|f\|$ such that

$$\left(\frac{\|f\|}{\lambda}\right)^T K \leq 1.$$

Then

$$\begin{aligned}
&\int_{\Omega_0}\left(\frac{|f|}{\lambda}\right)^p d\mu + \operatorname*{esssup}_{\Omega_\infty}\frac{|f|}{\lambda}\\
&\quad\leq \left(\frac{\|f\|}{\lambda}\right)^T\left(\frac{\lambda}{\|f\|}\right)^T\left(\int_{\Omega_0}\left(\frac{|f|}{\lambda}\right)^p d\mu + \operatorname*{esssup}_{\Omega_\infty}\frac{|f|}{\lambda}\right)\\
&\quad\leq\left(\frac{\|f\|}{\lambda}\right)^T\left(\int_{\Omega_0}\left(\frac{\lambda}{\|f\|}\right)^p\left(\frac{|f|}{\lambda}\right)^p d\mu + \operatorname*{esssup}_{\Omega_\infty}\frac{|f|}{\|f\|}\right)\\
&\quad\leq\left(\frac{\|f\|}{\lambda}\right)^T\left(\int_{\Omega_0}\left(\frac{|f|}{\|f\|}\right)^p d\mu + \operatorname*{esssup}_{\Omega_\infty}\frac{|f|}{\|f\|}\right)\leq 1.
\end{aligned}$$

This is a contradiction with the definition of $\|f\|$. So we have $K \geq 1$. We will prove that $K \leq 1$. Choose a sequence $\lambda_n \searrow \|f\|$. Then

$$\int_{\Omega_0}\left(\frac{|f|}{\lambda_n}\right)^p d\mu + \operatorname*{esssup}_{\Omega_\infty}\frac{|f|}{\lambda_n} \leq 1,$$

hence

$$\int_{\Omega_0}\left(\frac{|f|}{\lambda_n}\right)^p d\mu \leq 1 - \operatorname*{esssup}_{\Omega_\infty}\frac{|f|}{\|f\|}$$

for all n. Using the fact that $\frac{|f|}{\lambda_n} \nearrow \frac{|f|}{\|f\|}$ and the Lebesgue Monotone Convergence Theorem, we obtain

$$\int_{\Omega_0}\left(\frac{|f|}{\|f\|}\right)^p d\mu + \operatorname*{esssup}_{\Omega_\infty}\frac{|f|}{\|f\|} \leq 1.$$

□

3 Reflexivity and uniform convexity of $L^{p(x)}$

Recall that a Banach space X is said to be *uniformly convex* if for every $\varepsilon \in (0,2]$ there exists a $\delta > 0$ such that

$$\left\| \frac{1}{2}(x+y) \right\| \le 1-\delta \quad \text{whenever } x, y \in S_X,\ \|x-y\| \ge \varepsilon.$$

Every uniformly convex space is reflexive (D.P. Milman [16] and B.J. Pettis [21]: for a short proof using the James characterization of reflexivity see, for example, M. Fabian et al. [9, Theorem 9.12]).

Lemma 3.1 *Assume that μ is nonatomic. Then the following statements are equivalent*:

(i) $L^{p(x)}$ *has absolutely continuous norm*,
(ii) $p_{\max} < \infty$.

Proof Assume that $p_{\max} = \infty$ and $\mu(\Omega_\infty) = 0$. Define

$$\Omega_n := \{x \in \Omega : n \le p(x) < n+1\}, \quad n \in \mathbb{N}.$$

Then there exists a subsequence of natural numbers $\{n_k\}_{k\in\mathbb{N}}$ such that $\mu(\Omega_{n_k}) > 0$. Let $c_k > 0$ be such that

$$\int_{\Omega_{n_k}} c_k^{p(x)}\, d\mu(x) = 1, \quad k \in \mathbb{N}.$$

We set

$$f(x) := \sum_{k=1}^{\infty} c_k \chi_{\Omega_{n_k}}(x), \quad x \in \Omega, \quad \text{and} \quad E_j := \bigcup_{k=j}^{\infty} \Omega_{n_k}.$$

Then $E_n \to \emptyset$. Now

$$\begin{aligned} \|f\|_p &= \inf\left\{ \lambda > 0 : \sum_{k=1}^{\infty} \int_{\Omega_{n_k}} \left(\frac{c_k}{\lambda}\right)^{p(x)} d\mu(x) \le 1 \right\} \\ &\le \inf\left\{ \lambda > 1 : \sum_{k=1}^{\infty} \left(\frac{1}{\lambda}\right)^{n_k} \le 1 \right\} \le 2, \end{aligned}$$

and so $f \in L^{p(x)}$.

To finish the proof of the implication (i)⇒(ii) it suffices to write

$$\|f\chi_{E_j}\|_p \ge \inf\left\{ \lambda > 0 : \int_{\Omega_{n_\ell}} \left(\frac{c_\ell}{\lambda}\right)^{p(x)} d\mu(x) \le 1 \right\} = 1.$$

If $\mu(\Omega_\infty) > 0$, choose $A \subset \Omega_\infty$ with $0 < \mu(A) < \infty$. Then $\|\chi_A\| > 0$ and, obviously, χ_A does not have an absolutely continuous norm.

To prove the converse implication, we assume that $p_{\max} < \infty$ and choose $f \in L^{p(x)}$ with $\|f\|_p = 1$ and $\varepsilon > 0$. Let $\{E_n\}$ be a sequence of sets such that $\mu(E_n) \searrow 0$. Choose $l \in \mathbb{N}$ with $\|f\chi_{E_l}\| \geq 1-\varepsilon$. Put $\varphi = f\chi_{\Omega\setminus E_l}$ and $\psi = f\chi_{E_l}$. By Lemma 2.2,

$$\int_\Omega \left|\frac{\varphi(x)}{\|\varphi\|}\right|^{p(x)} d\mu(x) = 1 \quad \text{and} \quad \int_\Omega \left|\frac{\psi(x)}{\|\psi\|}\right|^{p(x)} d\mu(x) = 1$$

and so $\|\varphi\|^{p_{\max}} \leq \int_\Omega |\varphi|^p d\mu$ and $\|\psi\|^{p_{\max}} \leq \int_\Omega |\psi|^p d\mu$. Moreover, $\int_\Omega |\varphi|^p d\mu + \int_\Omega |\psi|^p d\mu \leq 1$. So we have

$$\|\psi\|^{p_{\max}} \leq \int_\Omega |\psi|^p d\mu \leq 1 - \int_\Omega |\varphi|^p d\mu \leq 1 - \|\varphi\|^{p_{\max}} \leq 1 - (1-\varepsilon)^{p_{\max}},$$

and therefore $\|\psi\| \leq (1-(1-\varepsilon)^{p_{\max}})^{1/p_{\max}}$. □

Remark 3.2 Note that the assumption that μ is nonatomic was needed only for the implication (i)⇒(ii).

Theorem 3.3 *Assume that μ is nonatomic. Then the following statements are equivalent*:

(i) $L^{p(x)}$ *is reflexive*,
(ii) *the spaces $L^{p(x)}$ and $L^{p'(x)}$ have absolutely continuous norm*,
(iii) $L^{p(x)}$ *is uniformly convex*,
(iv) $1 < p_{\min} \leq p_{\max} < \infty$.

Proof As mentioned above, the space $L^{p(x)}$ is a Banach function space. Therefore, it is reflexive if and only if it has absolutely continuous norm and its associate space has absolutely continuous norm (see C. Bennett and R. Sharpley [2, Corollary 4.4, p. 23]). This shows (i)⇔(ii).

We shall show (ii)⇒(iv). By symmetry, it suffices to show that if p is essentially unbounded, then the space $L^{p(x)}$ does not have an absolutely continuous norm. But this immediately follows from Lemma 3.1.

Since every uniformly convex space is reflexive, we have (iii)⇒(i).

To round the proof off, it just remains to show (iv)⇒(iii). Fix $\varepsilon \in (0,1)$ and $u, v \in S_{L^{p(x)}}$. We aim to show that there exists a $\delta > 0$ such that

$$\left\|\frac{1}{2}(u+v)\right\| > 1-\delta \quad \text{implies} \quad \|u-v\| \leq \varepsilon.$$

To this end, define s, t by putting

$$s := \frac{1}{2}(u+v) \quad \text{and} \quad t := \frac{1}{2}(u-v),$$

set

$$S := \{x \in \Gamma : |t(x)| < \varepsilon|s(x)|\} \quad \text{and} \quad T := \{x \in \Gamma : |t(x)| \geq \varepsilon|s(x)|\},$$

and note that $\mu(\Omega \setminus \Gamma) = 0$ where

$$\Gamma = \{x \in \Omega : p_{\min} \leq p(x) \leq p_{\max}\}.$$

Then

$$\int_S |t(x)|^{p(x)}\, d\mu(x) \leq \int_S \varepsilon^{p(x)} |s(x)|^{p(x)}\, d\mu(x) \leq \int_\Omega \varepsilon^{p(x)} |s(x)|^{p(x)}\, d\mu(x)$$
$$\leq \varepsilon^{p_{\min}} \int_\Omega |s(x)|^{p(x)}\, d\mu(x) = \varepsilon^{p_{\min}}. \tag{3.1}$$

Since, for $t \in (1, \infty)$, the function $\lambda \mapsto |\lambda|^t$ is strictly convex on $\mathbb{R}$, we have

$$\frac{1}{2}\big(|\lambda + 1|^t + |\lambda - 1|^t\big) > |\lambda|^t, \quad \lambda \in \mathbb{R}. \tag{3.2}$$

Next we observe that the function

$$f : (p, \lambda) \mapsto \frac{1}{2}(|\lambda + 1|^p + |\lambda - 1|^p) - |\lambda|^p, \quad p \in (1, \infty), \lambda \in \mathbb{R},$$

is continuous and strictly positive on $(1, \infty) \times \mathbb{R}$. Hence, there exists an $\alpha > 0$ such that $f(p, \lambda) \geq \alpha$ for every $p \in [p_{\max}, p_{\min}]$ and $\lambda \in [-1/\varepsilon, 1/\varepsilon]$. Therefore

$$\frac{1}{2}(|\lambda + 1|^{p(x)} + |\lambda - 1|^{p(x)}) - |\lambda|^{p(x)} \geq \alpha \tag{3.3}$$

for every $x \in \Gamma$ and $\lambda \in [-1/\varepsilon, 1/\varepsilon]$. An appeal to (3.3) reveals that

$$\frac{1}{2}(|s(x) + t(x)|^{p(x)} + |s(x) - t(x)|^{p(x)}) \geq |s(x)|^{p(x)} + \alpha |t(x)|^{p(x)}, \quad x \in T,$$

while by (3.2),

$$\frac{1}{2}(|s(x) + t(x)|^{p(x)} + |s(x) - t(x)|^{p(x)}) \geq |s(x)|^{p(x)}, \quad x \in S,$$

(consider $\lambda = s(x)/t(x)$). Consequently,

$$1 = \int_\Omega \frac{1}{2}(|s(x) + t(x)|^{p(x)} + |s(x) - t(x)|^{p(x)})\, d\mu(x)$$
$$\geq \int_\Omega |s(x)|^{p(x)}\, d\mu(x) + \int_T \alpha |t(x)|^{p(x)}\, d\mu(x).$$

It follows that

$$\int_T |t(x)|^{p(x)}\, d\mu(x) < \varepsilon^{p_{\min}} \quad \text{provided} \quad \int_\Omega |s(x)|^{p(x)}\, d\mu(x) > 1 - \alpha\varepsilon^{p_{\min}}.$$

Put $\Delta = \alpha\varepsilon^{p_{\min}}$ and assume that

$$\left\| \frac{1}{2}(u + v) \right\| > 1 - \Delta.$$

Then $\int_\Omega |s(x)|^{p(x)}\, d\mu(x) > 1-\Delta$ and invoking (3.1), we obtain

$$\int_\Omega |t(x)|^{p(x)}\, d\mu(x) < 2\varepsilon^{p_{\min}}.$$

Hence,

$$\|u-v\| = \|2t\| \le 2(2\varepsilon^{p_{\min}})^{\frac{1}{p_{\max}}},$$

and this suffices to complete the proof. □

Remark 3.4 We provide relatively simple proofs of the equivalence of assertions (i)–(iv) although some separate implications of Theorem 3.3 are not new. The characterization of reflexivity was shown by O. Kováčik and J. Rákosník in [13] while uniform convexity of $L^{p(x)}$-spaces was treated by other authors, too; see for example the paper [10] by X. Fan and D. Zhao (in particular, Theorem 1.10) and the references therein. Note that the assumption of μ nonatomic was needed only in the proof of the implication (ii)⇒(iv).

The uniform convexity of classical reflexive Lebesgue L^p-spaces (with constant exponent) was shown by J.A. Clarkson in [4] who established this result using the so-called "Clarkson inequalities". The idea of the proof of (iv)⇒(iii) (still for the classical L^p-spaces) goes back to E.J. McShane [24], cf. also M. Fabian et al. [9, Theorem 9.3]. The present proof follows J. Malý's reasoning in the classical case. The idea of his proof can be found in J. Lukeš [15, Theorem 21.9].

4 The Radon–Nikodym and the Daugavet properties of $L^{p(x)}$

A Banach space X is said to have the *Radon–Nikodym property*, shortly RNP, if given a finite measure space $(\Omega, \mathcal{S}, \mu)$ and a vector measure $\nu : \mathcal{S} \to X$ of finite variation and absolutely continuous with respect to μ, then there exists a Bochner integrable function $g : \Omega \to X$ such that

$$\nu(E) = \int_E g\, d\mu \quad \text{for any } E \in \mathcal{S}.$$

Note that the Radon–Nikodym property is hereditary to closed subspaces (see J. Diestel and J.J. Uhl [6, Theorem III.3.2]).

The next lemma is a variation of the Dunford theorem (see [7] or J. Diestel and J.J. Uhl [6, Theorem III.1.6], and its proof follows the lines of the proof presented therein).

Lemma 4.1 *Let X be a Banach function space on (Ω, μ) with absolutely continuous norm and let $\Omega = \bigcup_{n=1}^\infty A_n$ be a union of measurable sets. If the subspaces $X_n := \{f \in X : f = 0 \text{ on } \Omega \setminus A_n\}$ have the Radon–Nikodym property, then X has the Radon–Nikodym property as well.*

Proof Let (Γ, Σ, κ) be a finite measure space and let $\nu : \Sigma \to X$ be a vector measure of finite variation which is absolutely continuous with respect to κ. Define projections $P_n : X \to X_n$ by $P_n(f) = f\chi_{A_n}$ and put $\nu_n = P_n(\nu)$. Then each ν_n is an X_n-valued vector measure of finite variation which is absolutely continuous with respect to κ and so (using the fact that each space X_n has the Radon–Nikodym property) there exists a Bochner integrable function $g_n : \Gamma \to X_n$ satisfying

$$\nu_n(E) = \int_\Gamma g_n \, d\kappa \quad \text{for each } E \in \Sigma.$$

Now, for $E \in \Sigma$ and $n \in \mathbb{N}$, we have

$$\int_E \left\| \sum_{k=1}^{n} g_k \right\|_X d\kappa \le |\nu|(E), \tag{4.1}$$

and so, for κ-almost all $\alpha \in \Gamma$, there exists a $g(\alpha) \in X$ such that $g(\alpha) = \sum_{k=1}^{\infty} g_k(\alpha)$. By (4.1) and Fatou's lemma, g is also Bochner integrable. Finally,

$$\nu(E) = \lim_{n\to\infty} \sum_{k=1}^{n} \nu_k(E) = \lim_{n\to\infty} \sum_{k=1}^{n} \int_E g_n \, d\kappa = \int_E g \, d\kappa.$$

□

Proposition 4.2 *Let X be a Banach function space with $X_a \neq X$. Then ℓ^∞ isomorphically embeds into X. In particular, X lacks the Radon–Nikodym property.*

Proof Choose $f \in X \setminus X_a$ with $\|f\|_X = 1$. There exist $\delta > 0$ and a decreasing sequence $\{A_n\}$ of subsets of Ω such that $\mu(A_n) \to 0$ and $1 \ge \|f\chi_{A_n}\|_X \searrow 2\delta$. Denote $E_n := \Omega \setminus A_n$, $n \in \mathbb{N}$. There exists an $i \in \mathbb{N}$ such that $\|f\chi_{E_i}\|_X > \delta$. Put $g_1 = f\chi_{E_i}$ and $f_1 = f\chi_{A_i}$. Note that f_1 and g_1 have disjoint supports and $1 \ge \|f_1\chi_{A_n}\|_X \searrow 2\delta$. So we can apply the preceding procedure to f_1 in place of f, and we obtain functions g_2 and f_2 with disjoints supports such that $1 \ge \|g_2\|_X \ge \delta$ and $1 \ge \|f_2\chi_{A_n}\|_X \searrow 2\delta$. Using it inductively we can obtain a sequence $\{g_n\}$ of functions with pairwise disjoint supports such that $1 \ge \|g_n\|_X \searrow \delta$ and $\|\sum g_n\|_X \le 1$. Set

$$Y := \left\{ g \in X : \text{ there exists } \{c_n\} \in \ \ell^\infty \text{ such that } g = \sum_{n=1}^{\infty} c_n g_n \right\}.$$

Choose $c = \{c_n\} \in \ell^\infty$ and $\varepsilon > 0$. There exists an $i \in \mathbb{N}$ such that $\|c\|_\infty \le |c_i| + \varepsilon$. Then

$$\begin{aligned}
\delta\|c\|_\infty \le \delta|c_i| + \delta\varepsilon \le \|c_i g_i\|_X + \delta\varepsilon \le \left\| \sum_{n=1}^{\infty} c_n g_n \right\|_X + \delta\varepsilon \\
\le \left\| \sum_{n=1}^{\infty} \|c\|_\infty g_n \right\|_X + \delta\varepsilon \le \|c\|_\infty \left\| \sum_{n=1}^{\infty} g_n \right\|_X + \delta\varepsilon \\
\le \|c\|_\infty + \delta\varepsilon.
\end{aligned}$$

Hence

$$\delta\|c\|_\infty \le \left\|\sum_{n=1}^{\infty} c_n g_n\right\|_X \le \|c\|_\infty.$$

The mapping $\{c_n\} \mapsto \sum_{n=1}^{\infty} c_n g_n$ is an isomorphic embedding of ℓ^∞ into X. Since the space ℓ^∞ fails to have the Radon–Nikodym property, and the RNP is hereditary to closed subspaces, we conclude that X cannot have the Radon–Nikodym property. □

Remark 4.3 By an unpublished result of H.P. Lotz, if X is a Banach lattice, then X^* has the RNP if and only if no sublattice of X^* is isomorphic to c_0 or $L^1([0,1])$; cf. P. Meyer–Nieberg [19, Theorem 5.4.14]. Concerning the $L^{p(x)}$-spaces, we get the following result.

Theorem 4.4 *Assume that μ is nonatomic. Then the following statements are equivalent*:

(i) $L^{p(x)}$ *has the Radon–Nikodym property*,
(ii) $\mu(\{x \in \Omega : p(x) = 1\}) = 0$ *and* $p_{\max} < \infty$,

Proof Assume that $\mu(\{x \in \Omega : p(x) = 1\}) > 0$. Then $L^{p(x)}$ contains the space $L^1(\Omega_1, \mathcal{S}|_{\Omega_1}, \mu|_{\Omega_1})$ which lacks the RNP. Since the RNP is hereditary to closed subspaces, $L^{p(x)}$ lacks it as well. In the case when $p_{\max} = \infty$, $L^{p(x)}$ does not have an absolutely continuous norm according to Lemma 3.1, and therefore by Lemma 4.2 lacks the RNP.

Assume now that (ii) holds. Decompose Ω into countably many pairwise disjoint sets $\{\Lambda_n\}_{n=1}^{\infty}$ with $\operatorname{essinf}\{p(t) : t \in \Lambda_n\} > 1$ for each $n \in \mathbb{N}$. To finish the proof, it is sufficient to use Theorem 3.3, Lemma 4.1 and the fact that reflexive spaces have the RNP. □

A Banach space X is said to have the *Daugavet property*, and is called a *Daugavet space*, if

$$\|\operatorname{Id} + T\| = 1 + \|T\|$$

for every compact operator $T : X \to X$. I.K. Daugavet (cf. [5]) has shown that the space $\mathcal{C}([0,1])$ possesses this property. Three years later, G.Ya. Lozanovskij proved in [14] that the space $L^1([0,1])$ also has the Daugavet property. The same result is contained in the paper by V.F. Babenko and S.A. Pichugov [1].

Later on, the Daugavet property of various Banach spaces was investigated. Particular attention was paid to function spaces. P. Chauveheid has shown in [3] that the space $\mathcal{C}(K)$ has the Daugavet property if and only if the compact space K has no isolated points and that the space $L^1(\mu)$ is Daugavet if μ is a nonatomic measure. In this case the space $L^\infty(\mu)$ is a Daugavet space. The same results can be found also in a paper by H. Kamowitz [12]. Since Daugavet spaces are not reflexive (cf. Corollary 2.5 of [26] by D. Werner), the spaces $L^p(\mu)$ for $p \in (1,\infty)$ lack the Daugavet property.

Moreover, miscellaneous conditions characterizing Daugavet spaces were established; see, for example, the survey paper of D. Werner [26] where the following geometric characterization of Daugavet spaces can be traced. For $\varepsilon > 0$ and $f \in S_{X^*}$ define a slice $S(f, \varepsilon) := \{x \in B_X : f(x) \geq 1 - \varepsilon\}$.

Theorem 4.5 *Let X be a Banach space. Then the following two statements are equivalent*:

(i) *X has the Daugavet property,*
(ii) *for every $\varepsilon > 0$, $f \in S_{X^*}$ and $x \in S_X$, there exists $y \in S(f, \varepsilon)$ such that $\|x + y\| \geq 2 - \varepsilon$.*

We will also need the following lemma.

Lemma 4.6 *Assume that $p < \infty$ μ-a.e. Let $f, g \in L^{p(x)}$, $\varepsilon > 0$ and $K > 0$. Then there exists $T > 1$ such that*

$$\|f + g\| \leq \max(\|f\|, \|g\|) + \varepsilon$$

whenever $\|f\| \leq K$, $\|g\| \leq K$, g vanishes outside the set $\{x \in \Omega : p(x) \geq T\}$ and

$$\{x \in \Omega : g(x) \neq 0\} \cap \{x \in \Omega : f(x) \neq 0\} = \emptyset.$$

Proof Denote $\beta := \max(\|f\|, \|g\|)$ and pick $T > 1$ for the moment arbitrarily. Then

$$\begin{aligned}
&\int_\Omega \frac{|f(x) + g(x)|^{p(x)}}{(\beta + \varepsilon)^{p(x)}} \, d\mu(x) \\
&\leq \int_\Omega \left(\frac{|f(x)|}{(\beta + \varepsilon)} \right)^{p(x)} d\mu(x) + \int_\Omega \left(\frac{|g(x)|}{(\beta + \varepsilon)} \right)^{p(x)} d\mu(x) \\
&= \int_\Omega \left(\frac{|f(x)|}{\|f\|} \cdot \frac{\|f\|}{(\beta + \varepsilon)} \right)^{p(x)} d\mu(x) + \int_\Omega \left(\frac{|g(x)|}{\|g\|} \cdot \frac{\|g\|}{(\beta + \varepsilon)} \right)^{p(x)} d\mu(x) \\
&\leq \frac{\|f\|}{\beta + \varepsilon} \int_\Omega \left(\frac{|f(x)|}{\|f\|} \right)^{p(x)} d\mu(x) + \left(\frac{\|g\|}{\beta + \varepsilon} \right)^T \int_\Omega \left(\frac{|g(x)|}{\|g\|} \right)^{p(x)} d\mu(x) \\
&\leq \frac{\|f\|}{\|f\| + \varepsilon} + \left(\frac{\|g\|}{\|g\| + \varepsilon} \right)^T \\
&\leq \frac{1}{1 + \varepsilon K^{-1}} + \frac{1}{(1 + \varepsilon K^{-1})^T}.
\end{aligned}$$

Choose now $T > 1$ in such a way that

$$\frac{1}{1 + \varepsilon K^{-1}} + \frac{1}{(1 + \varepsilon K^{-1})^T} \leq 1.$$

Then,

$$\|f + g\| \leq \beta + \varepsilon = \max(\|f\|, \|g\|) + \varepsilon. \qquad \square$$

Theorem 4.7 *Assume that μ is a nonatomic measure on Ω. The following statements are equivalent*:

(i) $L^{p(x)}$ *has the Daugavet property*,
(ii) $\mu(\{x \in \Omega : 1 < p(x) < \infty\}) = 0$.

Proof For the proof of (i) $\Rightarrow$(ii), assume that $L^{p(x)}$ is a Daugavet space and that there exists an interval $[a,b] \subset (1,\infty)$ such that $\mu(p^{-1}([a,b])) > 0$. Denote $A := p^{-1}([a,b])$, $q := p|_A$,

$$Y := \{g \in L^{p(x)} : g = 0 \text{ on } \Omega \setminus A\} \quad \text{and} \quad X := L^{q(x)}(A, \mathcal{S}|_A, \mu|_A).$$

Then Y is a closed subspace of $L^{p(x)}$ isomorphic to X. By Theorem 3.3, X is reflexive, and consequently, so is Y.

Hence, Y does not possess the Daugavet property. According to Theorem 4.5, there exist $0 < \varepsilon < 1$, $F \in S_{Y^*}$ and $f \in S_Y$ such that, for every $g \in B_Y$ satisfying $F(g) \geq 1 - \varepsilon$, we have $\|f + g\|_Y < 2 - \varepsilon$. Let P_A be the projection of X onto Y defined as

$$P_A : \varphi \mapsto \varphi\chi_A, \quad \varphi \in X.$$

If $G := F \circ P_A$ (the composition of operators), then $G \in X^*$ and $\|G\| = \|F\| = 1$. Using Lemma 4.6, there exists $T > 1$ such that for every $h \in B_X$

$$\|f + P_A(h) + P_B(h)\| \leq \max(\|f + P_A(h)\|, \|P_B(h)\|) + \frac{\varepsilon}{3}$$

where

$$B := p^{-1}([T,\infty)) \quad \text{and} \quad P_B : \varphi \mapsto \varphi\chi_B, \quad \varphi \in X.$$

Set further

$$C := \Omega_0 \setminus (A \cup B) \quad \text{and} \quad P_C : \varphi \mapsto \varphi\chi_C, \quad \varphi \in X$$

and

$$P_\infty : \varphi \mapsto \varphi\chi_{\Omega_\infty}, \quad \varphi \in X.$$

We may assume that $T > \beta$. For any $\varphi \in X$,

$$\varphi = P_A(\varphi) + P_B(\varphi) + P_C(\varphi) + P_\infty(\varphi) \quad \text{on } \Omega.$$

Since our aim is to show that X does not possess the Daugavet property, it suffices to find $\gamma > 0$ in such a way that for every $s \in B_X$ satisfying $G(s) \geq 1 - \gamma$ we have $\|f + s\| \leq 2 - \frac{\varepsilon}{3}$.

Fix now $\omega > 0$ and $s \in B_X$ such that $G(s) \geq 1 - \omega$. Then

$$1 \geq \|s\| \geq \|P_A(s) + P_C(s) + P_\infty(s)\| \geq \|P_A(s)\| \geq 1 - \omega.$$

Hence

$$\int_\Omega |P_C(s)|^p \, d\mu + \int_\Omega |P_A(s)|^p \, d\mu + \operatorname*{esssup}_{\Omega_\infty} |P_\infty(s)| \leq 1$$

and

$$\int_{\Omega}\left|\frac{P_A(s)}{1-\omega}\right|^p d\mu \geq 1.$$

It follows that

$$\int_{\Omega}|P_C(s)|^p\,d\mu + \operatorname*{esssup}_{\Omega_\infty}|P_\infty(s)| \leq 1 - \int_{\Omega}|P_A(s)|^p\,d\mu \leq 1-(1-\omega)^\beta.$$

Analogously, using Lemma 2.2,

$$\begin{aligned}1 &= \int_{\Omega}\left|\frac{P_C(s)}{\|P_C(s)+P_\infty(s)\|}\right|^p d\mu + \operatorname*{esssup}_{\Omega_\infty}\frac{|P_\infty(s)|}{\|P_C(s)+P_\infty(s)\|}\\ &\leq \frac{1}{\|P_C(s)+P_\infty(s)\|^T}\left(\int_{\Omega}|P_C(s)|^p\,d\mu + \operatorname*{esssup}_{\Omega_\infty}|P_\infty(s)|\right).\end{aligned}$$

Summing up, we get

$$\|P_C(s)+P_\infty(s)\|^T \leq 1-(1-\omega)^\beta.$$

Choose now $\gamma > 0$ in such a way that

$$\big(1-(1-\gamma)^\beta\big)^{1/T} \leq \frac{\varepsilon}{3}.$$

Finally,

$$\begin{aligned}\|f+s\| &\leq \|f+P_A(s)+P_B(s)\| + \|P_C(s)+P_\infty(s)\|\\ &\leq \max(\|f+P_A(s)\|, \|P_B(s)\|) + \frac{\varepsilon}{3}+\frac{\varepsilon}{3}\\ &\leq \max(2-\varepsilon, 1) + \frac{\varepsilon}{3}+\frac{\varepsilon}{3} \leq 2-\frac{\varepsilon}{3}.\end{aligned}$$

For the implication (ii) ⇒(i) it is sufficient to use the fact that in this case the space $L^{p(x)}$ is isometrically isomorphic to the space $L^1(\Omega_0) \oplus_1 L^\infty(\Omega_\infty)$ (the direct sum of $L^1(\Omega_0)$ and $L^\infty(\Omega_\infty)$ equipped with the ℓ^1-norm) and the fact that an ℓ^1-sum of Daugavet spaces is Daugavet space as well (see P. Wojtaszczyk [27, Theorem 1]). □

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